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EXISTENCE AND BOUNDEDNESS OF PARAMETRIZED MARCINKIEWICZ INTEGRAL WITH ROUGH KERNEL ON CAMPANATO SPACES

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Abstract. Let $g(f)$, $S(f)$, $g_\lambda^*(f)$ be the Littlewood-Paley g function, Lusin area function, and Littlewood-Paley g_λ^* function of f, respectively. In 1990 Chen Jiecheng and Wang Silei showed that if, for a BMO function f, one of the above functions is finite for a single point in $\mathbb{R}^n$, then it is finite a.e. on $\mathbb{R}^n$, and BMO boundedness holds. Recently, Sun Yongzhong extended this result to the case of Campanato spaces (i.e. Morrey spaces, BMO, and Lipschitz spaces). One of us improved his g_λ^* result further, and treated parametrized Marcinkiewicz functions with Lipschitz kernel $\mu^\rho(f)$, $\mu_S^\rho(f)$ and $\mu_\lambda^{*,\rho}(f)$. In this paper, we show that the same results hold also in the case of rough kernel satisfying L^p-Dini type condition.

§1. Introduction

In this note we study the existence and boundedness property of parametrized Marcinkiewicz functions with rough kernel, on Campanato spaces. First, we recall the definition of Littlewood-Paley's functions (generalized ones) in the n-dimensional Euclidean space $\mathbb{R}^n$ $(n \geq 2)$.

Let ψ be a function ψ on $\mathbb{R}^n$ such that there exist positive constants C_0, C_1, δ, η and γ satisfying

(i) $\psi \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \psi(x)\,dx = 0$;

(ii) $|\psi(x)| \leq C_0(1+|x|)^{-n-\delta}$;

(iii) $|\psi(x+h) - \psi(x)| \leq C_1|h|^\gamma(1+|x|)^{-n-\eta}$ for $|h| \leq |x|/2$.

For this ψ, we define Littlewood-Paley's g, Lusin's area functions and Littlewood-Paley's g_λ^* function as follows. Here and hereafter, $\psi_t(x)$ de-

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notes $t^{-n}\psi(x/t)$.

$$g(f)(x) = \left(\int_0^\infty \frac{|\psi_t * f(x)|^2}{t}\, dt\right)^{1/2},$$

$$S(f)(x) = \left(\int_{\Gamma(x)} |\psi_t * f(y)|^2 \frac{dy\, dt}{t^{n+1}}\right)^{1/2},$$

where $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ \,;\, |x-y| < t\}$.

$$g^*_\lambda(f)(x) = \left(\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} |\psi_t * f(y)|^2 \frac{dy\, dt}{t^{n+1}}\right)^{1/2}$$

where $\lambda > 1$. L^p boundedness of these operators are known like as the classical Littlewood-Paley's g-functions. That is, g and S are L^p bounded for $1 < p < \infty$, and g^*_λ is L^p bounded for $1 < p < \infty$ if $\lambda > \max\{1, 2/p\}$ (see for example Torchinsky [21, pp. 309–318]). Here and hereafter, the letter C denotes a constant depending on main parameters and may vary at each occurrence.

Stein's generalization of the Marcinkiewicz function is as follows [17]: Let $\Omega(x)$ be a function on $\mathbb{R}^n$ which satisfies the following two conditions:

(i) $\Omega(x)$ is homogeneous of degree zero and continuous on the unit sphere S^{n-1}, and satisfies for some $0 < \beta \le 1$

$$|\Omega(x') - \Omega(y')| \le C|x'-y'|^\beta, \quad x', y' \in S^{n-1}.$$

(ii) $\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0$, where $d\sigma$ is the surface Lebesgue measure on S^{n-1}.

Define $\mu(f)(x)$ by

$$\mu(f)(x) = \left(\int_0^\infty \frac{|\psi_t * f(x)|^2}{t}\, dt\right)^{1/2},$$

where $\psi(x) = \frac{\Omega(x)}{|x|^{n-1}} \chi_{\{|x|\le 1\}}$.

In their work on Marcinkiewicz integral, A. Torchinsky and S. Wang [22] introduced the Marcinkiewicz functions $\mu_S(f)$ and $\mu^*_\lambda(f)$ corresponding to the S function and g^*_λ function. They gave L^p boundedness of $\mu_S(f)$ and $\mu^*_\lambda(f)$ for $p \ge 2$. On the other hand, in the connection of $\mu(f)$ a parametrized Marcinkiewicz function $\mu^\rho(f)$ was considered by L. Hörmander

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[11]. It corresponds to the case $\psi(x) = \Omega(x)|x|^{\rho-n}\chi_{\{|x|\le 1\}}$. Thus, Sakamoto and Yabuta have considered in [28] parametrized $\mu_S^{\rho}(f)$ and $\mu_{\lambda}^{*,\rho}(f)$, where $\psi(x) = \Omega(x)|x|^{\rho-n}\chi_{\{|x|\le 1\}}$, for $\rho \in \mathbb{C}$ with $\operatorname{Re}\rho = \sigma > 0$. Introducing these parametrized operators and using a Hilbert space valued version of the complex interpolation theorem of analytic families of operators, they could show that in the case $n \ge 3$, $\mu_S(f)$ and $\mu_{\lambda}^{*}(f)$ are L^p bounded for $p > 2n/(n+2)$ and $\lambda > \max\{1, 2/p\}$, respectively. L^p boundedness for parametrized operators $\mu_S^{\rho}(f)$ and $\mu_{\lambda}^{*,\rho}(f)$ are well discussed in [16], [28], and further developed by many authors. Recently we have shown the L^p boundedness under weaker condition on kernels than Lipschitz smoothness. To state it, we introduce the following smoothness for kernels. Let $\omega_q(\delta)$ be the L^q modulus of continuity of Ω $(1 \le q < \infty)$, defined by

$$\omega_q(\delta) = \sup_{|\gamma|\le\delta}\left(\int_{S^{n-1}} |\Omega(\gamma x') - \Omega(x')|^q \, d\sigma(x')\right)^{1/q},$$

where γ is a rotation on S^{n-1}, and $|\gamma| = \sup_{x'\in S^{n-1}} |\gamma x' - x'|$. If $\omega_q(\delta)$ satisfies

$$\int_0^1 \frac{\omega_q(\delta)}{\delta}\, d\delta < \infty,$$

we say that Ω satisfies L^q-Dini condition. If $\omega_q(\delta)$ satisfies

$$\int_0^1 \frac{\omega_q(\delta)}{\delta}(1 + |\log\delta|)^{\beta}\, d\delta < \infty,$$

we say that Ω satisfies L^q-$\log\beta$-Dini condition. If $\omega_q(\delta)$ satisfies

$$\int_0^1 \frac{\omega_q(\delta)}{\delta^{1+\beta}}\, d\delta < \infty,$$

we say that Ω satisfies L^q-β-Dini condition.

L^p boudedness results are as follows.

THEOREM A. (i) *Let $\sigma > 0$, $\max\{1, \frac{2n}{n+2\sigma}\} < p < 2$, and $\lambda > 2/p$. Let $\Omega \in L^2(S^{n-1})$ satisfy the cancellation condition $\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0$ and L^2-$\log\beta$-Dini condition for some $\beta > 1$. Then, there exist C_1, $C_2 > 0$, independent of f, such that*

$$\|\mu_S^{\rho}(f)\|_p \le C_1 \|\mu_{\lambda}^{*,\rho}(f)\|_p \le C_2 \|f\|_p.$$

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(ii) *Let* $\sigma > 0$, $2 \le p < \infty$, *and* $\lambda > 1$. *Let* $\Omega \in L\log^+ L(S^{n-1})$ *satisfy the cancellation condition* $\int_{S^{n-1}} \Omega(x')\,d\sigma(x') = 0$. *Then, there exist* C_1, $C_2 > 0$, *independent of* f, *such that*

$$\|\mu_S^\rho(f)\|_p \le C_1 \|\mu_\lambda^{*,\rho}(f)\|_p \le C_2 \|f\|_p.$$

(See [8] for the proof of Theorem A (i) and [6] for the proof of Theorem A (ii).)

As for μ^ρ, we have

Theorem B. *Let* $\sigma > 0$, $1 < p < \infty$. *Let* $\Omega \in H^1(S^{n-1})$ *(the Hardy space on* S^{n-1}*) satisfy the cancellation condition* $\int_{S^{n-1}} \Omega(x')\,d\sigma(x') = 0$. *Then, there exists* $C > 0$, *independent of* f, *such that*

$$\|\mu^\rho(f)\|_p \le C\|f\|_p.$$

(See [4] for the proof of Theorem B in the case $\rho \equiv 1$. The proof of Theorem B for $\rho \in \mathbb{C}$ with $\sigma > 0$ can be obtained by the same way as in [4].) We note the following:

$$\begin{aligned} \mathrm{Lip}_\alpha(S^{n-1})\ (0 < \alpha \le 1) \subsetneq L^q(S^{n-1})\ (q > 1) \subsetneq L\log^+ L(S^{n-1}) \\ \subsetneq H^1(S^{n-1}). \end{aligned}$$

It is also known that, if $\Omega \in L^1(S^{n-1})$ satisfies L^1-Dini condition, then $\Omega \in L\log^+ L(S^{n-1})$, see [1].

We recall also the definition of Campanato spaces [14].

Definition 1.1. For $1 \le p < \infty$ and $-n/p \le \alpha \le 1$, the Campanato space $\mathcal{E}^{\alpha,p}$ is defined by the set of functions for which

$$\|f\|_{\mathcal{E}^{\alpha,p}} = \sup_{x_0 \in \mathbb{R}^n} \sup_B \frac{1}{|B|^{\alpha/n}} \left(\frac{1}{|B|} \int_B |f(x) - f_B|^p\,dx \right)^{1/p} < \infty,$$

where B moves over all balls centered at x_0, and f_B is the average of f over B, $(1/|B|)\int_B f(t)\,dt$.

It is known that for $0 < \alpha \le 1$, $\mathcal{E}^{\alpha,p} = \mathrm{Lip}_\alpha$: the Banach space of Lipschitz continuous functions of exponent α, and the norms are equivalent. If $\alpha = 0$, $\mathcal{E}^{\alpha,p}$ coincides with BMO: the space of functions of bounded mean oscillation. And if $\alpha < 0$, $\mathcal{E}^{\alpha,p}$ is equivalent to the Morrey space $L^{p,n+p\alpha}$. It is also easily checked that $\|f\|_{\alpha,p} \le C \sup_B \inf_{a\in\mathbb{C}} |B|^{-\alpha/n}\big(|B|^{-1}\int_B |f(x) -$

$a|^p\,dx\big)^{1/p}$ $(-n/p \le \alpha \le 1)$, and hence these norms are equivalent. We note that balls can be replaced by cubes with sides parallel to the coordinate axes and the norms are equivalent.

In 1984, Wang Silei [24] showed that the BMO boundedness of Littlewood-Paley's g-function follows from its finiteness on a set of positive measure. Since then, many authors considered such problems in BMO, Lipschitz spaces, and Morrey spaces i.e. in Campanato spaces. In 1990, Wang Silei and Chen Jiecheng [25] showed that the BMO boundedness follows from its finiteness at only one point for Littlewood-Paley's g-function, Lusin's area function and Littlewood-Paley's g^*-function, and Marcinkiewicz function. Recently, Sun Yongzhong [20] improves and extends their results to the case of Campanato spaces. Further, one of us [29] improves Sun's result and also treats the case of parametrized Marcinkiewicz integrals. In this paper, we improve the results in [29], i.e. we treat parametrized Marcinkiewicz integrals with more rough kernels.

Our results are as follows, where $\rho \in \mathbb{C}$ and $\operatorname{Re}\rho = \sigma$.

THEOREM 1. *Let $\sigma > 0$, and suppose that $\Omega \in L^q(S^{n-1})$ and satisfies L^q-log 1-Dini condition for some $q > 1$ and the cancellation condition. Then, if $f \in \mathrm{BMO}(\mathbb{R}^n)$ and $\mu^\rho(f)(x)$ is finite for a point $x_0 \in \mathbb{R}^n$, it follows $\mu^\rho(f)(x) < \infty$ a.e. on $\mathbb{R}^n$, and there is a constant C independent of f, such that*

$$\|\mu^\rho(f)\|_{\mathrm{BMO}(\mathbb{R}^n)} \le C\|f\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

THEOREM 2. *Let $\sigma > 0$, and suppose that $\Omega \in L^1(S^{n-1})$ and satisfies L^1-β-Dini condition for some $0 < \beta \le 1$ and the cancellation condition. Then, if $f \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$ for $0 < \alpha < \beta$ and $\mu^\rho(f)(x)$ is finite for a point $x_0 \in \mathbb{R}^n$, it follows $\mu^\rho(f)(x) < \infty$ a.e. on $\mathbb{R}^n$, and there is a constant C independent of f, such that*

$$\|\mu^\rho(f)\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \le C\|f\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}.$$

THEOREM 3. *Let $\sigma > 0$, $1 < p < \infty$ and $-n/p \le \alpha < 0$. Moreover, suppose $\Omega \in L^{p'}(S^{n-1})$ and satisfies $L^{p'}$-Dini condition and the cancellation condition. Then, if $f \in \mathcal{E}^{\alpha,p}$ and $\mu^\rho(f)(x)$ is finite for a point $x_0 \in \mathbb{R}^n$, it follows $\mu^\rho(f)(x) < \infty$ a.e. on $\mathbb{R}^n$, and there is a constant C independent of f, such that*

$$\|\mu^\rho(f)\|_{\mathcal{E}^{\alpha,p}} \le C\|f\|_{\mathcal{E}^{\alpha,p}}.$$

THEOREM 4. *Let $\sigma > 0$, and suppose that $\Omega \in L^q(S^{n-1})$ for some $q > 1$ and satisfies the cancellation condition. Then, if $f \in \mathrm{BMO}(\mathbb{R}^n)$ and $\mu_S^\rho(f)(x)$ is finite for a point $x_0 \in \mathbb{R}^n$, it follows $\mu_S^\rho(f)(x) < \infty$ a.e. on $\mathbb{R}^n$, and there is a constant C independent of f, such that*

$$\|\mu_S^\rho(f)\|_{\mathrm{BMO}(\mathbb{R}^n)} \le C\|f\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

THEOREM 5. *Let $\sigma > 0$, $0 < \alpha < 1$, and suppose that $\Omega \in L\log^+ L(S^{n-1})$ and satisfies the cancellation condition if $0 < \alpha < 1/2$, and $\Omega \in L^1(S^{n-1})$ and satisfies L^1-β-Dini condition for some $0 < \beta \le 1$ and the cancellation condition if $1/2 \le \alpha < 1$. Then, if $f \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$ for $0 < \alpha < 1/2$ or $1/2 \le \alpha < \min\{\beta, \sigma\}$ and $\mu_S^\rho(f)(x)$ is finite for a point $x_0 \in \mathbb{R}^n$, it follows $\mu_S^\rho(f)(x) < \infty$ a.e. on $\mathbb{R}^n$, and there is a constant C independent of f, such that*

$$\|\mu_S^\rho(f)\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \le C\|f\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}.$$

THEOREM 6. *Let $1 < p < \infty$, $-n/p \le \alpha < 0$. Suppose the number σ and the kernel Ω satisfy one of the following conditions*:

(i) *$\sigma > -\alpha$, $\max\{1, \frac{2n}{n+2\sigma}\} < p$, $\Omega \in L^{\max\{2,p'\}}(S^{n-1})$ and Ω satisfies the cancellation condition. In the case $p < 2$, Ω moreover satisfies L^2-$\log\beta$-Dini condition for some $\beta > 1$.*

(ii) *$\sigma > n/2$, $\Omega \in L^2(S^{n-1})$ and Ω satisfies L^2-$\log\beta$-Dini condition for some $\beta > 1$ and the cancellation condition.*

Then, if $f \in \mathcal{E}^{\alpha,p}$ and $\mu_S^\rho(f)(x)$ is finite for a point $x_0 \in \mathbb{R}^n$, it follows $\mu_S^\rho(f)(x) < \infty$ a.e. on $\mathbb{R}^n$, and there is a constant C independent of f, such that

$$\|\mu_S^\rho(f)\|_{\mathcal{E}^{\alpha,p}} \le C\|f\|_{\mathcal{E}^{\alpha,p}}.$$

THEOREM 7. *Let $\sigma > 0$, $\lambda > 1$, and suppose that $\Omega \in L^q(S^{n-1})$ for some $q > 1$ and satisfies the cancellation condition. Then, if $f \in \mathrm{BMO}(\mathbb{R}^n)$ and $\mu_\lambda^{*,\rho}(f)(x)$ is finite for a point $x_0 \in \mathbb{R}^n$, it follows $\mu_\lambda^{*,\rho}(f)(x) < \infty$ a.e. on $\mathbb{R}^n$, and there is a constant C independent of f, such that*

$$\|\mu_\lambda^{*,\rho}(f)\|_{\mathrm{BMO}(\mathbb{R}^n)} \le C\|f\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

THEOREM 8. *Let $\sigma > 0$, $0 < \alpha < 1$, and suppose that $\Omega \in L\log^+ L(S^{n-1})$ and satisfies the cancellation condition if $0 < \alpha < 1/2$, and $\Omega \in L^1(S^{n-1})$ and satisfies L^1-β-Dini condition for some $0 < \beta \le 1$*

and the cancellation condition if $1/2 \le \alpha < 1$. *Suppose that* $f \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$ *for* $0 < \alpha < 1/2$ *or* $1/2 \le \alpha < \min\{\beta, \sigma\}$ *and* $\mu_\lambda^{*,\rho}(f)(x)$ *is finite for a point* $x_0 \in \mathbb{R}^n$, $\lambda > \lambda_0$, *where* $\lambda_0 = 1$ *for* $0 < \alpha < 1/2$, *and* $\lambda_0 = 1 + 2\alpha/n$ *for* $1/2 \le \alpha < 1$. *Then* $\mu_\lambda^{*,\rho}(f)(x) < \infty$ a.e. *on* $\mathbb{R}^n$, *and there is a constant* C *independent of* f, *such that*

$$\|\mu_\lambda^{*,\rho}(f)\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \le C\|f\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}.$$

THEOREM 9. *Let* $1 < p < \infty$, $-n/p \le \alpha < 0$. *Suppose the positive numbers* σ, λ *and the kernel* Ω *satisfy one of the following conditions*:

(i) $\sigma > -\alpha$, $\max\{1, \frac{2n}{n+2\sigma}\} < p$, $\lambda > \max\{1, 2/p\}$, $\Omega \in L^{\max\{2,p'\}}(S^{n-1})$ *and* Ω *satisfies the cancellation condition. In the case* $p < 2$, Ω *moreover satisfies* L^2-$\log\beta$-*Dini condition for some* $\beta > 1$.

(ii) $\sigma > n/2$, $\lambda > 2$, $\Omega \in L^2(S^{n-1})$ *and* Ω *satisfies* L^2-$\log\beta$-*Dini condition for some* $\beta > 1$ *and the cancellation condition.*

Then, if $f \in \mathcal{E}^{\alpha,p}$ *and* $\mu_\lambda^{*,\rho}(f)(x)$ *is finite for a point* $x_0 \in \mathbb{R}^n$, *it follows* $\mu_\lambda^{*,\rho}(f)(x) < \infty$ a.e. *on* $\mathbb{R}^n$, *and there is a constant* C *independent of* f, *such that*

$$\|\mu_\lambda^{*,\rho}(f)\|_{\mathcal{E}^{\alpha,p}} \le C\|f\|_{\mathcal{E}^{\alpha,p}}.$$

To prove the above theorems we use the following key lemmas.

LEMMA 1.1. *Let* $1 \le p < \infty$. *If* $\delta > 0$ *and* $-n/p \le \alpha < \min\{1, \delta/p\}$, *then there exists* $C > 0$ *such that for any ball* $B = B(x, r)$ *and any* $f \in \mathcal{E}^{\alpha,p}$

$$\left(\int_{\mathbb{R}^n} \frac{|f(y) - f_B|^p}{(r + |y - x|)^{n+\delta}}\, dy\right)^{1/p} \le C r^{\alpha - \delta/p}\|f\|_{\mathcal{E}^{\alpha,p}}.$$

This can be proved easily by modifying the proof of Lemma 2.3 in [9]. We need also the following lemma, which is an extension of the result obtained by Kurtz and Wheeden in 1979 [13], and whose proof is found in [8].

