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# An $H^3(G, \mathbb{T})$-valued index of symmetry-protected topological phases with on-site finite group symmetry for two-dimensional quantum spin systems

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**Abstract**

We consider symmetry-protected topological phases with on-site finite group $G$ symmetry $\beta$ for two-dimensional quantum spin systems. We show that they have $H^3(G, \mathbb{T})$-valued invariant.

## Contents

## 1. Introduction

The notion of symmetry-protected topological phases was introduced by Gu and Wen [GW]. It is defined as follows: We consider the set of all Hamiltonians with some symmetry which have a unique gapped ground state in the bulk and can be smoothly deformed into a common trivial gapped Hamiltonian without closing the gap. We say two such Hamiltonians are equivalent if they can be smoothly deformed into each other without breaking the symmetry. We call an equivalence class of this classification a symmetry-protected topological (SPT) phase. Based on tensor network or quantum field theory analysis [CGLW, MGSC], it is conjectured that SPT phases with on-site finite group $G$ symmetry for

$\nu$-dimensional quantum spin systems have an $H^{\nu+1}(G, \mathbb{T})$-valued invariant. We proved that conjecture affirmatively in [O1] for $\nu = 1$. In this paper, we show that the conjecture is also true for $\nu = 2$.

We start by summarising the standard setup of 2-dimensional quantum spin systems on the 2-dimensional lattice $\mathbb{Z}^2$ [BR1, BR2]. We will freely use the basic notation in Section A. Throughout this paper, we fix some $2 \le d \in \mathbb{N}$. We denote the algebra of $d \times d$ matrices by $\mathrm{M}_d$.

For each subset $\Gamma$ of $\mathbb{Z}^2$, we denote the set of all finite subsets in $\Gamma$ by $\mathfrak{S}_\Gamma$. We introduce the Euclidean metric on $\mathbb{Z}^2$, inherited from $\mathbb{R}^2$. We denote by $\mathrm{d}(S_1, S_2)$ the distance between $S_1, S_2 \subset \mathbb{Z}^2$. For a subset $\Gamma$ of $\mathbb{Z}^2$ and $r \in \mathbb{R}_{\ge 0}$, $\hat{\Gamma}(r)$ denotes all the points in $\mathbb{R}^2$ whose distance from $\Gamma$ is less than or equal to $r$. We also set $\Gamma(r) := \hat{\Gamma}(r) \cap \mathbb{Z}^2$. When we take a complement of $\Gamma \subset \mathbb{Z}^2$, it means $\Gamma^c := \mathbb{Z}^2 \setminus \Gamma$. For each $n \in \mathbb{N}$, we denote $[-n, n]^2 \cap \mathbb{Z}^2$ by $\Lambda_n$.

For each $z \in \mathbb{Z}^2$, let $\mathcal{A}_{\{z\}}$ be an isomorphic copy of $\mathrm{M}_d$, and for any finite subset $\Lambda \subset \mathbb{Z}^2$, we set $\mathcal{A}_\Lambda = \bigotimes_{z \in \Lambda} \mathcal{A}_{\{z\}}$. For finite $\Lambda$, the algebra $\mathcal{A}_\Lambda$ can be regarded as the set of all bounded operators acting on the Hilbert space $\bigotimes_{z \in \Lambda} \mathbb{C}^d$. We use this identification freely. If $\Lambda_1 \subset \Lambda_2$, the algebra $\mathcal{A}_{\Lambda_1}$ is naturally embedded in $\mathcal{A}_{\Lambda_2}$ by tensoring its elements with the identity. For an infinite subset $\Gamma \subset \mathbb{Z}^2$, $\mathcal{A}_\Gamma$ is given as the inductive limit of the algebras $\mathcal{A}_\Lambda$ with $\Lambda \in \mathfrak{S}_\Gamma$. We call $\mathcal{A}_\Gamma$ the quantum spin system on $\Gamma$. For simplicity, we denote the 2-dimensional quantum spin system $\mathcal{A}_{\mathbb{Z}^2}$ by $\mathcal{A}$. We also set $\mathcal{A}_{\mathrm{loc}} := \bigcup_{\Lambda \in \mathfrak{S}_{\mathbb{Z}^2}} \mathcal{A}_\Lambda$. For a subset $\Gamma_1$ of $\Gamma \subset \mathbb{Z}^2$, the algebra $\mathcal{A}_{\Gamma_1}$ can be regarded as a subalgebra of $\mathcal{A}_\Gamma$. With this identification, for $A \in \mathcal{A}_{\Gamma_1}$ we occasionally use the same symbol $A$ to denote $A \otimes \mathbb{I}_{\mathcal{A}_{\Gamma \setminus \Gamma_1}} \in \mathcal{A}_\Gamma$. Similarly, an automorphism $\gamma$ on $\mathcal{A}_{\Gamma_1}$ can be naturally regarded as an automorphism $\gamma \otimes \mathrm{id}_{\mathcal{A}_{\Gamma \setminus \Gamma_1}}$ on $\mathcal{A}_\Gamma$. We use this identification freely, and with a slight abuse of notation we occasionally denote $\gamma \otimes \mathrm{id}_{\mathcal{A}_{\Gamma \setminus \Gamma_1}}$ by $\gamma$. Similarly, for disjoint $\Gamma_-, \Gamma_+ \subset \mathbb{Z}^2$ and $\alpha_\pm \in \mathrm{Aut}\,\mathcal{A}_{\Gamma_\pm}$, we occasionally write $\alpha_- \otimes \alpha_+$ to denote $\left(\alpha_- \otimes \mathrm{id}_{\Gamma_-^c}\right)\left(\alpha_+ \otimes \mathrm{id}_{\Gamma_+^c}\right)$, under the given identification.

Throughout this paper we fix a finite group $G$ and a unitary representation $U$ on $\mathbb{C}^d$. Let $\Gamma \subset \mathbb{Z}^2$ be a nonempty subset. For each $g \in G$, there exists a unique automorphism $\beta_g^\Gamma$ on $\mathcal{A}_\Gamma$ such that

$$\beta_g^\Gamma (A) = \mathrm{Ad}\left(\bigotimes_I U(g)\right)(A), \quad A \in \mathcal{A}_I, \ g \in G, \tag{1.1}$$

for any finite subset $I$ of $\Gamma$. We call the group homomorphism $\beta^\Gamma : G \to \mathrm{Aut}\,\mathcal{A}_\Gamma$ the on-site action of $G$ on $\mathcal{A}_\Gamma$ given by $U$. For simplicity, we denote $\beta_g^{\mathbb{Z}^2}$ by $\beta_g$.

A mathematical model of a quantum spin system is fully specified by its interaction $\Phi$. A uniformly bounded interaction on $\mathcal{A}$ is a map $\Phi : \mathfrak{S}_{\mathbb{Z}^2} \to \mathcal{A}_{\mathrm{loc}}$ such that

$$\Phi(X) = \Phi(X)^* \in \mathcal{A}_X, \quad X \in \mathfrak{S}_{\mathbb{Z}^2}, \tag{1.2}$$

and

$$\sup_{X \in \mathfrak{S}_{\mathbb{Z}^2}} \|\Phi(X)\| < \infty. \tag{1.3}$$

It is of finite range, with interaction length less than or equal to $R \in \mathbb{N}$ if $\Phi(X) = 0$ for any $X \in \mathfrak{S}_{\mathbb{Z}^2}$ whose diameter is larger than $R$. An on-site interaction – that is, an interaction with $\Phi(X) = 0$ unless $X$ consists of a single point – is said to be trivial. An interaction $\Phi$ is $\beta$-invariant if $\beta_g(\Phi(X)) = \Phi(X)$ for any $X \in \mathfrak{S}_{\mathbb{Z}^2}$. For a uniformly bounded and finite-range interaction $\Phi$ and $\Lambda \in \mathfrak{S}_{\mathbb{Z}^2}$, define the local Hamiltonian

$$(H_\Phi)_\Lambda := \sum_{X \subset \Lambda} \Phi(X) \tag{1.4}$$

and denote the dynamics

$$\tau_t^{(\Lambda)\Phi}(A) := e^{it(H_\Phi)_\Lambda} A e^{-it(H_\Phi)_\Lambda}, \quad t \in \mathbb{R}, \ A \in \mathcal{A}. \tag{1.5}$$

By the uniform boundedness and finite-rangeness of $\Phi$, for each $A \in \mathcal{A}$ the following limit exists

$$\lim_{\Lambda \to \mathbb{Z}^\nu} \tau_t^{(\Lambda),\Phi}(A) =: \tau_t^\Phi(A), \quad t \in \mathbb{R}, \tag{1.6}$$

which defines the dynamics $\tau^\Phi$ on $\mathcal{A}$ [BR2]. For a uniformly bounded and finite-range interaction $\Phi$, a state $\varphi$ on $\mathcal{A}$ is called a $\tau^\Phi$-ground state if the inequality $-i\,\varphi(A^*\delta_\Phi(A)) \geq 0$ holds for any element $A$ in the domain $\mathcal{D}(\delta_\Phi)$ of the generator $\delta_\Phi$. Let $\varphi$ be a $\tau^\Phi$-ground state, with a Gelfand–Naimark–Segal (GNS) triple $(\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi)$. Then there exists a unique positive operator $H_{\varphi,\Phi}$ on $\mathcal{H}_\varphi$ such that $e^{itH_{\varphi,\Phi}}\pi_\varphi(A)\Omega_\varphi = \pi_\varphi\left(\tau_\Phi^t(A)\right)\Omega_\varphi$, for all $A \in \mathcal{A}$ and $t \in \mathbb{R}$. We call this $H_{\varphi,\Phi}$ the bulk Hamiltonian associated with $\varphi$.

**Definition 1.1.** We say that an interaction $\Phi$ has a unique gapped ground state if (i) the $\tau^\Phi$-ground state, which we denote as $\omega_\Phi$, is unique and (ii) there exists a $\gamma > 0$ such that $\sigma\left(H_{\omega_\Phi,\Phi}\right)\setminus\{0\} \subset [\gamma,\infty)$, where $\sigma\left(H_{\omega_\Phi,\Phi}\right)$ is the spectrum of $H_{\omega_\Phi,\Phi}$. We denote by $\mathcal{P}_{UG}$ the set of all uniformly bounded finite-range interactions with unique gapped ground state. We denote by $\mathcal{P}_{UG\beta}$ the set of all uniformly bounded finite-range $\beta$-*invariant* interactions with unique gapped ground state.

In this paper we consider a classification problem of a subset of $\mathcal{P}_{UG\beta}$, models with short-range entanglement. To describe the models with short-range entanglement, we need to explain the classification problem of unique gapped ground-state phases without symmetry. For $\Gamma \subset \mathbb{Z}^2$, we denote by $\Pi_\Gamma : \mathcal{A} \to \mathcal{A}_\Gamma$ the conditional expectation with respect to the trace state. Let $f : (0,\infty) \to (0,\infty)$ be a continuous decreasing function with $\lim_{t\to\infty} f(t) = 0$. For each $A \in \mathcal{A}$, define

$$\|A\|_f := \|A\| + \sup_{N \in \mathbb{N}} \left( \frac{\|A - \Pi_{\Lambda_N}(A)\|}{f(N)} \right). \tag{1.7}$$

We denote by $\mathcal{D}_f$ the set of all $A \in \mathcal{A}$ such that $\|A\|_f < \infty$.

The classification of unique gapped ground-state phases $\mathcal{P}_{UG}$ without symmetry is the following:

**Definition 1.2.** Two interactions $\Phi_0, \Phi_1 \in \mathcal{P}_{UG}$ are equivalent if there is a path of interactions $\Phi : [0,1] \to \mathcal{P}_{UG}$ satisfying the following:

1. $\Phi(0) = \Phi_0$ and $\Phi(1) = \Phi_1$.
2. For each $X \in \mathfrak{S}_{\mathbb{Z}^2}$, the map $[0,1] \ni s \to \Phi(X;s) \in \mathcal{A}_X$ is $C^1$. We denote by $\dot{\Phi}(X;s)$ the corresponding derivatives. The interaction obtained by differentiation is denoted by $\dot{\Phi}(s)$, for each $s \in [0,1]$.
3. There is a number $R \in \mathbb{N}$ such that $X \in \mathfrak{S}_{\mathbb{Z}^2}$ and $\operatorname{diam} X \geq R$ imply $\Phi(X;s) = 0$, for all $s \in [0,1]$.
4. Interactions are bounded as follows:

$$C_b^\Phi := \sup_{s \in [0,1]} \sup_{X \in \mathfrak{S}_{\mathbb{Z}^2}} \left( \|\Phi(X;s)\| + \|\dot{\Phi}(X;s)\| \right) < \infty. \tag{1.8}$$

5. Setting

$$b(\varepsilon) := \sup_{Z \in \mathfrak{S}_{\mathbb{Z}^2}} \sup_{s,s_0 \in [0,1], 0 < |s-s_0| < \varepsilon} \left\| \frac{\Phi(Z;s) - \Phi(Z;s_0)}{s - s_0} - \dot{\Phi}(Z;s_0) \right\| \tag{1.9}$$

for each $\varepsilon > 0$, we have $\lim_{\varepsilon \to 0} b(\varepsilon) = 0$.
6. There exists a $\gamma > 0$ such that $\sigma\left(H_{\omega_{\Phi(s)},\Phi(s)}\right)\setminus\{0\} \subset [\gamma,\infty)$ for all $s \in [0,1]$, where $\sigma\left(H_{\omega_{\Phi(s)},\Phi(s)}\right)$ is the spectrum of $H_{\omega_{\Phi(s)},\Phi(s)}$.

7. There exists $0 < \eta < 1$ satisfying the following: Set $\zeta(t) := e^{-t^\eta}$. Then for each $A \in \mathcal{D}_\zeta$, $\omega_{\Phi(s)}(A)$ is differentiable with respect to $s$, and there is a constant $C_\zeta$ such that

$$\left| \frac{d}{ds} \omega_{\Phi(s)}(A) \right| \le C_\zeta \, \|A\|_\zeta \, , \tag{1.10}$$

for any $A \in \mathcal{D}_\zeta$. (Recall definition (1.7)).

We write $\Phi_0 \sim \Phi_1$ if $\Phi_0$ and $\Phi_1$ are equivalent. If $\Phi_0, \Phi_1 \in \mathcal{P}_{UG\beta}$ and we can take the path in $\mathcal{P}_{UG\beta}$ – that is, so that $\beta_g(\Phi(X; s)) = \Phi(X; s)$, $g \in G$, for all $s \in [0, 1]$ – then we say $\Phi_0$ and $\Phi_1$ are $\beta$-equivalent and write $\Phi_0 \sim_\beta \Phi_1$.

