THE ABSTRACT GROUP $G^{3,7,16}$

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1. Introduction

A representation of degree 3 is used in § **4** to establish the necessary and sufficient condition

$$\cos\frac{2\pi}{p}+\cos\frac{2\pi}{q}+\cos\frac{2\pi}{r}<0$$

for the finiteness of the group $(3, p \mid q, r)$ defined by

$$A^3 = D^p = (AD)^q = (A^{-1}D)^r = E. \qquad \text{(1.1)}$$

In § **7**, this condition is seen to be related to a conjecture (**8**, p. 267) that the inequality

$$\cos\frac{4\pi}{m}+\cos\frac{4\pi}{n}<\tfrac{1}{2}$$

is a sufficient condition for the finiteness of the group $G^{3,m,n}$ defined by

$$B^m = C^n = (BC)^2 = (B^3C^2)^2 = (B^2C^3)^2 = E. \qquad \text{(1.2)}$$

In § **9** we summarise the work of Leech and Mennicke (**14**), who established the truth of this conjecture by finding the order of $G^{3,7,16}$. In § **11** we consider a topological application, increasing the number of known members in certain families of regular maps. Finally, in § **12**, the group $G^{3,7,16}$ is seen to be derivable in a rather natural manner from the direct product of two Abelian groups of order 8 and type (1, 1, 1).

This paper grew out of a lecture given at the American Mathematical Society's Summer Institute on the Theory of Groups at Pasadena (August, 1960).

2. The Groups $(l, p \mid q, r)$ and $G^{l,m,n}$

The group

$$A^l = D^p = (AD)^q = (A^{-1}D)^r = E \qquad \text{(2.1)}$$

is conveniently denoted by

$$(l, p \mid q, r).$$

Clearly, we can interchange l and p, or q and r, without altering the group. Moreover, when $l = 3$, the three periods p, q, r can be permuted in all the six possible ways (**3**, p. 78).

Let us extend $(l, p \mid q, r)$ by an involutory element R_3 that transforms each

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of the generators into its inverse, so that

$$R_3^2 = (AR_3)^2 = (R_3D)^2 = E.$$

Defining $R_2 = AR_3$, $R_1 = R_3D$, we obtain the group

$$R_1^2 = R_2^2 = R_3^2 = (R_2R_3)^l = (R_3R_1)^p = (R_1R_2)^q = (R_1R_3R_2R_3)^r = E, \quad\text{......(2.2)}$$

which is thus seen to contain $(l, p \mid q, r)$ as a subgroup of index 2, generated by R_2R_3 and R_3R_1.

When $l = p$, we can extend (2.2) by an involutory element T that interchanges R_1 and R_2 (transforming each into the other) and commutes with R_3, so that

$$T^2 = E, \quad R_1 = TR_2T, \quad R_3T = TR_3.$$

The extension is

$$T^2 = R_2^2 = R_3^2 = (R_2R_3)^l = (R_3T)^2 = (TR_2)^{2q} = (TR_2R_3)^{2r} = E.$$

By writing

$$T = AB, \qquad R_2 = ABC, \qquad R_3 = BC,$$

$$A = R_2R_3, \qquad B = R_3R_2T, \qquad C = TR_2,$$

we can express it in the more agreeable form

$$A^l = B^m = C^n = (AB)^2 = (BC)^2 = (CA)^2 = (ABC)^2 = E, \quad\text{.........(2.3)}$$

where $m = 2r$ and $n = 2q$.

The group (2.3) is denoted by $G^{l,\,m,\,n}$. When $l = 3$, it has the more concise presentation (1.2) (**3**, p. 113). Our conclusion is that $G^{l,\,2q,\,2r}$ contains $(l, l \mid q, r)$ as a subgroup of index 4. On the other hand, some of the most remarkable groups $G^{l,\,m,\,n}$ have *odd* values for m or n:

$$G^{3,\,5,\,5} \cong LF(2, 2^2) \cong LF(2, 5) \qquad G^{3,\,3,\,4} \cong PGL(2, 3),$$

$$G^{3,\,7,\,9} \cong LF(2, 2^3), \qquad G^{3,\,5,\,10} \cong PGL(2, 5),$$

$$G^{5,\,5,\,5} \cong LF(2, 11), \qquad G^{3,\,7,\,12} \cong G^{3,\,7,\,14} \cong PGL(2, 13),$$

$$G^{3,\,9,\,9} \cong LF(2, 19), \qquad G^{4,\,5,\,9} \cong PGL(2, 19),$$

$$G^{3,\,7,\,15} \cong LF(2, 29), \qquad G^{3,\,8,\,11} \cong PGL(2, 23)$$

(**3**, p. 148). Because l, m, n enter symmetrically, we usually write $G^{l,\,m,\,n}$ with $l \leqq m \leqq n$.

The relations (2.3) are easily seen to imply $A^2B^2C^2 = E$, so that, when n is even, A and B generate a subgroup of index 2:

$$A^l = B^m = (AB)^2 = (A^2B^2)^{\frac{1}{2}n} = E. \quad\text{.....................(2.4)}$$

In fact, since $(BCA)^{-1}A.BCA = A^{-1}(BC)^{-2} = A^{-1}$ and

$$BCA.B(BCA)^{-1} = (CA)^{-2}B^{-1} = B^{-1},$$

we may describe $G^{l,\,m,\,n}$ (n even) as being derived from (2.4) by adjoining an involutory element BCA that transforms the generators A and B into their inverses.