LEMMA 1.2. *Let* $1 \le q < \infty$ *and* $\rho = \sigma + i\tau$ $(\sigma, \tau \in \mathbb{R})$ *with* $\sigma > 0$. *Suppose that* Ω *is homogeneous of degree zero and satisfies the* L^q-*Dini condition. Then, there exists* $C > 0$ *such that for any* $R > 0$ *and* $|y| < \frac{1}{2}R$,

$$\left(\int_{R<|x|<2R} \left| \frac{\Omega(x-y)}{|x-y|^{n-\rho}} - \frac{\Omega(x)}{|x|^{n-\rho}} \right|^q dx\right)^{1/q}$$
$$\le C(1 + |\tau|)\, R^{n/q-(n-\sigma)} \left\{ \|\Omega\|_{L^q(S^{n-1})} \frac{|y|}{R} + \int_{|y|/2R}^{|y|/R} \frac{\omega_q(\delta)}{\delta}\, d\delta \right\},$$

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where C is independent of R and y.

We shall use the following elementary lemma, whose proof we omit.

LEMMA 1.3. (1) *Let $\sigma > 0$. Then there exists $C > 0$ such that*

$$\int_{t-r}^{t} s^{\sigma-1}\, ds \le Crt^{\sigma-1}, \quad 0 < r < t.$$

(2) *Let $\sigma \le 0$ and $a > 1$. Then there exists $C > 0$ such that*

$$\int_{t-r}^{t} s^{\sigma-1}\, ds \le Crt^{\sigma-1}, \quad 0 < ar < t.$$

In the next section, we prepare several lemmas to prove Theorems 1–9, and in Section 3, we shall prove them. Lemmas 2.1, 2.2, 2.3 are for the proofs of Theorems 1, 2, 3. Lemmas 2.4, 2.5, 2.6 are for the proofs of Theorems 4, 5, 6. And Lemmas 2.7, 2.8, 2.9, 2.10, 2.11, 2.12 are for the proofs of Theorems 7, 8, 9.

§2. Lemmas

In this section, r is a temporarily fixed positive number. For a ball $B = B(x_0, r)$ and a function f we set always $f_1 = f_{4B}$, $f_2 = (f(y) - f_{4B})\chi_{4B}$ and $f_3 = (f(y) - f_{4B})\chi_{(4B)^c}$. To proceed as in the proof of Theorem 3 in Yabuta [29], we introduce auxiliary operators depending on r as follows. Relating to $\mu^\rho(f)$, we define the following.

DEFINITION 2.1.

$$\mu_0^\rho(f)(x) = \left(\int_0^r \left|\frac{1}{t^\rho}\int_{|y-x|\le t} \frac{\Omega(y-x)}{|y-x|^{n-\rho}} f(y)\, dy\right|^2 \frac{dt}{t}\right)^{1/2},$$

and

$$\mu_\infty^\rho(f)(x) = \left(\int_r^\infty \left|\frac{1}{t^\rho}\int_{|y-x|\le t} \frac{\Omega(y-x)}{|y-x|^{n-\rho}} f(y)\, dy\right|^2 \frac{dt}{t}\right)^{1/2}.$$

Relating to $\mu_S^\rho(f)$, we define the following.

DEFINITION 2.2.

$$\mu_{S,0}^\rho(f)(x) = \left(\int_0^r \int_{|u-x|\le t} \left|\frac{1}{t^\rho}\int_{|y-u|\le t} \frac{\Omega(u-y)f(y)}{|u-y|^{n-\rho}}\, dy\right|^2 \frac{du\, dt}{t^{n+1}}\right)^{1/2},$$

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and

$$\mu^{\rho}_{S,\infty}(f)(x) = \left(\int_r^\infty \int_{|u-x|\le t} \left| \frac{1}{t^\rho} \int_{|y-u|\le t} \frac{\Omega(u-y)f(y)}{|u-y|^{n-\rho}}\,dy \right|^2 \frac{du\,dt}{t^{n+1}}\right)^{1/2}.$$

Relating to $\mu^{*,\rho}_{\lambda}(f)$, we define the following.

DEFINITION 2.3.

$$\begin{aligned}
&\mu^{*;\rho}_{\lambda,0}(f)(x)\\
&= \left(\int_0^r \int_{\mathbb{R}^n} \left| \frac{1}{t^\rho} \int_{|y-u|\le t} \frac{\Omega(u-y)f(y)}{|u-y|^{n-\rho}}\,dy \right|^2 \left(\frac{t}{t+|u-x|}\right)^{\lambda n} \frac{du\,dt}{t^{n+1}}\right)^{1/2},\\
&\mu^{*;\rho}_{\lambda,\infty}(f)(x)\\
&= \left(\int_r^\infty \int_{\mathbb{R}^n} \left| \frac{1}{t^\rho} \int_{|y-u|\le t} \frac{\Omega(u-y)f(y)}{|u-y|^{n-\rho}}\,dy \right|^2 \left(\frac{t}{t+|u-x|}\right)^{\lambda n} \frac{du\,dt}{t^{n+1}}\right)^{1/2},\\
&\mu^{*;\rho}_{\lambda,0,0}(f)(x)\\
&= \left(\int_0^r \int_{|u-x|\le 8r} \left| \frac{1}{t^\rho} \int_{|y-u|\le t} \frac{\Omega(u-y)f(y)}{|u-y|^{n-\rho}}\,dy \right|^2 \left(\frac{t}{t+|u-x|}\right)^{\lambda n} \frac{du\,dt}{t^{n+1}}\right)^{1/2},
\end{aligned}$$

and

$$\begin{aligned}
&\mu^{*;\rho}_{\lambda,0,\infty}(f)(x)\\
&= \left(\int_0^r \int_{|u-x|> 8r} \left| \frac{1}{t^\rho} \int_{|y-u|\le t} \frac{\Omega(u-y)f(y)}{|u-y|^{n-\rho}}\,dy \right|^2 \left(\frac{t}{t+|u-x|}\right)^{\lambda n} \frac{du\,dt}{t^{n+1}}\right)^{1/2}.
\end{aligned}$$

First, we prepare three lemmas to prove Theorems 1, 2, and 3.

LEMMA 2.1. *Let $1 \le p < \infty$. Let $\Omega \in L^{p'}(S^{n-1})$, $\int_{S^{n-1}} \Omega(x)\,d\sigma(x) = 0$, $-n/p \le \alpha < 1$, and $\rho = \sigma + i\tau$ ($\sigma > 0$, $\tau \in \mathbb{R}$). Then, if $f \in \mathcal{E}^{\alpha,p}$ and $\mu^\rho(f)(x_0) < +\infty$ for some $x_0 \in \mathbb{R}^n$, there exists $C > 0$ such that for any ball $B = B(x_0, r)$*

$$\mu^\rho_\infty(f_2)(x_0) \le C(\mu^\rho(f)(x_0) + \|\Omega\|_{L^{p'}(S^{n-1})} r^\alpha \|f\|_{\mathcal{E}^{\alpha,p}}).$$

In the case $0 < \alpha < 1$, $\Omega \in L^{p'}(S^{n-1})$ and $\|\Omega\|_{L^{p'}(S^{n-1})}$ can be replaced by $\Omega \in L^1(S^{n-1})$ and $\|\Omega\|_{L^1(S^{n-1})}$.

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Proof. By assumption we have

$$\left(\int_r^{2r}\left|\frac{1}{t^\rho}\int_{|y-x_0|\le t}\frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}f(y)\,dy\right|^2\frac{dt}{t}\right)^{1/2}\le \mu^\rho(f)(x_0)<+\infty.$$

Hence, for some $r\le t_0\le 2r$ we get

$$\frac{r}{t_0}\left|\frac{1}{t_0^\rho}\int_{|y-x_0|\le t_0}\frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}f(y)\,dy\right|\le \mu^\rho(f)(x_0).$$

Since, in the above integral, the integration domain is contained in $|y-x_0|\le 4r$, we see, using the cancellation property of Ω, that the above integral is equal to

$$\int_{|y-x_0|\le t_0}\frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}(f(y)-f_{4B})\chi_{4B}\,dy.$$

Hence

$$\left|\int_{|y-x_0|\le t_0}\frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}(f(y)-f_{4B})\chi_{4B}\,dy\right|\le Cr^\sigma\mu^\rho(f)(x_0).$$

Thus for $t>r$ we have

$$\begin{aligned}
&\left|\int_{|y-x_0|\le t}\frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}f_2(y)\,dy\right|\\
&\le\left|\int_{|y-x_0|\le t_0}\frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}f_2(y)\,dy\right|\\
&\qquad+\int_{t_0<|y-x_0|<\min\{t,4r\}}\frac{|\Omega(y-x_0)||f(y)-f_{4B}|}{|y-x_0|^{n-\sigma}}\,dy\\
&\le Cr^\sigma\mu^\rho(f)(x_0)+Cr^{\sigma-n}\int_{|y-x_0|<4r}|\Omega(y-x_0)||f(y)-f_{4B}|\,dy\\
&\le Cr^\sigma\mu^\rho(f)(x_0)+Cr^{\sigma-n}\left(\int_{|y-x_0|<4r}|\Omega(y-x_0)|^{p'}\,dy\right)^{1/p'}\\
&\qquad\qquad\times\left(\int_{|y-x_0|<4r}|f(y)-f_{4B}|^p\,dy\right)^{1/p}\\
&\le Cr^\sigma\mu^\rho(f)(x_0)\\
&\qquad+Cr^{\sigma-n}r^{n/p'}\|\Omega\|_{L^{p'}(S^{n-1})}\left(\int_{|y-x_0|<4r}|f(y)-f_{4B}|^p\,dy\right)^{1/p}\\
&\le Cr^\sigma\mu^\rho(f)(x_0)+C\|\Omega\|_{L^{p'}(S^{n-1})}r^{\sigma+\alpha}\|f\|_{\mathcal{E}^{\alpha,p}}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}\mu_\infty^\rho(f_2)(x_0) &= \left(\int_r^\infty \left|\frac{1}{t^\rho}\int_{|y-x_0|\le t}\frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}f_2(y)\,dy\right|^2\frac{dt}{t}\right)^{1/2}\\ &\le Cr^\sigma(\mu^\rho(f)(x_0)+\|\Omega\|_{L^{p'}(S^{n-1})}r^\alpha\|f\|_{\mathcal{E}^{\alpha,p}})\left(\int_r^\infty\frac{dt}{t^{2\sigma+1}}\right)^{1/2}\\ &\le C(\mu^\rho(f)(x_0)+\|\Omega\|_{L^{p'}(S^{n-1})}r^\alpha\|f\|_{\mathcal{E}^{\alpha,p}}).\end{aligned}$$

In the case $0<\alpha<1$, for $|y-x_0|<4r$ and $|z-x_0|<4r$, we have $|y-z|\le|y-x_0|+|x_0-z|<8r$, and hence

$$\begin{aligned}|f(y)-f_{4B}| &= \left|\frac{1}{|4B|}\int_{4B}(f(y)-f(z))\,dz\right|\\ &\le \|f\|_{\mathrm{Lip}_\alpha}\frac{1}{|4B|}\int_{4B}|y-z|^\alpha\,dz\le C\|f\|_{\mathrm{Lip}_\alpha}r^\alpha.\end{aligned}$$

Thus,

$$\begin{aligned}&\int_{t_0<|y-x_0|<\min(t,4r)}\frac{|\Omega(y-x_0)||f(y)-f_{4B}|}{|y-x_0|^{n-\sigma}}\,dy\\ &\le C\|f\|_{\mathrm{Lip}_\alpha}r^\alpha\int_{|y-x_0|<4r}\frac{|\Omega(y-x_0)|}{|y-x_0|^{n-\sigma}}\,dy\\ &\le C\|f\|_{\mathrm{Lip}_\alpha}r^\alpha\|\Omega\|_{L^1(S^{n-1})}r^\sigma.\end{aligned}$$

Hence, pursueing the rest process in the above proof, we obtain the desired conclusion. □

LEMMA 2.2. *Let* $\Omega\in L^1(S^{n-1})$ *and* $\rho\in\mathbb{C}$. *Then, for any* $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, *for any ball* $B=B(x_0,r)$ *and any* $x\in B$

$$\mu_0^\rho(f_3)(x)=0.$$

Proof. For $|x-x_0|\le r$ and $|x-y|\le t\le r$, we have $|x_0-y|\le 2r$, and hence the integration domain with respect to y has no intersection with the support of f_3 in the expression of $\mu_0^\rho(f_3)$. So, we have $\mu_0^\rho(f_3)=0$ for $x\in B$. □

LEMMA 2.3. *Suppose one of the following three conditions is satisfied*:

(i) $\sigma>0$, $\alpha=0$, $\Omega\in L^q(S^{n-1})$ *and* Ω *satisfies* L^q-log 1-*Dini condition for some* $q>1$ *and the cancellation condition.*

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(ii) $\sigma > 0$, $0 < \beta \le 1$, $0 < \alpha < \min\{\sigma,\beta\} \le 1$, $1 \le p < \infty$, $\Omega \in L^1(S^{n-1})$ *and* Ω *satisfies* L^1-β-*Dini condition and the cancellation condition.*

(iii) $\sigma > 0$, $1 < p < \infty$ *and* $-n/p \le \alpha < 0$. *Moreover, suppose* $\Omega \in L^{p'}(S^{n-1})$ *and satisfies* $L^{p'}$-*Dini condition.*

Then there exists $C > 0$ *such that for any ball* $B = B(x_0, r)$ *and any* $f \in \mathcal{E}^{\alpha,p}$ *satisfying* $\mu^\rho_\infty(f_3)(x_0) < +\infty$, *it holds*

$$\mu^\rho_\infty(f_3)(x) < +\infty \quad \textit{and} \quad |\mu^\rho_\infty(f_3)(x) - \mu^\rho_\infty(f_3)(x_0)| \le Cr^\alpha \|f\|_{\mathcal{E}^{\alpha,p}}$$

for any $x \in B$.

Proof. For any $x \in B$ we have

$$\begin{aligned}
&|\mu^\rho_\infty(f_3)(x) - \mu^\rho_\infty(f_3)(x_0)| \\
&\le \Big(\int_r^\infty \Big|\frac{1}{t^\rho}\int_{|y-x|<t} \frac{\Omega(y-x)f_3(y)\,dy}{|y-x|^{n-\rho}} \\
&\qquad\qquad - \frac{1}{t^\rho}\int_{|y-x_0|<t} \frac{\Omega(y-x_0)f_3(y)\,dy}{|y-x_0|^{n-\rho}}\Big|^2 \frac{dt}{t}\Big)^{1/2} \\
&\le \Big(\int_r^\infty \Big(\frac{1}{t^\sigma}\int_{\substack{|y-x|<t\\|y-x_0|\ge t}} \frac{|\Omega(y-x)f_3(y)|\,dy}{|y-x|^{n-\sigma}}\Big)^2 \frac{dt}{t}\Big)^{1/2} \\
&\quad + \Big(\int_r^\infty \Big(\frac{1}{t^\sigma}\int_{\substack{|y-x|\ge t\\|y-x_0|<t}} \frac{|\Omega(y-x_0)f_3(y)|\,dy}{|y-x_0|^{n-\sigma}}\Big)^2 \frac{dt}{t}\Big)^{1/2} \\
&\quad + \Big(\int_r^\infty \frac{1}{t^{2\sigma}}\Big(\int_{\substack{|y-x|<t\\|y-x_0|<t}} \Big|\frac{\Omega(y-x)}{|y-x|^{n-\rho}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}\Big||f_3(y)|\,dy\Big)^2 \frac{dt}{t}\Big)^{1/2} \\
&=: I_1(x) + I_2(x) + I_3(x).
\end{aligned}$$

(i) In this case, $\mathcal{E}^{0,p} = \mathrm{BMO}$ $(1 \le p < \infty)$, and the norms are equivalent. Take a positive constant η with $0 < \eta < q' - 1$. For $y \in (4B)^c$, $|y - x| < t$, $x \in B$, we have $t > |y - x_0| - |x - x_0| > 3r$. Noting this, we get by Hölder's inequality, Lemma 1.3 and Lemma 1.1

$$\begin{aligned}
&\int_{\substack{|y-x|<t\\|y-x_0|\ge t}} \frac{|\Omega(y-x)f_3(y)|\,dy}{|y-x|^{n-\sigma}} \\
&\le \Big(\int_{\max\{3r,t-r\}<|y-x|<t} \frac{|\Omega(y-x)|^q\,dy}{|y-x|^{(n-\sigma)q-(n+\eta)q/q'}}\Big)^{1/q} \Big(\int_{\mathbb{R}^n} \frac{|f_3(y)|^{q'}\,dy}{|y-x|^{n+\eta}}\Big)^{1/q'}
\end{aligned}$$

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$$\leq \|\Omega\|_{L^q(S^{n-1})}\left(\int_{\max\{3r,t-r\}}^{t} s^{(n+\eta)q/q'-(n-\sigma)q+n-1}\,ds\right)^{1/q}\|f\|_{\mathrm{BMO}}r^{-\eta/q'}$$
$$\leq C\|\Omega\|_{L^q(S^{n-1})}\|f\|_{\mathrm{BMO}}r^{-\eta/q'}r^{1/q}t^{\eta/q'-1/q+\sigma}.$$

Thus, we have, noting $\eta/q'-1/q=(\eta-(q'-1))/q'<0$,

$$I_1(x)\leq C\|f\|_{\mathrm{BMO}}r^{-n/q'+1/q}\left(\int_{3r}^{\infty} t^{2\eta/q'-2/q-1}\,dt\right)^{1/2}\leq C\|f\|_{\mathrm{BMO}}.$$

Similarly, we have the same estimate for $I_2(x)$.

As for $I_3(x)$, we have by Minkowski's inequality and Hölder's inequality

$$\begin{aligned}
I_3(x) &\leq \int_{\mathbb{R}^n}\left|\frac{\Omega(y-x)}{|y-x|^{n-\rho}}-\frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}\right||f_3(y)|\left(\int_r^{\infty}\chi_{|y-x_0|<t}\frac{dt}{t^{2\sigma+1}}\right)^{1/2}dy\\
&\leq C\int_{|y-x_0|>4r}\left|\frac{\Omega(y-x)}{|y-x|^{n-\rho}}-\frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}\right|\frac{|f_3(y)|}{|y-x_0|^{\sigma}}\,dy\\
&=\sum_{j=2}^{\infty}\int_{2^jr\leq|y-x_0|<2^{j+1}r}\left|\frac{\Omega(y-x)}{|y-x|^{n-\rho}}-\frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}\right|\frac{|f_3(y)|}{|y-x_0|^{\sigma}}\,dy\\
&\leq\sum_{j=2}^{\infty}\left(\int_{2^jr\leq|y-x_0|<2^{j+1}r}\left|\frac{\Omega(y-x)}{|y-x|^{n-\rho}}-\frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}\right|^q dy\right)^{1/q}\\
&\qquad\times\left(\int_{2^jr\leq|y-x_0|<2^{j+1}r}\frac{|f_3(y)|^{q'}}{|y-x_0|^{q'\sigma}}\,dy\right)^{1/q'}.
\end{aligned}$$

Now, since $\alpha=0$ and $f\in\mathrm{BMO}$, we see easily

$$\begin{aligned}
&\left(\frac{1}{|2^{j+1}B|}\int_{2^jr\leq|y-x_0|<2^{j+1}r}\frac{|f_3(y)|^{q'}}{|y-x_0|^{q'\sigma}}\,dy\right)^{1/q'}\\
&\quad\leq\frac{1}{(2^jr)^{\sigma}}\left(\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f(y)-f_{4B}|^{q'}\,dy\right)^{1/q'}\\
&\quad\leq C\frac{\|f\|_{\mathrm{BMO}}}{(2^jr)^{\sigma}}j\leq C\|f\|_{\mathrm{BMO}}r^{-\sigma}2^{-j\sigma}j.
\end{aligned}$$

Hence, using Lemma 1.2, we get

$$\begin{aligned}
I_3(x)\leq C\sum_{j=2}^{\infty}(2^jr)^{n/q-(n-\sigma)}&\left(\|\Omega\|_{L^q(S^{n-1})}\frac{|x-x_0|}{2^jr}+\int_{\frac{|x-x_0|}{2^{j+1}r}}^{\frac{|x-x_0|}{2^jr}}\frac{\omega_q(\delta)}{\delta}\,d\delta\right)\\
&\times\|f\|_{\mathrm{BMO}}2^{-j\sigma}r^{-\sigma}(2^jr)^{n/q'}j
\end{aligned}$$

$$\le C\|f\|_{\mathrm{BMO}}\sum_{j=2}^{\infty}\Big(\|\Omega\|_{L^q(S^{n-1})}2^{-j}j+j\int_{\frac{|x-x_0|}{2^{j+1}r}}^{\frac{|x-x_0|}{2^j r}}\frac{\omega_q(\delta)}{\delta}\,d\delta\Big)$$

$$\le C\|f\|_{\mathrm{BMO}}\Big(\|\Omega\|_{L^q(S^{n-1})}+\int_0^1\frac{\omega_q(\delta)}{\delta}(1+|\log\delta|)\,d\delta\Big).$$

Summing up the estimates for $I_1(x)$, $I_2(x)$, $I_3(x)$, we obtain the desired estimate in the case (i).