The reason we require these conditions is that we rely on the result in [MO]. The object we classify in this paper is the following:

**Definition 1.3.** Fix a trivial interaction $\Phi_0 \in \mathcal{P}_{UG}$. We denote by $\mathcal{P}_{SL\beta}$ the set of all $\Phi \in \mathcal{P}_{UG\beta}$ such that $\Phi \sim \Phi_0$. Connected components of $\mathcal{P}_{SL\beta}$ with respect to $\sim_\beta$ are the SPT phases.

Because we have $\Phi_0 \sim \tilde{\Phi}_0$ for any trivial $\Phi_0, \tilde{\Phi}_0 \in \mathcal{P}_{UG}$, the set $\mathcal{P}_{SL\beta}$ does not depend on the choice of $\Phi_0$.

**Remark 1.4.** From the automorphic equivalence (Theorem 5.1), $\Phi \sim \Phi_0$ means that the ground state of $\Phi$ has a short-range entanglement. This is because the automorphisms in Theorem 5.1 can be regarded as a version of a quantum circuit with finite depth, which is regarded as a quantum circuit that does not create long-range entanglement [BL].

The main result of this paper is as follows:

**Theorem 1.5.** *There is an $H^3(G, \mathbb{T})$-valued index on $\mathcal{P}_{SL\beta}$, which is an invariant of the classification $\sim_\beta$ of $\mathcal{P}_{SL\beta}$.*

The paper is organised as follows. In Section 2, we define the $H^3(G, \mathbb{T})$-valued index for a class of states which are created from a fixed product state via 'factorisable' automorphisms, satisfying some additional condition. This additional condition is the existence of the set of automorphisms which (i) do not move the state and (ii) are almost like $\beta$-action restricted to the upper half-plane, except for some 1-dimensional perturbation. In Section 3, we show that the existence of such set of automorphisms is guaranteed in a suitable situation. Furthermore, in Section 4 we show the stability of the index – that is, a suitably $\beta$-invariant automorphism does not change this index. Finally, in Section 5 we show our main theorem, Theorem 1.5, and that in our setting of Theorem 1.5, all the conditions required in Sections 2, 3 and 4 are satisfied. Although the index is defined in terms of GNS representations, in some good situation, we can calculate it without going through GNS representation; this is shown in Section 6. Reviews of this article can be found in [O3, O4].

## 2. The $H^3(G, \mathbb{T})$-valued index in 2-dimensional systems

In this section, we associate an $H^3(G, \mathbb{T})$-index for some class of states. It will turn out later that this class includes SPT phases. For a nontrivial example of this index, see [O3]. It is also shown there that if a state is of product form of two states on half-planes, then our index is trivial. From the construction to follow, one can easily see that the group structure of $H^3(G, \mathbb{T})$, which is a simple pointwise multiplication, shows up when we tensor two systems.

### 2.1. An overview

We consider states of the form $\omega = \omega_0 \circ \alpha$, where $\omega_0$ is a pure infinite tensor product state (see definition (2.18)) and $\alpha$ an automorphism satisfying some factorisation property (2.8). In equation (2.8), $\alpha_L, \alpha_R$

are automorphisms localised to the left and right infinite planes $H_L, H_R$, and $\Theta$ is localised in $(C_\theta)^c$, where $C_\theta$ is defined by definition (2.2). We then have $\omega \simeq (\omega_L \alpha_L \otimes \omega_R \alpha_R) \circ \Theta$ with pure states $\omega_L, \omega_R$ on the left and right infinite planes. We further assume that the effective excitation caused by $(\beta_g^U)^{-1}$ (see formula (2.5)) on $\omega$ is localised around the $x$-axis, in the sense that for any $0 < \theta < \frac{\pi}{2}$, there are automorphisms $\eta_g^L, \eta_g^R$ localised in $C_\theta \cap H_L, C_\theta \cap H_R$ such that $\omega \circ (\beta_g^U)^{-1}$ is equivalent to $\omega \circ (\eta_g^L \otimes \eta_g^R)$. This corresponds to thinking of $\mathcal{T}(\theta, (\tilde{\beta}_g))$ (definition (2.22)) and $\mathrm{IG}(\omega, \theta)$ (definition (2.24)). Setting $\gamma_g^R := \eta_g^R \beta_g^{UR}, \gamma_g^L := \eta_g^L \beta_g^{UL}$, with $\beta_g^{UR}, \beta_g^{UL}$ in formula (2.5), the condition given is $\omega \simeq \omega \circ (\gamma_g^L \otimes \gamma_g^R)$. Repeated use of this formula gives us $\omega \simeq \omega \circ (\gamma_g^L \gamma_h^L (\gamma_{gh}^L)^{-1} \otimes \gamma_g^R \gamma_h^R (\gamma_{gh}^R)^{-1})$. Substituting the factorisation of $\omega$, we then have

$$(\omega_L \alpha_L \otimes \omega_R \alpha_R) \circ \Theta \simeq (\omega_L \alpha_L \otimes \omega_R \alpha_R) \circ \Theta \left( \gamma_g^L \gamma_h^L \left( \gamma_{gh}^L \right)^{-1} \otimes \gamma_g^R \gamma_h^R \left( \gamma_{gh}^R \right)^{-1} \right).$$

However, because conjugation by $\beta_g^U$ does not change the support of automorphisms, we see that this combination $\gamma_g^R \gamma_h^R (\gamma_{gh}^R)^{-1}$ is localised in $C_\theta \cap H_R$. As a result, $\gamma_g^R \gamma_h^R (\gamma_{gh}^R)^{-1}$ – and also $\gamma_g^L \gamma_h^L (\gamma_{gh}^L)^{-1}$ – commutes with $\Theta$. Letting them commute, we obtain

$$\omega_L \alpha_L \otimes \omega_R \alpha_R \simeq (\omega_L \alpha_L \otimes \omega_R \alpha_R) \circ \left( \gamma_g^L \gamma_h^L \left( \gamma_{gh}^L \right)^{-1} \otimes \gamma_g^R \gamma_h^R \left( \gamma_{gh}^R \right)^{-1} \right),$$

from which we can conclude $\omega_R \alpha_R \simeq \omega_R \alpha_R \gamma_g^R \gamma_h^R (\gamma_{gh}^R)^{-1}$. This means that $\alpha_R \gamma_g^R \gamma_h^R (\gamma_{gh}^R)^{-1} \alpha_R^{-1}$ is implementable by some unitary $u(g, h)$ unitary in the GNS representation $\pi_R$ of $\omega_R$ (equation (2.19); see equation (2.27)). On the other hand, substituting the factorisation of $\omega$ to $\omega \simeq \omega \circ (\gamma_g^L \otimes \gamma_g^R)$ implies

$$(\omega_L \alpha_L \otimes \omega_R \alpha_R) \simeq (\omega_L \alpha_L \otimes \omega_R \alpha_R) \circ \Theta \circ \left( \gamma_g^L \otimes \gamma_g^R \right) \circ \Theta^{-1}, \tag{2.1}$$

from which we can derive the implementability of $\Theta \circ (\gamma_g^L \otimes \gamma_g^R) \circ \Theta^{-1}$ in the representation $\pi_L \alpha_L \otimes \pi_R \alpha_R$ by some unitary $W_g$ (see equation (2.26)). Using the definitions of $W_g$ and $u(g, h)$, we can see that they satisfy some nontrivial relation (2.52), with some $U(1)$-phase $c_R(g, h.k)$. In fact, this is quite a similar situation to that of cocycle actions [J]. As in [J], we can show that this $U(1)$-phase $c_R(g, h.k)$ is a 3-cocycle and obtain an $H^3(G, \mathbb{T})$-index. The rest of this section is devoted to the proof that our index is independent of the choice of objects we introduced to define it. All of them follow from the fact that the difference of $W_g$ and $u(g, h)$ caused by the different choice of objects can be implemented by some unitary, and the proof is rather straightforward.

## 2.2. Definitions and the setting

For $0 < \theta < \frac{\pi}{2}$, a (double) cone $C_\theta$ is defined by

$$C_\theta := \left\{ (x, y) \in \mathbb{Z}^2 \mid |y| \le \tan \theta \cdot |x| \right\}. \tag{2.2}$$

Note that it consists of the left part $x \le -1$ and the right part $0 \le x$. For $0 < \theta_1 < \theta_2 \le \frac{\pi}{2}$, we use the notation $\mathcal{C}_{(\theta_1, \theta_2]} := C_{\theta_2} \setminus C_{\theta_1}$ and $\mathcal{C}_{[0, \theta_1]} := C_{\theta_1}$. Left, right, upper and lower half-planes are denoted by $H_L, H_R, H_U$ and $H_D$:

$$H_L := \left\{ (x, y) \in \mathbb{Z}^2 \mid x \le -1 \right\}, \qquad H_R := \left\{ (x, y) \in \mathbb{Z}^2 \mid 0 \le x \right\}, \tag{2.3}$$

$$H_U := \left\{ (x, y) \in \mathbb{Z}^2 \mid 0 \le y \right\}, \qquad H_D := \left\{ (x, y) \in \mathbb{Z}^2 \mid y \le -1 \right\}. \tag{2.4}$$

We use the notation

$$\beta_g := \beta_g^{\mathbb{Z}^2}, \quad \beta_g^U := \beta_g^{H_U}, \quad \beta_g^{RU} := \beta_g^{H_R \cap H_U}, \quad \beta_g^{LU} := \beta_g^{H_L \cap H_U}. \tag{2.5}$$

For each subset $S$ of $\mathbb{Z}^2$, we set

$$S_\sigma := S \cap H_\sigma, \quad S_\zeta := S \cap H_\zeta, \quad S_{\sigma\zeta} := S \cap H_\sigma \cap H_\zeta, \quad \sigma = L, R, \ \zeta = U, D. \tag{2.6}$$

We occasionally write $\mathcal{A}_{S,\sigma}, \mathcal{A}_{S,\zeta}, \mathcal{A}_{S,\sigma,\zeta}$ to denote $\mathcal{A}_{S_\sigma}, \mathcal{A}_{S_\zeta}, \mathcal{A}_{S_{\sigma\zeta}}$. For an automorphism $\alpha$ on $\mathcal{A}$ and $0 < \theta < \frac{\pi}{2}$, we denote by $\mathfrak{D}_\alpha^\theta$ a set of all triples $(\alpha_L, \alpha_R, \Theta)$ with

$$\alpha_L \in \mathrm{Aut}\left(\mathcal{A}_{H_L}\right), \quad \alpha_R \in \mathrm{Aut}\left(\mathcal{A}_{H_R}\right), \quad \Theta \in \mathrm{Aut}\left(\mathcal{A}_{(C_\theta)^c}\right) \tag{2.7}$$

decomposing $\alpha$ as
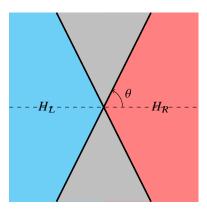
$$\alpha = (\text{inner}) \circ (\alpha_L \otimes \alpha_R) \circ \Theta. \tag{2.8}$$

For $(\alpha_L, \alpha_R, \Theta) \in \mathfrak{D}_\alpha^{(\theta)}$, we set

$$\alpha_0 := \alpha_L \otimes \alpha_R. \tag{2.9}$$

The class of automorphisms which allow such decompositions for any directions is denoted by

$$\mathrm{QAut}\,(\mathcal{A}) := \left\{ \alpha \in \mathrm{Aut}(\mathcal{A}) \mid \mathfrak{D}_\alpha^\theta \neq \emptyset \text{ for all } 0 < \theta < \frac{\pi}{2} \right\}. \tag{2.10}$$



The automorphism $\Theta$ in equation (2.8) acts nontrivially only on $C_\theta^c$, the gray area.

Furthermore, for each

$$0 < \theta_{0.8} < \theta_1 < \theta_{1.2} < \theta_{1.8} < \theta_2 < \theta_{2.2} < \theta_{2.8} < \theta_3 < \theta_{3.2} < \frac{\pi}{2}, \tag{2.11}$$

we consider decompositions of $\alpha \in \mathrm{Aut}(\mathcal{A})$ such that

$$\alpha = (\text{inner}) \circ \left( \alpha_{[0,\theta_1]} \otimes \alpha_{(\theta_1,\theta_2]} \otimes \alpha_{(\theta_2,\theta_3]} \otimes \alpha_{(\theta_3,\frac{\pi}{2}]} \right) \circ \left( \alpha_{(\theta_{0.8},\theta_{1.2}]} \otimes \alpha_{(\theta_{1.8},\theta_{2.2}]} \otimes \alpha_{(\theta_{2.8},\theta_{3.2}]} \right), \tag{2.12}$$

with

$$\alpha_X := \bigotimes_{\sigma=L,R,\,\zeta=D,U} \alpha_{X,\sigma,\zeta}, \qquad \alpha_{[0,\theta_1]} := \bigotimes_{\sigma=L,R} \alpha_{[0,\theta_1],\sigma}, \qquad \alpha_{\left(\theta_3,\frac{\pi}{2}\right]} := \bigotimes_{\zeta=D,U} \alpha_{\left(\theta_3,\frac{\pi}{2}\right],\zeta},$$

$$\alpha_{X,\sigma,\zeta} \in \mathrm{Aut}\left(\mathcal{A}_{C_{X,\sigma,\zeta}}\right), \qquad \alpha_{X,\sigma} := \bigotimes_{\zeta=U,D} \alpha_{X,\sigma,\zeta}, \qquad \alpha_{X,\zeta} := \bigotimes_{\sigma=L,R} \alpha_{X,\sigma,\zeta},$$

$$\alpha_{[0,\theta_1],\sigma} \in \mathrm{Aut}\left(\mathcal{A}_{C_{[0,\theta_0],\sigma}}\right), \qquad \alpha_{\left(\theta_3,\frac{\pi}{2}\right],\zeta} \in \mathrm{Aut}\left(\mathcal{A}_{C_{\left(\theta_3,\frac{\pi}{2}\right],\zeta}}\right),$$

$$(2.13)$$

for

$$X = (\theta_1, \theta_2], (\theta_2, \theta_3], (\theta_{0.8}, \theta_{1.2}], (\theta_{1.8}, \theta_{2.2}], (\theta_{2.8}, \theta_{3.2}], \quad \sigma = L, R, \ \zeta = D, U. \qquad (2.14)$$