3. Unitary Reflections of Period Two

In complex affine 3-space, consider two covertical trihedra formed by triads of lines

$$e_1e_2e_3, \quad e^1e^2e^3,$$

such that the three planes e_1e^1, e_2e^2, e_3e^3, all pass through one line.

The plane at infinity meets the trihedra in two perspective triangles which, according to von Staudt's theorem (**7**, p. 74), are related by a unique projective polarity. The same kind of analytic proof (*ibid.*, pp. 201, 218) shows that these triangles are also related by a unique antiprojective polarity, which allows us to use the trihedra to set up dual bases for affine coordinates with a generalised unitary metric.

The line e_1 and plane e^2e^3 determine an affine reflection (of period 2) such that, if P' is the image of P, the line PP' is parallel to e_1 and the segment PP' is bisected by e^2e^3 (**9**, p. 225). The generalised unitary metric makes this line and plane perpendicular; accordingly we call the affine reflection a *unitary reflection*, namely the reflection in the plane $x_1 = 0$.

For a group generated by three unitary reflections R_1, R_2, R_3, we may conveniently take the mirrors to be $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, so that R_j reverses the sign of x_j and leaves invariant all the x^k except x^j, which it transforms into (say)

$$\sum c_k x^k.$$

(Of course, c_k depends on the " fixed " number j as well as on k.)

If the unitary metric is given by the Hermitian form

$$\sum\sum a_{jk}x^j\bar{x}^k \quad (a_{jk} = \bar{a}_{kj}),$$

this reflection R_j transforms

$$x_j = \sum a_{jk}\bar{x}^k$$

into

$$-x_j = a_{jj}\overline{\sum c_k x^k} + \sum_{k \neq j} a_{jk}\bar{x}^k.$$

By addition,

$$0 = a_{jj}(\bar{x}^j + \sum \bar{c}_k\bar{x}^k) + 2\sum_{k \neq j} a_{jk}\bar{x}^k.$$

Since this holds for all points (x), the coefficients of $\bar{x}^j$ and $\bar{x}^k$ must each vanish:

$$a_{jj}(1+\bar{c}_j) = 0, \quad a_{jj}\bar{c}_k + 2a_{jk} = 0 \quad (k \neq j).$$

But $a_{jj} \neq 0$ (since otherwise every $a_{jk} = 0$); hence

$$1+\bar{c}_j = 0, \quad \bar{c}_k = -2a_{jk}/a_{jj},$$

that is,

$$c_j = -1, \quad c_k = -2a_{kj}/a_{jj} \quad (k \neq j).$$

Thus R_j leaves invariant every x^k except x^j, which it transforms into

$$\sum c_k x^k = x^j - 2\sum_{k=1}^{n} a_{kj}x^k/a_{jj}$$
$$= x^j - 2\bar{x}_j/a_{jj}$$

(cf. **8**, p. 245, where j and k were interchanged).

If the group is irreducible, the invariant form is unique (apart from the obvious possibility of multiplying by a constant). If the group is finite (as well as irreducible), the unique form is definite, say positive definite (**1**, p. 253). In this case $a_{jj}>0$. By writing $a_{jj}^{-\frac{1}{2}}x^j$ for x^j, we may suppose without essential loss of generality that $a_{jj} = 1$ and

$$c_k = -2a_{kj} \quad (k \neq j).$$

4. Automorphs of a Ternary Hermitian Form

In the case of the form

$$x^1\bar{x}^1+x^2\bar{x}^2+x^3\bar{x}^3-\tfrac{1}{2}(ax^2\bar{x}^3+\bar{a}x^3\bar{x}^2+bx^3\bar{x}^1+\bar{b}x^1\bar{x}^3+cx^1\bar{x}^2+\bar{c}x^2\bar{x}^1) \quad \ldots(4.1)$$

we have

$$a_{jj}=1, \quad a_{23} = -\tfrac{1}{2}a, \quad a_{31} = -\tfrac{1}{2}b, \quad a_{12} = -\tfrac{1}{2}c.$$

The three reflections, expressed as matrices (ready for multiplication from left to right), are

$$R_1 = \begin{pmatrix} -1 & 0 & 0 \\ \bar{c} & 1 & 0 \\ b & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & c & 0 \\ 0 & -1 & 0 \\ 0 & \bar{a} & 1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & \bar{b} \\ 0 & 1 & a \\ 0 & 0 & -1 \end{pmatrix},$$

so that

$$R_2R_3 = \begin{pmatrix} 1 & c & ca+\bar{b} \\ 0 & -1 & -a \\ 0 & \bar{a} & a\bar{a}-1 \end{pmatrix}, \quad R_3R_1 = \begin{pmatrix} b\bar{b}-1 & 0 & \bar{b} \\ ab+\bar{c} & 1 & a \\ -b & 0 & -1 \end{pmatrix}, \quad R_1R_2 = \begin{pmatrix} -1 & -c & 0 \\ \bar{c} & c\bar{c}-1 & 0 \\ b & bc+\bar{a} & 1 \end{pmatrix},$$

$$R_1R_3R_2R_3 = \begin{pmatrix} -1 & -(\overline{ab}+c) & -a(\overline{ab}+c) \\ \bar{c} & \overline{abc}+a\bar{a}+c\bar{c}-1 & a(\overline{abc}+a\bar{a}+c\bar{c}-2) \\ b & \bar{a}(b\bar{b}-1)+bc & a\bar{a}(b\bar{b}-1)+abc+1 \end{pmatrix}.$$