(ii) In this case, $\mathcal{E}^{\alpha,p}=\mathrm{Lip}_\alpha$ $(1\le p<\infty)$, and the norms are equivalent. For $|y-x_0|\ge 4r$ and $|z-x_0|<4r$, we have $|y-z|\le|y-x|+|x-x_0|+|x_0-z|\le|y-x|+5r$, and hence we have $|f_3(y)|\le|f(y)-f_{4B}|\le 5\cdot 2^\alpha\|f\|_{\mathrm{Lip}_\alpha}(|y-x|^\alpha+r^\alpha)$. Also, $|y-x|<t$ and $|y-x_0|\ge t>r$ implies $t-r\le|y-x_0|-|x_0-x|\le|x-y|<t$. So, for I_1 we have using Lemma 1.3

$$I_1(x)\le C\|f\|_{\mathrm{Lip}_\alpha}\bigg(\int_r^\infty\bigg(\frac{1}{t^\sigma}\int_{\substack{|y-x|<t\\|y-x_0|\ge t}}\frac{|\Omega(y-x)|(|y-x|^\alpha+r^\alpha)\,dy}{|y-x|^{n-\sigma}}\bigg)^2\frac{dt}{t}\bigg)^{1/2}$$

$$\le C\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\mathrm{Lip}_\alpha}\bigg(\int_r^\infty\bigg(\frac{1}{t^\sigma}\int_{t-r}^t\frac{(s^\alpha+r^\alpha)s^{n-1}\,ds}{s^{n-\sigma}}\bigg)^2\frac{dt}{t}\bigg)^{1/2}$$

$$\le C\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\mathrm{Lip}_\alpha}\bigg(\int_r^\infty\bigg(\frac{1}{t^\sigma}(rt^{\sigma+\alpha-1}+r^\alpha rt^{\sigma-1})\bigg)^2\frac{dt}{t}\bigg)^{1/2}$$

$$\le C\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\mathrm{Lip}_\alpha}r^\alpha.$$

Similarly, we can get the same estimate for $I_2(x)$.

As for $I_3(x)$, we see easily as before that $|f_3(y)|\le C\|f\|_{\mathrm{Lip}_\alpha}t^\alpha$ for $|y-x|<t$, $|y-x_0|<t$ and $t>r$. Using Lemma 1.2 we have

$$J(r,t):=\int_{\substack{|y-x|<t\\|y-x_0|<t}}\bigg|\frac{\Omega(y-x)}{|y-x|^{n-\rho}}-\frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}\bigg||f_3(y)|\,dy$$

$$\le C\|f\|_{\mathrm{Lip}_\alpha}t^\alpha\int_{4r<|y-x_0|<2t}\bigg|\frac{\Omega(y-x)}{|y-x|^{n-\rho}}-\frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}\bigg|\,dy$$

$$\le C\|f\|_{\mathrm{Lip}_\alpha}t^\alpha\sum_{k=2}^{[\log_2\frac{t}{r}]+1}\int_{2^kr\le|y-x_0|<2^{k+1}r}\bigg|\frac{\Omega(y-x)}{|y-x|^{n-\rho}}-\frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}\bigg|\,dy$$

$$\le C\|f\|_{\mathrm{Lip}_\alpha}t^\alpha\sum_{k=2}^{[\log_2\frac{t}{r}]+1}(2^kr)^{n-(n-\sigma)}\bigg(\|\Omega\|_{L^1(S^{n-1})}\frac{r}{2^kr}+\int_{r/(2^{k+1}r)}^{r/(2^kr)}\frac{\omega_1(\delta)}{\delta}\,d\delta\bigg)$$

$$\le C\|f\|_{\mathrm{Lip}_\alpha}t^\alpha\sum_{k=2}^{[\log_2\frac{t}{r}]+1}r^\sigma\bigg(\|\Omega\|_{L^1(S^{n-1})}2^{(\sigma-1)k}+2^{(\sigma-\beta)k}\int_{2^{-k-1}}^{2^{-k}}\frac{\omega_1(\delta)}{\delta^{1+\beta}}\,d\delta\bigg).$$

To continue estimating $J(r,t)$, we treat the following four cases.

(a) $\sigma > 1$. Since $\sigma > 1$, we see that

$$\sum_{k=2}^{[\log_2 \frac{t}{r}]+1} (2^{(\sigma-1)k} + 2^{(\sigma-\beta)k}) \le C\Big(\Big(\frac{t}{r}\Big)^{\sigma-1} + \Big(\frac{t}{r}\Big)^{\sigma-\beta}\Big),$$

and hence we have

$$J(r,t) \le C\|f\|_{\mathrm{Lip}_\alpha}(rt^{\sigma+\alpha-1} + r^{\beta}t^{\sigma+\alpha-\beta}).$$

Thus, noting $\alpha < \beta \le 1$, we have

$$I_3(x) \le C\|f\|_{\mathrm{Lip}_\alpha}\Big(\int_r^\infty (r^2t^{2\alpha-2} + r^{2\beta}t^{2\alpha-2\beta})\frac{dt}{t}\Big)^{1/2} \le C\|f\|_{\mathrm{Lip}_\alpha} r^{\alpha}.$$

(b) $\sigma = 1$. In this case, we have

$$\begin{aligned} J(r,t) &\le C\|f\|_{\mathrm{Lip}_\alpha} t^{\alpha} r^{\sigma}\Big(\|\Omega\|_{L^1(S^{n-1})}\log_2\frac{t}{r} + \Big(\frac{t}{r}\Big)^{\sigma-\beta}\int_0^1 \frac{\omega_1(\delta)}{\delta^{1+\beta}}\,d\delta\Big) \\ &\le C\|f\|_{\mathrm{Lip}_\alpha} t^{\alpha} r^{\sigma}\Big(\log_2\frac{t}{r} + \Big(\frac{t}{r}\Big)^{\sigma-\beta}\Big). \end{aligned}$$

Thus, noting $\alpha < \beta \le 1$, we have

$$\begin{aligned} I_3(x) &\le C\|f\|_{\mathrm{Lip}_\alpha}\Big(\int_r^\infty t^{2\alpha}\Big(\Big(\frac{r}{t}\Big)^{2\sigma}\log_2^2\frac{t}{r} + \Big(\frac{r}{t}\Big)^{2\beta}\Big)\frac{dt}{t}\Big)^{1/2} \\ &\le C\|f\|_{\mathrm{Lip}_\alpha} r^{\alpha}\Big(\int_1^\infty \Big(\frac{1}{s^{2\sigma-2\alpha+1}}\log_2^2 s + \frac{1}{s^{2\beta-2\alpha+1}}\Big)ds\Big)^{1/2} \\ &= C\|f\|_{\mathrm{Lip}_\alpha} r^{\alpha}. \end{aligned}$$

(c) $0 < \beta < \sigma < 1$. In this case, we have

$$J(r,t) \le C\|f\|_{\mathrm{Lip}_\alpha} t^{\alpha} r^{\sigma}\Big(\|\Omega\|_{L^1(S^{n-1})} + \Big(\frac{t}{r}\Big)^{\sigma-\beta}\int_0^1 \frac{\omega_1(\delta)}{\delta^{1+\beta}}\,d\delta\Big).$$

Thus, noting $\alpha < \beta < \sigma$, we have

$$\begin{aligned} I_3(x) &\le C\|f\|_{\mathrm{Lip}_\alpha}\Big(\int_r^\infty t^{2\alpha}\Big(\Big(\frac{r}{t}\Big)^{2\sigma} + \Big(\frac{r}{t}\Big)^{2\beta}\Big)\frac{dt}{t}\Big)^{1/2} \\ &\le C\|f\|_{\mathrm{Lip}_\alpha} r^{\alpha}\Big(\int_1^\infty \Big(\frac{1}{s^{2\sigma-2\alpha+1}} + \frac{1}{s^{2\beta-2\alpha+1}}\Big)ds\Big)^{1/2} \\ &= C\|f\|_{\mathrm{Lip}_\alpha} r^{\alpha}. \end{aligned}$$

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(d) $0 < \sigma \le \beta \le 1$ and $\sigma < 1$. In this case, we have

$$J(r,t) \le C\|f\|_{\mathrm{Lip}_\alpha} t^\alpha r^\sigma \biggl(\|\Omega\|_{L^1(S^{n-1})} + \int_0^1 \frac{\omega_1(\delta)}{\delta^{1+\beta}}\, d\delta \biggr).$$

Thus, noting $\alpha < \sigma$, we have

$$\begin{aligned} I_3(x) &\le C\|f\|_{\mathrm{Lip}_\alpha} \biggl(\int_r^\infty t^{2\alpha} \Bigl(\frac{r}{t}\Bigr)^{2\sigma} \frac{dt}{t} \biggr)^{1/2} \\ &\le C\|f\|_{\mathrm{Lip}_\alpha} r^\alpha \biggl(\int_1^\infty \frac{1}{s^{2\sigma-2\alpha+1}}\, ds \biggr)^{1/2} = C\|f\|_{\mathrm{Lip}_\alpha} r^\alpha. \end{aligned}$$

Summing up the estimates for $I_1(x)$, $I_2(x)$, $I_3(x)$, we obtain the desired estimate in the case (ii).

(iii) Take a positive constant η with $0 < \eta < p-1$. For $y \in (4B)^c$, $|y-x| < t$, $x \in B$, we have $t > |y-x_0| - |x-x_0| > 3r$. Noting this, we get by Hölder's inequality, Lemma 1.3 and Lemma 1.1

$$\begin{aligned} &\int_{\substack{|y-x|<t \\ |y-x_0|\ge t}} \frac{|\Omega(y-x) f_3(y)|\, dy}{|y-x|^{n-\sigma}} \\ &\le \biggl(\int_{\max\{3r,t-r\}<|y-x|<t} \frac{|\Omega(y-x)|^{p'}\, dy}{|y-x|^{(n-\sigma)p'-(n+\eta)p'/p}} \biggr)^{1/p'} \biggl(\int_{\mathbb{R}^n} \frac{|f_3(y)|^p\, dy}{|y-x|^{n+\eta}} \biggr)^{1/p} \\ &\le \|\Omega\|_{L^{p'}(S^{n-1})} \biggl(\int_{\max\{3r,t-r\}}^t s^{(n+\eta)p'/p-(n-\sigma)p'+n-1}\, ds \biggr)^{1/p'} \|f\|_{\mathcal{E}^{\alpha,p}} r^{\alpha-\eta/p} \\ &\le C\|\Omega\|_{L^{p'}(S^{n-1})} \|f\|_{\mathcal{E}^{\alpha,p}} r^{\alpha-\eta/p} r^{1/p'} t^{\eta/p-1/p'+\sigma}. \end{aligned}$$

Thus, we have, noting $\eta/p - 1/p' = (\eta-(p-1))/p < 0$,

$$I_1(x) \le C\|f\|_{\mathcal{E}^{\alpha,p}} r^{\alpha-\eta/p+1/p'} \biggl(\int_{3r}^\infty t^{2\eta/p-2/p'-1}\, dt \biggr)^{1/2} \le C\|f\|_{\mathcal{E}^{\alpha,p}} r^\alpha.$$

Similarly, we have the same estimate for $I_2(x)$.

As for $I_3(x)$, we have by Minkowski's inequality and Hölder's inequality

$$\begin{aligned} I_3(x) &\le \int_{\mathbb{R}^n} \biggl| \frac{\Omega(y-x)}{|y-x|^{n-\rho}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}} \biggr| |f_3(y)| \biggl(\int_r^\infty \chi_{|y-x_0|<t} \frac{dt}{t^{2\sigma+1}} \biggr)^{1/2} dy \\ &\le C \int_{|y-x_0|>4r} \biggl| \frac{\Omega(y-x)}{|y-x|^{n-\rho}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}} \biggr| \frac{|f_3(y)|}{|y-x_0|^\sigma}\, dy \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=2}^{\infty} \int_{2^j r \le |y-x_0| < 2^{j+1} r} \left| \frac{\Omega(y-x)}{|y-x|^{n-\rho}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}} \right| \frac{|f_3(y)|}{|y-x_0|^{\sigma}} \, dy \\
&\le \sum_{j=2}^{\infty} \left(\int_{2^j r \le |y-x_0| < 2^{j+1} r} \left| \frac{\Omega(y-x)}{|y-x|^{n-\rho}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}} \right|^{p'} dy \right)^{1/p'} \\
&\qquad \times \left(\int_{2^j r \le |y-x_0| < 2^{j+1} r} \frac{|f_3(y)|^p}{|y-x_0|^{p\sigma}} \, dy \right)^{1/p}.
\end{aligned}
$$

Now, since $\alpha < 0$ and $f \in \mathcal{E}^{\alpha,p}$, we see easily

$$
\begin{aligned}
&\left(\frac{1}{|2^{j+1}B|} \int_{2^j r \le |y-x_0| < 2^{j+1} r} \frac{|f_3(y)|^p}{|y-x_0|^{p\sigma}} \, dy \right)^{1/p} \\
&\le \frac{1}{(2^j r)^{\sigma}} \left(\frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |f(y) - f_{4B}|^p \, dy \right)^{1/p} \\
&\le C \|f\|_{\mathcal{E}^{\alpha,p}} r^{\alpha-\sigma} 2^{-j\sigma}.
\end{aligned}
$$

Hence, using Lemma 1.2, we get

$$
\begin{aligned}
I_3(x) &\le C \sum_{j=2}^{\infty} (2^j r)^{n/p' - (n-\sigma)} \left(\|\Omega\|_{L^{p'}(S^{n-1})} \frac{|x-x_0|}{2^j r} + \int_{\frac{|x-x_0|}{2^{j+1} r}}^{\frac{|x-x_0|}{2^j r}} \frac{\omega_{p'}(\delta)}{\delta} \, d\delta \right) \\
&\qquad \times \|f\|_{\mathcal{E}^{\alpha,p}} 2^{-j\sigma} r^{\alpha-\sigma} (2^j r)^{n/p} \\
&\le C r^{\alpha} \|f\|_{\mathcal{E}^{\alpha,p}} \sum_{j=2}^{\infty} \left(\|\Omega\|_{L^{p'}(S^{n-1})} 2^{-j} + \int_{\frac{|x-x_0|}{2^{j+1} r}}^{\frac{|x-x_0|}{2^j r}} \frac{\omega_{p'}(\delta)}{\delta} \, d\delta \right) \\
&\le C r^{\alpha} \|f\|_{\mathcal{E}^{\alpha,p}} \left(\|\Omega\|_{L^{p'}(S^{n-1})} + \int_0^1 \frac{\omega_{p'}(\delta)}{\delta} \, d\delta \right).
\end{aligned}
$$

Summing up the estimates for $I_1(x)$, $I_2(x)$, $I_3(x)$, we obtain the desired estimate in the case (iii). ∎

Next, we prepare three lemmas for the proofs of Theorems 4, 5, and 6. As for $\mu^{\rho}_{S,\infty}(f_2)$ we have

LEMMA 2.4. *Let $\rho = \sigma + i\tau$ with $\sigma > 0$ and $\tau \in \mathbb{R}$.*

(a) *Let $\max\{1, \frac{2n}{n+2\sigma}\} < p < +\infty$ and $-n/p \le \alpha < 0$. Moreover, $\Omega \in L^{p_0'}(S^{n-1})$ for $p_0 = \min\{p, 2\}$. Then, for any $f \in \mathcal{E}^{\alpha,p}$, any ball $B = B(x_0, r)$ and any $x \in \mathbb{R}^n$*

$$
\mu^{\rho}_{S,\infty}(f_2)(x) \le C \|\Omega\|_{L^{p_0'}(S^{n-1})} r^{\alpha} \|f\|_{\mathcal{E}^{\alpha,p}}.
$$

(b) *Let* $1 \le p < \infty$, $\alpha = 0$ *and* $\Omega \in L^q(S^{n-1})$ *for some* $q > 1$. *Then, for any* $f \in \mathcal{E}^{\alpha,p}$, *any ball* $B = B(x_0, r)$ *and any* $x \in \mathbb{R}^n$

$$\mu^{\rho}_{S,\infty}(f_2)(x) \le C\|\Omega\|_{L^q(S^{n-1})}\|f\|_{\mathcal{E}^{\alpha,p}}.$$

(c) *Let* $1 \le p < \infty$, $0 < \alpha < 1$, *and* $\Omega \in L^1(S^{n-1})$. *Then, for any* $f \in \mathcal{E}^{\alpha,p}$, *any ball* $B = B(x_0, r)$ *and any* $x \in \mathbb{R}^n$

$$\mu^{\rho}_{S,\infty}(f_2)(x) \le C\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\mathcal{E}^{\alpha,p}} r^{\alpha}.$$

Since one can prove this lemma more easily than the corresponding lemma for $\mu^{*,\rho}_{\lambda,\infty}(f_2)(x)$, we omit the proof of this lemma.

Lemma 2.5. *Let* $\Omega \in L^1(S^{n-1})$, $\rho \in \mathbb{C}$. *Then, for any* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, *for any ball* $B = B(x_0, r)$ *and any* $x \in B$

$$\mu^{\rho}_{S,0}(f_3)(x) = \left(\int_0^r \int_{|u-x|\le t} \left| \frac{1}{t^{\rho}} \int_{|y-u|\le t} \frac{\Omega(u-y) f_3(y)}{|u-y|^{n-\rho}}\, dy \right|^2 \frac{du\, dt}{t^{n+1}} \right)^{1/2} = 0.$$

Proof. For $|x - x_0| \le r$, $|u - x| \le t \le r$ and $|u - y| \le t \le r$, we have $|x_0 - y| \le |x_0 - x| + |x - u| + |u - y| \le 3r$, and hence the integration domain with respect to y has no intersection with the support of f_3 in the expression of $\mu^{\rho}_{S,0}(f_3)$. So, we have $\mu^{\rho}_{S,0}(f_3) = 0$ for $x \in B$. $\square$

Lemma 2.6. *Suppose one of the following five conditions is satisfied*:

(i) $\sigma > 0$, $\alpha = 0$, $1 \le p < \infty$. *Moreover,* $\Omega \in L^q(S^{n-1})$ *for some* $q > 1$ *and satisfies the cancellation condition.*

(ii) $\sigma > 0$, $0 < \beta \le 1$, $0 < \alpha < \min\{1/2, \min\{\beta, \sigma\}\}$, *and* $1 \le p < \infty$. *Moreover,* $\Omega \in L^1(S^{n-1})$ *and satisfies the cancellation condition.*

(iii) $\sigma > 0$, $0 < \beta \le 1$, $1/2 \le \alpha < \min\{\beta, \sigma\}$, *and* $1 \le p < \infty$. *Moreover,* $\Omega \in L^1(S^{n-1})$ *and satisfies* L^1*-*β*-Dini condition and the cancellation condition.*

(iv) $\max\{1, \frac{2n}{n+2\sigma}\} < p < +\infty$, $-n/p < \alpha < 0$, *and* $\sigma > -\alpha$. *Moreover,* $\Omega \in L^{p_0'}(S^{n-1})$ *and satisfies the cancellation condition, where* $p_0 = \max\{2, p\}$.