The class of automorphisms on $\mathcal{A}$ which allow such decompositions for any directions $\theta_{0.8}, \theta_1, \theta_{1.2}, \theta_{1.8}, \theta_2, \theta_{2.2}, \theta_{2.8}, \theta_3, \theta_{3.2}$ (satisfying formula (2.11)) is denoted by SQAut($\mathcal{A}$). Note that SQAut($\mathcal{A}$) $\subset$ QAut($\mathcal{A}$). The set of all $\alpha \in$ SQAut($\mathcal{A}$) with each of $\alpha_I$ in the decompositions required to commute with $\beta_g^U$, $g \in G$, is denoted by GSQAut($\mathcal{A}$):

GSQAut($\mathcal{A}$)

$$:= \left\{ \alpha \in \mathrm{SQAut}(\mathcal{A}) \,\middle|\, \begin{array}{c} \text{for any } \theta_{0.8}, \theta_1, \theta_{1.2}, \theta_{1.8}, \theta_2, \theta_{2.2}, \theta_{2.8}, \theta_3, \theta_{3.2} \text{ satisfying formula (2.11),} \\ \text{there is a decomposition (2.12), (2.13), (2.14) satisfying} \\ \alpha_I \circ \beta_g^U = \beta_g^U \circ \alpha_I, \ g \in G, \\ \text{for all } I = [0,\theta_1], (\theta_1,\theta_2], (\theta_2,\theta_3], \left(\theta_3, \frac{\pi}{2}\right], (\theta_{0.8},\theta_{1.2}], (\theta_{1.8},\theta_{2.2}], (\theta_{2.8},\theta_{3.2}] \end{array} \right\}.$$

$$(2.15)$$

We also define

$$\mathrm{HAut}(\mathcal{A}) := \left\{ \alpha \in \mathrm{Aut}(\mathcal{A}) \,\middle|\, \begin{array}{c} \text{for any } 0 < \theta < \frac{\pi}{2}, \text{ there exist } \alpha_\sigma \in \mathrm{Aut}\left(\mathcal{A}_{(C_\theta)_\sigma}\right), \ \sigma = L, R, \\ \text{such that } \alpha = (\text{inner}) \circ (\alpha_L \otimes \alpha_R) \end{array} \right\}. \quad (2.16)$$

In Section 5, we will see that quasilocal automorphisms corresponding to paths in symmetric gapped phases belong to the following set:

$$\mathrm{GUQAut}(\mathcal{A}) := \left\{ \gamma \in \mathrm{Aut}(\mathcal{A}) \,\middle|\, \begin{array}{c} \text{there are } \gamma_H \in \mathrm{HAut}(\mathcal{A}), \gamma_C \in \mathrm{GSQAut}(\mathcal{A}), \\ \text{such that } \gamma = \gamma_C \circ \gamma_H \end{array} \right\}. \qquad (2.17)$$

We fix a reference state $\omega_0$ as follows: We fix a unit vector $\xi_x \in \mathbb{C}^d$ and let $\rho_{\xi_x}$ be the vector state on $\mathrm{M}_d$ given by $\xi_x$, for each $x \in \mathbb{Z}^2$. Then our reference state $\omega_0$ is given by

$$\omega_0 := \bigotimes_{x \in \mathbb{Z}^2} \rho_{\xi_x}. \qquad (2.18)$$

Throughout this section this $\omega_0$ is fixed. Let $(\mathcal{H}_0, \pi_0, \Omega_0)$ be a GNS triple of $\omega_0$. Because of the product structure of $\omega_0$, it is decomposed as

$$\mathcal{H}_0 = \mathcal{H}_L \otimes \mathcal{H}_R, \quad \pi_0 = \pi_L \otimes \pi_R, \quad \Omega_0 = \Omega_L \otimes \Omega_R, \qquad (2.19)$$

where $(\mathcal{H}_\sigma, \pi_\sigma, \Omega_\sigma)$ is a GNS triple of $\omega_\sigma := \omega_0|_{\mathcal{A}_{H_\sigma}}$ for $\sigma = L, R$. As $\omega_0|_{\mathcal{A}_{H_\sigma}}$ is pure, $\pi_\sigma$ is irreducible. What we consider in this section is the set of states created via elements in $\mathrm{QAut}(\mathcal{A})$ from our reference state $\omega_0$:

$$\mathcal{SL} := \{\omega_0 \circ \alpha \mid \alpha \in \mathrm{QAut}(\mathcal{A})\}. \tag{2.20}$$

Because any pure product states can be transformed to each other via an automorphism of product form $\tilde{\alpha} = \bigotimes_{x \in \mathbb{Z}^2} \tilde{\alpha}_x$, and $\tilde{\alpha}\alpha$ belongs to $\mathrm{QAut}(\mathcal{A})$ for any $\alpha \in \mathrm{QAut}(\mathcal{A})$, $\mathcal{SL}$ does not depend on the choice of $\omega_0$. For each $\omega \in \mathcal{SL}$, we set

$$\mathrm{EAut}(\omega) := \{\alpha \in \mathrm{QAut}(\mathcal{A}) \mid \omega = \omega_0 \circ \alpha\}. \tag{2.21}$$

By the definition of $\mathcal{SL}$, $\mathrm{EAut}(\omega)$ is not empty.

For $0 < \theta < \frac{\pi}{2}$ and a set of automorphisms $(\tilde{\beta}_g)_{g \in G} \subset \mathrm{Aut}(\mathcal{A})$, we introduce a set

$$\mathcal{T}(\theta, (\tilde{\beta}_g)) := \left\{ (\eta_g^\sigma)_{g \in G, \, \sigma = L, R} \middle| \begin{array}{c} \eta_g^\sigma \in \mathrm{Aut}\left(\mathcal{A}_{(C_\theta)_\sigma}\right), \\ \tilde{\beta}_g = (\text{inner}) \circ \left(\eta_g^L \otimes \eta_g^R\right) \circ \beta_g^U, \\ \text{for all } g \in G, \, \sigma = L, R \end{array} \right\}. \tag{2.22}$$

In a word, it is a set of decompositions of $\tilde{\beta}_g \circ (\beta_g^U)^{-1}$ into tensors of $\mathrm{Aut}(\mathcal{A}_{(C_\theta)_L})$, $\mathrm{Aut}(\mathcal{A}_{(C_\theta)_R})$ modulo inner automorphisms. For $(\eta_g^\sigma)_{g \in G, \, \sigma = L, R} \in \mathcal{T}(\theta, (\tilde{\beta}_g))$, we set

$$\eta_g := \eta_g^L \otimes \eta_g^R, \quad g \in G. \tag{2.23}$$

The following set of automorphisms is the key ingredient for the definition of our index: For $\omega \in \mathcal{SL}$ and $0 < \theta < \frac{\pi}{2}$, we set

$$\mathrm{IG}(\omega, \theta) := \left\{ (\tilde{\beta}_g)_{g \in G} \in \mathrm{Aut}(\mathcal{A})^{\times G} \middle| \begin{array}{c} \omega \circ \tilde{\beta}_g = \omega \text{ for all } g \in G \\ \text{and } \mathcal{T}(\theta, (\tilde{\beta}_g)) \neq \emptyset \end{array} \right\}. \tag{2.24}$$

We also set

$$\mathrm{IG}(\omega) := \cup_{0 < \theta < \frac{\pi}{2}} \mathrm{IG}(\omega, \theta). \tag{2.25}$$

In this section we associate some third cohomology $h(\omega)$ for each $\omega \in \mathcal{SL}$ with $\mathrm{IG}(\omega) \neq \emptyset$.

## 2.3. Derivation of elements in $Z^3(G, \mathbb{T})$

In this subsection, we derive 3-cocycles out of $\omega, \alpha, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_L, \alpha_R, \Theta)$.

**Lemma 2.1.** *Set* $\omega \in \mathcal{SL}, \alpha \in \mathrm{EAut}(\omega), 0 < \theta < \frac{\pi}{2}, (\tilde{\beta}_g) \in \mathrm{IG}(\omega, \theta), (\eta_g^\sigma) \in \mathcal{T}(\theta, (\tilde{\beta}_g)), (\alpha_L, \alpha_R, \Theta) \in \mathfrak{D}_\alpha^\theta$. *Then the following hold:*

(i) *There are unitaries* $W_g, g \in G$, *on* $\mathcal{H}_0$ *such that*

$$\mathrm{Ad}(W_g) \circ \pi_0 = \pi_0 \circ \alpha_0 \circ \Theta \circ \eta_g \beta_g^U \circ \Theta^{-1} \circ \alpha_0^{-1}, \quad g \in G, \tag{2.26}$$

*with the notation of definitions (2.9) and (2.23).*

(ii) *There exists a unitary $u_\sigma(g, h)$ on $\mathcal{H}_\sigma$, for each $\sigma = L, R$ and for $g, h \in G$, such that*

$$\text{Ad}\left(u_\sigma(g, h)\right) \circ \pi_\sigma = \pi_\sigma \circ \alpha_\sigma \circ \eta_g^\sigma \beta_g^{\sigma U} \eta_h^\sigma \left(\beta_g^{\sigma U}\right)^{-1} \left(\eta_{gh}^\sigma\right)^{-1} \circ \alpha_\sigma^{-1} \tag{2.27}$$

*and*

$$\text{Ad}\left(u_L(g, h) \otimes u_R(g, h)\right) \pi_0 = \pi_0 \circ \alpha_0 \circ \eta_g \beta_g^U \eta_h \left(\beta_g^U\right)^{-1} \left(\eta_{gh}\right)^{-1} \circ \alpha_0^{-1}. \tag{2.28}$$

*Furthermore, $u_\sigma(g, h)$ commutes with any element of $\pi_\sigma \circ \alpha_\sigma \left(\mathcal{A}_{((C_\theta)^c)_\sigma}\right)$.*

**Definition 2.2.** For $\omega \in \mathcal{SL}, \alpha \in \text{EAut}(\omega), 0 < \theta < \frac{\pi}{2}, (\tilde{\beta}_g) \in \text{IG}(\omega, \theta), (\eta_g^\sigma)_{g \in G, \sigma = L, R} \in \mathcal{T}\left(\theta, (\tilde{\beta}_g)\right), (\alpha_L, \alpha_R, \Theta) \in \mathfrak{D}_\alpha^\theta$, we denote by

$$\text{IP}\left(\omega, \alpha, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_L, \alpha_R, \Theta)\right) \tag{2.29}$$

the set of $((W_g)_{g \in G}, (u_\sigma(g, h))_{g, h \in G, \sigma = L, R})$ with $W_g \in \mathcal{U}(\mathcal{H}_0)$ and $u_\sigma(g, h) \in \mathcal{U}(\mathcal{H}_\sigma)$ satisfying

$$\text{Ad}(W_g) \circ \pi_0 = \pi_0 \circ \alpha_0 \circ \Theta \circ \eta_g \beta_g^U \circ \Theta^{-1} \circ \alpha_0^{-1}, \quad g \in G, \tag{2.30}$$

and

$$\text{Ad}\left(u_\sigma(g, h)\right) \circ \pi_\sigma = \pi_\sigma \circ \alpha_\sigma \circ \eta_g^\sigma \beta_g^{\sigma U} \eta_h^\sigma \left(\beta_g^{\sigma U}\right)^{-1} (\eta_{gh}^\sigma)^{-1} \circ \alpha_\sigma^{-1}, \quad g, h \in G, \; \sigma = L, R. \tag{2.31}$$

(Here we used the notation of definition (2.9) and (2.23).) By Lemma 2.1, it is nonempty.

*Proof.* For a GNS triple $(\mathcal{H}_0, \pi_0 \circ \alpha, \Omega_0)$ of $\omega = \omega_0 \circ \alpha$, there are unitaries $\tilde{W}_g$ on $\mathcal{H}_0$ such that

$$\text{Ad}\left(\tilde{W}_g\right) \circ \pi_0 \circ \alpha = \pi_0 \circ \alpha \circ \tilde{\beta}_g, \quad g \in G, \tag{2.32}$$

because $\omega \circ \tilde{\beta}_g = \omega$.

Because $(\eta_g^\sigma)_{g \in G, \sigma = L, R} \in \mathcal{T}(\theta, (\tilde{\beta}_g))$ and $(\alpha_L, \alpha_R, \Theta) \in \mathfrak{D}_\alpha^\theta$, there are unitaries $v_g, V \in \mathcal{U}(\mathcal{A})$ such that

$$\tilde{\beta}_g = \text{Ad}\left(v_g\right) \circ \left(\eta_g^L \otimes \eta_g^R\right) \circ \beta_g^U, \quad \alpha = \text{Ad}\, V \circ \alpha_0 \circ \Theta. \tag{2.33}$$

Substituting these, we have

$$\begin{aligned}
\text{Ad}\left(\tilde{W}_g \pi_0(V)\right) \pi_0 \circ \alpha_0 \circ \Theta &= \pi_0 \circ \alpha \tilde{\beta}_g = \pi_0 \circ \alpha \circ \text{Ad}\left(v_g\right) \circ \eta_g \beta_g^U \\
&= \text{Ad}\left(\left(\pi_0 \circ \alpha(v_g)\right) \pi_0(V)\right) \pi_0 \circ \alpha_0 \circ \Theta \circ \eta_g \beta_g^U.
\end{aligned} \tag{2.34}$$

Therefore, setting $W_g := \pi_0(V)^*(\pi_0 \circ \alpha(v_g^*))\tilde{W}_g \pi_0(V) \in \mathcal{U}(\mathcal{H}_0)$, we obtain equation (2.26).