The characteristic equations for these four products reduce (after removal of the trivial factor $\lambda-1$) to

$$(\lambda+1)^2-a\bar{a}\lambda = 0, \quad (\lambda+1)^2-b\bar{b}\lambda = 0, \quad (\lambda+1)^2-c\bar{c}\lambda = 0,$$

$$(\lambda+1)^2-(ab+\bar{c})(\overline{ab}+c)\lambda = 0.$$

Comparing them with the characteristic equation

$$(\lambda+1)^2-\left(2\cos\frac{\pi}{n}\right)^2\lambda = 0$$

for an ordinary rotation of period n, we see that the reflections satisfy the relations (2.2) if and only if

$$a\bar{a} = 4\cos^2\frac{\pi}{l}, \quad b\bar{b} = 4\cos^2\frac{\pi}{p}, \quad c\bar{c} = 4\cos^2\frac{\pi}{q},$$

$$(ab+\bar{c})(\overline{ab}+c) = 4\cos^2\frac{\pi}{r}. \quad \ldots\ldots(4.2)$$

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If l, p, q, r are all greater than 2, we can solve these equations for a, b, c. The form (4.1), having determinant

$$\frac{1}{8}\begin{vmatrix} 2 & -c & -\bar{b} \\ -\bar{c} & 2 & -a \\ -b & -\bar{a} & 2 \end{vmatrix} = \tfrac{1}{8}(8-2a\bar{a}-2b\bar{b}-2c\bar{c}-abc-\overline{abc})$$

$$= \tfrac{1}{8}\{(a\bar{a}-2)(b\bar{b}-2)-(c\bar{c}-2)-(ab+\bar{c})(\overline{ab}+c)+2\}$$

$$= \tfrac{1}{4}\left(2\cos\frac{2\pi}{l}\cos\frac{2\pi}{p}-\cos\frac{2\pi}{q}-\cos\frac{2\pi}{r}\right),$$

is positive definite if and only if

$$2\cos\frac{2\pi}{l}\cos\frac{2\pi}{p}>\cos\frac{2\pi}{q}+\cos\frac{2\pi}{r}. \qquad \text{......................(4.3)}$$

Accordingly, this inequality is a *necessary* (though possibly not *sufficient*) condition for the finiteness of the group (2.2) and of its subgroup (2.1), which is $(l, p \mid q, r)$.

5. The Groups $(3, p \mid q, r)$

When l and p are both greater than 3, this necessary condition is certainly not sufficient; for it admits the group $(4, 4 \mid 3, 3)$, which is known to be infinite (**3**, p. 83).

On the other hand, we shall find a more satisfactory state of affairs when $l = 3$. In fact, the condition

$$\cos\frac{2\pi}{p}+\cos\frac{2\pi}{q}+\cos\frac{2\pi}{r}<0, \qquad \text{............................(5.1)}$$

for the finiteness of $(3, p \mid q, r)$, is not only necessary but also sufficient, provided we make the obvious restriction that, if $p = 2$, $q = r$. To see this, we merely have to examine all the solutions of (5.1) and verify that each of the groups so obtained is finite. Because p, q, r enter symmetrically, we lose no generality by assuming that $p \leqq q \leqq r$.

When $p = 2$, we have the polyhedral groups

$$(3, 2 \mid 2, 2) \cong S_3, \quad (3, 2 \mid 3, 3) \cong A_4,$$
$$(3, 2 \mid 4, 4) \cong S_4, \quad (3, 2 \mid 5, 5) \cong A_5$$

(**9**, pp. 270-276). For every r there is a group $(3, 3 \mid 3, r)$ of order $3r^2$ (**11**, p. 208; **3**, p. 83). It was observed by Miller (**15**, pp. 364-368) that $(3, 3 \mid 4, 4)$ is Klein's simple group $LF(2, 7)$ of order 168, and that $(3, 3 \mid 4, 5)$ has order 1080. The list of solutions of (5.1) is now complete.

Since the condition (5.1) is necessary, all other groups $(3, p \mid q, r)$ are infinite. Apparently the only cases in which this was previously known are when p, q, r are multiples of 3, 4, 6 (in any order) (**4**, p. 250).