(v) $1 < p < \infty$, $-n/p \le \alpha < 0$, *and* $\sigma > n/2$. *Moreover,* $\Omega \in L^2(S^{n-1})$ *and satisfies* L^2*-*$\log \beta$ *Dini condition for some* $\beta > 1$ *and the cancellation condition.*

Then there exists $C > 0$ such that for any ball $B = B(x_0, r)$ and any $f \in \mathcal{E}^{\alpha,p}$ satisfying $\mu^{\rho}_{S,\infty}(f_3)(x_0) < +\infty$, it holds

$$\mu^{\rho}_{S,\infty}(f_3)(x) < +\infty \quad \textit{and} \quad |\mu^{\rho}_{S,\infty}(f_3)(x) - \mu^{\rho}_{S,\infty}(f_3)(x_0)| \le Cr^{\alpha}\|f\|_{\mathcal{E}^{\alpha,p}}$$

for any $x \in B$, where $f_3(x) = (f(x) - f_{4B})\chi_{(4B)^c}$.

Since one can prove this lemma more easily than the corresponding lemma for $\mu^{*,\rho}_{\lambda,\infty}(f_3)(x)$, we omit the proof of this lemma.

Finally, we prepare six lemmas for the proofs of Theorems 7, 8, and 9.

LEMMA 2.7. *Let $\rho = \sigma + i\tau$ ($\sigma > 0$, $\tau \in \mathbb{R}$), and $\lambda > 1$.*

(a) *Let $\max\{1, \frac{2n}{n+2\sigma}\} < p < +\infty$ and $-n/p \le \alpha < 0$. Moreover, $\Omega \in L^{p_0'}(S^{n-1})$ for $p_0 = \min\{p, 2\}$. Then, for any $f \in \mathcal{E}^{\alpha,p}$, any ball $B = B(x_0, r)$ and any $x \in \mathbb{R}^n$*

$$\mu^{*,\rho}_{\lambda,\infty}(f_2)(x) \le C\|\Omega\|_{L^{p_0'}(S^{n-1})} r^{\alpha}\|f\|_{\mathcal{E}^{\alpha,p}}.$$

(b) *Let $1 \le p < \infty$, $\alpha = 0$ and $\Omega \in L^q(S^{n-1})$ for some $q > 1$. Then, for any $f \in \mathcal{E}^{\alpha,p}$, any ball $B = B(x_0, r)$ and any $x \in \mathbb{R}^n$*

$$\mu^{*,\rho}_{\lambda,\infty}(f_2)(x) \le C\|\Omega\|_{L^q(S^{n-1})}\|f\|_{\mathcal{E}^{\alpha,p}}.$$

(c) *Let $1 \le p < \infty$, $0 < \alpha < 1$, and $\Omega \in L^1(S^{n-1})$. Then, for any $f \in \mathcal{E}^{\alpha,p}$, any ball $B = B(x_0, r)$ and any $x \in \mathbb{R}^n$*

$$\mu^{*,\rho}_{\lambda,\infty}(f_2)(x) \le C\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\mathcal{E}^{\alpha,p}} r^{\alpha}.$$

Proof. Proof of (a).

(i) The case $0 < \sigma < n$ and $\max\{1, \frac{2n}{n+2\sigma}\} < p < +\infty$. Since the support of f_2 is contained in $4B = \{|y - x_0| < 4r\}$ we see easily

$$\begin{aligned}
&\left|\int_{r<|y-u|\le t} \frac{\Omega(u-y)f_2(y)}{|u-y|^{n-\rho}}\,dy\right| \\
&\le \left(\int_{r<|y-u|\le t} \frac{|\Omega(u-y)|^{p'}\chi_{4B}}{|u-y|^{p'(n-\sigma)}}\,dy\right)^{1/p'} \left(\int_{|y-x_0|\le 4r} |f(y) - f_{4B}|^p\,dy\right)^{1/p} \\
&\le C\|\Omega\|_{L^{p'}(S^{n-1})}\left(\int_{\max\{r,|u-x_0|-4r\}}^{\min\{t,|u-x_0|+4r\}} s^{-(n-\sigma)p'+n-1}\,ds\right)^{1/p'} r^{\alpha+n/p}\|f\|_{\mathcal{E}^{\alpha,p}} \\
&\le C\|\Omega\|_{L^{p_0'}(S^{n-1})}\left(r\max\{t^{n-(n-\sigma)p'-1}, r^{n-(n-\sigma)p'-1}\}\right)^{1/p'} r^{\alpha+n/p}\|f\|_{\mathcal{E}^{\alpha,p}}.
\end{aligned}$$

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In the last inequality, we use the fact: $L^{p_0'}(S^{n-1}) \subset L^{p'}(S^{n-1})$ by $p_0' \geq p'$ and $\|\Omega\|_{L^{p'}(S^{n-1})} \leq C\|\Omega\|_{L^{p_0'}(S^{n-1})}$. In the case $n-(n-\sigma)p'-1 \geq 0$, we have $\max\{t^{n-(n-\sigma)p'-1}, r^{n-(n-\sigma)p'-1}\} = t^{n-(n-\sigma)p'-1}$, and hence

$$\begin{aligned}
&\left(\int_r^\infty \int_{\mathbb{R}^n} \left|\frac{1}{t^\rho}\int_{r<|y-u|\leq t} \frac{\Omega(u-y)f_2(y)}{|u-y|^{n-\rho}}\,dy\right|^2 \left(1+\frac{|u-x|}{t}\right)^{-\lambda n}\frac{du\,dt}{t^{n+1}}\right)^{1/2}\\
&\quad\leq C\|\Omega\|_{L^{p'}(S^{n-1})}\|f\|_{\mathcal{E}^{\alpha,p}} r^{\alpha+\frac{n}{p}}\, r^{\frac{1}{p'}}\\
&\qquad\times\left(\int_r^\infty\int_{\mathbb{R}^n}\left(1+\frac{|u-x|}{t}\right)^{-\lambda n}\frac{du}{t^n}\frac{dt}{t^{2n(1-\frac{1}{p'})+\frac{2}{p'}+1}}\right)^{1/2}\\
&\quad\leq C\|\Omega\|_{L^{p'}(S^{n-1})}\|f\|_{\mathcal{E}^{\alpha,p}} r^{\alpha+\frac{n}{p}+\frac{1}{p'}}\left(\int_r^\infty \frac{dt}{t^{2n(1-\frac{1}{p'})+\frac{2}{p'}+1}}\right)^{1/2}\\
&\quad\leq Cr^\alpha\|\Omega\|_{L^{p_0'}(S^{n-1})}\|f\|_{\mathcal{E}^{\alpha,p}}.
\end{aligned}$$

And in the case $n-(n-\sigma)p'-1 < 0$, we have $\max\{t^{n-(n-\sigma)p'-1}, r^{n-(n-\sigma)p'-1}\} = r^{n-(n-\sigma)p'-1}$, and hence

$$\begin{aligned}
&\left(\int_r^\infty \int_{\mathbb{R}^n} \left|\frac{1}{t^\rho}\int_{r<|y-u|\leq t} \frac{\Omega(u-y)f_2(y)}{|u-y|^{n-\rho}}\,dy\right|^2 \left(1+\frac{|u-x|}{t}\right)^{-\lambda n}\frac{du\,dt}{t^{n+1}}\right)^{1/2}\\
&\quad\leq C\|\Omega\|_{L^{p'}(S^{n-1})}\|f\|_{\mathcal{E}^{\alpha,p}} r^{\alpha+\frac{n}{p}}\, r^{\frac{1}{p'}}\, r^{\frac{n}{p'}-(n-\sigma)-\frac{1}{p'}}\\
&\qquad\times\left(\int_r^\infty\int_{\mathbb{R}^n}\left(1+\frac{|u-x|}{t}\right)^{-\lambda n}\frac{du}{t^n}\frac{dt}{t^{2\sigma+1}}\right)^{1/2}\\
&\quad\leq C\|\Omega\|_{L^{p'}(S^{n-1})}\|f\|_{\mathcal{E}^{\alpha,p}} r^{\alpha+\sigma}\left(\int_r^\infty \frac{dt}{t^{2\sigma+1}}\right)^{1/2}\\
&\quad\leq Cr^\alpha\|\Omega\|_{L^{p_0'}(S^{n-1})}\|f\|_{\mathcal{E}^{\alpha,p}}.
\end{aligned}$$

So, we need only to show

$$\begin{aligned}
I &:= \left(\int_r^\infty \int_{\mathbb{R}^n} \left|\frac{1}{t^\rho}\int_{|y-u|\leq r} \frac{\Omega(u-y)f_2(y)}{|u-y|^{n-\rho}}\,dy\right|^2 \left(1+\frac{|u-x|}{t}\right)^{-\lambda n}\frac{du\,dt}{t^{n+1}}\right)^{1/2}\\
&\leq Cr^\alpha\|\Omega\|_{L^{p_0'}(S^{n-1})}\|f\|_{\mathcal{E}^{\alpha,p}}.
\end{aligned}$$

Since $p > \frac{2n}{n+2\sigma}$, we have

$$\frac{n}{2(n-\sigma)} - \left(1-\frac{n}{n-\sigma}\left(1-\frac{1}{p}\right)\right) = \frac{n+2\sigma}{2p(n-\sigma)}\left(p-\frac{2n}{n+2\sigma}\right) > 0.$$

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So, we take $p_0 = \min\{2, p\}$ and choose a real number a so that

$$\min\left\{1, \frac{n}{2(n-\sigma)}\right\} > a > 1 - \frac{n}{(n-\sigma)p_0'}, \quad \text{where} \quad \frac{1}{p_0} + \frac{1}{p_0'} = 1.$$

Then, noting $0 < (n-\sigma)(1-a)p_0' < n$ we have by Hölder's inequality

$$\begin{aligned}
&\left|\int_{|y-u|\le r} \frac{\Omega(u-y)f_2(y)}{|u-y|^{n-\rho}}\,dy\right| \\
&\le \left(\int_{|y-u|\le r} \frac{|\Omega(u-y)|^{p_0'}\,dy}{|u-y|^{(n-\sigma)(1-a)p_0'}}\right)^{1/p_0'} \left(\int_{|y-u|\le r} \frac{|f_2(y)|^{p_0}\,dy}{|u-y|^{(n-\sigma)ap_0}}\right)^{1/p_0} \\
&\le C\|\Omega\|_{L^{p_0'}(S^{n-1})} r^{\frac{n}{p_0'}-(n-\sigma)(1-a)} \left(\int_{|y-u|\le r} \frac{|f_2(y)|^{p_0}\,dy}{|u-y|^{(n-\sigma)ap_0}}\right)^{1/p_0}.
\end{aligned}$$

Hence by Minkowski's inequality $(2/p_0 \ge 1)$ and by using $2a(n-\sigma) < n$ we get

$$\begin{aligned}
&I/\|\Omega\|_{L^{p_0'}(S^{n-1})} \le Cr^{\frac{n}{p_0'}-(n-\sigma)(1-a)} \\
&\quad\times \left(\int_r^\infty \left(\int_{4B}\left(\int_{|y-u|\le r} \frac{\left(\frac{t}{t+|u-x|}\right)^{\lambda n} du}{|u-y|^{2(n-\sigma)a}}\right)^{p_0/2} |f_2(y)|^{p_0}\,dy\right)^{2/p_0} \frac{dt}{t^{2\sigma+n+1}}\right)^{1/2} \\
&\le Cr^{\frac{n}{p_0'}-(n-\sigma)(1-a)} r^{\frac{n}{2}-(n-\sigma)a} \left(\int_r^\infty \left(\int_{4B} |f(y)-f_{4B}|^{p_0}\,dy\right)^{2/p_0} \frac{dt}{t^{2\sigma+n+1}}\right)^{1/2} \\
&\le Cr^{\frac{n}{p_0'}+\frac{n}{2}-(n-\sigma)} r^{\alpha+\frac{n}{p_0}} \|f\|_{\mathcal{E}^{\alpha,p_0}} r^{-\sigma-\frac{n}{2}} \le Cr^\alpha \|f\|_{\mathcal{E}^{\alpha,p}}.
\end{aligned}$$

(ii) The case $\sigma \ge n$. In this case we see easily

$$\begin{aligned}
&\left|\int_{|y-u|\le t} \frac{\Omega(u-y)f_2(y)}{|u-y|^{n-\rho}}\,dy\right| \le t^{\sigma-n} \int_{|y-u|\le t} |\Omega(u-y)||f_2(y)|\,dy \\
&\le t^{\sigma-n} \left(\int_{|y-u|\le t} |\Omega(u-y)|^{p'}\,dy\right)^{1/p'} \left(\int_{|y-x_0|\le 4r} |f(y)-f_{4B}|^p\,dy\right)^{1/p} \\
&\le Ct^{\sigma-n+n/p'} \|\Omega\|_{L^{p_0'}(S^{n-1})} r^{\alpha+n/p} \|f\|_{\mathcal{E}^{\alpha,p}}.
\end{aligned}$$

Using this, we get

$$
\begin{aligned}
&\mu^{*,\rho}_{\lambda,\infty}(f_2)(x)/(\|\Omega\|_{L^{p_0'}(S^{n-1})}\|f\|_{\mathcal{E}^{\alpha,p}})\\
&\quad\le Cr^{\alpha+n/p}\Big(\int_r^\infty\int_{\mathbb{R}^n} t^{2(-n+\frac{n}{p'})}\Big(1+\frac{|u-x|}{t}\Big)^{-\lambda n}\frac{du\,dt}{t^{n+1}}\Big)^{1/2}\\
&\quad\le Cr^{\alpha+n/p}\Big(\int_r^\infty\Big(\int_{\mathbb{R}^n}\Big(1+\frac{|u-x|}{t}\Big)^{-\lambda n}\frac{du}{t^n}\Big)\frac{dt}{t^{2n(1-\frac{1}{p'})+1}}\Big)^{1/2}\\
&\quad\le Cr^{\alpha+n/p}\,r^{-n(1-\frac{1}{p'})}\le Cr^{\alpha}.
\end{aligned}
$$

Proof of (b). In this case, $\mathcal{E}^{\alpha,p}=\mathrm{BMO}$ $(1\le p<\infty)$, and the norms are equivalent. Choose a positive number q_0 such that $q_0\le q$ and $(n-\sigma)q_0<n$. Then, using the Hölder inequality, the increasingness of $s^{n-(n-\sigma)q_0}$ and the decreasingness of s^{-1}, we have

$$
\begin{aligned}
&\Big|\int_{r<|y-u|\le t}\frac{\Omega(u-y)f_2(y)}{|u-y|^{n-\rho}}\,dy\Big|\\
&\quad\le\Big(\int_{r<|y-u|\le t}\frac{|\Omega(u-y)|^{q_0}\chi_{4B}}{|u-y|^{(n-\sigma)q_0}}\,dy\Big)^{1/q_0}\Big(\int|f_2(y)|^{q_0'}\,dy\Big)^{1/q_0'}\\
&\quad\le C\|\Omega\|_{L^{q_0}(S^{n-1})}\Big(\int_{\max\{r,|u-x_0|-4r\}}^{\min\{t,|u-x_0|+4r\}}s^{-(n-\sigma)q_0+n-1}\,ds\Big)^{1/q_0}\|f\|_{\mathrm{BMO}}r^{n/q_0'}\\
&\quad\le C\|\Omega\|_{L^{q_0}(S^{n-1})}\|f\|_{\mathrm{BMO}}r^{n/q_0'}\,t^{n/q_0-(n-\sigma)}\Big(\int_{\max\{r,|u-x_0|-4r\}}^{\min\{t,|u-x_0|+4r\}}s^{-1}\,ds\Big)^{1/q_0}\\
&\quad\le C\|\Omega\|_{L^{q_0}(S^{n-1})}\|f\|_{\mathrm{BMO}}r^{n/q_0'}\,t^{n/q_0-(n-\sigma)}(8r\cdot r^{-1})^{1/q_0}\\
&\quad\le C\|\Omega\|_{L^{q}(S^{n-1})}\|f\|_{\mathrm{BMO}}r^{n/q_0'}\,t^{\sigma-n/q_0'}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\Big(\int_r^\infty\int_{\mathbb{R}^n}\Big|\frac{1}{t^\rho}\int_{r<|y-u|\le t}\frac{\Omega(u-y)f_2(y)}{|u-y|^{n-\rho}}\,dy\Big|^2\Big(1+\frac{|u-x|}{t}\Big)^{-\lambda n}\frac{du\,dt}{t^{n+1}}\Big)^{1/2}\\
&\quad\le C\|\Omega\|_{L^{q}(S^{n-1})}\|f\|_{\mathrm{BMO}}r^{n/q_0'}\\
&\qquad\times\Big(\int_r^\infty\Big(\int_{\mathbb{R}^n}\Big(1+\frac{|u-x|}{t}\Big)^{-\lambda n}\frac{du}{t^n}\Big)t^{-2n/q_0'}\frac{dt}{t}\Big)^{1/2}\\
&\quad\le C\|\Omega\|_{L^{q}(S^{n-1})}\|f\|_{\mathrm{BMO}}r^{n/q_0'}\,r^{-n/q_0'}\le C\|\Omega\|_{L^{q}(S^{n-1})}\|f\|_{\mathrm{BMO}}.
\end{aligned}
$$

Further,

$$\begin{aligned}
&\left|\int_{|y-u|\le r} \frac{\Omega(u-y)f_2(y)}{|u-y|^{n-\rho}}\,dy\right| \\
&\le \left(\int_{|y-u|\le r} \frac{|\Omega(u-y)|^{q_0}\chi_{4B}}{|u-y|^{(n-\sigma)q_0}}\,dy\right)^{1/q_0}\left(\int |f_2(y)|^{q_0'}\,dy\right)^{1/q_0'} \\
&\le C\|\Omega\|_{L^{q_0}(S^{n-1})} r^{\sigma-n+n/q_0}\|f\|_{\mathrm{BMO}} r^{n/q_0'} \\
&\le C\|\Omega\|_{L^{q}(S^{n-1})}\|f\|_{\mathrm{BMO}} r^{\sigma}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left(\int_r^\infty\int_{\mathbb{R}^n}\left|\frac{1}{t^\rho}\int_{|y-u|\le r}\frac{\Omega(u-y)f_2(y)}{|u-y|^{n-\rho}}\,dy\right|^2\left(1+\frac{|u-x|}{t}\right)^{-\lambda n}\frac{du\,dt}{t^{n+1}}\right)^{1/2} \\
&\le C\|\Omega\|_{L^q(S^{n-1})}\|f\|_{\mathrm{BMO}} r^\sigma\left(\int_r^\infty\left(\int_{\mathbb{R}^n}\left(1+\frac{|u-x|}{t}\right)^{-\lambda n}\frac{du}{t^n}\right)t^{-2\sigma}\frac{dt}{t}\right)^{1/2} \\
&\le C\|\Omega\|_{L^q(S^{n-1})}\|f\|_{\mathrm{BMO}}.
\end{aligned}$$

This completes the proof of (b).