Using equation (2.26), we have

$$\text{Ad}\left(W_g W_h W_{gh}^*\right) \pi_0 = \pi_0 \circ \alpha_0 \circ \Theta \circ \eta_g \beta_g^U \eta_h \left(\beta_g^U\right)^{-1} \eta_{gh}^{-1} \Theta^{-1} \alpha_0^{-1}. \tag{2.35}$$

Note that because conjugation by $\beta_g^U$ does not change the support of automorphisms, $\eta_g \beta_g^U \eta_h (\beta_g^U)^{-1} \eta_{gh}^{-1}$ belongs to $\mathrm{Aut}(\mathcal{A}_{C_\theta})$. On the other hand, $\Theta$ belongs to $\mathrm{Aut}(\mathcal{A}_{(C_\theta)^c})$. Therefore, they commute and we obtain

$$
\begin{aligned}
\mathrm{Ad}\left(W_g W_h W_{gh}^*\right) \pi_0 &= \text{equation (2.35)} = \pi_0 \circ \alpha_0 \circ \eta_g \beta_g^U \eta_h \left(\beta_g^U\right)^{-1} \eta_{gh}^{-1} \alpha_0^{-1} \\
&= \bigotimes_{\sigma = L, R} \pi_\sigma \circ \alpha_\sigma \circ \eta_g^\sigma \beta_g^{\sigma U} \eta_h^\sigma \left(\beta_g^{\sigma U}\right)^{-1} \left(\eta_{gh}^\sigma\right)^{-1} \circ \alpha_\sigma^{-1}.
\end{aligned}
\tag{2.36}
$$

From this and the irreducibility of $\pi_R$, we see that $\mathrm{Ad}(W_g W_h W_{gh}^*)$ gives rise to a $*$-isomorphism $\tau$ on $\mathcal{B}(\mathcal{H}_R)$. It is implemented by some unitary $u_R(g, h)$ on $\mathcal{H}_R$ by the Wigner theorem, and we obtain

$$
\begin{aligned}
\mathbb{I}_{\mathcal{H}_L} \otimes \left(\mathrm{Ad}\left(u_R(g, h)\right) \circ \pi_R(A)\right) &= \mathbb{I}_{\mathcal{H}_L} \otimes \tau \left(\pi_R(A)\right) = \mathrm{Ad}\left(W_g W_h W_{gh}^*\right) \left(\mathbb{I}_{\mathcal{H}_L} \otimes \pi_R(A)\right) \\
&= \mathbb{I}_{\mathcal{H}_L} \otimes \pi_R \circ \alpha_R \circ \eta_g^R \beta_g^{RU} \eta_h^R \left(\beta_g^{RU}\right)^{-1} \left(\eta_{gh}^R\right)^{-1} \circ \alpha_R^{-1}(A),
\end{aligned}
\tag{2.37}
$$

for any $A \in \mathcal{A}_{H_R}$. Hence we obtain equation (2.27) for $\sigma = R$.

To see that $u_R(g, h)$ belongs to $\left(\pi_R \circ \alpha_R \left(\mathcal{A}_{((C_\theta)^c)_R}\right)\right)'$, set $A \in \mathcal{A}_{((C_\theta)^c)_R}$. Then because $\eta_g^R \beta_g^{RU} \eta_h^R (\beta_g^{RU})^{-1} (\eta_{gh}^R)^{-1}$ belongs to $\mathrm{Aut}\left(\mathcal{A}_{(C_\theta)_R}\right)$, we have

$$
\mathrm{Ad}\left(u_R(g, h)\right) \pi_R \left(\alpha_R(A)\right) = \pi_R \alpha_R \eta_g^R \beta_g^{RU} \eta_h^R \left(\beta_g^{RU}\right)^{-1} \left(\eta_{gh}^R\right)^{-1} \alpha_R^{-1} \alpha_R(A) = \pi_R \alpha_R(A).
\tag{2.38}
$$

This proves that $u_R(g, h)$ belongs to $\left(\pi_R \circ \alpha_R \left(\mathcal{A}_{((C_\theta)^c)_R}\right)\right)'$. An analogous statement for $u_L(g, h)$ can be shown exactly the same way. The last statement of (ii), equation (2.28), is trivial from equation (2.27). $\qquad\square$

**Lemma 2.3.** *Set* $\omega \in \mathcal{SL}, \alpha \in \mathrm{EAut}(\omega), 0 < \theta < \frac{\pi}{2}, (\tilde{\beta}_g) \in \mathrm{IG}(\omega, \theta), (\eta_g^\sigma) \in \mathcal{T}(\theta, (\tilde{\beta}_g)),$ $(\alpha_L, \alpha_R, \Theta) \in \mathfrak{D}_\alpha^\theta.$ *Let* $((W_g), (u_R(g, h)))$ *be an element of* $\mathrm{IP}(\omega, \alpha, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_L, \alpha_R, \Theta)).$
*Then the following hold:*

(i) *For any* $g, h, k \in G,$

$$
\begin{aligned}
&\mathrm{Ad}\left(W_g \left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R(h, k)\right) W_g^*\right) \circ \pi_0 \\
&= \pi_0 \circ \left(\mathrm{id}_{\mathcal{A}_{H_L}} \otimes \alpha_R \eta_g^R \beta_g^{RU} \left(\eta_h^R \beta_h^{RU} \eta_k^R \left(\beta_h^{RU}\right)^{-1} \left(\eta_{hk}^R\right)^{-1}\right) \left(\eta_g^R \beta_g^{RU}\right)^{-1} \alpha_R^{-1}\right).
\end{aligned}
\tag{2.39}
$$

(ii) *For any* $g, h \in G,$

$$
\mathrm{Ad}\left((u_L(g, h) \otimes u_R(g, h)) W_{gh}\right) = \mathrm{Ad}\left(W_g W_h\right)
\tag{2.40}
$$

*on* $\mathcal{B}(\mathcal{H}_0).$

(iii) *For any* $g, h, k \in G,$

$$
\mathrm{Ad}(W_g) \left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R(h, k)\right) \in \mathbb{C}\mathbb{I}_{\mathcal{H}_L} \otimes \mathcal{B}(\mathcal{H}_R).
\tag{2.41}
$$

(iv) *For any* $g, h, k, f \in G,$

$$
\mathrm{Ad}\left(W_g W_h\right) \left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R(k, f)\right) = \left(\mathrm{Ad}\left((\mathbb{I}_{\mathcal{H}_L} \otimes u_R(g, h)) W_{gh}\right)\right) \left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R(k, f)\right).
\tag{2.42}
$$

*Proof.* We use the notation from definitions (2.9) and (2.23).

(i) Substituting equations (2.30) and (2.31), we have

$$\mathrm{Ad}\left(W_g\left(\mathbb{I}_{\mathcal{H}_L}\otimes u_R(h,k)\right)W_g^*\right)\circ\pi_0$$

$$= \pi_0\circ\alpha_0\circ\Theta\circ\eta_g\beta_g^U\circ\Theta^{-1}\circ\alpha_0^{-1}\circ\left(\mathrm{id}_{\mathcal{A}_{H_L}}\otimes\alpha_R\circ\eta_h^R\beta_h^{RU}\eta_k^R\left(\beta_h^{RU}\right)^{-1}\left(\eta_{hk}^R\right)^{-1}\circ\alpha_R^{-1}\right)$$

$$\circ\alpha_0\circ\Theta\circ\left(\eta_g\beta_g^U\right)^{-1}\circ\Theta^{-1}\circ\alpha_0^{-1}$$

$$= \pi_0\circ\alpha_0\circ\Theta\circ\eta_g\beta_g^U\circ\Theta^{-1}\circ\left(\mathrm{id}_{\mathcal{A}_{H_L}}\otimes\eta_h^R\beta_h^{RU}\eta_k^R\left(\beta_h^{RU}\right)^{-1}\left(\eta_{hk}^R\right)^{-1}\right)\circ\Theta\circ\left(\eta_g\beta_g^U\right)^{-1}\circ\Theta^{-1}\circ\alpha_0^{-1}.$$

$$\tag{2.43}$$

Because $\eta_h^R\beta_h^{RU}\eta_k^R(\beta_h^{RU})^{-1}(\eta_{hk}^R)^{-1}$ belongs to $\mathrm{Aut}\left(\mathcal{A}_{(C_\theta)_R}\right)$, it commutes with $\Theta\in\mathrm{Aut}\left(\mathcal{A}_{(C_\theta)^c}\right)$. Hence we obtain

equation (2.43)

$$= \pi_0\circ\alpha_0\circ\Theta\circ\eta_g\beta_g^U\circ\left(\mathbb{I}_{\mathcal{A}_{H_L}}\otimes\eta_h^R\beta_h^{RU}\eta_k^R\left(\beta_h^{RU}\right)^{-1}\left(\eta_{hk}^R\right)^{-1}\right)\circ\left(\eta_g\beta_g^U\right)^{-1}\circ\Theta^{-1}\circ\alpha_0^{-1}$$

$$= \pi_0\circ\alpha_0\circ\Theta\circ\left(\mathbb{I}_{\mathcal{A}_{H_L}}\otimes\eta_g^R\beta_g^{RU}\circ\eta_h^R\beta_h^{RU}\eta_k^R\left(\beta_h^{RU}\right)^{-1}\left(\eta_{hk}^R\right)^{-1}\circ\left(\eta_g^R\beta_g^{RU}\right)^{-1}\right)\circ\Theta^{-1}\circ\alpha_0^{-1}.$$

$$\tag{2.44}$$

Again, the term in parentheses in the last line is localised at $(C_\theta)_R$, and it commutes with $\Theta$. Therefore, we have

$$\mathrm{Ad}\left(W_g\left(\mathbb{I}_{\mathcal{H}_L}\otimes u_R(h,k)\right)W_g^*\right)\circ\pi_0$$

$$= \pi_0\circ\left(\mathrm{id}_{\mathcal{A}_{H_L}}\otimes\alpha_R\circ\eta_g^R\beta_g^{RU}\circ\eta_h^R\beta_h^{RU}\eta_k^R\left(\beta_h^{RU}\right)^{-1}\left(\eta_{hk}^R\right)^{-1}\circ\left(\eta_g^R\beta_g^{RU}\right)^{-1}\circ\alpha_R^{-1}\right). \tag{2.45}$$

(ii) Again by equations (2.30) and (2.31), we have

$$\mathrm{Ad}\left((u_L(g,h)\otimes u_R(g,h))W_{gh}\right)\circ\pi_0 = \pi_0\circ\alpha_0\circ\eta_g\beta_g^U\eta_h\left(\beta_g^U\right)^{-1}\left(\eta_{gh}\right)^{-1}\circ\Theta\circ\eta_{gh}\beta_{gh}^U\circ\Theta^{-1}\circ\alpha_0^{-1}$$

$$= \pi_0\circ\alpha_0\circ\Theta\circ\eta_g\beta_g^U\eta_h\left(\beta_g^U\right)^{-1}\left(\eta_{gh}\right)^{-1}\circ\eta_{gh}\beta_{gh}^U\circ\Theta^{-1}\circ\alpha_0^{-1}$$

$$= \pi_0\circ\alpha_0\circ\Theta\circ\eta_g\beta_g^U\eta_h\beta_h^U\circ\Theta^{-1}\circ\alpha_0^{-1}$$

$$= \mathrm{Ad}\left(W_gW_h\right)\circ\pi_0.$$

$$\tag{2.46}$$

Here, for the second equality we again used the commutativity of $\eta$s and $\Theta$, due to their disjoint support. Because $\pi_0$ is irreducible, we obtain equation (2.40).

(iii) For any $A\in\mathcal{A}_{H_L}$, we have

$$\Theta^{-1}\circ\alpha_0^{-1}\left(A\otimes\mathbb{I}_{\mathcal{A}_{H_R}}\right) = \Theta^{-1}\circ\left(\alpha_L^{-1}(A)\otimes\mathbb{I}_{\mathcal{A}_{H_R}}\right)\in\Theta^{-1}\left(\mathcal{A}_{H_L}\otimes\mathbb{C}\mathbb{I}_{\mathcal{A}_{H_R}}\right)\subset\mathcal{A}_{H_L\cup(C_\theta^c)_R}, \tag{2.47}$$

because $\Theta\in\mathrm{Aut}\left(\mathcal{A}_{(C_\theta)^c}\right)$. Therefore, $\eta_g^R\in\mathrm{Aut}\left(\mathcal{A}_{(C_\theta)_R}\right)$ acts trivially on it and we have

$$\left(\beta_g^U\right)^{-1}\left(\eta_g\right)^{-1}\circ\Theta^{-1}\circ\alpha_0^{-1}\left(A\otimes\mathbb{I}_{\mathcal{A}_{H_R}}\right)\in\mathcal{A}_{H_L\cup(C_\theta^c)_R}. \tag{2.48}$$

As $\Theta$ preserves $\mathcal{A}_{H_L \cup (C_\theta^c)_R}$,

$$\Theta \circ \left(\beta_g^U\right)^{-1} \left(\eta_g\right)^{-1} \circ \Theta^{-1} \circ \alpha_0^{-1} \left(A \otimes \mathbb{I}_{\mathcal{A}_{H_R}}\right) \tag{2.49}$$

also belongs to $\mathcal{A}_{H_L \cup (C_\theta^c)_R}$. As a result,

$$\mathrm{Ad}\left(W_g^*\right)\left(\pi_L(A) \otimes \mathbb{I}_{\mathcal{H}_R}\right) = \pi_0 \circ \alpha_0 \circ \Theta \circ \left(\beta_g^U\right)^{-1} \left(\eta_g\right)^{-1} \circ \Theta^{-1} \circ \alpha_0^{-1} \left(A \otimes \mathbb{I}_{\mathcal{A}_{H_R}}\right) \tag{2.50}$$

belongs to $\pi_L(\mathcal{A}_{H_L}) \otimes \pi_R \circ \alpha_R(\mathcal{A}_{(C_\theta^c)_R})$, and hence commutes with $\mathbb{I}_{\mathcal{H}_L} \otimes u_R(h, k)$. Hence $\mathrm{Ad}(W_g)\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R(h, k)\right)$ commutes with any elements in $\pi_L(\mathcal{A}_L) \otimes \mathbb{C}\mathbb{I}_{\mathcal{H}_R}$. Because $\pi_L$ is irreducible, $\mathrm{Ad}(W_g)\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R(h, k)\right)$ belongs to $\mathbb{C}\mathbb{I}_{\mathcal{H}_L} \otimes \mathcal{B}(\mathcal{H}_R)$.

(iv) By (iii), $\mathrm{Ad}\left(W_{gh}\right)\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R(k, f)\right)$ belongs to $\mathbb{C}\mathbb{I}_{\mathcal{H}_L} \otimes \mathcal{B}(\mathcal{H}_R)$. Therefore, from (ii), we have

$$\begin{aligned}
\mathrm{Ad}\left(W_g W_h\right)\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R(k, f)\right) &= \mathrm{Ad}\left(\left(u_L(g, h) \otimes u_R(g, h)\right) W_{gh}\right)\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R(k, f)\right) \\
&= \mathrm{Ad}\left(\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R(g, h)\right) W_{gh}\right)\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R(k, f)\right),
\end{aligned} \tag{2.51}$$

obtaining (iv). $\qquad\square$

With this preparation, we may obtain some element of $Z^3(G, \mathbb{T})$ from $\left((W_g), (u_\sigma(g, h))\right)$.