6. The Groups (3, 3 | q, r)

Evidently, most of the finite groups (3, p | q, r) are covered by taking $p = 3$. In fig. 1, these groups (3, 3 | q, r) are plotted as points with Cartesian co-ordinates (q, r). The finite groups that fulfil the given relations are marked as little circles (with the order of each written underneath), the infinite groups are marked as black dots, and the cases of collapse as crosses. The pattern

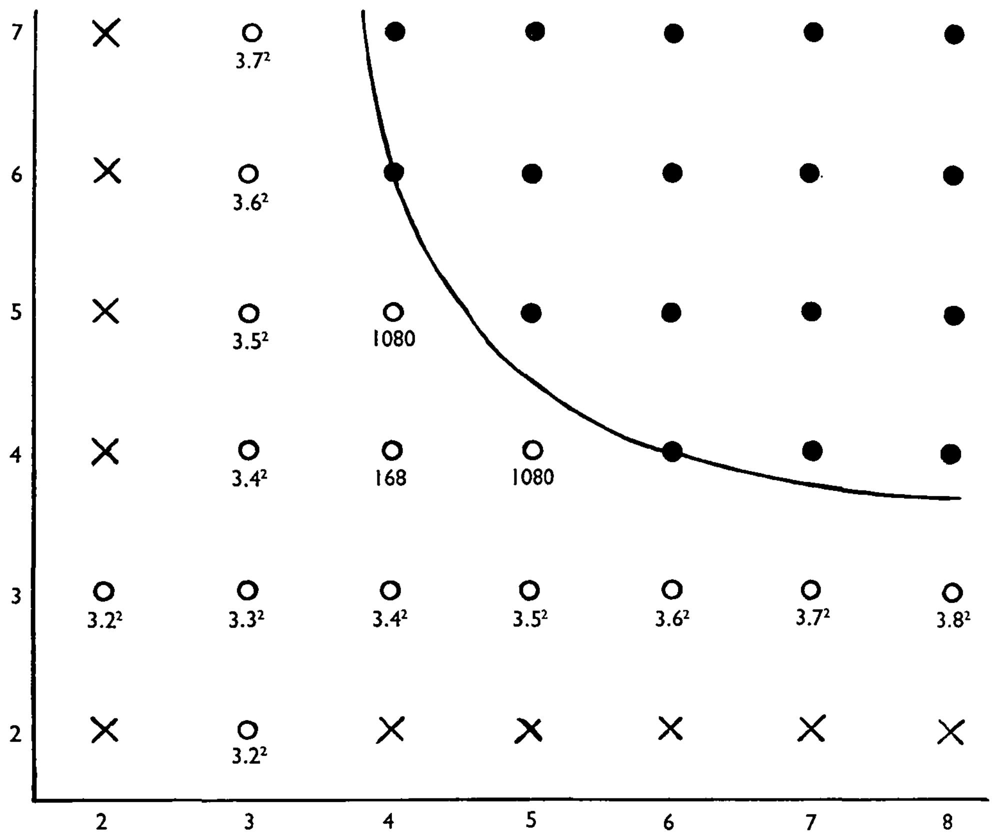

FIG. 1.

along the lines $q = 3$ and $r = 3$ has been continued far enough to indicate that it goes on for ever. These lines are the asymptotes of the curve

$$\cos\frac{2\pi}{q}+\cos\frac{2\pi}{r}=\tfrac{1}{2},$$

which passes through the points (4, 6), (6, 4) and separates all the remaining black dots from all the little circles and crosses. Instead of this non-algebraic curve we could have used the hyperbola

$$(q-3)(r-3) = 3,$$

which happens to separate the lattice points in the same manner.

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Since (3, 3 | 4, 6) is "only just infinite," one might have expected the representation described in § **4** to remain faithful. Setting $l = p = 3$, $q = 4$, $r = 6$ in (4.2), we obtain

$$a\bar{a} = b\bar{b} = 1, \quad c\bar{c} = 2, \quad (1+\bar{c})(1+c) = 3.$$

Setting $a = b = 1$, $c = \sqrt{-2}$ in the matrices for R_2R_3 and R_3R_1, we deduce the generators

$$A = \begin{pmatrix} 1 & \sqrt{-2} & 1+\sqrt{-2} \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 0 & 1 \\ 1-\sqrt{-2} & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix},$$

which leave invariant the semidefinite Hermitian form

$$x^1\bar{x}^1 + x^2\bar{x}^2 + x^3\bar{x}^3 - \tfrac{1}{2}(x^2\bar{x}^3 + x^3\bar{x}^2 + x^3\bar{x}^1 + x^1\bar{x}^3 + \sqrt{-2}x^1\bar{x}^2 - \sqrt{-2}x^2\bar{x}^1).$$

Graham Higman has pointed out that the commutator of these matrices is of period 4. Hence the group generated by them is not (3, 3 | 4, 6), though it may be (3, 3 | 4, 6; 4) (**4**, p. 150).

7. The Groups $G^{l,\,m,\,n}$ with m and n even

As we remarked in § **2**, the group $(l, l \mid q, r)$, defined by (2.1) with $p = l$, is a subgroup of index four in $G^{l,\,2q,\,2r}$. Setting

$$p = l, \quad q = \tfrac{1}{2}m, \quad r = \tfrac{1}{2}n$$

in (4.3), we deduce that the inequality

$$\cos\frac{4\pi}{m} + \cos\frac{4\pi}{n} < 1 + \cos\frac{4\pi}{l} \qquad \text{(7.1)}$$

is a necessary condition for the finiteness of $G^{l,\,m,\,n}$ with m and n both even. Moreover, when $l = 3$, this necessary condition

$$\cos\frac{4\pi}{m} + \cos\frac{4\pi}{n} < \tfrac{1}{2} \qquad \text{(7.2)}$$

is also sufficient. We naturally ask whether it is permissible to omit the restriction "m and n both even".