Proof of (c). In the case $0<\alpha<1$, we have $\mathcal{E}^{\alpha,p}=\mathrm{Lip}_\alpha$ $(1\le p<\infty)$, and the norms are equivalent. And we have as before $|f_2(y)|\le C\|f\|_{\mathrm{Lip}_\alpha} r^\alpha$ for $y\in 4B$. Choose a positive number γ such that $\gamma<\min\{1,\sigma\}$. Then, as in the proof of (b), we have

$$\begin{aligned}
&\left|\int_{r<|y-u|\le t}\frac{\Omega(u-y)f_2(y)}{|u-y|^{n-\rho}}\,dy\right| \\
&\le C\|f\|_{\mathrm{Lip}_\alpha} r^\alpha\int_{r<|y-u|\le t}\frac{|\Omega(u-y)|\chi_{4B}}{|u-y|^{n-\sigma}}\,dy \\
&\le C\|f\|_{\mathrm{Lip}_\alpha} r^\alpha\|\Omega\|_{L^1(S^{n-1})}\int_{\max\{r,|u-x_0|-4r\}}^{\min\{t,|u-x_0|+4r\}} s^{-(n-\sigma)+n-1}\,ds \\
&\le C\|f\|_{\mathrm{Lip}_\alpha} r^\alpha\|\Omega\|_{L^1(S^{n-1})} t^{\sigma-\gamma}\int_{\max\{r,|u-x_0|-4r\}}^{\min\{t,|u-x_0|+4r\}} s^{\gamma-1}\,ds \\
&\le C\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\mathrm{Lip}_\alpha} r^{\alpha+\gamma}t^{\sigma-\gamma}.
\end{aligned}$$

The rest of the proof proceeds in the same line as in the proof of (b). We omit the details. ∎

As for $\mu_\lambda^{*,\rho}(f_2)$, we need

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LEMMA 2.8. *Let $1 \le p < \infty$. Let $\Omega \in L^1(S^{n-1})$, $\rho = \sigma + i\tau$ ($\sigma > 0$, $\tau \in \mathbb{R}$), $\lambda > 1$ and $-n/p \le \alpha < 1$. Then, for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, any ball $B = B(x_0, r)$ and any $x \in B$*

$$\mu^{*,\rho}_{\lambda,0,\infty}(f_2)(x) = 0.$$

Proof. For $|y - x_0| \le 4r$, $|y - u| \le t \le r$ and $|x - x_0| \le r$, we have $|u-x| \le |u-y|+|y-x_0|+|x_0-x| \le 6r$, and hence the integration u-domain of the above integral is empty. □

Next we investigate $\mu^{*,\rho}_{\lambda}(f_3)$.

LEMMA 2.9. *Let $\rho = \sigma + i\tau$ ($\sigma > 0$, $\tau \in \mathbb{R}$).*

(a) *Let $\max\{1, \frac{2n}{n+2\sigma}\} < p < \infty$, $p_0 = \min\{2, p\}$, and $\Omega \in L^{p_0'}(S^{n-1})$. Suppose $\lambda > \max\{1, 2/p\}$ and $-n/p \le \alpha < 0$. Then, for any $f \in \mathcal{E}^{\alpha,p}$, for any ball $B = B(x_0, r)$ and any $x \in B$*

$$\begin{aligned}
&\mu^{*,\rho}_{\lambda,0}(f_3)(x)\\
&= \left(\int_0^r \int_{\mathbb{R}^n} \left| \frac{1}{t^\rho} \int_{|y-u|\le t} \frac{\Omega(u-y) f_3(y)}{|u-y|^{n-\rho}}\, dy \right|^2 \left(\frac{t}{t+|u-x|}\right)^{\lambda n} \frac{du\, dt}{t^{n+1}}\right)^{1/2}\\
&\le C r^\alpha \|\Omega\|_{L^{p_0'}(S^{n-1})} \|f\|_{\mathcal{E}^{\alpha,p}}.
\end{aligned}$$

(b) *Let $1 \le p < \infty$, $\alpha = 0$, $\lambda > 1$, and $\Omega \in L^q(S^{n-1})$ for some $q > 1$. Then, for any $f \in \mathcal{E}^{\alpha,p}$, for any ball $B = B(x_0, r)$ and any $x \in B$*

$$\mu^{*,\rho}_{\lambda,0}(f_3)(x) \le C r^\alpha \|\Omega\|_{L^q(S^{n-1})} \|f\|_{\mathcal{E}^{\alpha,p}}.$$

(c) *Let $1 \le p < \infty$, $0 < \alpha < 1$, $\lambda > 1 + 2\alpha/n$, and $\Omega \in L^1(S^{n-1})$. Then, for any $f \in \mathcal{E}^{\alpha,p}$, for any ball $B = B(x_0, r)$ and any $x \in B$*

$$\mu^{*,\rho}_{\lambda,0}(f_3)(x) \le C r^\alpha \|\Omega\|_{L^1(S^{n-1})} \|f\|_{\mathcal{E}^{\alpha,p}}.$$

Proof. Proof of (a).

(i) The case $0 < \sigma < n$ and $\max\{1, \frac{2n}{n+2\sigma}\} < p < +\infty$. Take p_0 and a as in the proof of Lemma 2.7. Then, by Hölder's inequality we have

$$\begin{aligned}
&\left| \int_{|y-u|\le t} \frac{\Omega(u-y) f_3(y)}{|u-y|^{n-\rho}}\, dy \right|\\
&\le \left(\int_{|y-u|\le t} \frac{|\Omega(u-y)|^{p_0'}\, dy}{|u-y|^{(n-\sigma)(1-a)p_0'}}\right)^{1/p_0'} \left(\int_{|y-u|\le t} \frac{|f_3(y)|^{p_0}\, dy}{|u-y|^{(n-\sigma)ap_0}}\right)^{1/p_0}\\
&\le C \|\Omega\|_{L^{p_0'}(S^{n-1})} t^{n/p_0' - (n-\sigma)(1-a)} \left(\int_{|y-u|\le t} \frac{|f_3(y)|^{p_0}\, dy}{|u-y|^{(n-\sigma)ap_0}}\right)^{f/1p_0}.
\end{aligned}$$

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Hence using Minkowski's inequality ($2/p_0 \geq 1$) and then noting $|u-x| \geq |y-x_0|-|y-u|-|x_0-x| > \frac{1}{4}(|y-x_0|+r)$ for $|u-y| \leq t \leq r$, $|y-x_0| > 4r$ and $|x_0-x| \leq r$, we have

$$
\begin{aligned}
&\mu^{*,\rho}_{\lambda,0}(f_3)(x)/\|\Omega\|_{L^{p_0'}(S^{n-1})} \\
&\leq C\Big(\int_0^r\Big(\int_{\mathbb{R}^n}\Big(\int_{|y-u|\leq t}\frac{|f_3(y)|^{p_0}\,dy}{|u-y|^{(n-\sigma)ap_0}}\Big)^{2/p_0}\frac{du}{\big(\frac{t+|u-x|}{t}\big)^{\lambda n}}\Big) \\
&\qquad\times t^{\frac{2n}{p_0'}-2(n-\sigma)(1-a)-2\sigma-n-1}\,dt\Big)^{1/2} \\
&\leq C\Big(\int_0^r\Big(\int_{(4B)^c}\Big(\int_{\mathbb{R}^n}\frac{\chi_{|y-u|\leq t}}{|u-y|^{2(n-\sigma)a}}\frac{du}{\big(\frac{t+|u-x|}{t}\big)^{\lambda n}}\Big)^{p_0/2}|f_3(y)|^{p_0}\,dy\Big)^{2/p_0} \\
&\qquad\times t^{\frac{2n}{p_0'}-2(n-\sigma)(1-a)-2\sigma-n-1}\,dt\Big)^{1/2} \\
&\leq C\Big(\int_0^r\Big(\int_{(4B)^c}\Big(\int_{\mathbb{R}^n}\frac{\chi_{|y-u|\leq t}\,du}{|u-y|^{2(n-\sigma)a}}\Big)^{p_0/2}\frac{|f_3(y)|^{p_0}\,dy}{(r+|y-x_0|)^{p_0\lambda n/2}}\Big)^{2/p_0} \\
&\qquad\times t^{\lambda n+\frac{2n}{p_0'}-2(n-\sigma)(1-a)-2\sigma-n-1}\,dt\Big)^{1/2} \\
&\leq C\Big(\int_0^r\Big(\int_{(4B)^c}\frac{|f(y)-f_{4B}|^{p_0}\,dy}{(r+|y-x_0|)^{p_0\lambda n/2}}\Big)^{2/p_0}t^{n-2(n-\sigma)a} \\
&\qquad\times t^{\lambda n+\frac{2n}{p_0'}-2(n-\sigma)(1-a)-2\sigma-n-1}\,dt\Big)^{1/2} \\
&\leq C\Big(\int_0^r t^{\lambda n+\frac{2n}{p_0'}-2(n-\sigma)-2\sigma-1}\,dt\Big)^{1/2} r^{\alpha-(\frac{p_0\lambda}{2}-1)\frac{n}{p_0}}\|f\|_{\mathcal{E}^{\alpha,p_0}} \\
&\leq Cr^{\frac{1}{2}(\lambda n+\frac{2n}{p_0'}-2n)}r^{\alpha-\frac{\lambda n}{2}+\frac{n}{p_0}}\|f\|_{\mathcal{E}^{\alpha,p_0}} \leq Cr^{\alpha}\|f\|_{\mathcal{E}^{\alpha,p}}.
\end{aligned}
$$

We have used here $\lambda n - \frac{2n}{p_0} > 0$, $\alpha < \big(\frac{\lambda}{2}-\frac{1}{p_0}\big)n$ and Lemma 1.1.

(ii) The case $\sigma \geq n$. In this case, we take $p_0 = \min\{2, p\}$ and $a = 0$. Then the reasoning in the step (i) still works.

Proof of (b). In this case, $\mathcal{E}^{\alpha,p} = \mathrm{BMO}$ ($1 \leq p < \infty$), and the norms are equivalent.

If $|u-x| \leq 2r$, $|x-x_0| \leq r$ and $|u-y| \leq t < r$, we have $|y-x_0| \leq$

$|x_0 - x| + |x - u| + |u - y| < 4r$, and hence

$$\mu_{\lambda,0}^{*,\rho}(f_3)(x) = \left(\int_0^r \int_{|u-x|>2r} \left| \frac{1}{t^\rho} \int_{|y-u|\le t} \frac{\Omega(u-y)f_3(y)}{|u-y|^{n-\rho}}\, dy \right|^2 \right.$$
$$\left. \times \left(\frac{t}{t+|u-x|}\right)^{\lambda n} \frac{du\, dt}{t^{n+1}}\right)^{1/2}.$$

Now choose the smallest integer j_0 satisfying $2^{j_0}t \ge |u - x_0| + 4r$ so that $B(u, 2^{j_0}t) \supset 4B$. Choose also the smallest integer j_1 satisfying $B(x_0, 2^{j_1+2}r) \supset B(u, 2^{j_0}t)$. Notice that $j_0 \ge j_1$. Then, we get

$$\begin{aligned} |f_{B(u,t)} - f_{4B}| &\le \sum_{j=1}^{j_0} |f_{B(u,2^{j-1}t)} - f_{B(u,2^j t)}| + |f_{B(u,2^{j_0}t)} - f_{B(x_0,2^{j_1+2}r)}| \\ &\quad + \sum_{j=1}^{j_1} |f_{B(x_0,2^{j+1}r)} - f_{B(x_0,2^{j+2}r)}| \\ &\le C(j_0 + 1 + (j_1+2))\|f\|_{\mathrm{BMO}} \le C\|f\|_{\mathrm{BMO}}\left(1 + \log\frac{|u-x_0|+4r}{t}\right). \end{aligned}$$

Choose positive numbers γ and q_0 such that $2\gamma < \lambda n - n$ and $1 < q_0 \le q$, and $(n-\sigma)q_0 < n$. Then, noting $1 + \log\frac{|u-x_0|+4r}{t} \le 1 + C\log\frac{|u-x|}{t} \le C\big(\frac{|u-x|}{t}\big)^\gamma$ for $|u - x| > 2r > 2t$, we have

$$\begin{aligned} &\left|\int_{|y-u|\le t} \frac{\Omega(u-y)f_3(y)}{|u-y|^{n-\rho}}\, dy\right| \\ &\le \left(\int_{|y-u|\le t} \frac{|\Omega(u-y)|^{q_0}\, dy}{|u-y|^{(n-\sigma)q_0}}\right)^{1/q_0} \left(\int_{|y-u|\le t} |f(y) - f_{4B}|^{q_0'} \chi_{(4B)^c}\, dy\right)^{1/q_0'} \\ &\le C\|\Omega\|_{L^{q_0}(S^{n-1})} t^{\sigma - n + \frac{n}{q_0}} \left\{ \left(\int_{|y-u|<t} |f(y) - f_{B(u,t)}|^{q_0'}\, dy\right)^{1/q_0'} \right. \\ &\qquad \left. + |f_{B(u,t)} - f_{4B}| \left(\int_{|y-u|<t} dy\right)^{1/q_0'} \right\} \\ &\le C\|\Omega\|_{L^q(S^{n-1})} t^\sigma \left(\frac{|u-x|}{t}\right)^\gamma \|f\|_{\mathrm{BMO}}. \end{aligned}$$

Therefore

$$\mu_{\lambda,0}^{*,\rho}(f_3)(x) \le C\|\Omega\|_{L^q(S^{n-1})}\|f\|_{\mathrm{BMO}} \left(\int_0^r \int_{|u-x|>2r} \left(\frac{t}{|u-x|}\right)^{\lambda n - 2\gamma} \frac{du\, dt}{t^{n+1}}\right)^{1/2}$$

$$\leq C\|\Omega\|_{L^q(S^{n-1})}\|f\|_{\mathrm{BMO}}\Big(\int_0^r t^{\lambda n-2\gamma-n-1}\,dt\int_{|u-x|>2r}\frac{du}{|u-x|^{\lambda n-2\gamma}}\Big)^{1/2}$$
$$\leq C\|\Omega\|_{L^q(S^{n-1})}\|f\|_{\mathrm{BMO}}.$$

Proof of (c). Since $0<\alpha<1$, $\mathcal{E}^{\alpha,p}=\mathrm{Lip}_\alpha$ $(1\leq p<\infty)$, and the norms are equivalent. For $y\in(4B)^c$, $x\in B$, $z\in 4B$ and $u\in\mathbb{R}^n$, we have $|y-z|\leq|y-x|+|x-x_0|+|x_0-z|\leq|y-x|+5r$ and $|y-x|\geq|y-x_0|-|x_0-x|\geq 4r-r=3r$, and hence $|y-z|\leq\frac{8}{3}|y-x|\leq\frac{8}{3}(|y-u|+|u-x|)$.

Thus, we have

$$|f_3(y)|\leq\frac{1}{|4B|}\int_{4B}|f(y)-f(z)|\,dz\leq C\|f\|_{\mathrm{Lip}_\alpha}(|y-u|+|u-x|)^\alpha,$$

and hence

$$\begin{aligned}&\Big|\int_{|y-u|\leq t}\frac{\Omega(u-y)f_3(y)}{|u-y|^{n-\rho}}\,dy\Big|\\&\leq C\|f\|_{\mathrm{Lip}_\alpha}\int_{|y-u|\leq t}\frac{|\Omega(u-y)|(|y-u|+|u-x|)^\alpha\,dy}{|u-y|^{n-\sigma}}\\&\leq C\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\mathrm{Lip}_\alpha}\int_0^t(s+|u-x|)^\alpha s^{\sigma-1}\,ds\\&\leq C\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\mathrm{Lip}_\alpha}(t+|u-x|)^\alpha t^\sigma.\end{aligned}$$

Thus, noting $\lambda n-2\alpha>n$ and $\alpha>0$ we have

$$\begin{aligned}\mu^{*,\rho}_{\lambda,0}(f_3)(x)&\leq C\Big(\int_0^r\Big(\int_{\mathbb{R}^n}(t+|u-x|)^{2\alpha}\Big(\frac{t}{t+|u-x|}\Big)^{\lambda n}\frac{du\,dt}{t^{n+1}}\Big)^{1/2}\\&\leq C\Big(\int_0^r\Big(\int_{\mathbb{R}^n}\Big(\frac{1}{1+\frac{|u-x|}{t}}\Big)^{\lambda n-2\alpha}\frac{du}{t^n}\,t^{2\alpha-1}\,dt\Big)^{1/2}\\&\leq Cr^\alpha\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\mathrm{Lip}_\alpha}.\end{aligned}$$

∎

LEMMA 2.10. *Let $1\leq p<\infty$. Let $\Omega\in L^1(S^{n-1})$, $\rho=\sigma+i\tau$ $(\sigma>0,\ \tau\in\mathbb{R})$, $\lambda>1$, and $0<\alpha<1$. Then, for any $f\in\mathcal{E}^{\alpha,p}$, for any ball $B=B(x_0,r)$ and any $x\in B$*

$$\mu^{*,\rho}_{\lambda,0,0}(f_3)(x)\leq Cr^\alpha\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\mathcal{E}^{\alpha,p}}.$$

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Proof. In this case, $\mathcal{E}^{\alpha,p} = \mathrm{Lip}_\alpha$ $(1 \le p < \infty)$, and the norms are equivalent. So, for $y \in (4B)^c$ we have $|f_3(y)| = |f(y) - f_{4B}| \le |f(y) - f(x_0)| + |f(x_0) - f_{4B}| \le \|f\|_{\mathrm{Lip}_\alpha}(|y-x_0|^\alpha + r^\alpha) \le C\|f\|_{\mathrm{Lip}_\alpha}|y-x_0|^\alpha$. For $|x-x_0| \le r$, $|u-y| \le t \le r$ and $|u-x| \le 8r$, we have $|y-x_0| \le |y-u| + |u-x| + |x-x_0| \le 10r$, and for $|x-x_0| \le r$, $|u-y| \le t \le r$ and $y \in (4B)^c$ we have $|u-x| \ge |y-x_0| - |u-y| - |x_0-x| > \frac{1}{2}|y-x_0| > 2r$. Hence we have

$$
\begin{aligned}
&\mu^{*,\rho}_{\lambda,0,0}(f_3)(x)\\
&\le C\bigg(\int_0^r\int_{2r<|u-x|\le 8r}\bigg|\frac{1}{t^\sigma}\int_{\substack{|y-u|\le t\\ 4r<|y-x_0|\le 10r}}\frac{|\Omega(u-y)||y-x_0|^\alpha}{|u-y|^{n-\sigma}}\,dy\bigg|^2\\
&\qquad\times\Big(\frac{t}{t+2r}\Big)^{\lambda n}\frac{du\,dt}{t^{n+1}}\bigg)^{1/2}\|f\|_{\mathrm{Lip}_\alpha}\\
&\le C\bigg(\int_0^r\int_{2r<|u-x|\le 8r}\bigg|\frac{1}{t^\sigma}\int_{|y-u|\le t}\frac{|\Omega(u-y)|r^\alpha}{|u-y|^{n-\sigma}}\,dy\bigg|^2 t^{\lambda n}r^{-\lambda n}\frac{du\,dt}{t^{n+1}}\bigg)^{1/2}\\
&\qquad\times\|f\|_{\mathrm{Lip}_\alpha}\\
&\le C\bigg(\int_0^r r^{2\alpha-\lambda n}r^n t^{\lambda n-n-1}\,dt\bigg)^{1/2}\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\mathrm{Lip}_\alpha}\\
&\le Cr^\alpha\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\mathrm{Lip}_\alpha}.
\end{aligned}
$$

□

LEMMA 2.11. *Suppose one of the following five conditions is satisfied:*

(i) $\sigma > 0$, $\lambda > 1$, $\alpha = 0$, $1 \le p < \infty$. *Moreover,* $\Omega \in L^q(S^{n-1})$ *for some* $q > 1$ *and satisfies the cancellation condition.*

(ii) $\sigma > 0$, $0 < \beta \le 1$, $0 < \alpha < \min\{1/2, \min\{\beta,\sigma\}\}$, $1 \le p < \infty$, *and* $\lambda > 1$. *Moreover,* $\Omega \in L^1(S^{n-1})$ *and satisfies the cancellation condition.*

(iii) $\sigma > 0$, $0 < \beta \le 1$, $1/2 \le \alpha < \min\{\beta,\sigma\}$, $1 \le p < \infty$, *and* $\lambda > 1 + 2\alpha/n$. *Moreover,* $\Omega \in L^1(S^{n-1})$ *and satisfies* L^1-β-*Dini condition and the cancellation condition.*

(iv) $\max\{1, \frac{2n}{n+2\sigma}\} < p < +\infty$, $-n/p < \alpha < 0$, $\sigma > -\alpha$ *and* $\lambda > 1$. *Moreover,* $\Omega \in L^{p_0'}(S^{n-1})$ *and satisfies the cancellation condition, where* $p_0 = \max\{2, p\}$.