**Lemma 2.4.** *Set* $\omega \in \mathcal{SL}, \alpha \in \mathrm{EAut}(\omega), 0 < \theta < \frac{\pi}{2}, (\tilde{\beta}_g) \in \mathrm{IG}(\omega, \theta), (\eta_g^\sigma) \in \mathcal{T}(\theta, (\tilde{\beta}_g)),$ $(\alpha_L, \alpha_R, \Theta) \in \mathfrak{D}_\alpha^\theta.$ *Let* $\left((W_g), (u_\sigma(g, h))\right)$ *be an element of* $\mathrm{IP}(\omega, \alpha, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_L, \alpha_R, \Theta)).$ *Then there is a* $c_R \in Z^3(G, \mathbb{T})$ *such that*

$$\mathbb{I}_{\mathcal{H}_L} \otimes u_R(g, h) u_R(gh, k) = c_R(g, h, k) \left(W_g \left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R(h, k)\right) W_g^*\right)\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R(g, hk)\right), \tag{2.52}$$

*for all* $g, h, k \in G.$

**Definition 2.5.** We denote this 3-cocycle $c_R$ in Lemma 2.4 by

$$c_R\left(\omega, \alpha, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_L, \alpha_R, \Theta), \left((W_g), (u_\sigma(g, h))\right)\right) \tag{2.53}$$

and its cohomology class by

$$\begin{aligned}
h^{(1)} &\left(\omega, \alpha, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_L, \alpha_R, \Theta), \left((W_g), (u_\sigma(g, h))\right)\right) \\
&:= \left[c_R\left(\omega, \alpha, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_L, \alpha_R, \Theta), \left((W_g), (u_\sigma(g, h))\right)\right)\right]_{H^3(G, \mathbb{T})}.
\end{aligned} \tag{2.54}$$

*Proof.* First we prove that there is a number $c_R(g, h, k) \in \mathbb{T}$ satisfying equation (2.52). From equation (2.31), we have

$$\mathrm{Ad}\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R(g, h) u_R(gh, k)\right) \pi_0 = \pi_L \otimes \pi_R \circ \alpha_R \circ \left(\eta_g^R \beta_g^{RU}\right)\left(\eta_h^R \beta_h^{RU}\right)\left(\eta_k^R \beta_k^{RU}\right)\left(\eta_{ghk}^R \beta_{ghk}^{RU}\right)^{-1} \alpha_R^{-1}. \tag{2.55}$$

On the other hand, using Lemma 2.3(i), we have that

$$\mathrm{Ad}\left(\left(W_g \left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R(h, k)\right) W_g^*\right)\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R(g, hk)\right)\right) \pi_0 \tag{2.56}$$

is also equal to the right-hand side of equation (2.55). Because $\pi_0$ is irreducible, this means that there is a number $c_R(g, h, k) \in \mathbb{T}$ satisfying equation (2.52).

Now let us check that this $c_R$ is a 3-cocycle. For any $g, h, k, f \in G$, by repeated use of equation (2.52), we get

$$
\begin{aligned}
\mathbb{I}_{\mathcal{H}_L} \otimes u_R(g,h) u_R(gh,k) u_R(ghk,f) &= \left[ \mathbb{I}_{\mathcal{H}_L} \otimes u_R(g,h) u_R(gh,k) \right] \cdot \left( \mathbb{I}_{\mathcal{H}_L} \otimes u_R(ghk,f) \right) \\
&= \left( c_R(g,h,k) \left( W_g \left( \mathbb{I}_{\mathcal{H}_L} \otimes u_R(h,k) \right) W_g^* \right) \left( \mathbb{I}_{\mathcal{H}_L} \otimes u_R(g,hk) \right) \right) \cdot \left( \mathbb{I}_{\mathcal{H}_L} \otimes u_R(ghk,f) \right) \\
&= \left( c_R(g,h,k) \left( W_g \left( \mathbb{I}_{\mathcal{H}_L} \otimes u_R(h,k) \right) W_g^* \right) \right) \cdot \left[ \mathbb{I}_{\mathcal{H}_L} \otimes u_R(g,hk) u_R(ghk,f) \right] \\
&= \left( c_R(g,h,k) \left( W_g \left( \mathbb{I}_{\mathcal{H}_L} \otimes u_R(h,k) \right) W_g^* \right) \right) \\
&\quad \cdot \left( c_R(g,hk,f) \left( W_g \left( \mathbb{I}_{\mathcal{H}_L} \otimes u_R(hk,f) \right) W_g^* \right) \left( \mathbb{I}_{\mathcal{H}_L} \otimes u_R(g,hkf) \right) \right) \\
&= c_R(g,h,k) c_R(g,hk,f) \left( W_g \left[ \mathbb{I}_{\mathcal{H}_L} \otimes u_R(h,k) u_R(hk,f) \right] W_g^* \right) \cdot \left( \mathbb{I}_{\mathcal{H}_L} \otimes u_R(g,hkf) \right) \\
&= c_R(g,h,k) c_R(g,hk,f) c_R(h,k,f) \\
&\quad W_g \left( W_h \left( \mathbb{I}_{\mathcal{H}_L} \otimes u_R(k,f) \right) W_h^* \left( \mathbb{I}_{\mathcal{H}_L} \otimes u_R(h,kf) \right) \right) W_g^* \cdot \left( \mathbb{I}_{\mathcal{H}_L} \otimes u_R(g,hkf) \right) \\
&= c_R(g,h,k) c_R(g,hk,f) c_R(h,k,f) \\
&\quad \cdot W_g W_h \left( \mathbb{I}_{\mathcal{H}_L} \otimes u_R(k,f) \right) W_h^* W_g^* \cdot \left[ \left( W_g \left( \mathbb{I}_{\mathcal{H}_L} \otimes u_R(h,kf) \right) W_g^* \right) \mathbb{I}_{\mathcal{H}_L} \otimes u_R(g,hkf) \right] \\
&= c_R(g,h,k) c_R(g,hk,f) c_R(h,k,f) \overline{c(g,h,kf)} \cdot \left\{ W_g W_h \left( \mathbb{I}_{\mathcal{H}_L} \otimes u_R(k,f) \right) W_h^* W_g^* \right\} \\
&\quad \cdot \left( \mathbb{I}_{\mathcal{H}_L} \otimes u_R(g,h) u_R(gh,kf) \right).
\end{aligned}
\tag{2.57}
$$

Here and in the following, we apply equation (2.52) for terms in $[\cdot]$ to get the succeeding equality. Applying Lemma 2.3(iv) to the $\{\cdot\}$ part of equation (2.57), we obtain

$$
\begin{aligned}
\text{equation (2.57)} &= c_R(g,h,k) c_R(g,hk,f) c_R(h,k,f) \overline{c(g,h,kf)} \\
&\quad \left( \text{Ad} \left( \left( \mathbb{I}_{\mathcal{H}_L} \otimes u_R(g,h) \right) W_{gh} \right) \right) \left( \mathbb{I}_{\mathcal{H}_L} \otimes u_R(k,f) \right) \left( \mathbb{I}_{\mathcal{H}_L} \otimes u_R(g,h) u_R(gh,kf) \right) \\
&= c_R(g,h,k) c_R(g,hk,f) c_R(h,k,f) \overline{c(g,h,kf)} \\
&\quad \left( \mathbb{I}_{\mathcal{H}_L} \otimes u_R(g,h) \right) \left[ W_{gh} \left( \mathbb{I}_{\mathcal{H}_L} \otimes u_R(k,f) \right) W_{gh}^* \left( \mathbb{I}_{\mathcal{H}_L} \otimes u_R(gh,kf) \right) \right] \\
&= c_R(g,h,k) c_R(g,hk,f) c_R(h,k,f) \overline{c(g,h,kf) c_R(gh,k,f)} \\
&\quad \left( \mathbb{I}_{\mathcal{H}_L} \otimes u_R(g,h) u_R(gh,k) u_R(ghk,f) \right).
\end{aligned}
\tag{2.58}
$$

Hence, we obtain

$$
c_R(g,h,k) c_R(g,hk,f) c_R(h,k,f) \overline{c(g,h,kf) c_R(gh,k,f)} = 1, \quad \text{for all } g, h, k, f \in G. \tag{2.59}
$$

This means $c_R \in Z^3(G, \mathbb{T})$. $\qquad\square$

### 2.4. The $H^3(G, \mathbb{T})$-valued index

From the previous subsection, we remark the following fact:

**Lemma 2.6.** *For any $\omega \in \mathcal{SL}$ with $\mathrm{IG}(\omega) \neq \emptyset$, there are*

$$
\alpha \in \mathrm{EAut}(\omega), \quad 0 < \theta < \frac{\pi}{2}, \quad (\tilde{\beta}_g) \in \mathrm{IG}(\omega, \theta), \quad (\eta_g^\sigma) \in \mathcal{T}(\theta, (\tilde{\beta}_g)), \quad (\alpha_L, \alpha_R, \Theta) \in \mathfrak{D}_\alpha^\theta,
$$
$$
((W_g), (u_R(g,h))) \in \mathrm{IP} \left( \omega, \alpha, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_L, \alpha_R, \Theta) \right).
$$
$$
\tag{2.60}
$$

*Proof.* Because $\mathrm{IG}(\omega) \neq \emptyset$, there is some $0 < \theta < \frac{\pi}{2}$ such that $\mathrm{IG}(\omega, \theta) \neq \emptyset$, and hence $(\tilde{\beta}_g) \in \mathrm{IG}(\omega, \theta)$ and $(\eta_g^\sigma) \in \mathcal{T}(\theta, (\tilde{\beta}_g))$ exist. Because $\omega \in \mathcal{SL}$, by definition there exists some $\alpha \in \mathrm{EAut}(\omega)$, and by the definition of $\mathrm{EAut}(\omega)$, there is some $(\alpha_L, \alpha_R, \Theta) \in \mathfrak{D}_\alpha^\theta$. The existence of $((W_g), (u_R(g, h))) \in \mathrm{IP}(\omega, \alpha, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_L, \alpha_R, \Theta))$ is given by Lemma 2.1. □

By Lemma 2.4, for $\omega \in \mathcal{SL}$ with $\mathrm{IG}(\omega) \neq \emptyset$ and each choice of (2.60), we can associate some element of $H^3(G, \mathbb{T})$:

$$h^{(1)} \left( \omega, \alpha, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_L, \alpha_R, \Theta), ((W_g), (u_\sigma(g, h))) \right). \tag{2.61}$$

In this subsection, we show that the third cohomology class does not depend on the choice of (2.60):

**Theorem 2.7.** *For any $\omega \in \mathcal{SL}$ with $\mathrm{IG}(\omega) \neq \emptyset$,*

$$h^{(1)} \left( \omega, \alpha, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_L, \alpha_R, \Theta), ((W_g), (u_\sigma(g, h))) \right)$$

*is independent of the choice of*

$$\alpha, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_L, \alpha_R, \Theta), ((W_g), (u_\sigma(g, h))).$$

**Definition 2.8.** Let $\omega \in \mathcal{SL}$ with $\mathrm{IG}(\omega) \neq \emptyset$. We denote the third cohomology given in Theorem 2.7 by

$$h(\omega) := h^{(1)} \left( \omega, \alpha, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_L, \alpha_R, \Theta), ((W_g), (u_\sigma(g, h))) \right).$$

First we show the independence from $((W_g), (u_\sigma(g, h)))$.

**Lemma 2.9.** *Set*

$$\omega \in \mathcal{SL}, \quad \alpha \in \mathrm{EAut}(\omega), \quad 0 < \theta < \frac{\pi}{2}, \quad (\tilde{\beta}_g) \in \mathrm{IG}(\omega, \theta), \quad (\eta_g^\sigma) \in \mathcal{T}(\theta, (\tilde{\beta}_g)), \quad (\alpha_L, \alpha_R, \Theta) \in \mathcal{D}_\alpha^\theta, \tag{2.62}$$

$$((W_g), (u_\sigma(g, h))), ((\tilde{W}_g), (\tilde{u}_\sigma(g, h))) \in \mathrm{IP} \left( \omega, \alpha, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_L, \alpha_R, \Theta) \right). \tag{2.63}$$

*Then we have*

$$h^{(1)} \left( \omega, \alpha, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_L, \alpha_R, \Theta), ((W_g), (u_\sigma(g, h))) \right)$$
$$= h^{(1)} \left( \omega, \alpha, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_L, \alpha_R, \Theta), ((\tilde{W}_g), (\tilde{u}_\sigma(g, h))) \right). \tag{2.64}$$

**Definition 2.10.** From this lemma and because there is always $((W_g), (u_R(g, h)))$ in $\mathrm{IP}(\omega, \alpha, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_L, \alpha_R, \Theta))$ by Lemma 2.1, we may define

$$h^{(2)} \left( \omega, \alpha, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_L, \alpha_R, \Theta) \right) := h^{(1)} \left( \omega, \alpha, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_L, \alpha_R, \Theta), ((W_g), (u_\sigma(g, h))) \right) \tag{2.65}$$

for any

$$\omega \in \mathcal{SL}, \quad \alpha \in \mathrm{EAut}(\omega), \quad 0 < \theta < \frac{\pi}{2}, \quad (\tilde{\beta}_g) \in \mathrm{IG}(\omega, \theta), \quad (\eta_g^\sigma) \in \mathcal{T}(\theta, (\tilde{\beta}_g)), \quad (\alpha_L, \alpha_R, \Theta) \in \mathcal{D}_\alpha^\theta, \tag{2.66}$$

independent of the choice of $((W_g), (u_\sigma(g, h)))$.