8. Allowing m or n to be odd

In fig. 2, which employs the same conventions as fig. 1, the groups $G^{3,\,m,\,n}$ are plotted as points with Cartesian coordinates (m, n). For simplicity, we omit the groups with m or n less than 5, as all of these collapse except

$$G^{3,\,2,\,6} \cong G^{3,\,6,\,2}, \quad \text{of order 12,}$$

$$G^{3,\,3,\,4} \cong G^{3,\,4,\,3}, \quad \text{of order 24,}$$

and

$$G^{3,\,4,\,6} \cong G^{3,\,6,\,4}, \quad \text{of order 48.}$$

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Since $(3, 3 \mid q, r)$ is a subgroup of index four in $G^{3,\,2q,\,2r}$, the points (q, r) in fig. 1 reappear as the points $(2q, 2r)$ in fig. 2. However, the interest of fig. 2 is enhanced by the "interpolated" points (m, n) for which at least one coordinate is odd. (In the places where these points are not marked, nothing is known.)

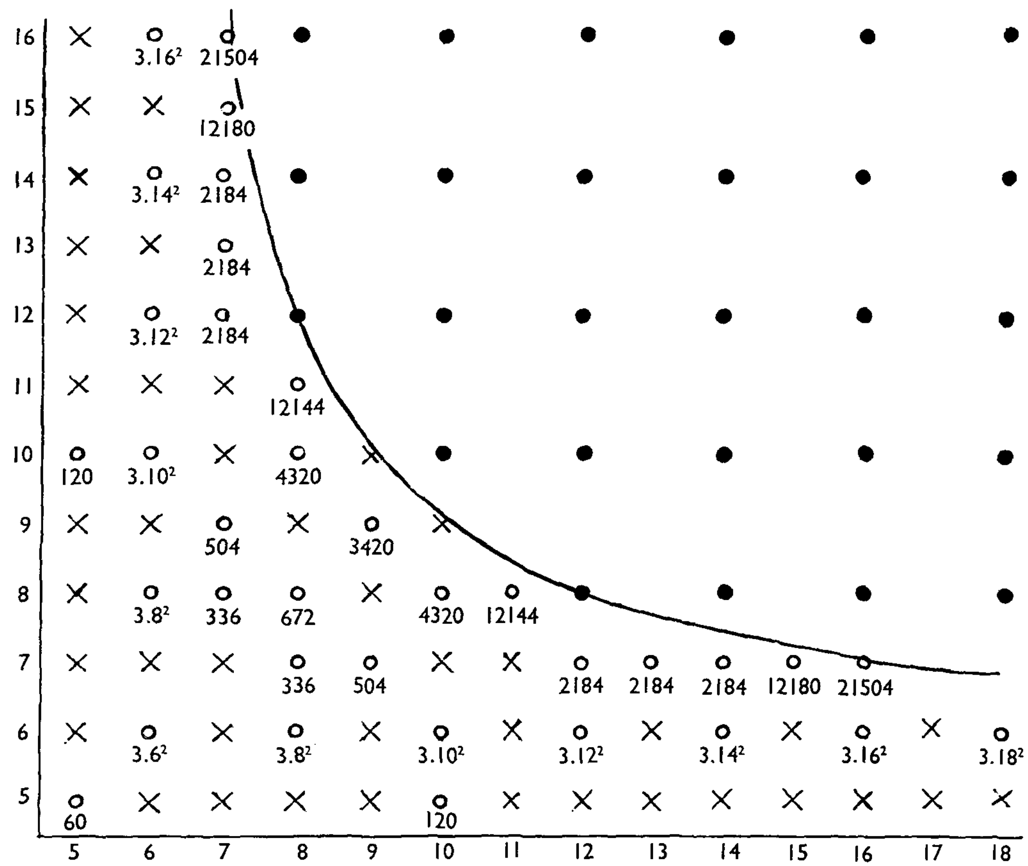

FIG. 2.

Incidentally, the curve

$$\cos\frac{4\pi}{m} + \cos\frac{4\pi}{n} = \tfrac{1}{2},$$

whose asymptotes are $m = 6$ and $n = 6$, can no longer be replaced by the hyperbola

$$(m-6)(n-6) = 12,$$

which passes through the point (9, 10) although the group $G^{3,\,9,\,10}$ collapses (3, p. 143).

Of the groups $G^{3,m,n}$ satisfying (7.2), the only one that was not known in 1939 is $G^{3,7,16} \cong G^{3,16,7}$. Since in every other case the inequality is a valid criterion regardless of the parity of m and n, the " principle of permanence " suggests that this remaining group $G^{3,7,16}$ should be finite. The work of Leech and Mennicke (**14**), shows that in fact its order is 21504.