(v) $1 < p < \infty$, $-n/p \le \alpha < 0$, $\sigma > n/2$, *and* $\lambda > 2$. *Moreover,* $\Omega \in L^2(S^{n-1})$ *and satisfies* L^2-$\log\beta$ *Dini condition for some* $\beta > 1$ *and the cancellation condition.*

Then there exists $C > 0$ such that for any ball $B = B(x_0, r)$ and any $f \in \mathcal{E}^{\alpha,p}$ satisfying $\mu^{*,\rho}_{\lambda,\infty}(f_3)(x_0) < +\infty$, it holds

$$\mu^{*,\rho}_{\lambda,\infty}(f_3)(x) < +\infty \quad \textit{and} \quad |\mu^{*,\rho}_{\lambda,\infty}(f_3)(x) - \mu^{*,\rho}_{\lambda,\infty}(f_3)(x_0)| \le Cr^{\alpha}\|f\|_{\mathcal{E}^{\alpha,p}}$$

for any $x \in B$, where $f_3(x) = (f(x) - f_{4B})\chi_{(4B)^c}$.

Proof. Let $x \in B$. Then, we see easily

$$\begin{aligned}
&|\mu^{*,\rho}_{\lambda,\infty}(f_3)(x) - \mu^{*,\rho}_{\lambda,\infty}(f_3)(x_0)| \\
&\le \Bigg(\int_r^{\infty}\int_{\mathbb{R}^n}\left|\frac{1}{t^{\rho}}\int_{|y-u|\le t}\frac{\Omega(u-y)f_3(y)}{|u-y|^{n-\rho}}\,dy\right|^2 \\
&\qquad\qquad \times \left|\left(1+\frac{|u-x|}{t}\right)^{-\lambda n} - \left(1+\frac{|u-x_0|}{t}\right)^{-\lambda n}\right|\frac{du\,dt}{t^{n+1}}\Bigg)^{1/2} \\
&\le \Bigg(\int_r^{\infty}\int_{|u-x_0|>8r}\left|\frac{1}{t^{\rho}}\int_{|y-u|\le t}\frac{\Omega(u-y)f_3(y)}{|u-y|^{n-\rho}}\,dy\right|^2 \\
&\qquad\qquad \times \left|\left(1+\frac{|u-x|}{t}\right)^{-\lambda n} - \left(1+\frac{|u-x_0|}{t}\right)^{-\lambda n}\right|\frac{du\,dt}{t^{n+1}}\Bigg)^{1/2} \\
&\quad + \Bigg(\int_r^{\infty}\int_{|u-x_0|\le 8r}\left|\frac{1}{t^{\rho}}\int_{|y-u|\le t}\frac{\Omega(u-y)f_3(y)}{|u-y|^{n-\rho}}\,dy\right|^2 \\
&\qquad\qquad \times \left|\left(1+\frac{|u-x|}{t}\right)^{-\lambda n} - \left(1+\frac{|u-x_0|}{t}\right)^{-\lambda n}\right|\frac{du\,dt}{t^{n+1}}\Bigg)^{1/2} \\
&=: I_1 + I_2.
\end{aligned}$$

(i) In this case, $\mathcal{E}^{0,s} = \mathrm{BMO}$ $(1 \le s < \infty)$, and the norms are equivalent. Firstly we note that for $\alpha = 0$

$$\left(\int_{|y-x_0|\le R}|f_3(y)|^s\,dy\right)^{1/s} \le C\|f\|_{\mathrm{BMO}}R^{n/s}\log\frac{R}{r} \quad (R > 4r).$$

This can be seen like as in Lemma 2.3. Choose $q_0 > 1$ such that $q_0' > \max\{q', \frac{n}{\sigma}, \frac{2}{\lambda-1+\frac{1}{n}}\}$. Then, since $|y - x_0| \le |y - u| + |u - x_0| \le t + |u - x_0|$

for $|y-u|<t$, we get by using the Hölder inequality

$$\begin{aligned}
&\Big|\int_{|y-u|\le t}\frac{\Omega(u-y)f_3(y)}{|u-y|^{n-\rho}}\,dy\Big|\\
&\le\Big(\int_{|y-u|\le t}\frac{|\Omega(u-y)|^{q_0}}{|u-y|^{q_0(n-\sigma)}}\,dy\Big)^{1/q_0}\Big(\int_{|y-x_0|\le t+|u-x_0|}|f_3(y)|^{q_0'}\,dy\Big)^{1/q_0'}\\
&\le C\|\Omega\|_{L^{q_0}(S^{n-1})}t^{\sigma-n+n/q_0}\|f\|_{\mathrm{BMO}}(t+|u-x_0|)^{n/q_0'}\log\frac{(t+|u-x_0|)}{r}.
\end{aligned}$$

By the mean value theorem, we get

$$\Big|\Big(1+\frac{|u-x|}{t}\Big)^{-\lambda n}-\Big(1+\frac{|u-x_0|}{t}\Big)^{-\lambda n}\Big|\le C\frac{|x-x_0|}{t}\Big(1+\frac{|u-x_0|}{t}\Big)^{-\lambda n-1},$$

for $|u-x_0|>8r$ and $x\in B$. Hence, we have

$$\begin{aligned}
I_1&\le C\|\Omega\|_{L^{q_0}(S^{n-1})}\|f\|_{\mathrm{BMO}}\Big(\int_r^\infty\int_{|u-x_0|>8r}t^{-2n/q_0'}(t+|u-x_0|)^{2n/q_0'}\\
&\qquad\times\log^2\frac{(t+|u-x_0|)}{r}\frac{r}{t}\Big(1+\frac{|u-x_0|}{t}\Big)^{-\lambda n-1}\frac{du\,dt}{t^{n+1}}\Big)^{1/2}\\
&\le C\|\Omega\|_{L^{q_0}(S^{n-1})}\|f\|_{\mathrm{BMO}}\Big(\int_r^\infty r\Big(\int_{\mathbb{R}^n}\Big(1+\frac{|u-x_0|}{t}\Big)^{-\lambda n-1+2n/q_0'}\\
&\qquad\times\log^2\frac{(t+|u-x_0|)}{r}\frac{du}{t^n}\Big)\frac{dt}{t^2}\Big)^{1/2}\\
&\le C\|\Omega\|_{L^{q_0}(S^{n-1})}\|f\|_{\mathrm{BMO}}\Big(\int_r^\infty r\Big(\log^2\frac{t}{r}+1\Big)\frac{dt}{t^2}\Big)^{1/2}\\
&\le C\|\Omega\|_{L^q(S^{n-1})}\|f\|_{\mathrm{BMO}}.
\end{aligned}$$

As for I_2, since $t+|u-x_0|\le 9t$ for $|u-x_0|\le 8r$ and $t>r$, we have, using the mean value theorem,

$$\Big|\Big(1+\frac{|u-x|}{t}\Big)^{-\lambda n}-\Big(1+\frac{|u-x_0|}{t}\Big)^{-\lambda n}\Big|\le C\frac{|x-x_0|}{t}.$$

Hence, we obtain

$$\begin{aligned}
I_2&\le C\|\Omega\|_{L^{q_0}(S^{n-1})}\|f\|_{\mathrm{BMO}}\Big(\int_r^\infty\int_{|u-x_0|\le 8r}t^{-2n/q_0'}\,t^{2n/q_0'}\frac{r}{t}\log^2\frac{t}{r}\,\frac{du\,dt}{t^{n+1}}\Big)^{1/2}\\
&\le C\|\Omega\|_{L^q(S^{n-1})}\|f\|_{\mathrm{BMO}}.
\end{aligned}$$

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Thus, we obtain the desired estimate in the case (i).

(ii) In this case, $0 < \alpha < 1/2$, and $\mathcal{E}^{\alpha,p} = \mathrm{Lip}_\alpha$ $(1 \le p < \infty)$, and the norms are equivalent. Let $x \in B$.

For $t > r$, $y \in (4B)^c$, $|y-u| < t$, $z \in 4B$ and $u \in \mathbb{R}^n$, we have $|y-z| \le |y-u| + |u-x_0| + |x_0 - z| \le C(t + |u-x_0|)$, and hence $|f_3(y)| \le C\|f\|_{\mathrm{Lip}_\alpha}(t+|u-x_0|)^\alpha$. So, we have

$$\left|\int_{|y-u|\le t} \frac{\Omega(u-y)f_3(y)}{|u-y|^{n-\rho}}\,dy\right| \le C\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\mathrm{Lip}_\alpha}(t+|u-x_0|)^\alpha t^\sigma.$$

We get, as in the estimate for I_1 in the case (i),

$$\begin{aligned} I_1 &\le C\biggl(\int_r^\infty\int_{|u-x_0|>8r}\left|\frac{1}{t^\rho}\int_{|y-u|\le t}\frac{\Omega(u-y)f_3(y)}{|u-y|^{n-\rho}}\,dy\right|^2 \\ &\qquad\times \frac{|x-x_0|}{t}\Bigl(1+\frac{|u-x_0|}{t}\Bigr)^{-\lambda n-1}\frac{du\,dt}{t^{n+1}}\biggr)^{1/2} \\ &\le C\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\mathrm{Lip}_\alpha} r^{1/2} \\ &\qquad\times\biggl(\int_r^\infty\int_{|u-x_0|>8r}\Bigl(1+\frac{|u-x_0|}{t}\Bigr)^{-\lambda n-1+2\alpha}\frac{du\,dt}{t^{n+2-2\alpha}}\biggr)^{1/2} \\ &\le C\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\mathrm{Lip}_\alpha} r^{1/2} \\ &\qquad\times\biggl(\int_r^\infty\biggl(\int_{\mathbb{R}^n}\Bigl(1+\frac{|u-x_0|}{t}\Bigr)^{-\lambda n-1+2\alpha}\frac{du}{t^n}\biggr)\frac{dt}{t^{2-2\alpha}}\biggr)^{1/2} \\ &\le C\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\mathrm{Lip}_\alpha} r^\alpha. \end{aligned}$$

In the above, we have used $\alpha < 1/2$.

Next, for $|u-x_0| \le 8r$ and $t > r$, we have $t + |u-x_0| \le 9t$, and so, $|f_3(y)| \le C\|f\|_{\mathrm{Lip}_\alpha} t^\alpha$. Hence, we obtain, as in the estimate for I_2 in the case (i),

$$\begin{aligned} I_2 &\le C\|f\|_{\mathrm{Lip}_\alpha} \\ &\qquad\times\biggl(\int_r^\infty\int_{|u-x_0|\le 8r}\left|\frac{1}{t^\sigma}\int_{|y-u|\le t}\frac{|\Omega(u-y)|t^\alpha}{|u-y|^{n-\sigma}}\,dy\right|^2\frac{|x-x_0|}{t}\frac{du\,dt}{t^{n+1}}\biggr)^{1/2} \\ &\le C\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\mathrm{Lip}_\alpha} r^{1/2}\biggl(\int_r^\infty\int_{|u-x_0|\le 8r} t^{2\alpha}\frac{du\,dt}{t^{n+2}}\biggr)^{1/2} \\ &\le C\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\mathrm{Lip}_\alpha} r^\alpha. \end{aligned}$$

Thus, summing the estimates for I_1 and I_2, we have

$$|\mu_{\lambda,\infty}^{*,\rho}(f_3)(x) - \mu_{\lambda,\infty}^{*,\rho}(f_3)(x_0)| \le Cr^\alpha \|\Omega\|_{L^1(S^{n-1})}\|f\|_{\mathcal{E}^{\alpha,p}} \quad \text{for any } x \in B.$$

(iii) Also, in this case, $\mathcal{E}^{\alpha,p} = \mathrm{Lip}_\alpha$ $(1 \le p < \infty)$, and the norms are equivalent. Let $x \in B$. Using the expression

$$\mu_{\lambda,\infty}^{*,\rho}(f_3)(x) = \left(\int_r^\infty \int_{\mathbb{R}^n} \left|\frac{1}{t^\rho}\int_{|x-u-y|\le t} \frac{\Omega(x-u-y)f_3(y)}{|x-u-y|^{n-\rho}}\,dy\right|^2 \left(\frac{t}{t+|u|}\right)^{\lambda n} \frac{du\,dt}{t^{n+1}}\right)^{1/2},$$

we see that

$$\begin{aligned}
&|\mu_{\lambda,\infty}^{*,\rho}(f_3)(x) - \mu_{\lambda,\infty}^{*,\rho}(f_3)(x_0)| \\
&\le \Bigg(\int_r^\infty \int_{\mathbb{R}^n} \Bigg|\frac{1}{t^\rho}\int_{|x-u-y|\le t} \frac{\Omega(x-u-y)f_3(y)}{|x-u-y|^{n-\rho}}\,dy \\
&\qquad - \frac{1}{t^\rho}\int_{|x_0-u-y|\le t} \frac{\Omega(x_0-u-y)f_3(y)}{|x_0-u-y|^{n-\rho}}\,dy\Bigg|^2 \left(1+\frac{|u|}{t}\right)^{-\lambda n} \frac{du\,dt}{t^{n+1}}\Bigg)^{1/2} \\
&\le \left(\int_r^\infty \int_{\mathbb{R}^n} \left|\frac{1}{t^\rho}\int_{\substack{|x-u-y|\le t\\ |x_0-u-y|>t}} \frac{\Omega(x-u-y)f_3(y)}{|x-u-y|^{n-\rho}}\,dy\right|^2 \left(1+\frac{|u|}{t}\right)^{-\lambda n} \frac{du\,dt}{t^{n+1}}\right)^{1/2} \\
&\quad + \left(\int_r^\infty \int_{\mathbb{R}^n} \left|\frac{1}{t^\rho}\int_{\substack{|x_0-u-y|\le t\\ |x-u-y|>t}} \frac{\Omega(x-u-y)f_3(y)}{|x_0-u-y|^{n-\rho}}\,dy\right|^2 \left(1+\frac{|u|}{t}\right)^{-\lambda n} \frac{du\,dt}{t^{n+1}}\right)^{1/2} \\
&\quad + \Bigg(\int_r^\infty \int_{\mathbb{R}^n} \left|\frac{1}{t^\rho}\int_{\substack{|x_0-u-y|\le t\\ |x-u-y|\le t}} \left(\frac{\Omega(x-u-y)}{|x-u-y|^{n-\rho}} - \frac{\Omega(x_0-u-y)}{|x_0-u-y|^{n-\rho}}\right) f_3(y)\,dy\right|^2 \\
&\qquad \times \left(1+\frac{|u|}{t}\right)^{-\lambda n} \frac{du\,dt}{t^{n+1}}\Bigg)^{1/2} \\
&=: L_1 + L_2 + L_3.
\end{aligned}$$

For $y \in (4B)^c$, $z \in 4B$, and $u \in \mathbb{R}^n$, we have $|y-z| \le |y-x+u|+|u|+|x-z| \le |y-x+u|+|u|+5r$, and hence $|f_3(y)| \le C\|f\|_{\mathrm{Lip}_\alpha}(|y-x+u|+|u|+r)^\alpha$. Thus for L_1, we have using Lemma 1.3

$$\begin{aligned}
&\left|\frac{1}{t^\rho}\int_{\substack{|x-u-y|\le t\\ |x_0-u-y|>t}} \frac{\Omega(x-u-y)f_3(y)}{|x-u-y|^{n-\rho}}\,dy\right| \\
&\qquad \le C\|f\|_{\mathrm{Lip}_\alpha} t^{-\sigma} \int_{\substack{|x-u-y|\le t\\ |x_0-u-y|>t}} \frac{|\Omega(x-u-y)|(|y-x+u|^\alpha + |u|^\alpha + r^\alpha)}{|x-u-y|^{n-\sigma}}\,dy
\end{aligned}$$

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$$
\begin{aligned}
&\le C\|f\|_{\mathrm{Lip}_\alpha} t^{-\sigma}\|\Omega\|_{L^1(S^{n-1})}\int_{t-r}^{t} s^{\sigma-1}(s^\alpha+|u|^\alpha+r^\alpha)\,ds\\
&\le C\|f\|_{\mathrm{Lip}_\alpha}\|\Omega\|_{L^1(S^{n-1})} t^{-\sigma} r t^{\sigma-1}(t^\alpha+|u|^\alpha+r^\alpha)\\
&\le C\|f\|_{\mathrm{Lip}_\alpha}\|\Omega\|_{L^1(S^{n-1})} r t^{-1}(t^\alpha+|u|^\alpha).
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
L_1 &\le C\|f\|_{\mathrm{Lip}_\alpha}\|\Omega\|_{L^1(S^{n-1})}\\
&\qquad\times\left(\int_r^\infty\int_{\mathbb{R}^n} r^2t^{-2}(t^{2\alpha}+|u|^{2\alpha})\Big(1+\frac{|u|}{t}\Big)^{-\lambda n}\frac{du\,dt}{t^{n+1}}\right)^{1/2}\\
&\le C\|f\|_{\mathrm{Lip}_\alpha}\|\Omega\|_{L^1(S^{n-1})} r\left(\int_r^\infty\left(\int_{\mathbb{R}^n}\Big(1+\frac{|u|}{t}\Big)^{-\lambda n+2\alpha}\frac{du}{t^n}\right)\frac{dt}{t^{3-2\alpha}}\right)^{1/2}\\
&\le C\|f\|_{\mathrm{Lip}_\alpha}\|\Omega\|_{L^1(S^{n-1})} r^\alpha.
\end{aligned}
$$

Similarly, we obtain the same estimate for L_2. We turn to the estimate for L_3. Since $\{|x_0-u-y|\le t,\ |x-u-y|\le t\}\subset\{4r<|x_0-u-y|\le 2t\}\cup\{|x_0-u-y|\le 4r\}$ and $\{|x_0-u-y|\le 4r\}\subset\{|x-u-y|\le 5r\}$ for $x\in B$, we see

$$
\begin{aligned}
L_3 \le{}& \left(\int_r^\infty\int_{\mathbb{R}^n}\left|\frac{1}{t^\rho}\int_{4r<|x_0-u-y|\le 2t}\left|\frac{\Omega(x-u-y)}{|x-u-y|^{n-\rho}}-\frac{\Omega(x_0-u-y)}{|x_0-u-y|^{n-\rho}}\right|\right.\right.\\
&\qquad\left.\left.\times|f_3(y)|\,dy\right|^2\Big(1+\frac{|u|}{t}\Big)^{-\lambda n}\frac{du\,dt}{t^{n+1}}\right)^{1/2}\\
&+\left(\int_r^\infty\int_{\mathbb{R}^n}\left|\frac{1}{t^\rho}\int_{|x_0-u-y|\le 4r}\left|\frac{\Omega(x_0-u-y)}{|x_0-u-y|^{n-\rho}}\right||f_3(y)|\,dy\right|^2\right.\\
&\qquad\left.\times\Big(1+\frac{|u|}{t}\Big)^{-\lambda n}\frac{du\,dt}{t^{n+1}}\right)^{1/2}\\
&+\left(\int_r^\infty\int_{\mathbb{R}^n}\left|\frac{1}{t^\rho}\int_{|x-u-y|\le 5r}\left|\frac{\Omega(x-u-y)}{|x-u-y|^{n-\rho}}\right||f_3(y)|\,dy\right|^2\right.\\
&\qquad\left.\times\Big(1+\frac{|u|}{t}\Big)^{-\lambda n}\frac{du\,dt}{t^{n+1}}\right)^{1/2}\\
=:{}& L_{31}+L_{32}+L_{33}.
\end{aligned}
$$

In this case, we have as before $|f_3(y)|\le C\|f\|_{\mathrm{Lip}_\alpha}(|y-x+u|+|u|+r)^\alpha\le$

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$C\|f\|_{\mathrm{Lip}_\alpha}(t+|u|)^\alpha$. Hence, if $\sigma>1$, as in the proof of Lemma 2.3, we have

$$\begin{aligned}
&\left|\frac{1}{t^\rho}\int_{4r<|x_0-u-y|\le 2t}\left|\frac{\Omega(x-u-y)}{|x-u-y|^{n-\rho}}-\frac{\Omega(x_0-u-y)}{|x_0-u-y|^{n-\rho}}\right||f_3(y)|\,dy\right|\\
&\quad\le C\|f\|_{\mathrm{Lip}_\alpha}(t+|u|)^\alpha t^{-\sigma}\\
&\qquad\qquad\times\int_{4r<|x_0-u-y|\le 2t}\left|\frac{\Omega(x-u-y)}{|x-u-y|^{n-\rho}}-\frac{\Omega(x_0-u-y)}{|x_0-u-y|^{n-\rho}}\right|dy\\
&\quad\le C\|f\|_{\mathrm{Lip}_\alpha}(t+|u|)^\alpha t^{-\sigma}r^\sigma\left(\left(\frac{t}{r}\right)^{\sigma-1}+\left(\frac{t}{r}\right)^{\sigma-\beta}\right)\\
&\quad\le C\|f\|_{\mathrm{Lip}_\alpha}(1+|u|/t)^\alpha t^\alpha\left(\frac{r}{t}+\left(\frac{r}{t}\right)^\beta\right).
\end{aligned}$$

Thus, noting $\beta>\alpha$ and $\lambda n>n+2\alpha$, we have

$$\begin{aligned}
L_{31}&\le C\|f\|_{\mathrm{Lip}_\alpha}\left(\int_r^\infty\left(\int_{\mathbb{R}^n}\left(1+\frac{|u|}{t}\right)^{-\lambda n+2\alpha}\frac{du}{t^n}\right)\left(\frac{r^2}{t^2}+\frac{r^{2\beta}}{t^{2\beta}}\right)\frac{dt}{t^{1-2\alpha}}\right)^{1/2}\\
&\le C\|f\|_{\mathrm{Lip}_\alpha}\left(r^2r^{2\alpha-2}+r^{2\beta}r^{2\alpha-2\beta}\right)^{1/2}\le C\|f\|_{\mathrm{Lip}_\alpha}r^\alpha.
\end{aligned}$$

The other three cases $\sigma=1$, $0<\beta<\sigma<1$, and $0<\sigma<\beta\le 1$ and $\sigma<1$ are treated in the same way as in the corresponding cases in Lemma 2.3, respectively.