*Proof.* Because

$$\text{Ad}(W_g) \circ \pi_0 = \pi_0 \circ \alpha_0 \circ \Theta \circ \eta_g \beta_g^U \circ \Theta^{-1} \circ \alpha_0^{-1} = \text{Ad}\left(\tilde{W}_g\right) \circ \pi_0, \tag{2.67}$$

$$\text{Ad}\left(u_R(g,h)\right) \circ \pi_R = \pi_R \circ \alpha_R \circ \eta_g^R \beta_g^{RU} \eta_h^R \left(\beta_g^{RU}\right)^{-1} \left(\eta_{gh}^R\right)^{-1} \circ \alpha_R^{-1} = \text{Ad}\left(\tilde{u}_R(g,h)\right) \circ \pi_R \tag{2.68}$$

and $\pi_0, \pi_R$ are irreducible, there are $b(g), a(g,h) \in \mathbb{T}, g, h \in G$, such that

$$W_g = b(g)\tilde{W}_g, \qquad \tilde{u}_R(g,h) = a(g,h)u_R(g,h). \tag{2.69}$$

Set

$$c_R := c_R\left(\omega, \alpha, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_L, \alpha_R, \Theta), ((W_g), (u_\sigma(g,h)))\right),$$

$$\tilde{c}_R := c_R\left(\omega, \alpha, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_L, \alpha_R, \Theta), ((\tilde{W}_g), (\tilde{u}_\sigma(g,h)))\right). \tag{2.70}$$

Then from the definition of these values and equation (2.69), we have

$$\begin{aligned}
a(g,h)a(gh,k)\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R(g,h)u_R(gh,k)\right) &= \mathbb{I}_{\mathcal{H}_L} \otimes \tilde{u}_R(g,h)\tilde{u}_R(gh,k) \\
&= \tilde{c}_R(g,h,k)\left(\tilde{W}_g\left(\mathbb{I}_{\mathcal{H}_L} \otimes \tilde{u}_R(h,k)\right)\tilde{W}_g^*\right)\left(\mathbb{I}_{\mathcal{H}_L} \otimes \tilde{u}_R(g,hk)\right) \\
&= \tilde{c}_R(g,h,k)a(h,k)a(g,hk)\left(W_g\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R(h,k)\right)W_g^*\right)\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R(g,hk)\right) \\
&= \tilde{c}_R(g,h,k)a(h,k)a(g,hk)\overline{c_R(g,h,k)}\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R(g,h)u_R(gh,k)\right). \tag{2.71}
\end{aligned}$$

Hence we have $\tilde{c}_R(g,h,k) = c_R(g,h,k)\overline{a(h,k)a(g,hk)}a(g,h)a(gh,k)$, and we get $[c_R]_{H^3(G,\mathbb{T})} = [\tilde{c}_R]_{H^3(G,\mathbb{T})}$, proving the claim. $\square$

Next we show the independence from $\alpha, (\alpha_L, \alpha_R, \Theta)$:

**Lemma 2.11.** *Set*

$$\omega \in \mathcal{SL}, \quad \alpha_1, \alpha_2 \in \text{EAut}(\omega), \quad 0 < \theta < \frac{\pi}{2}, \quad (\tilde{\beta}_g) \in \text{IG}(\omega, \theta), \quad (\eta_g^\sigma) \in \mathcal{T}\left(\theta, (\tilde{\beta}_g)\right), \tag{2.72}$$

$$\left(\alpha_{L,1}, \alpha_{R,1}, \Theta_1\right) \in \mathcal{D}_{\alpha_1}^\theta, \quad \left(\alpha_{L,2}, \alpha_{R,2}, \Theta_2\right) \in \mathcal{D}_{\alpha_2}^\theta. \tag{2.73}$$

*Then we have*

$$h^{(2)}\left(\omega, \alpha_1, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_{L,1}, \alpha_{R,1}, \Theta_1)\right) = h^{(2)}\left(\omega, \alpha_2, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_{L,2}, \alpha_{R,2}, \Theta_2)\right). \tag{2.74}$$

**Definition 2.12.** From this lemma and because there are always $\alpha \in \text{EAut}(\omega)$ and $(\alpha_L, \alpha_R, \Theta) \in \mathcal{D}_\alpha^\theta$ for $\omega \in \mathcal{SL}$ and $0 < \theta < \frac{\pi}{2}$ by the definition, we may define

$$h^{(3)}\left(\omega, \theta, (\tilde{\beta}_g), (\eta_g^\sigma)\right) := h^{(2)}\left(\omega, \alpha, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_L, \alpha_R, \Theta)\right) \tag{2.75}$$

for any

$$\omega \in \mathcal{SL}, \quad 0 < \theta < \frac{\pi}{2}, \quad (\tilde{\beta}_g) \in \text{IG}(\omega, \theta), \quad (\eta_g^\sigma) \in \mathcal{T}\left(\theta, (\tilde{\beta}_g)\right), \tag{2.76}$$

independent of the choice of $\alpha, (\alpha_L, \alpha_R, \Theta)$.

*Proof.* By Lemma 2.1, there are

$$\left((W_{g,1}), (u_{\sigma,1}(g,h))\right) \in \mathrm{IP}\left(\omega, \alpha_1, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_{L,1}, \alpha_{R,1}, \Theta_1)\right). \tag{2.77}$$

For each $i = 1, 2$, we have $\Theta_i \in \mathrm{Aut}\, \mathcal{A}_{C_\theta^c}$ and

$$\alpha_i = (\text{inner}) \circ \alpha_{0,i} \circ \Theta_i, \tag{2.78}$$

setting

$$\alpha_{0,i} := \alpha_{L,i} \otimes \alpha_{R,i}. \tag{2.79}$$

Because $\omega_0 \circ \alpha_1 = \omega = \omega_0 \circ \alpha_2$, we have $\omega_0 \circ \alpha_2 \circ \alpha_1^{-1} = \omega_0$. Therefore, there is a unitary $\tilde{V}$ on $\mathcal{H}_0$ such that $\pi_0 \circ \alpha_2 \circ \alpha_1^{-1} = \mathrm{Ad}\,(\tilde{V}) \circ \pi_0$. Substituting equation (2.78) into this, we see that there is a unitary $V$ on $\mathcal{H}_0$ satisfying

$$\pi_0 \circ \alpha_{0,2} \circ \Theta_2 = \mathrm{Ad}\,(V) \circ \pi_0 \circ \alpha_{0,1} \circ \Theta_1. \tag{2.80}$$

From this, we obtain

$$\begin{aligned}
\mathrm{Ad}&\left(V W_{g,1} V^*\right) \circ \pi_0 \\
&= \mathrm{Ad}\left(V W_{g,1}\right) \pi_0 \circ \alpha_{0,1} \circ \Theta_1 \circ \Theta_2^{-1} \circ \alpha_{0,2}^{-1} \\
&= \mathrm{Ad}\,(V) \circ \pi_0 \circ \alpha_{0,1} \circ \Theta_1 \circ \eta_g \beta_g^U \circ \Theta_1^{-1} \circ \alpha_{0,1}^{-1} \circ \alpha_{0,1} \circ \Theta_1 \circ \Theta_2^{-1} \circ \alpha_{0,2}^{-1} \\
&= \pi_0 \circ \alpha_{0,2} \circ \Theta_2 \circ \Theta_1^{-1} \circ \alpha_{0,1}^{-1} \circ \alpha_{0,1} \circ \Theta_1 \circ \eta_g \beta_g^U \circ \Theta_1^{-1} \circ \alpha_{0,1}^{-1} \circ \alpha_{0,1} \circ \Theta_1 \circ \Theta_2^{-1} \circ \alpha_{0,2}^{-1} \\
&= \pi_0 \circ \alpha_{0,2} \circ \Theta_2 \circ \eta_g \beta_g^U \circ \Theta_2^{-1} \circ \alpha_{0,2}^{-1},
\end{aligned} \tag{2.81}$$

for all $g \in G$. Furthermore, we have

$$\begin{aligned}
\mathrm{Ad}&\left(V\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_{R,1}(g,h)\right) V^*\right) \circ \pi_0 = \mathrm{Ad}\left(V\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_{R,1}(g,h)\right)\right) \circ \pi_0 \circ \alpha_{0,1} \circ \Theta_1 \circ \Theta_2^{-1} \circ \alpha_{0,2}^{-1} \\
&= \mathrm{Ad}\,(V) \circ \pi_0 \circ \left(\mathrm{id}_{\mathcal{A}_{H_L}} \otimes \alpha_{R,1} \eta_g^R \beta_g^{RU} \eta_h^R \left(\beta_g^{RU}\right)^{-1} \left(\eta_{gh}^R\right)^{-1} \alpha_{R,1}^{-1}\right) \alpha_{0,1} \circ \Theta_1 \circ \Theta_2^{-1} \circ \alpha_{0,2}^{-1} \\
&= \pi_0 \circ \alpha_{0,2} \circ \Theta_2 \circ \Theta_1^{-1} \circ \alpha_{0,1}^{-1} \left(\mathrm{id}_{\mathcal{A}_{H_L}} \otimes \alpha_{R,1} \eta_g^R \beta_g^{RU} \eta_h^R \left(\beta_g^{RU}\right)^{-1} \left(\eta_{gh}^R\right)^{-1} \alpha_{R,1}^{-1}\right) \circ \alpha_{0,1} \circ \Theta_1 \circ \Theta_2^{-1} \circ \alpha_{0,2}^{-1} \\
&= \pi_0 \circ \alpha_{0,2} \circ \Theta_2 \circ \Theta_1^{-1} \circ \left(\mathrm{id}_{\mathcal{A}_{H_L}} \otimes \eta_g^R \beta_g^{RU} \eta_h^R \left(\beta_g^{RU}\right)^{-1} \left(\eta_{gh}^R\right)^{-1}\right) \circ \Theta_1 \circ \Theta_2^{-1} \circ \alpha_{0,2}^{-1}. \tag{2.82}
\end{aligned}$$

Now, because $\eta_g^R \beta_g^{RU} \eta_h^R \left(\beta_g^{RU}\right)^{-1} \left(\eta_{gh}^R\right)^{-1}$ is an automorphism on $\mathcal{A}_{C_\theta}$ and $\Theta_2 \circ \Theta_1^{-1}$ is an automorphism on $\mathcal{A}_{C_\theta^c}$, they commute. Therefore, we have

$$\begin{aligned}
\mathrm{Ad}&\left(V\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_{R,1}(g,h)\right) V^*\right) \circ \pi_0 \\
&= \text{equation (2.83)} = \pi_0 \circ \alpha_{0,2} \circ \left(\mathrm{id}_{\mathcal{A}_{H_L}} \otimes \eta_g^R \beta_g^{RU} \eta_h^R \left(\beta_g^{RU}\right)^{-1} \left(\eta_{gh}^R\right)^{-1}\right) \circ \alpha_{0,2}^{-1} \\
&= \pi_L \otimes \left(\pi_R \circ \alpha_{R,2} \eta_g^R \beta_g^{RU} \eta_h^R \left(\beta_g^{RU}\right)^{-1} \left(\eta_{gh}^R\right)^{-1} \left(\alpha_{R,2}\right)^{-1}\right). \tag{2.83}
\end{aligned}$$

From this equality and the fact that $\pi_L$ is irreducible, we see that $V\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_{R,1}(g,h)\right) V^*$ is of the form $\mathbb{I}_{\mathcal{H}_L} \otimes u_{R,2}(g,h)$ with some unitary $u_{R,2}(g,h)$ on $\mathcal{H}_R$. This $u_{R,2}(g,h)$ satisfies

$$\mathrm{Ad}\left(u_{R,2}(g,h)\right) \circ \pi_R = \pi_R \circ \alpha_{R,2} \eta_g^R \beta_g^{RU} \eta_h^R \left(\beta_g^{RU}\right)^{-1} \left(\eta_{gh}^R\right)^{-1} \left(\alpha_{R,2}\right)^{-1}. \tag{2.84}$$

Analogously, we obtain a unitary $u_{L,2}(g,h)$ on $\mathcal{H}_L$ such that

$$V\left(u_{L,1}(g,h) \otimes \mathbb{I}_{\mathcal{H}_R}\right) V^* = u_{L,2}(g,h) \otimes \mathbb{I}_{\mathcal{H}_R}, \tag{2.85}$$

$$\mathrm{Ad}\left(u_{L,2}(g,h)\right) \circ \pi_L = \pi_L \circ \alpha_{L,2} \eta_g^L \beta_g^{LU} \eta_h^L \left(\beta_g^{LU}\right)^{-1} \left(\eta_{gh}^L\right)^{-1} \left(\alpha_{L,2}\right)^{-1}. \tag{2.86}$$

From equations (2.81), (2.84) and (2.85), we see that

$$\left(\left(VW_{g,1}V^*\right), \left(u_{\sigma,2}(g,h)\right)\right) \in \mathrm{IP}\left(\omega, \alpha_2, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_{L,2}, \alpha_{R,2}, \Theta)\right). \tag{2.87}$$

Set

$$c_{R,1} := c_R\left(\omega, \alpha_1, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_{L,1}, \alpha_{R,1}, \Theta_1), \left((W_{g,1}), (u_{\sigma,1}(g,h))\right)\right),$$
$$c_{R,2} := c_R\left(\omega, \alpha_2, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_{L,2}, \alpha_{R,2}, \Theta_2), \left(VW_{g,1}V^*\right), \left(u_{\sigma,2}(g,h)\right)\right). \tag{2.88}$$

It suffices to show that $c_{R,1} = c_{R,2}$. This can be checked directly as follows:

$$V\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_{R,1}(g,h) u_{R,1}(gh,k)\right) V^* = \mathbb{I}_{\mathcal{H}_L} \otimes u_{R,2}(g,h) u_{R,2}(gh,k)$$
$$= c_{R,2}(g,h,k)\left(VW_{g,1}V^*\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_{R,2}(h,k)\right) VW_{g,1}^*V^*\right)\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_{R,2}(g,hk)\right)$$
$$= c_{R,2}(g,h,k) V\left(W_{g,1}\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_{R,1}(h,k)\right) W_{g,1}^*\right)\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_{R,1}(g,hk)\right) V^*$$
$$= c_{R,2}(g,h,k)\overline{c_{R,1}(g,h,k)} V\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_{R,1}(g,h) u_{R,1}(gh,k)\right) V^*. \tag{2.89}$$

$\square$

**Lemma 2.13.** *Set*

$$\omega \in \mathcal{SL}, \quad 0 < \theta < \frac{\pi}{2}, \quad (\tilde{\beta}_g) \in \mathrm{IG}(\omega, \theta), \quad (\eta_g^\sigma), \left(\tilde{\eta}_g^\sigma\right) \in \mathcal{T}(\theta, (\tilde{\beta}_g)). \tag{2.90}$$

*Then we have*

$$h^{(3)}\left(\omega, \theta, (\tilde{\beta}_g), (\eta_g^\sigma)\right) = h^{(3)}\left(\omega, \theta, (\tilde{\beta}_g), \left(\tilde{\eta}_g^\sigma\right)\right). \tag{2.91}$$

**Definition 2.14.** From this lemma and the definition of $\mathrm{IG}(\omega, \theta)$, we may define

$$h^{(4)}\left(\omega, \theta, (\tilde{\beta}_g)\right) := h^{(3)}\left(\omega, \theta, (\tilde{\beta}_g), (\eta_g^\sigma)\right) \tag{2.92}$$

for any

$$\omega \in \mathcal{SL}, \quad 0 < \theta < \frac{\pi}{2}, \quad (\tilde{\beta}_g) \in \mathrm{IG}(\omega, \theta), \quad (\eta_g^\sigma) \in \mathcal{T}(\theta, (\tilde{\beta}_g)), \tag{2.93}$$

independent of the choice of $(\eta_g^\sigma)$.