9. Summary of the Work of Leech and Mennicke

In the case of $G^{3,7,16}$, the subgroup (2.4) is

$$A^3 = B^7 = (AB)^2 = (A^{-1}B^2)^8 = E$$

or, in terms of $P = B$ and $Q = AB^{-2}$,

$$P^7 = Q^8 = (QP^2)^3 = (QP^3)^2 = E \quad \text{.................(9.1)}$$

(Sinkov, 1937, p. 582) or, in terms of Q and $S = P^3$ (so that $QP^2 = P^{-3}Q^{-1}P^{-1} = S^{-1}Q^{-1}S^2$),

$$Q^8 = S^7 = (QS)^2 = (Q^{-1}S)^3 = E.$$

Thus (8, 7 | 2, 3) occurs in $G^{3,7,16}$ as a subgroup of index 2. For the sake of agreement with Leech and Mennicke (**14**), let us abandon our attempted consistency of notation and call the generators A and B instead of Q and S, so that now

$$A^8 = B^7 = (AB)^2 = (A^1B^{-3}) = E. \quad \text{....................(9.2)}$$

As a first step towards identifying this with Sinkov's group of order 10752, Leech and Mennicke observe that the seven elements

$$\begin{array}{llll} a = A^4, & c = B^{-1}aB, & e = B^{-1}cB, & g = B^{-1}eB, \\ & b = B^{-1}gB, & d = B^{-1}bB, & f = B^{-1}dB, \end{array}$$

which are transformed by A into

$$\begin{array}{llll} a, & f, & fd, & e, \\ & fga, & b, & ca, \end{array}$$

generate a normal subgroup whose quotient group (4, 7 | 2, 3) is derived from (9.2) by setting $A^4 = E$. The substitution $S_2 = AB$, $S_7 = B^{-2}$ serves to identify this quotient group with $LF(2, 7)$, of order 168:

$$S_2^2 = S_7^7 = (S_7S_2)^3 = (S_7^4S_2)^4 = E$$

(**1**, p. 422; **3**, p. 54 †).

Leech and Mennicke then prove that the seven elements $a, \ldots, g$ are involutory, mutually commutative, and have the identity for their product, but do not satisfy any relations independent of these. Thus the group $\{a, \ldots, g\}$, which occurs in (8, 7 | 2, 3) as a normal subgroup of index 168, is the Abelian group of order 64 and type (1, 1, 1, 1, 1, 1): the direct product of six groups of order 2.

† In the sentence " Fig. xv . . . " the number 7 should be 6, since in fact Fig. xv shows {4, 6 | 3}, not {4, 7 | 3}.

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It follows that the order of $(8, 7 \mid 2, 3)$ is

$$168 \,.\, 64 = 10752,$$

and that the order of $G^{3,\,7,\,16}$ is 21504 (see fig. 2).

10. Enumerating Cosets

During the time when the finiteness of $(8, 7 \mid 2, 3)$ was a mere conjecture, various attempts were made to determine its order by enumerating cosets of a suitable subgroup. Leech attempted an enumeration on EDSAC 2 at Cambridge, using the octahedral subgroup generated by A^2 and $A^{-1}B$, but was unsuccessful. Storage space for 800 cosets was exhausted without any sign that the process was drawing to a close. When he and Mennicke had established the order 24 . 448, it became clear that the number of cosets is really only 448. To see why the machine had failed, J. A. Todd performed the same enumeration by the old-fashioned method of pencil and paper. With extraordinary perseverance he continued until he had defined about 950 apparently distinct cosets, and then at last the inevitable end came: two differently numbered cosets were identified, and the consequent identification of other pairs brought the number down to 448, as predicted. He found later that, by re-defining the successive cosets in a different order, he could reduce the number of redundant cosets from about five hundred to as few as 63. Still later, after correcting "an extremely subtle programme error", Haselgrove and Leech persuaded the Mercury at Manchester to enumerate the 448 cosets in 42 minutes. (The last 15 minutes were occupied with the reduction from about two thousand cosets to 448.) On the other hand, Todd's *tour de force* can hardly have taken less than thirty hours.

11. Compound Tessellations and Maps

For any two positive integers p and q satisfying $(p-2)(q-2)>4$ there is, in the hyperbolic plane, a *regular tessellation* $\{p, q\}$ consisting of regular p-gons $\{p\}$, q surrounding each vertex (**10**, pp. 53, 54). Any two adjacent edges of a face $\{p\}$ belong to a *Petrie polygon*: an infinite zig-zag in which every two adjacent edges, but no three, belong to a face. Fig. 1 of Leech and Mennicke (**14**, p. 26) is a conformal representation of part of $\{3, 7\}$. Their broken line marked A crosses eight edges belonging to a Petrie polygon. In fact, the generator A of the infinite group

$$B^7 = (AB)^2 = (A^{-1}B)^3 = E \quad \text{......................(11.1)}$$

is represented as a translation that shifts this Petrie polygon two steps along itself. Their whole figure consists of 56 faces (some bisected) of $\{3, 7\}$, forming altogether a $\{14\}$ which serves as a fundamental region for the normal subgroup generated by the translations a, b, c, d, e, f, g. Since the $\{14\}$, like a face of the $\{3, 7\}$, has angle $2\pi/7$, it is one of the faces of a tessellation $\{14, 7\}$ *inscribed* in the $\{3, 7\}$.

This state of affairs is analogous to the way a cube $\{4, 3\}$ can be inscribed in the regular dodecahedron $\{5, 3\}$. The 20 vertices of the dodecahedron, each used twice, are the vertices of 5 such cubes forming a compound polyhedron

$$2\{5, 3\}[5\{4, 3\}]$$

(6, p. 50 †). Similarly, the infinitely many vertices of the $\{3, 7\}$, each used twice, are the vertices of 24 $\{14, 7\}$'s forming a *compound tessellation*

$$2\{3, 7\}[24\{14, 7\}]\{7, 3\}.$$

The " $\{7, 3\}$ " at the end signifies that each face of a $\{14, 7\}$ is concentric with a face of a $\{7, 3\}$ (namely, the dual of the original $\{3, 7\}$).