As for L_{32}, we have as above $|f_3(y)|\le C\|f\|_{\mathrm{Lip}_\alpha}(t+|u|)^\alpha$. Using this and noting $\alpha<\sigma$ and $\lambda n>n+2\alpha$, we get

$$\begin{aligned}
L_{32}&\le C\|f\|_{\mathrm{Lip}_\alpha}\left(\int_r^\infty\int_{\mathbb{R}^n}\left|\frac{1}{t^\rho}\int_{|x_0-u-y|\le 4r}\left|\frac{\Omega(x_0-u-y)}{|x_0-u-y|^{n-\rho}}\right|(t+|u|)^\alpha\,dy\right|^2\right.\\
&\qquad\qquad\left.\times\left(1+\frac{|u|}{t}\right)^{-\lambda n}\frac{du\,dt}{t^{n+1}}\right)^{1/2}\\
&\le C\|f\|_{\mathrm{Lip}_\alpha}\|\Omega\|_{L^1(S^{n-1})}\left(\int_r^\infty\int_{\mathbb{R}^n}t^{-2\sigma}r^{2\sigma}(t+|u|)^{2\alpha}\left(1+\frac{|u|}{t}\right)^{-\lambda n}\frac{du\,dt}{t^{n+1}}\right)^{1/2}\\
&\le C\|f\|_{\mathrm{Lip}_\alpha}\|\Omega\|_{L^1(S^{n-1})}r^\sigma\left(\int_r^\infty\left(\int_{\mathbb{R}^n}\left(1+\frac{|u|}{t}\right)^{-\lambda n+2\alpha}\frac{du}{t^n}\right)\frac{dt}{t^{1+2\sigma-2\alpha}}\right)^{1/2}\\
&\le C\|f\|_{\mathrm{Lip}_\alpha}\|\Omega\|_{L^1(S^{n-1})}r^\sigma r^{-(2\sigma-2\alpha)/2}=C\|f\|_{\mathrm{Lip}_\alpha}\|\Omega\|_{L^1(S^{n-1})}r^\alpha.
\end{aligned}$$

Similarly we get the same estimate for L_{33}. Summing up the estimates for L_{31}, L_{32}, L_{33}, we obtain the desired estimate for L_3. Finally summing up the estimates for L_1, L_2, L_3, we get

$$|\mu^{*,\rho}_{\lambda,\infty}(f_3)(x)-\mu^{*,\rho}_{\lambda,\infty}(f_3)(x_0)|\le C\|f\|_{\mathrm{Lip}_\alpha}r^\alpha.$$

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(iv) Using the Minkowski inequality and the cancellation property of Ω we see

$$\begin{aligned}
&|\mu^{*,\rho}_{\lambda,\infty}(f_3)(x) - \mu^{*,\rho}_{\lambda,\infty}(f_3)(x_0)| \\
&\le \Big(\int_r^\infty\int_{\mathbb{R}^n}\Big|\frac{1}{t^\rho}\int_{|y-u|\le t}\frac{\Omega(u-y)(f(y)-f_{4B})}{|u-y|^{n-\rho}}\,dy\Big|^2 \\
&\qquad\qquad \times\Big|\Big(1+\frac{|u-x|}{t}\Big)^{-\lambda n}-\Big(1+\frac{|u-x_0|}{t}\Big)^{-\lambda n}\Big|\frac{du\,dt}{t^{n+1}}\Big)^{1/2} \\
&\quad + \Big(\int_r^\infty\int_{\mathbb{R}^n}\Big|\frac{1}{t^\rho}\int_{|y-u|\le t}\frac{\Omega(u-y)f_2(y)}{|u-y|^{n-\rho}}\,dy\Big|^2 \\
&\qquad\qquad \times\Big|\Big(1+\frac{|u-x|}{t}\Big)^{-\lambda n}-\Big(1+\frac{|u-x_0|}{t}\Big)^{-\lambda n}\Big|\frac{du\,dt}{t^{n+1}}\Big)^{1/2} \\
&\le \Big(\int_r^\infty\int_{\mathbb{R}^n}\Big|\frac{1}{t^\rho}\int_{|y-u|\le t}\frac{\Omega(u-y)(f(y)-f_{B(u,t)})}{|u-y|^{n-\rho}}\,dy\Big|^2 \\
&\qquad\qquad \times\Big|\Big(1+\frac{|u-x|}{t}\Big)^{-\lambda n}-\Big(1+\frac{|u-x_0|}{t}\Big)^{-\lambda n}\Big|\frac{du\,dt}{t^{n+1}}\Big)^{1/2} \\
&\quad + \sqrt{2}(\mu^{*,\rho}_{\lambda,\infty}(f_2)(x) + \mu^{*,\rho}_{\lambda,\infty}(f_2)(x_0)) \\
&=: J_1 + J_2.
\end{aligned}$$

Now, we note that for $\alpha < 0$ and non-negative integer j

$$\Big(\int_{|y-u|\le t/2^j}|f(y)-f_{B(u,t)}|^p\,dy\Big)^{1/p} \le C\|f\|_{\mathcal{E}^{\alpha,p}}t^{\alpha+\frac{n}{p}}2^{-j(\alpha+\frac{n}{p})}.$$

This can be seen like as in Lemma 2.3. So, choosing a non-negative number γ satisfying $n/p-\sigma<\gamma<\alpha+n/p$, we have by Hölder's inequality

$$\begin{aligned}
&\Big|\int_{|y-u|\le t}\frac{\Omega(u-y)(f(y)-f_{B(u,t)})}{|u-y|^{n-\rho}}\,dy\Big| \\
&\le \Big(\int_{|y-u|\le t}\frac{|\Omega(u-y)|^{p'}}{|u-y|^{(n-\sigma-\gamma)p'}}\,dy\Big)^{1/p'}\Big(\int_{|y-u|\le t}\frac{|f(y)-f_{B(u,t)}|^p}{|u-y|^{p\gamma}}\,dy\Big)^{1/p} \\
&\le C\|\Omega\|_{L^{p'}(S^{n-1})}t^{\sigma+\gamma-n+\frac{n}{p'}}\Big(\sum_{j=0}^\infty\int_{\frac{t}{2^{j+1}}<|y-u|\le\frac{t}{2^j}}\frac{|f(y)-f_{B(u,t)}|^p}{|u-y|^{p\gamma}}\,dy\Big)^{1/p} \\
&\le C\|\Omega\|_{L^{p'}(S^{n-1})}t^{\sigma+\gamma-\frac{n}{p}}\Big(\sum_{j=0}^\infty\frac{1}{(\frac{t}{2^{j+1}})^{p\gamma}}\int_{|y-u|\le\frac{t}{2^j}}|f(y)-f_{B(u,t)}|^p\,dy\Big)^{1/p}
\end{aligned}$$

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$$\le C\|\Omega\|_{L^{p'}(S^{n-1})}\|f\|_{\mathcal{E}^{\alpha,p}}t^{\sigma+\alpha}.$$

Hence, as in the proof of I_1 in the case (i)

$$\begin{aligned} J_1 &\le C\|\Omega\|_{L^{p'}(S^{n-1})}\|f\|_{\mathcal{E}^{\alpha,p}}\biggl(\int_r^\infty\int_{|u-x_0|>8r} t^{2\alpha}\frac{r}{t}\Bigl(1+\frac{|u-x_0|}{t}\Bigr)^{-\lambda n-1}\frac{du\,dt}{t^{n+1}} \\ &\qquad + \int_r^\infty\int_{|u-x_0|\le 8r} t^{2\alpha}\frac{r}{t}\frac{du\,dt}{t^{n+1}}\biggr)^{1/2} \\ &\le C\|\Omega\|_{L^{p'}(S^{n-1})}\|f\|_{\mathcal{E}^{\alpha,p}}\biggl(r\int_r^\infty\int_{\mathbb{R}^n}\Bigl(1+\frac{|u-x_0|}{t}\Bigr)^{-\lambda n-1}\frac{du}{t^n}\frac{dt}{t^{2-2\alpha}} \\ &\qquad + r\int_r^\infty Cr^n\frac{dt}{t^{n-2\alpha+2}}\biggr)^{1/2} \\ &\le C\|\Omega\|_{L^{p'}(S^{n-1})}\|f\|_{\mathcal{E}^{\alpha,p}}r^\alpha. \end{aligned}$$

As for J_2, we have by Lemma 2.7 (a)

$$J_2 \le C\|\Omega\|_{L^{p'_0}(S^{n-1})}\|f\|_{\mathcal{E}^{\alpha,p}}r^\alpha.$$

Thus, we have the desired estimate in this case (iv).

(v) We use the expression in (iii), and estimate L_1, L_2, L_3. For $|x-u-y|\le t$ and $|y-x_0|>4r$, we see $t+|u|\ge|x-u-y|+|u|\ge|x-y|\ge|x_0-y|-|x_0-x|\ge\frac{3}{4}|y-x_0|$. Choose $\eta>0$ such that $0<\eta<\min\{1/2,\lambda/2-n,\sigma-n/2\}$. Then, by the Minkowski inequality, Lemma 1.3 (noting $\sigma-n/2>0$) and Lemma 1.1 we have

$$\begin{aligned} L_1 &\le \biggl(\int_r^\infty\int_{\mathbb{R}^n}\biggl(\frac{1}{t^\sigma}\int_{t-r<|x-u-y|\le t}\frac{|\Omega(x-u-y)f_3(y)|}{|x-u-y|^{n-\sigma}}\,dy\biggr)^2 \\ &\qquad \times\Bigl(\frac{t}{t+|u|}\Bigr)^{2n+2\eta}\frac{du\,dt}{t^{n+1}}\biggr)^{1/2} \\ &\le \int_{\mathbb{R}^n}\biggl(\int_r^\infty\int_{t-r<|x-u-y|\le t}\frac{|\Omega(x-u-y)|^2}{|x-u-y|^{2n-2\sigma}}\Bigl(\frac{t}{|y-x_0|}\Bigr)^{2n+2\eta}\frac{du\,dt}{t^{n+2\sigma+1}}\biggr)^{1/2} \\ &\qquad \times|f_3(y)|\,dy \\ &\le \int_{\mathbb{R}^n}\biggl(\int_r^\infty\int_{t-r<|x-u-y|\le t}\frac{|\Omega(x-u-y)|^2}{|x-u-y|^{2n-2\sigma}}\frac{du\,dt}{t^{2\sigma-n-2\eta+1}}\biggr)^{1/2}\frac{|f_3(y)|}{|y-x_0|^{n+\eta}}\,dy \\ &\le C\|\Omega\|_{L^2(S^{n-1})}\int_{\mathbb{R}^n}\biggl(\int_r^\infty\int_{t-r}^t s^{2\sigma-n-1}\,ds\,\frac{dt}{t^{2\sigma-n-2\eta+1}}\biggr)^{1/2}\frac{|f_3(y)|}{|y-x_0|^{n+\eta}}\,dy \end{aligned}$$

$$\le C\|\Omega\|_{L^2(S^{n-1})}\int_{\mathbb{R}^n}\Big(\int_r^\infty rt^{2\sigma-n-1}\frac{dt}{t^{2\sigma-n-2\eta+1}}\Big)^{1/2}\frac{|f_3(y)|}{|y-x_0|^{n+\eta}}\,dy$$
$$\le C\|\Omega\|_{L^2(S^{n-1})}r^\eta\int_{\mathbb{R}^n}\frac{|f_3(y)|}{|y-x_0|^{n+\eta}}\,dy\le C\|\Omega\|_{L^2(S^{n-1})}\|f\|_{\mathcal{E}^{\alpha,p}}r^\alpha.$$

We have used above the fact $\|f\|_{\mathcal{E}^{\alpha,1}}r^\alpha\le\|f\|_{\mathcal{E}^{\alpha,p}}r^\alpha$ for $p>1$.

Similarly we have the same estimate for L_2.

Next, we estimate L_{31}, L_{32}, L_{33}. To estimate L_{31}, we choose $\gamma>0$ such that $1<\gamma<\beta$, and note that for $|x_0-u-y|<2t$ and $|y-x_0|>4r$, we have $t+|u|\ge(|x_0-u-y|+|u|)/2\ge|x_0-y|/2\ge 2r$, and so

$$\Big(\frac{t}{t+|u|}\Big)^{\lambda n}\le C\Big(\frac{t}{t+|u|}\Big)^{2n}\Big(\frac{1+\log\frac{t}{r}}{1+\log\frac{t+|u|}{r}}\Big)^{2\gamma}$$
$$\le C\Big(\frac{t}{|y-x_0|}\Big)^{2n}\Big(1+\log\frac{t}{r}\Big)^{2\gamma}\Big(1+\log\frac{|y-x_0|}{r}\Big)^{-2\gamma}.$$

(This follows from an elementary inequality $\big(\log\frac{e^{1/\gamma}t}{r}\big)/\big(\log\frac{e^{1/\gamma}s}{r}\big)>t^\gamma/s^\gamma$ for $0<r<t<s$ and $\gamma>0$).

Hence, using the Minkowski inequality we get

$$L_{31}\le C\int_{\mathbb{R}^n}\Big(\int_r^\infty\frac{1}{t^{2\sigma}}\int_{4r<|x_0-u-y|\le 2t}\Big|\frac{\Omega(x-u-y)}{|x-u-y|^{n-\rho}}-\frac{\Omega(x_0-u-y)}{|x_0-u-y|^{n-\rho}}\Big|^2$$
$$\times t^{2n}\Big(1+\log\frac{t}{r}\Big)^{2\gamma}\frac{du\,dt}{t^{n+1}}\Big)^{1/2}\frac{|f_3(y)|}{|y-x_0|^n\log^\gamma\frac{|y-x_0|}{r}}\,dy.$$

Now, using an elementary inequality $\int_A^\infty\frac{\log^b}{s^{1+a}}\,ds\le C\frac{\log^b A}{A^a}$ for $A\ge1$, $a,b>0$ (This can be seen by integrating by parts $[b]+1$-times if b is not an integer and b-times if b is an integer) and Lemma 1.2, we obtain

$$\Big(\int_r^\infty\frac{1}{t^{2\sigma}}\int_{4r<|x_0-u-y|\le 2t}\Big|\frac{\Omega(x-u-y)}{|x-u-y|^{n-\rho}}-\frac{\Omega(x_0-u-y)}{|x_0-u-y|^{n-\rho}}\Big|^2$$
$$\times t^{2n}\Big(1+\log\frac{t}{r}\Big)^{2\gamma}\frac{du\,dt}{t^{n+1}}\Big)^{1/2}$$
$$\le\Big(\int_{4r<|x_0-u-y|}\Big|\frac{\Omega(x-u-y)}{|x-u-y|^{n-\rho}}-\frac{\Omega(x_0-u-y)}{|x_0-u-y|^{n-\rho}}\Big|^2$$
$$\times\int_{|x_0-u-y|/2}^\infty\frac{(1+\log\frac{t}{r})^{2\gamma}}{t^{2\sigma-n+1}}\,dt\,du\Big)^{1/2}$$

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$$\le \Bigg(\int_{4r<|x_0-u-y|}\bigg|\frac{\Omega(x-u-y)}{|x-u-y|^{n-\rho}}-\frac{\Omega(x_0-u-y)}{|x_0-u-y|^{n-\rho}}\bigg|^2 \times \frac{(1+\log\frac{|x_0-u-y|}{r})^{2\gamma}}{|x_0-u-y|^{2\sigma-n}}\,du\Bigg)^{1/2}$$

$$\le \Bigg(\sum_{k=2}^{\infty}\int_{2^kr\le|x_0-u-y|<2^{k+1}r}\bigg|\frac{\Omega(x-u-y)}{|x-u-y|^{n-\rho}}-\frac{\Omega(x_0-u-y)}{|x_0-u-y|^{n-\rho}}\bigg|^2 \times \frac{(1+\log\frac{|x_0-u-y|}{r})^{2\gamma}}{|x_0-u-y|^{2\sigma-n}}\,du\Bigg)^{1/2}$$

$$\le \sum_{k=2}^{\infty}\Bigg(\int_{2^kr\le|x_0-u-y|<2^{k+1}r}\bigg|\frac{\Omega(x-u-y)}{|x-u-y|^{n-\rho}}-\frac{\Omega(x_0-u-y)}{|x_0-u-y|^{n-\rho}}\bigg|^2 \times \frac{(1+\log\frac{|x_0-u-y|}{f}r)^{2\gamma}}{|x_0-u-y|^{2\sigma-n}}\,du\Bigg)^{1/2}$$

$$\le C\sum_{k=2}^{\infty}\frac{(1+(k+1)\log 2)^{\gamma}}{(2^kr)^{\sigma-\frac{n}{2}}}(2^{k+1}r)^{\frac{n}{2}-(n-\sigma)} \times \left(\|\Omega\|_{L^2(S^{n-1})}\frac{|x-x_0|}{2^kr}+\int_{\frac{|x-x_0|}{2^{k+1}r}}^{\frac{|x-x_0|}{2^kr}}\frac{\omega_2(\delta)}{\delta}\,d\delta\right)$$

$$\le C\sum_{k=2}^{\infty}\left(\|\Omega\|_{L^2(S^{n-1})}\frac{(1+k)^{\gamma}}{2^k}+\frac{(1+k)^{\gamma}}{(1+k)^{\beta}}\int_{\frac{|x-x_0|}{2^{k+1}r}}^{\frac{|x-x_0|}{2^kr}}\frac{\omega_2(\delta)}{\delta}(1+|\log\delta|)^{\beta}\,d\delta\right)$$

$$\le C\left(\|\Omega\|_{L^2(S^{n-1})}+\int_0^1\frac{\omega_2(\delta)}{\delta}(1+|\log\delta|)^{\beta}\,d\delta\right).$$

Hence, we have

$$\begin{aligned}
L_{31} &\le C\int_{\mathbb{R}^n}\frac{|f_3(y)|}{|y-x_0|^n\log^{\gamma}\frac{|y-x_0|}{r}}\,dy \\
&\le C\sum_{k=2}^{\infty}\int_{2^kr\le|y-x_0|<2^{k+1}r}\frac{|f(y)-f_{4B}|\,dy}{|y-x_0|^n\log^{\gamma}\frac{|y-x_0|}{r}} \\
&\le C\sum_{k=2}^{\infty}\frac{1}{r^n2^{nk}k^{\gamma}}\int_{|y-x_0|<2^{k+1}}|f(y)-f_{4B}|\,dy \\
&\le C\sum_{k=2}^{\infty}\frac{1}{r^n2^{nk}k^{\gamma}}\|f\|_{\mathcal{E}^{\alpha,1}}r^{\alpha}(2^{k+1}r)^n \le C\left(\sum_{k=2}^{\infty}\frac{1}{k^{\gamma}}\right)\|f\|_{\mathcal{E}^{\alpha,p}}r^{\alpha}.
\end{aligned}$$