*Proof.* There are $\alpha \in \mathrm{EAut}(\omega)$ and $(\alpha_L, \alpha_R, \Theta) \in \mathcal{D}_\alpha^\theta$ for $\omega \in \mathcal{SL}$ by the definition. We set $\alpha_0 := \alpha_L \otimes \alpha_R$ and $\eta_g := \eta_g^L \otimes \eta_g^R$, $\tilde{\eta}_g := \tilde{\eta}_g^L \otimes \tilde{\eta}_g^R$. By Lemma 2.1, there is some

$$((W_g), (u_\sigma(g, h))) \in \mathrm{IP}\left(\omega, \alpha, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_L, \alpha_R, \Theta)\right). \tag{2.94}$$

Because $(\eta_g^\sigma), \left(\tilde{\eta}_g^\sigma\right) \in \mathcal{T}(\theta, (\tilde{\beta}_g))$, we have

$$\tilde{\beta}_g = (\text{inner}) \circ \left(\eta_g^L \otimes \eta_g^R\right) \circ \beta_g^U = (\text{inner}) \circ \left(\tilde{\eta}_g^L \otimes \tilde{\eta}_g^R\right) \circ \beta_g^U. \tag{2.95}$$

From this, we obtain

$$\tilde{\eta}_g^L \circ \left(\eta_g^L\right)^{-1} \otimes \tilde{\eta}_g^R \circ \left(\eta_g^R\right)^{-1} = (\text{inner}), \tag{2.96}$$

hence there are unitaries $v_g^\sigma \in \mathcal{A}_{H_\sigma}$, $\sigma = L, R$, such that

$$\tilde{\eta}_g^\sigma \circ \left(\eta_g^\sigma\right)^{-1} = \mathrm{Ad}\left(v_g^\sigma\right). \tag{2.97}$$

Because $\tilde{\eta}_g^\sigma, \eta_g^\sigma$ are automorphisms on $\mathcal{A}_{C_\theta, \sigma}$, $v_g^\sigma$ belongs to $\mathcal{A}_{C_\theta, \sigma}$. (See Lemma B.1.) Setting $v_g := v_g^L \otimes v_g^R$, we obtain $\tilde{\eta}_g = \mathrm{Ad}\left(v_g\right) \circ \eta_g$.

Set

$$\tilde{W}_g := \left(\left(\pi_L \alpha_L \left(v_g^L\right)\right) \otimes \left(\pi_R \alpha_R \left(v_g^R\right)\right)\right) W_g, \tag{2.98}$$

$$\tilde{u}_\sigma(g, h) := \pi_\sigma\left(\alpha_\sigma\left(v_g^\sigma \cdot \left(\eta_g^\sigma \beta_g^{\sigma U}\right)\left(v_h^\sigma\right)\right)\right) \cdot u_\sigma(g, h) \cdot \pi_\sigma\left(\alpha_\sigma\left(\left(v_{gh}^\sigma\right)^*\right)\right), \tag{2.99}$$

for each $g, h \in G$ and $\sigma = L, R$. We claim that

$$\left((\tilde{W}_g), (\tilde{u}_\sigma(g, h))\right) \in \mathrm{IP}\left(\omega, \alpha, \theta, (\tilde{\beta}_g), \left(\tilde{\eta}_g^\sigma\right), (\alpha_L, \alpha_R, \Theta)\right). \tag{2.100}$$

First, we have

$$
\begin{aligned}
\pi_0 \circ \alpha_0 &\circ \Theta \circ \tilde{\eta}_g \beta_g^U \circ \Theta^{-1} \circ \alpha_0^{-1} = \pi_0 \circ \alpha_0 \circ \Theta \circ \mathrm{Ad}\left(v_g\right) \circ \eta_g \beta_g^U \circ \Theta^{-1} \circ \alpha_0^{-1} \\
&= \pi_0 \circ \alpha_0 \circ \mathrm{Ad}\left(v_g\right) \circ \Theta \circ \eta_g \beta_g^U \circ \Theta^{-1} \circ \alpha_0^{-1} \\
&= \mathrm{Ad}\left(\left(\pi_L \alpha_L\left(v_g^L\right)\right) \otimes \left(\pi_R \alpha_R\left(v_g^R\right)\right)\right) \pi_0 \circ \alpha_0 \circ \Theta \circ \eta_g \beta_g^U \circ \Theta^{-1} \circ \alpha_0^{-1} = \mathrm{Ad}\left(\tilde{W}_g\right) \circ \pi_0.
\end{aligned}
\tag{2.101}
$$

For the first equality, we substituted $\tilde{\eta}_g = \mathrm{Ad}\left(v_g\right) \circ \eta_g$, and for the second equality, we used the fact that $v_g^\sigma$ belongs to $\mathcal{A}_{C_\theta, \sigma}$, while $\Theta$ is an automorphism on $\mathcal{A}_{(C_\theta)^c, \sigma}$. The last equality follows from the definition of $W_g$. On the other hand, we have

$$
\begin{aligned}
\pi_\sigma &\circ \alpha_\sigma \circ \tilde{\eta}_g^\sigma \beta_g^{\sigma U} \tilde{\eta}_h^\sigma \left(\beta_g^{\sigma U}\right)^{-1} \left(\tilde{\eta}_{gh}^\sigma\right)^{-1} \circ \alpha_\sigma^{-1} \\
&= \pi_\sigma \circ \alpha_\sigma \circ \mathrm{Ad}\left(v_g^\sigma\right) \circ \eta_g^\sigma \beta_g^{\sigma U} \mathrm{Ad}\left(v_h^\sigma\right) \circ \eta_h^\sigma \left(\beta_g^{\sigma U}\right)^{-1} (\eta_{gh}^\sigma)^{-1} \mathrm{Ad}\left(v_{gh}^{\sigma *}\right) \circ \alpha_\sigma^{-1} \\
&= \mathrm{Ad}\left(\pi_\sigma \circ \alpha_\sigma\left(\left(v_g^\sigma\right) \eta_g^\sigma \beta_g^{\sigma U}\left(v_h^\sigma\right)\right)\right) \pi_\sigma \circ \alpha_\sigma \eta_g^\sigma \beta_g^{\sigma U} \eta_h^\sigma \left(\beta_g^{\sigma U}\right)^{-1} (\eta_{gh}^\sigma)^{-1} \circ \alpha_\sigma^{-1} \circ \mathrm{Ad}\left(\alpha_\sigma\left(v_{gh}^{\sigma *}\right)\right) \\
&= \mathrm{Ad}\left(\pi_\sigma \circ \alpha_\sigma\left(\left(v_g^\sigma\right) \eta_g^\sigma \beta_g^{\sigma U}\left(v_h^\sigma\right)\right)\right) \circ \mathrm{Ad}\left(u_\sigma(g, h)\right) \pi_\sigma \circ \mathrm{Ad}\left(\alpha_\sigma\left(v_{gh}^{\sigma *}\right)\right) = \mathrm{Ad}\left(\tilde{u}_\sigma(g, h)\right) \circ \pi_\sigma,
\end{aligned}
\tag{2.102}
$$

for all $g, h \in G$. For the first equality, we substituted $\tilde{\eta}_g = \mathrm{Ad}\,(v_g) \circ \eta_g$. The third equality is the definition of $u(g, h)$. Hence we have proven formula (2.100).

Set

$$c_R := c_R\left(\omega, \alpha, \theta, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_L, \alpha_R, \Theta), ((W_g), (u_\sigma(g, h)))\right),$$

$$\tilde{c}_R := c_R\left(\omega, \alpha, \theta, (\tilde{\beta}_g), \left(\tilde{\eta}_g^\sigma\right), (\alpha_L, \alpha_R, \Theta), ((\tilde{W}_g), (\tilde{u}_\sigma(g, h)))\right). \tag{2.103}$$

In order to show the statement of the lemma, it suffices to show that $c_R = \tilde{c}_R$. Substituting the definition of $\tilde{u}_R$, we obtain

$$\tilde{u}_R(g, h)\tilde{u}_R(gh, k)$$
$$= \pi_R\left(\alpha_R\left(v_g^R\left(\eta_g^R\beta_g^{RU}\right)(v_h^R)\right)\right) \cdot u_R(g, h) \cdot \pi_R\left(\alpha_R\left(\left(v_{gh}^R\right)^*\right)\right)$$
$$\pi_R\left(\alpha_R\left(v_{gh}^R\left(\eta_{gh}^R\beta_{gh}^{RU}\right)(v_k^R)\right)\right) \cdot u_R(gh, k) \cdot \pi_R\left(\alpha_R\left(\left(v_{ghk}^R\right)^*\right)\right)$$
$$= \pi_R\left(\alpha_R\left(v_g^R\left(\eta_g^R\beta_g^{RU}\right)(v_h^R)\right)\right)$$
$$\quad \cdot \left[u_R(g, h) \cdot \pi_R\left(\alpha_R\left(\left(\eta_{gh}^R\beta_{gh}^{RU}\right)(v_k^R)\right)\right)\right] u_R(gh, k)\, \pi_R\left(\alpha_R\left(\left(v_{ghk}^R\right)^*\right)\right)$$
$$= \pi_R\left(\alpha_R\left(v_g^R\left(\eta_g^R\beta_g^{RU}\right)(v_h^R)\right)\right)$$
$$\quad \cdot \left[\mathrm{Ad}\,(u_R(g, h))\left(\pi_R\left(\alpha_R\left(\left(\eta_{gh}^R\beta_{gh}^{RU}\right)(v_k^R)\right)\right)\right) \cdot u_R(g, h)\right] u_R(gh, k) \cdot \pi_R\left(\alpha_R\left(\left(v_{ghk}^R\right)^*\right)\right)$$
$$= \pi_R\left(\alpha_R\left(v_g^R\left(\eta_g^R\beta_g^{RU}\right)(v_h^R)\right)\right)$$
$$\quad \cdot \left(\pi_R\left(\alpha_R \circ \eta_g^R\beta_g^{RU}\eta_h^R\left(\beta_g^{RU}\right)^{-1}\left(\eta_{gh}^R\right)^{-1} \circ \alpha_R^{-1}\alpha_R\left(\left(\eta_{gh}^R\beta_{gh}^{RU}\right)(v_k^R)\right)\right)\right)$$
$$\quad \cdot u_R(g, h)\, u_R(gh, k) \cdot \pi_R\left(\alpha_R\left(\left(v_{ghk}^R\right)^*\right)\right)$$
$$= \pi_R\left(\alpha_R\left(v_g^R \cdot \left(\eta_g^R\beta_g^{RU}\right)(v_h^R) \cdot \eta_g^R\beta_g^{RU}\eta_h^R\beta_h^{RU}(v_k^R)\right)\right) \cdot u_R(g, h)\, u_R(gh, k) \cdot \pi_R\left(\alpha_R\left(\left(v_{ghk}^R\right)^*\right)\right). \tag{2.104}$$

For the fourth equality, we used the definition of $u_R$. From this equation, applying equation (2.52) to the $[\cdot]$ part, we have

$$\mathbb{I}_{\mathcal{H}_L} \otimes \tilde{u}_R(g, h)\tilde{u}_R(gh, k)$$
$$= \mathbb{I}_{\mathcal{H}_L} \otimes \pi_R\left(\alpha_R\left(v_g^R \cdot \left(\eta_g^R\beta_g^{RU}\right)(v_h^R) \cdot \eta_g^R\beta_g^{RU}\eta_h^R\beta_h^{RU}(v_k^R)\right)\right)$$
$$\quad \cdot \left[u_R(g, h)\, u_R(gh, k)\right] \cdot \pi_R\left(\alpha_R\left(\left(v_{ghk}^R\right)^*\right)\right)$$
$$= c_R(g, h, k)\left(\mathbb{I}_{\mathcal{H}_L} \otimes \pi_R\left(\alpha_R\left(v_g^R \cdot \left(\eta_g^R\beta_g^{RU}\right)(v_h^R) \cdot \eta_g^R\beta_g^{RU}\eta_h^R\beta_h^{RU}(v_k^R)\right)\right)\right)$$
$$\quad \left\{W_g\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R(h, k)\right)W_g^*\right\}\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R(g, hk)\right) \cdot \pi_R\left(\alpha_R\left(\left(v_{ghk}^R\right)^*\right)\right). \tag{2.105}$$

Now from the definition of $\tilde{u}_R$, the $\{\cdot\}$ part becomes

$$W_g\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R(h, k)\right)W_g^*$$
$$= \mathrm{Ad}(W_g) \circ \pi_0 \circ \left(\mathrm{id}_L \otimes \alpha_R\left(\left(v_h^R \cdot \eta_h^R\beta_h^{RU}(v_k^R)\right)^*\right)\right)$$
$$\quad \cdot \mathrm{Ad}(W_g)\left(\mathbb{I}_{\mathcal{H}_L} \otimes \tilde{u}_R(h, k)\right) \cdot \left(\mathrm{Ad}(W_g)\pi_0\left(\mathrm{id}_L \otimes \alpha_R\left(v_{hk}^R\right)\right)\right). \tag{2.106}$$

Because $v_g^R$ belongs to $\mathcal{A}_{C_\theta,R}$ and $\eta_g^R$ is an automorphism on $\mathcal{A}_{C_\theta,R}$ while $\Theta$ is an automorphism on $\mathcal{A}_{(C_\theta)^c}$ and $\beta_g^U\left(\mathcal{A}_{C_\theta,R}\right) = \mathcal{A}_{C_\theta,R}$, we have

$$\mathrm{Ad}(W_g) \circ \pi_0 \circ \left(\mathrm{id}_L \otimes \alpha_R\left(\left(v_h^R \cdot \eta_h^R \beta_h^{RU}\left(v_k^R\right)\right)^*\right)\right)$$

$$= \pi_0 \circ \alpha_0 \circ \Theta \circ \eta_g \beta_g^U \circ \Theta^{-1} \circ \alpha_0^{-1} \circ \left(\mathrm{id}_L \otimes \alpha_R\left(\left(v_h^R \cdot \eta_h^R \beta_h^{RU}\left(v_k^R\right)\right)^*\right)\right)$$

$$= \pi_0\left(\mathrm{id}_L \otimes \alpha_R \circ \eta_g^R \beta_g^{RU}\left(\left(v_h^R \cdot \eta_h^R \beta_h^{RU}\left(v_k^R\right)\right)^*\right)\right), \quad \text{and}$$

$$\mathrm{Ad}(W_g) \circ \pi_0\left(\mathrm{id}_L \otimes \alpha_R\left(v_{hk}^R\right)\right) = \pi_0\left(\mathrm{id}_L \otimes \alpha_R \circ \eta_g^R \beta_g^{RU}\left(v_{hk}^R\right)\right). \tag{2.107}$$

Substituting this into equation (2.106), we obtain

$$W_g\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R(h,k)\right) W_g^*$$

$$= \pi_0\left(\mathrm{id}_L \otimes \alpha_R \circ \eta_g^R \beta_g^{RU}\left(\left(v_h^R \cdot \eta_h^R \beta_h^{RU}\left(v_k^R\right)\right)^*\right)\right)$$

$$\cdot \mathrm{Ad}(W_g)\left(\mathbb{I}_{\mathcal{H}_L} \otimes \tilde{u}_R(h,k)\right) \cdot \pi_0\left(\mathrm{id}_L \otimes \alpha_R \circ \eta_g^R \beta_g^{RU}\left(v_{hk}^R\right)\right). \tag{2.108}$$