Leech and Mennicke (**14**, p. 27) mention also an "intermediate" tessellation $\{3, 14\}$, which is inscribed in the $\{3, 7\}$ while the $\{14, 7\}$ is inscribed in it. The corresponding compounds are

$$\{3, 7\}[8\{3, 14\}]2\{3, 7\} \quad \text{and} \quad 2\{3, 14\}[3\{14, 7\}]\{14, 3\}.$$

Their conclusion, that when $A^8 = E$ the group generated by $a, \ldots, g$ is of order 64, shows that 64 suitably chosen faces of the $\{14, 7\}$ form together a fundamental region for the normal subgroup of (11.1) generated by A^8 and its conjugates. However, these 64 $\{14\}$'s cannot be arranged to form a regular polygon!

The subgroup generated by A^4 and its conjugates can be interpreted as the fundamental group of the surface of genus 3 that is obtained by suitably identifying pairs of sides of one $\{14\}$ (**10**, p. 110). Hence the quotient group $(4, 7 \mid 2, 3)$, which is the simple group of order 168, is the group of direct symmetry operations of the regular map $\{3, 7\}_8$, which consists of 56 triangles covering that surface (**16**, p. 479). (The subscript 8 indicates that $\{3, 7\}_8$ is derived from $\{3, 7\}$ by identifying all pairs of edges that differ by 8 steps along a Petrie polygon. Thus the Petrie polygon of $\{3, 7\}_8$ is an "octagon". Two Petrie polygons of the dual map $\{7, 3\}_8$ are shown very clearly in a drawing by Klein, (**12**, p. 449.)

Somewhat similarly, the subgroup generated by A^8 and its conjugates can be interpreted as the fundamental group of the surface of genus 129 that is obtained by identifying all pairs of edges of $\{3, 7\}$ that differ by 16 steps along a Petrie polygon. Hence the quotient group $(8, 7 \mid 2, 3)$, of order 10752, is the group of direct symmetry operations of a new regular map $\{3, 7\}_{16}$ which consists of 3584 triangles covering that surface. Collecting these 56 . 64 triangles in sets of 56, as in fig. 1 of Leech and Mennicke, we obtain, on the same surface of genus 129, a map of type $\{14, 7\}$ (" uniform," but not regular) having 128 vertices, 448 edges, and 64 faces. Any one of these 64 $\{14\}$'s will serve as a fundamental region for the Abelian group generated by $a, \ldots, g$. In other words, just as the elliptic map $\{3, 5\}_5$ (**10**, 1947, p. 111) can be derived

† The first edition of *Regular Polytopes* is out of print, but a second edition (paperback) is being published by the Macmillan Company, New York.

from the icosahedron $\{3, 5\} = \{3, 5\}_{10}$ by identifying pairs of opposite vertices, so $\{3, 7\}_8$ can be derived from $\{3, 7\}_{16}$ by identifying its 24 . 64 vertices in sets of 64. Just as a Petrie polygon of $\{3, 5\}_5$ may be regarded as one half of a Petrie polygon of the icosahedron, so a Petrie polygon of $\{3, 7\}_8$ may be regarded as one half of a Petrie polygon of $\{3, 7\}_{16}$.

This information enables us to improve the tables of groups and regular maps in some earlier papers. In fact, the table of (**3**, pp. 146-148) should have the following further entries:

(Table I)	$(8, 7 \mid 2, 3)$	10752	-0.168		
(Table II)	$(2, 3, 7;\ 8)$	10752			
(Table III)	$G^{3,7,16}$	21504	256	129	-0.0473

Similarly, fig. vi of (**2**, p. 41) should have little circles at the points (8, 7), (7, 8); table I (p. 61) should have the extra entries

$\{7, 8 \mid 3\}$	1536	5376	1344	1249	10752
$\{8, 7 \mid 3\}$	1344	5376	1536	1249	

and table II:

$\{4,\ 6 \mid, 2\}$	12	24	8	3	$S_4 \times S_2$	48
$\{5,\ 6 \mid, 2\}$	24	60	20	9	$A_5 \times S_2$	120
$\{3, 11 \mid, 4\}$	2024	3036	552	231	$LF(2, 23)$	6072
$\{3,\ 7 \mid, 8\}$	3584	5376	1536	129		10752
$\{3,\ 9 \mid, 5\}$	12180	18270	4060	1016	$LF(2, 29) \times A_3$	36540

Finally, table 8 of (**10**, p. 141) should be continued as follows:

$\{7, 3\}_{16}$	3584	5376	1536	-256	129	$G^{3,7,16}$	21504
$\{16, 3\}_7$	3584	5376	672	-1120	—		
$\{16, 7\}_3$	1536	5376	672	-3168	—		

(and, of course, the heading of table 9 should read " The regular maps of genus 2 ").