In the fourth inequality in the above, we have used the inequality

$$\left(\int_{|y-x_0|\le R} |f_3(y)|^p\,dy\right)^{1/p} \le C\|f\|_{\mathcal{E}^{\alpha,p}} r^\alpha R^{n/p}.$$

As for L_{32}, we have, in the same way as in the estimate for L_1,

$$\begin{aligned}
L_{32} &\le \left(\int_r^\infty \int_{\mathbb{R}^n} \left(\frac{1}{t^\sigma}\int_{|x_0-u-y|\le 4r} \frac{|\Omega(x_0-u-y)f_3(y)|}{|x_0-u-y|^{n-\sigma}}\,dy\right)^2 \right.\\
&\qquad \left.\times\left(\frac{t}{t+|u|}\right)^{2n+2\eta}\frac{du\,dt}{t^{n+1}}\right)^{1/2}\\
&\le \int_{\mathbb{R}^n}\left(\int_r^\infty\int_{|x_0-u-y|\le 4r}\frac{|\Omega(x_0-u-y)|^2}{|x_0-u-y|^{2n-2\sigma}}\left(\frac{t}{|y-x_0|}\right)^{2n+2\eta}\frac{du\,dt}{t^{n+2\sigma+1}}\right)^{1/2}\\
&\qquad \times |f_3(y)|\,dy\\
&\le \int_{\mathbb{R}^n}\left(\int_r^\infty\int_{|x_0-u-y|\le 4r}\frac{|\Omega(x_0-u-y)|^2}{|x_0-u-y|^{2n-2\sigma}}\frac{du\,dt}{t^{2\sigma-n-2\eta+1}}\right)^{1/2}\frac{|f_3(y)|}{|y-x_0|^{n+\eta}}\,dy\\
&\le C\|\Omega\|_{L^2(S^{n-1})}\int_{\mathbb{R}^n}\left(\int_r^\infty\int_0^{4r} s^{2\sigma-n-1}\,ds\,\frac{dt}{t^{2\sigma-n-2\eta+1}}\right)^{1/2}\frac{|f_3(y)|}{|y-x_0|^{n+\eta}}\,dy\\
&\le C\|\Omega\|_{L^2(S^{n-1})} r^\eta \|f\|_{\mathcal{E}^{\alpha,1}} r^{\alpha-\eta} \le C\|\Omega\|_{L^2(S^{n-1})}\|f\|_{\mathcal{E}^{\alpha,p}} r^\alpha
\end{aligned}$$

Similarly we get the same estimate for L_{33}. Summing up the estimates for L_{31}, L_{32}, L_{33}, we obtain the desired estimate for L_3. Finally summing up the estimates for L_1, L_2, L_3, we get

$$|\mu^{*,\rho}_{\lambda,\infty}(f_3)(x) - \mu^{*,\rho}_{\lambda,\infty}(f_3)(x_0)| \le C\|f\|_{\mathcal{E}^{\alpha,p}} r^\alpha.$$

□

LEMMA 2.12. *Let* $\lambda > 1$, $1 \le p < \infty$, $0 < \alpha < \min\{1, (\lambda-1)n/2+1/2\}$, *and* $\Omega \in L^1(S^{n-1})$. *Then there exists* $C > 0$ *such that for any ball* $B = B(x_0, r)$ *and any* $f \in \mathcal{E}^{\alpha,p}$ *satisfying* $\mu^{*,\rho}_{\lambda,0,\infty}(f_3)(x_0) < +\infty$, *it holds*

$$\mu^{*,\rho}_{\lambda,0,\infty}(f_3)(x) < +\infty \quad \textit{and}$$
$$|\mu^{*,\rho}_{\lambda,0,\infty}(f_3)(x) - \mu^{*,\rho}_{\lambda,0,\infty}(f_3)(x_0)| \le Cr^\alpha\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\mathcal{E}^{\alpha,p}}$$

for any $x \in B$.

Proof. For $x \in B$ we see easily

$$\mu^{*,\rho}_{\lambda,0,\infty}(f_3)(x) \le \left(\int_0^r \int_{|u-x_0|>8r} \left| \frac{1}{t^\rho} \int_{|y-u|\le t} \frac{\Omega(u-y) f_3(y)}{|u-y|^{n-\rho}}\, dy \right|^2 \times \left(1 + \frac{|u-x|}{t}\right)^{-\lambda n} \frac{du\, dt}{t^{n+1}} \right)^{1/2} + \left(\int_0^r \int_{|u-x|\le 9r} \left| \frac{1}{t^\rho} \int_{|y-u|\le t} \frac{\Omega(u-y) f_3(y)}{|u-y|^{n-\rho}}\, dy \right|^2 \times \left(1 + \frac{|u-x|}{t}\right)^{-\lambda n} \frac{du\, dt}{t^{n+1}} \right)^{1/2}.$$

We see by Lemma 2.10 (its variant replaced $8r$ by $9r$) that the second term in the right-hand side of the above inequality is bounded by $Cr^\alpha \|\Omega\|_{L^1(S^{n-1})}$ $\|f\|_{\mathcal{E}^{\alpha,p}}$. Hence, we have

$$\begin{aligned} \mu^{*,\rho}_{\lambda,0,\infty}(f_3)(x) &\le Cr^\alpha \|\Omega\|_{L^1(S^{n-1})} \|f\|_{\mathcal{E}^{\alpha,p}} + \mu^{*,\rho}_{\lambda,0,\infty}(f_3)(x_0) \\ &\quad + \left(\int_0^r \int_{|u-x_0|>8r} \left| \frac{1}{t^\rho} \int_{|y-u|\le t} \frac{\Omega(u-y) f_3(y)}{|u-y|^{n-\rho}}\, dy \right|^2 \times \left| \left(1 + \frac{|u-x|}{t}\right)^{-\lambda n} - \left(1 + \frac{|u-x_0|}{t}\right)^{-\lambda n} \right| \frac{du\, dt}{t^{n+1}} \right)^{1/2} \\ &= Cr^\alpha \|\Omega\|_{L^1(S^{n-1})} \|f\|_{\mathcal{E}^{\alpha,p}} + \mu^{*,\rho}_{\lambda,0,\infty}(f_3)(x_0) + I, \ \textit{say}. \end{aligned}$$

As in the proof of Lemma 2.9 (c), we have for every $0 < t < r$ and $u \in \mathbb{R}^n$ with $|u - x_0| > 8r$

$$\begin{aligned} \left| \int_{|y-u|\le t} \frac{\Omega(u-y) f_3(y)}{|u-y|^{n-\rho}}\, dy \right| &\le C \|\Omega\|_{L^1(S^{n-1})} \|f\|_{\mathrm{Lip}_\alpha} (t + |u - x_0|)^\alpha t^\sigma \\ &\le C \|\Omega\|_{L^1(S^{n-1})} \|f\|_{\mathrm{Lip}_\alpha} |u - x_0|^\alpha t^\sigma. \end{aligned}$$

By the mean value theorem, we get

$$\begin{aligned} &\left| \left(1 + \frac{|u-x|}{t}\right)^{-\lambda n} - \left(1 + \frac{|u-x_0|}{t}\right)^{-\lambda n} \right| \\ &\quad \le C \frac{|x-x_0|}{t} \left(1 + \frac{|u-x_0|}{t}\right)^{-\lambda n - 1} \le C \frac{|x-x_0| t^{\lambda n}}{|u-x_0|^{\lambda n + 1}}. \end{aligned}$$

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Hence, we have

$$\begin{aligned} I &\le C\Big(\int_0^r\int_{|u-x_0|>8r}\Big(\frac{1}{t^\sigma}\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\mathrm{Lip}_\alpha}|u-x_0|^\alpha t^\sigma\Big)^2 \\ &\qquad\times\frac{|x-x_0|t^{\lambda n}}{|u-x_0|^{\lambda n+1}}\frac{du\,dt}{t^{n+1}}\Big)^{1/2} \\ &\le C\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\mathrm{Lip}_\alpha}r^{1/2} \\ &\qquad\times\Big(\int_0^r\Big(\int_{|u-x_0|>8r}|u-x_0|^{2\alpha-\lambda n-1}\,du\Big)t^{\lambda n-n-1}\,dt\Big)^{1/2} \\ &\le C\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\mathrm{Lip}_\alpha}r^\alpha. \end{aligned}$$

Thus, we have

$$\mu^{*,\rho}_{\lambda,0,\infty}(f_3)(x)\le\mu^{*,\rho}_{\lambda,0,\infty}(f_3)(x_0)+Cr^\alpha\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\mathcal{E}^{\alpha,p}}\ \text{ for any } x\in B.$$

Reversing the roles of $\mu^{*,\rho}_{\lambda,0,\infty}(f_3)(x_0)$ and $\mu^{*,\rho}_{\lambda,0,\infty}(f_3)(x)$, we have

$$\mu^{*,\rho}_{\lambda,0,\infty}(f_3)(x_0)\le\mu^{*,\rho}_{\lambda,0,\infty}(f_3)(x)+Cr^\alpha\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\mathcal{E}^{\alpha,p}}\ \text{ for any } x\in B,$$

and hence we have

$$|\mu^{*,\rho}_{\lambda,0,\infty}(f_3)(x)-\mu^{*,\rho}_{\lambda,0,\infty}(f_3)(x_0)|\le Cr^\alpha\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\mathcal{E}^{\alpha,p}}\ \text{ for any } x\in B.$$

□

§3. Proofs of Theorems 1–9

Now, we will prove the theorems.

Proof of Theorem 1. We follow the ideas by Kurtz [12] and Sun [20]. Let $r>0$ and $B=B(x_0,r)$. Set $f_1=f_{4B}$, $f_2=(f-f_{4B})\chi_{4B}$ and $f_3=(f-f_{4B})\chi_{(4B)^c}$. Then, $f=f_1+f_2+f_3$ and $\mu^\rho(f_1)=0$.

By assumption, $\mu^\rho(f)(x_0)<\infty$. So, we have $\mu^\rho_\infty(f)(x_0)\le\mu^\rho(f)(x_0)<\infty$. Using Lemma 2.1 we have $\mu^\rho_\infty(f_3)(x_0)\le\mu^\rho_\infty(f)(x_0)+\mu^\rho_\infty(f_2)(x_0)<\infty$. Hence by Lemmas 2.2 and 2.3 (i) we have for $x\in B$

$$\mu^\rho(f_3)(x)\le\mu^\rho_0(f_3)(x)+\mu^\rho_\infty(f_3)(x)\le Cr^\alpha\|f\|_{\mathrm{BMO}}+\mu^\rho_\infty(f_3)(x_0)<\infty,$$

and

$$|\mu^\rho(f_3)(x)-\mu^\rho(f_3)(x_0)|=|\mu^\rho_\infty(f_3)(x)-\mu^\rho_\infty(f_3)(x_0)|\le C\|f\|_{\mathrm{BMO}}.$$

Using L^2-boundedness of μ^ρ (Theorem B) we have $\|\mu^\rho(f_2)\|_{L^2} \le C\|f_2\|_{L^2}$, and from this it follows that $\mu^\rho(f_2)(x) < \infty$ for almost all $x \in B$. Thus, we have $\mu^\rho(f)(x) \le \mu^\rho(f_2)(x) + \mu^\rho(f_3)(x) < \infty$ for almost all $x \in B$. Since r is arbitrary, we see that $\mu^\rho(f)(x) < \infty$ for almost all $x \in \mathbb{R}^n$.

Let $E = \{x \in \mathbb{R}^n\ ;\ \mu^\rho(f)(x) < \infty\}$. We have only to show that for any ball $B = B(x_0, r)$ with center $x_0 \in E$,

$$\int_B |\mu^\rho(f)(x) - (\mu^\rho(f))_B|\,dx \le C|B|\|f\|_{\mathrm{BMO}}.$$

Set $f = f_1 + f_2 + f_3$ as above. Noting $\mu^\rho(f_1) = 0$, and using $\|\mu^\rho(f_2)\|_{L^2} \le C\|f_2\|_{L^2} \le C|B|^{1/2}\|f\|_{\mathcal{E}^{0,2}} \le C|B|^{1/2}\|f\|_{\mathrm{BMO}}$ and the above inequality for $\mu^\rho(f_3)$, we have

$$\begin{aligned}
\frac{1}{|B|}\int_B |\mu^\rho(f)(x) - (\mu^\rho(f))_B|\,dx &\le \frac{2}{|B|}\int_B |\mu^\rho(f)(x) - \mu^\rho(f_3)(x_0)|\,dx \\
&= \frac{2}{|B|}\int_B |\mu^\rho(f_2+f_3)(x) - \mu^\rho(f_3)(x) + \mu^\rho(f_3)(x) - \mu^\rho(f_3)(x_0)|\,dx \\
&\le \frac{2}{|B|}\int_B |\mu^\rho(f_2)(x)|\,dx + \frac{2}{|B|}\int_B |\mu^\rho(f_3)(x) - \mu^\rho(f_3)(x_0)|\,dx \\
&\le C\left(\frac{1}{|B|}\int_{4B} |f_2(x)|^2\,dx\right)^{1/2} + C\|f\|_{\mathcal{E}^{0,1}} \le C\|f\|_{\mathrm{BMO}}.
\end{aligned}$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. The proof is the same as that of Theorem 1. We use Lemmas 2.1, 2.2, 2.3 (ii) and Theorem B. We note that, since Ω satisfies L^1-β-Dini condition, it satisfies L^1-Dini condition and hence $\Omega \in L\log^+ L(S^{n-1}) \subsetneq H^1(S^{n-1})$. So, we can use Theorem B. □

Proof of Theorem 3. The proof is the same as that of Theorem 1. We use Lemmas 2.1, 2.2, 2.3 (iii), and Theorem B. □

Proof of Theorem 4. The proof is the same as that of Theorem 1. We use $\mu^\rho_{S,0}$ and $\mu^\rho_{S,\infty}$, Lemmas 2.4, 2.5, 2.6 (i), and Theorem A (ii) ($p = 2$). □

Proof of Theorem 5. The proof is the same as that of Theorem 1. We use $\mu^\rho_{S,0}$ and $\mu^\rho_{S,\infty}$, Lemmas 2.4, 2.5, 2.6 (ii), (iii) and Theorem A (ii) ($p = 2$). In the case $1/2 \le \alpha < 1$, as is noted in the proof of Theorem 2, we see $\Omega \in L\log^+ L(S^{n-1})$, and we can apply Theorem A (ii). □

Proof of Theorem 6. (i) Since $\mathcal{E}^{\alpha,p} = L^p(\mathbb{R}^n)$ for $\alpha = n/p$, by Theorem A we have only to treat the case $n/p < \alpha < 0$. So, the proof is the same as that of Theorem 1. We use $\mu^{\rho}_{S,0}$ and $\mu^{\rho}_{S,\infty}$, Lemmas 2.4, 2.5, 2.6 (iv) and Theorem A.

(ii) The proof is the same as the above case (i). We use 2.6 (v) in place of 2.6 (iv). Note that if $1 < p < 2$, the condition $\sigma > n/2$ implies $\frac{2n}{n+2\sigma} < 1$, and so we can apply Theorem A. □

Proof of Theorem 7. The proof is the same as that of Theorem 1. We use $\mu^{*,\rho}_{\lambda,0}$ and $\mu^{*,\rho}_{\lambda,\infty}$, Lemmas 2.7, 2.9 (b), 2.11 (i), and Theorem A (ii) ($p = 2$). □

Proof of Theorem 8. We follow the proof of Theorem 1 by modifying it. Let $r > 0$ and $B = B(x_0, r)$. Set $f_1 = f_{4B}$, $f_2 = (f - f_{4B})\chi_{4B}$ and $f_3 = (f - f_{4B})\chi_{(4B)^c}$. Then, $f = f_1 + f_2 + f_3$ and $\mu^{*,\rho}_{\lambda}(f_1) = 0$.

(i) The case $0 < \alpha < 1/2$. By assumption, $\mu^{*,\rho}_{\lambda}(f)(x_0) < \infty$. So, we have $\mu^{*,\rho}_{\lambda,\infty}(f)(x_0) + \mu^{*,\rho}_{\lambda,0,\infty}(f)(x_0) \leq 2\mu^{*,\rho}_{\lambda}(f)(x_0) < \infty$. Using Lemmas 2.7 and 2.8 we have $\mu^{*,\rho}_{\lambda,\infty}(f_3)(x_0) + \mu^{*,\rho}_{\lambda,0,\infty}(f_3)(x_0) \leq \mu^{*,\rho}_{\lambda,\infty}(f)(x_0) + \mu^{*,\rho}_{\lambda,0,\infty}(f)(x_0) + \mu^{*,\rho}_{\lambda,\infty}(f_2)(x_0) + \mu^{*,\rho}_{\lambda,0,\infty}(f_2)(x_0) < \infty$. Hence by Lemmas 2.10, 2.11 and 2.12 we have for $x \in B$

$$\begin{aligned}\mu^{*,\rho}_{\lambda}(f_3)(x) &\leq \mu^{*,\rho}_{\lambda,0,0}(f_3)(x) + \mu^{*,\rho}_{\lambda,0,\infty}(f_3)(x) + \mu^{*,\rho}_{\lambda,\infty}(f_3)(x)\\ &\leq 3Cr^{\alpha}\|f\|_{\mathcal{E}^{\alpha,p}} + \mu^{*,\rho}_{\lambda,0,\infty}(f_3)(x_0) + \mu^{*,\rho}_{\lambda,\infty}(f_3)(x_0) < \infty,\end{aligned}$$

and

$$\begin{aligned}|\mu^{*,\rho}_{\lambda}(f_3)(x) - \mu^{*,\rho}_{\lambda}(f_3)(x_0)| &\leq |\mu^{*,\rho}_{\lambda,0,0}(f_3)(x) - \mu^{*,\rho}_{\lambda,0,0}(f_3)(x_0)|\\ &\quad + |\mu^{*,\rho}_{\lambda,0,\infty}(f_3)(x) - \mu^{*,\rho}_{\lambda,0,\infty}(f_3)(x_0)| + |\mu^{*,\rho}_{\lambda,\infty}(f_3)(x) - \mu^{*,\rho}_{\lambda,\infty}(f_3)(x_0)|\\ &\leq 4Cr^{\alpha}\|f\|_{\mathcal{E}^{\alpha,p}}.\end{aligned}$$

Using L^p-boundedness of $\mu^{*,\rho}_{\lambda}$ (Theorem A) we have $\|\mu^{*,\rho}_{\lambda}(f_2)\|_{L^p} \leq C\|f_2\|_{L^p}$, and from this it follows that $\mu^{*,\rho}_{\lambda}(f_2)(x) < \infty$ for almost all $x \in B$. Thus, we have $\mu^{*,\rho}_{\lambda}(f)(x) \leq \mu^{*,\rho}_{\lambda}(f_2)(x) + \mu^{*,\rho}_{\lambda}(f_3)(x) < \infty$ for almost all $x \in B$. Since r is arbitrary, we see that $\mu^{*,\rho}_{\lambda}(f)(x) < \infty$ for almost all $x \in \mathbb{R}^n$.

The rest of the proof is the same as that of Theorem 1. We omit it.

(ii) The case $1/2 \leq \alpha < 1$. In this case, the proof is simpler than the case (i), and the same as that of Theorem 1. We use $\mu^{*,\rho}_{\lambda,0}$ and $\mu^{*,\rho}_{\lambda,\infty}$, Lemmas 2.7, 2.9 and 2.11 (iii). As is noted in the proof of Theorem 5, we see $\Omega \in L\log^+ L(S^{n-1})$, and we can apply Theorem A (ii). This completes the proof of Theorem 8. ∎

Proof of Theorem 9. (i) The proof is the same as that of Theorem 6 (i). We use $\mu^{*,\rho}_{\lambda,0}$ and $\mu^{*,\rho}_{\lambda,\infty}$, Lemmas 2.7, 2.9, 2.11 (iv), and Theorem A.

(ii) The proof is the same as the above case (i). We use 2.11 (v) in place of 2.11 (iv). Note that if $1 < p < 2$, the condition $\sigma > n/2$ implies $\frac{2n}{n+2\sigma} < 1$, and so we can apply Theorem A. ∎

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