ON CHARACTERIZING CLASSES OF FUNCTIONS IN TERMS OF ASSOCIATED SETS

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Let κ be a class of real valued functions defined on an interval which need not be bounded. The class κ is said to be characterized in terms of associated sets if there exists a family of sets of real numbers P such that $f \epsilon \kappa$ if and only if for every real number α the sets $E^{\alpha} \equiv \{x: f(x) < \alpha\}$ and $E_{\alpha} \equiv \{x: f(x) > \alpha\}$ are members of P. Many classes of functions have been characterized in terms of associated sets. The chart below summarizes a few such characterizations.

f is	if and only if for all real α
continuous	E^{α} and E_{α} are open
measurable	E^{α} and E_{α} are measurable
in Baire class ξ (ξ a countable ordinal)	E^{α} and E_{α} are in additive Borel class ξ if ξ is finite, $\xi+1$ if ξ is infinite
approximately continuous	E^{α} (E_{α}) is of type F_{σ}, and each point of such a set is a point of density of the set
a Darboux function in Baire class 1	E^{α} (E_{α}) is of type F_{σ} and bilaterally c-dense-in-itself

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In his study of derivatives, Zahorski [8], among other things, posed the problem of characterizing the class of derivatives in terms of associated sets. He was able to show that the class of bounded derivatives does not admit of such a characterization. The purpose of this note is to give a simple condition on a class κ which is necessary for κ to admit a characterization in terms of associated sets. We use this result to prove that certain classes, including the class of derivatives, do not admit of such characterization.

THEOREM. Let κ be a class of functions characterized in terms of associated sets. Then for each $f \in \kappa$ and each homeomorphism h of the real line R onto itself, the function $h \circ f$ is in κ.

Proof. Let P be the family of associated sets. Let α be a real number, $f \in \kappa$, and h a homeomorpnism. For definiteness, suppose h is increasing. Then the set $\{x: (h \circ f)(x) < \alpha\} = \{x: f(x) < h^{-1}(\alpha)\}$. Since $f \in \kappa$, this set is a member of P. The same is true of the set $\{x: (h \circ f)(x) > \alpha\}$. Thus, the associated sets of the function $h \circ f$ are all members of P. Therefore $h \circ f \in \kappa$.

COROLLARY 1. None of the classes listed below can be characterized in terms of associated sets.

(i) the class of finite derivatives;

(ii) the class of derivatives (possibly infinite) of continuous functions;

(iii) the class of finite approximate derivatives;

(iv) the class of approximate derivatives (possibly infinite), of approximately continuous functions.

Proof. (i) and (ii). The only homeomorphisms h with the property that $h \circ f$ is a derivative for every derivative f are those homeomorphisms of the form $h(x) = ax + b$ [2; p.89].

(iii) and (iv). According to the theorem of Khintchine [3] and [4], if F is an increasing approximately differentiable function, then f is, in fact, differentiable. Since an approximately differentiable function whose approximate derivative is

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everywhere positive must be increasing, the approximate derivative of such a function must actually be the ordinary derivative (for (iv) we use here a theorem of Tolstoff's [5]).

Let, now, f be a positive bounded derivative which is not approximately continuous. Then [7] f^2 is not a derivative, but f^2 is positive. If f^2 were the approximate derivative of some function F then we would be able to infer from the above remarks that f^2 would, in fact, be the derivative of F. Thus, if h is the homeomorphism defined by

$$h(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases},$$

then $h \circ f$ is not an approximate derivative.

Remark. Certain necessary conditions for a function to belong to classes (i), (ii), and (iii) above, in terms of associated sets, can be found in Weil [6] and Zahorski [8].

COROLLARY 2. None of the classes below, each defined on a compact interval $[a, b]$ can be characterized in terms of associated sets.

(i) the class of functions of bounded variation;

(ii) the class of functions satisfying Lusin's condition (N);

(iii) the class of Lebesgue integrals of Lebesgue integrable functions;

(iv) the class of Riemann integrals of Riemann integrable functions;

(v) the class of Denjoy integrals of Denjoy integrable functions;

(vi) the class of Denjoy-Khintchine integrals of Denjoy-Khintchine integrable functions.

Here, the statement that F is an integral means there exists a function f such that $F(x) = \int_a^x f(t)dt$ for all $x \in [a, b]$.

Proof. (i). This follows from the fact that the function f

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given by $f(x) = (x-a)^2 \left| \sin \frac{1}{x-a} \right|$, $f(a) = 0$, is of bounded variation on $[a, b]$, but the function $g(x) = \sqrt{f(x)}$, is not.

(ii) through (vi). A function in any of these classes satisfies Lusin's condition (N), and the identity function is in each of these classes. On the other hand, there is a homeomorphism h, which does not satisfy Lusin's condition (N) on any interval. The composition of h with the identity function on $[a, b]$ is, therefore, not in any of the classes, so none of these classes can be characterized in terms of associated sets.

In conclusion, we mention that the converse to the theorem is not valid. To see this, let κ be the class of Darboux functions defined on the real line R. It is clear that if h is any homeomorphism and if $f \in \kappa$, then $h \circ f \in \kappa$. Now, if $f \in \kappa$ and α is a real number, then each set of the form $E^{\alpha} \equiv \{x: f(x) < \alpha\}$ and $E_{\alpha} \equiv \{x: f(x) > \alpha\}$ is bilaterally c-dense-in-itself. This means that if $x \in E^{\alpha}$, for example, and $\delta > 0$, then each of the sets $E^{\alpha} \cap (x, x+\delta)$ and $E^{\alpha} \cap (x-\delta, x)$ has the power of the continuium. On the other hand, for every real number α and every set E which is bilaterally c-dense-in-itself, there exists an $f \in \kappa$ such that $\{x: f(x) < \alpha\} = E^{\alpha}$. (We need only define f to be any function which takes the value α on the complement of E, and on each relative interval of E, f takes on all values in the interval $(-\infty, \alpha)$.) The analogous assertion for the set E_{α} is likewise valid.

It follows that if there is a family of sets P which characterizes κ in terms of associated sets, then P must be the family each of whose members is bilaterally c-dense-in-itself. But there exist functions which are not in κ yet each of whose associated sets is bilaterally c-dense-in-itself. For example, let A and B be disjoint sets such that $R = A \cup B$ and A and B are each c-dense in R (Lemma 4.1 of [1]). Then the function f given by

$$f(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in B \end{cases}$$

is not in κ, while each associated set is c-dense-in-itself.

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