12. The Structure of $G^{3,7,16}$

As we remarked in § **9**, the seven elements

$$a, \quad b, \quad c, \quad d, \quad e, \quad f, \quad g$$

of $(8, 7 \mid 2, 3)$ are transformed by A into

$$a, \quad afg, \quad f, \quad b, \quad df, \quad ac, \quad e$$

and by B into

$$c, \quad d, \quad e, \quad f, \quad g, \quad a, \quad b.$$

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It follows that the fourteen products

$$p_1 = abf,\ p_2 = acd,\ p_3 = cef,\ p_4 = aeg,\ p_5 = bcg,\ p_6 = bde,\ p_7 = dfg,$$
$$q_1 = acg,\ q_2 = aef,\ q_3 = cdf,\ q_4 = abd,\ q_5 = bfg,\ q_6 = deg,\ q_7 = bce$$

are transformed by A into

$$\begin{array}{ccccccc} p_6, & p_1, & p_2, & p_5, & p_4, & p_3, & p_7, \\ q_2, & q_3, & q_6, & q_5, & q_4, & q_1, & q_7, \end{array}$$

and by B into

$$\begin{array}{ccccccc} p_2, & p_3, & p_4, & p_5, & p_6, & p_7, & p_1, \\ q_7, & q_1, & q_2, & q_3, & q_4, & q_5, & q_6. \end{array}$$

In other words, A and B transform the p's according to the permutations

$$(6321)(45) \quad \text{and} \quad (1234567), \qquad \text{(12.1)}$$

and transform the q's according to the respectively inverse permutations

$$(1236)(54) \quad \text{and} \quad (7654321). \qquad \text{(12.2)}$$

Either pair of permutations provides a representation of the group (4, 7 | 2, 3), which is $LF(2, 7)$ (**3**, p. 84).

Since the seven p's satisfy the relations

$$p_i^2 = E,\quad p_i p_j = p_j p_i,\quad p_1p_2p_3p_4p_5p_6p_7 = E,$$
$$p_1p_2p_4 = p_2p_3p_5 = p_3p_4p_6 = p_4p_5p_7 = p_5p_6p_1 = p_6p_7p_2 = p_7p_1p_3 = E$$

(for instance, $p_1p_2p_4 = a^2 \,.\, abcdefg = E$), they form with E the Abelian group of order 8 and type (1, 1, 1): the direct product of three groups of order 2, say those generated by p_1, p_2, p_3. The seven q's, satisfying exactly similar relations, form with E another such group of order 8. Each group is transformed by A and B (albeit in slightly different ways) according to automorphisms which generate $LF(2, 7)$. It is interesting to compare the p's and q's with the basic units of the algebra of Cayley numbers which, though neither commutative nor associative, satisfy the relations

$$e_1^2 = e_2^2 = \ldots = e_7^2 = e_1e_2e_4 = e_2e_3e_5 = \ldots = e_7e_1e_3 = -1$$

(**5**, p. 561) and therefore have $LF(2, 7)$ for their group of automorphisms.

The Abelian group of order 64, which Leech and Mennicke obtained, can now be recognised as the direct product of these two groups of order 8; in fact,

$$a = p_7q_7,\qquad b = p_4q_3,\qquad c = p_1q_6,\qquad d = p_5q_2,$$
$$e = p_2q_5,\qquad f = p_6q_1,\qquad g = p_3q_4.$$

It is interesting to compare this with the work of Sinkov (**17**, p. 584), who showed that the relations (9.1), along with either

$$Q^4PQ^4P^4Q^4P^2 = E \quad \text{or} \quad Q^4PQ^4P^2Q^4P^4 = E,$$

determine a group of order 1344. (His Q and P are our A and B^{-2}.) Leech

(1962, § 4.2) observed that these relations are equivalent to $p_i = E$ and $q_i = E$, respectively (for any i). Hence Sinkov's group of order 1344, which is, of course, a factor group of $(8, 7 \mid 2, 3)$, is the holomorph of the Abelian group $\{q_i\}$ or $\{p_i\}$, of order 8 (**1**, pp. 111-117).

The remark at the end of § **2** shows that we can derive $G^{3,7,16}$ from $(8, 7 \mid 2, 3)$ by adjoining an involutory element that transforms A and B into their inverses. However, these generators were the A and B of (2.4), not the A and B of (9.2). Making the necessary adjustment, we can assert that the new element transforms the A and B of (9.2) into

$$B^{-3}A^{-1}B^3 \quad \text{and} \quad B^{-1}.$$

Accordingly, it transforms

$$a,\quad b,\quad c,\quad d,\quad e,\quad f,\quad g$$

into

$$g,\quad f,\quad e,\quad d,\quad c,\quad b,\quad a,$$

and transforms

$$p_1,\quad p_2,\quad p_3,\quad p_4,\quad p_5,\quad p_6,\quad p_7$$

into

$$q_5,\quad q_6,\quad q_7,\quad q_1,\quad q_2,\quad q_3,\quad q_4.$$

We see now that the group $(8, 7 \mid 2, 3)$, of order 8 . 8 . 168, is derived from the direct product $\{p_i\}\times\{q_i\}$ by adjoining elements A and B which perform the automorphisms (12.1) and (12.2) while satisfying

$$A^4 = p_7q_7, \quad B^7 = (AB)^2 = (A^{-1}B)^3 = E.$$

Moreover, the group $G^{3,7,16}$ of order 2 . 8 . 8 . 168 is derived from $(8, 7 \mid 2, 3)$ by adjoining an involutory element that interchanges the p's and q's.

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