Substituting this to the $\{\}$ part and the $\{\cdot\}$ part of equation (2.105), we obtain

$$\mathbb{I}_{\mathcal{H}_L} \otimes \tilde{u}_R(g,h)\tilde{u}_R(gh,k)$$

$$= c_R(g,h,k)\left(\mathbb{I}_{\mathcal{H}_L} \otimes \pi_R\left(\alpha_R\left(v_g^R \cdot \left(\eta_g^R \beta_g^{RU}\right)\left(v_h^R\right) \cdot \eta_g^R \beta_g^{RU} \eta_h^R \beta_h^{RU}\left(v_k^R\right)\right)\right)\right)$$

$$\pi_0\left(\mathrm{id}_L \otimes \alpha_R \circ \eta_g^R \beta_g^{RU}\left(\left(v_h^R \cdot \eta_h^R \beta_h^{RU}\left(v_k^R\right)\right)^*\right)\right)$$

$$\cdot \mathrm{Ad}(W_g)\left(\mathbb{I}_{\mathcal{H}_L} \otimes \tilde{u}_R(h,k)\right) \cdot \pi_0\left(\mathrm{id}_L \otimes \alpha_R \circ \eta_g^R \beta_g^{RU}\left(v_{hk}^R\right)\right)$$

$$\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R(g,hk) \cdot \pi_R\left(\alpha_R\left(\left(v_{ghk}^R\right)^*\right)\right)\right)$$

$$= c_R(g,h,k)\left(\mathbb{I}_{\mathcal{H}_L} \otimes \pi_R\left(\alpha_R\left(v_g^R\right)\right)\right) \mathrm{Ad}(W_g)\left(\mathbb{I}_{\mathcal{H}_L} \otimes \tilde{u}_R(h,k)\right) \cdot \pi_0\left(\mathrm{id}_L \otimes \alpha_R \circ \eta_g^R \beta_g^{RU}\left(v_{hk}^R\right)\right)$$

$$\pi_0 \circ \left(\mathrm{id}_L \otimes \alpha_R\left(\left(v_g^R \cdot \eta_g^R \beta_g^{RU}\left(v_{hk}^R\right)\right)^*\right)\right)$$

$$\cdot \left(\mathbb{I}_{\mathcal{H}_L} \otimes \tilde{u}_R(g,hk) \cdot\right) \pi_0\left(\mathrm{id}_L \otimes \alpha_R\left(v_{ghk}^R\right) \alpha_R\left(\left(v_{ghk}^R\right)^*\right)\right)$$

$$= c_R(g,h,k) \mathrm{Ad}\left(\left(\mathbb{I}_{\mathcal{H}_L} \otimes \pi_R\left(\alpha_R\left(v_g^R\right)\right)\right) W_g\right)\left(\mathbb{I}_{\mathcal{H}_L} \otimes \tilde{u}_R(h,k)\right) \cdot \left(\mathbb{I}_{\mathcal{H}_L} \otimes \tilde{u}_R(g,hk)\right)$$

$$= c_R(g,h,k)\left\{\mathrm{Ad}\left(\pi_L\left(\alpha_L\left(v_g^{L*}\right) \otimes \mathbb{I}_{\mathcal{H}_R}\right)\right) \mathrm{Ad}\,\tilde{W}_g\left(\mathbb{I}_{\mathcal{H}_L} \otimes \tilde{u}_R(h,k)\right)\right\} \cdot \left(\mathbb{I}_{\mathcal{H}_L} \otimes \tilde{u}_R(g,hk)\right). \tag{2.109}$$

Because of Lemma 2.3(iii), the $\{\cdot\}$ part of the last equation is equal to $\mathrm{Ad}\,\tilde{W}_g\left(\mathbb{I}_{\mathcal{H}_L} \otimes \tilde{u}_R(h,k)\right)$. Hence we obtain

$$\mathbb{I}_{\mathcal{H}_L} \otimes \tilde{u}_R(g,h)\tilde{u}_R(gh,k) = c_R(g,h,k)\,\mathrm{Ad}\,\tilde{W}_g\left(\mathbb{I}_{\mathcal{H}_L} \otimes \tilde{u}_R(h,k)\right) \cdot \left(\mathbb{I}_{\mathcal{H}_L} \otimes \tilde{u}_R(g,hk)\right). \tag{2.110}$$

This proves $c_R = \tilde{c}_R$, completing the proof. $\qquad \square$

**Lemma 2.15.** *Set*

$$\omega \in \mathcal{SL}, \quad 0 < \theta < \frac{\pi}{2}, \quad \left(\tilde{\beta}_g^{(1)}\right), \left(\tilde{\beta}_g^{(2)}\right) \in \mathrm{IG}(\omega,\theta). \tag{2.111}$$

*Then we have*

$$h^{(4)}\left(\omega, \theta, \left(\tilde{\beta}_g^{(1)}\right)\right) = h^{(4)}\left(\omega, \theta, \left(\tilde{\beta}_g^{(2)}\right)\right). \tag{2.112}$$

**Definition 2.16.** From this lemma we may define

$$h^{(5)}(\omega, \theta) := h^{(4)}\left(\omega, \theta, (\tilde{\beta}_g)\right) \tag{2.113}$$

for any

$$\omega \in \mathcal{SL}, \quad 0 < \theta < \frac{\pi}{2}, \quad \text{with } \mathrm{IG}(\omega, \theta) \neq \emptyset, \tag{2.114}$$

independent of the choice of $(\tilde{\beta}_g)$.

*Proof.* By the definition of $\mathrm{IG}(\omega, \theta)$, there are

$$\left(\eta_{g,i}^{\sigma}\right)_{g \in G, \, \sigma = L, R} \in \mathcal{T}\left(\theta, \left(\tilde{\beta}_g^{(i)}\right)\right), \quad \text{for } i = 1, 2. \tag{2.115}$$

We set $\eta_{g,i} := \eta_{g,i}^L \otimes \eta_{g,i}^R$, for $i = 1, 2$. There are $\alpha \in \mathrm{EAut}(\omega)$ and $(\alpha_L, \alpha_R, \Theta) \in \mathcal{D}_\alpha^\theta$ for $\omega \in \mathcal{SL}$ by the definition. Setting $\alpha_0 := \alpha_L \otimes \alpha_R$, we have $\alpha = (\text{inner}) \circ \alpha_0 \circ \Theta$. By Lemma 2.1, there is some

$$\left((W_{g,1}), \left(u_\sigma^{(1)}(g, h)\right)\right) \in \mathrm{IP}\left(\omega, \alpha, \theta, \left(\tilde{\beta}_g^{(1)}\right), \left(\eta_{g,1}^\sigma\right), (\alpha_L, \alpha_R, \Theta)\right). \tag{2.116}$$

Set

$$K_g^\sigma := \eta_{g,2}^\sigma \circ \left(\eta_{g,1}^\sigma\right)^{-1} \in \mathrm{Aut}\left(\mathcal{A}_{C_\theta, \sigma}\right), \quad \text{for } \sigma = L, R, \, g \in G, \qquad K_g := K_g^L \otimes K_g^R \in \mathrm{Aut}\left(\mathcal{A}_{C_\theta}\right). \tag{2.117}$$

We claim that there are unitaries $V_g^\sigma$, $g \in G$, $\sigma = L, R$, on $\mathcal{H}_\sigma$ such that

$$\mathrm{Ad}\left(V_g^\sigma\right) \circ \pi_\sigma = \pi_\sigma \circ \alpha_\sigma \circ K_g^\sigma \circ (\alpha_\sigma)^{-1}. \tag{2.118}$$

To see this, note that

$$\omega = \omega \circ \tilde{\beta}_g^{(i)} = \omega_0 \circ \alpha \circ \tilde{\beta}_g^{(i)} \sim_{\text{q.e.}} \omega_0 \circ \alpha_0 \circ \Theta \circ \left(\eta_{g,i}^L \otimes \eta_{g,i}^R\right) \circ \beta_g^U, \quad i = 1, 2. \tag{2.119}$$

Therefore, we have

$$\omega_0 \circ \alpha_0 \circ \Theta \circ \left(\eta_{g,1}^L \otimes \eta_{g,1}^R\right) \sim_{\text{q.e.}} \omega \circ \left(\beta_g^U\right)^{-1} \sim_{\text{q.e.}} \omega_0 \circ \alpha_0 \circ \Theta \circ \left(\eta_{g,2}^L \otimes \eta_{g,2}^R\right), \tag{2.120}$$

and then using the facts that $\Theta \in \mathrm{Aut}\left(\mathcal{A}_{C_\theta^c}\right)$ and $K_g \in \mathrm{Aut}\left(\mathcal{A}_{C_\theta}\right)$,

$$\omega_0 \sim_{\text{q.e.}} \omega_0 \circ \alpha_0 \circ \Theta \circ K_g \circ \Theta^{-1} \circ \alpha_0^{-1} = \omega_0 \circ \alpha_0 \circ K_g \circ (\alpha_0)^{-1} = \bigotimes_{\sigma = L, R} \omega_\sigma \circ \alpha_\sigma K_g^\sigma (\alpha_\sigma)^{-1}. \tag{2.121}$$

This implies that $\omega_\sigma$ and $\omega_\sigma \circ \alpha_\sigma K_g^\sigma (\alpha_\sigma)^{-1}$ are quasiequivalent. Because $\pi_\sigma$ is irreducible, this implies the existence of a unitary $V_g^\sigma$ on $\mathcal{H}_\sigma$ satisfying equation (2.118), proving the claim.

Next we claim that there are unitaries $v_{g,h}^\sigma$ on $\mathcal{H}_\sigma$, for $g, h \in G$ and $\sigma = L, R$, such that

$$\mathrm{Ad}_{W_{g,1}}\left(\mathbb{I}_{\mathcal{H}_L} \otimes V_h^R\right) = \mathbb{I}_{\mathcal{H}_L} \otimes v_{g,h}^R, \qquad \mathrm{Ad}_{W_{g,1}}\left(V_h^L \otimes \mathbb{I}_{\mathcal{H}_R}\right) = v_{g,h}^L \otimes \mathbb{I}_{\mathcal{H}_R} \qquad (2.122)$$

and

$$\mathrm{Ad}\left(V_g^\sigma v_{g,h}^\sigma u_\sigma^{(1)}(g,h)\left(V_{gh}^\sigma\right)^*\right)\pi_\sigma = \pi_\sigma \circ \alpha_\sigma \circ \eta_{g,2}^\sigma \beta_g^{\sigma U}\eta_{h,2}^\sigma \left(\beta_g^{\sigma U}\right)^{-1}\left(\eta_{gh,2}^\sigma\right)^{-1}\circ \alpha_\sigma^{-1}, \qquad (2.123)$$

for any $g, h \in G$ and $\sigma = L, R$. To see this, first we calculate

$$\mathrm{Ad}\left(W_{g,1}\left(\mathbb{I}_{\mathcal{H}_L}\otimes V_h^R\right)\left(W_{g,1}\right)^*\right)\circ \pi_0 = \mathrm{Ad}\left(W_{g,1}\left(\mathbb{I}_{\mathcal{H}_L}\otimes V_h^R\right)\right)\pi_0 \circ \alpha_0 \circ \Theta \circ \left(\eta_{g,1}\beta_g^U\right)^{-1}\circ \Theta^{-1}\circ \alpha_0^{-1}$$

$$= \pi_0 \circ \alpha_0 \circ \Theta \circ \eta_{g,1}\beta_g^U \circ \Theta^{-1}\circ \alpha_0^{-1}\circ \left(\mathrm{id}_L \otimes \alpha_R \circ K_h^R \circ (\alpha_R)^{-1}\right)\circ \alpha_0 \circ \Theta \circ \left(\eta_{g,1}\beta_g^U\right)^{-1}\circ \Theta^{-1}\circ \alpha_0^{-1}$$

$$= \pi_0 \circ \alpha_0 \circ \Theta \circ \eta_{g,1}\beta_g^U \circ \Theta^{-1}\circ \left(\mathrm{id}_L \otimes K_h^R\right)\circ \Theta \circ \left(\eta_{g,1}\beta_g^U\right)^{-1}\circ \Theta^{-1}\circ \alpha_0^{-1}$$

$$= \pi_0 \circ \alpha_0 \circ \Theta \circ \eta_{g,1}\beta_g^U \circ \left(\mathrm{id}_L \otimes K_h^R\right)\circ \left(\eta_{g,1}\beta_g^U\right)^{-1}\circ \Theta^{-1}\circ \alpha_0^{-1}$$

$$= \pi_0 \circ \alpha_0 \circ \Theta \circ \left(\mathrm{id}_L \otimes \eta_{g,1}^R \beta_g^{RU}K_h^R \left(\eta_{g,1}^R \beta_g^{RU}\right)^{-1}\right)\circ \Theta^{-1}\circ \alpha_0^{-1}$$

$$= \pi_0 \circ \left(\mathrm{id}_L \otimes \alpha_R \circ \eta_{g,1}^R \beta_g^{RU}K_h^R \left(\eta_{g,1}^R \beta_g^{RU}\right)^{-1}\alpha_R^{-1}\right). \qquad (2.124)$$

In the fourth and sixth equalities, we used the fact that $K_h^R, \eta_{g,1}^R \beta_g^{RU}K_h^R \left(\eta_{g,1}^R \beta_g^{RU}\right)^{-1}\in \mathrm{Aut}\left(\mathcal{A}_{C_\theta}\right)$ and $\Theta \in \mathrm{Aut}\left(\mathcal{A}_{C_\theta^c}\right)$ commute, in order to remove $\Theta$. Equation (2.124) and the fact that $\pi_L$ is irreducible imply that there is a unitary $v_{g,h}^R$ satisfying equation (2.122). The same argument implies the existence of $v_{g,h}^L$ satisfying equation (2.122).

For this $v_{g,h}^R$, we would like to show equation (2.123). Rewriting

$$\eta_{g,2}^\sigma \beta_g^{\sigma U}\eta_{h,2}^\sigma \left(\beta_g^{\sigma U}\right)^{-1}\left(\eta_{gh,2}^\sigma\right)^{-1}$$

$$= K_g^\sigma \circ \left(\eta_{g,1}^\sigma \beta_g^{\sigma U}K_h^\sigma \left(\eta_{g,1}^\sigma \beta_g^{\sigma U}\right)^{-1}\right)\circ \eta_{g,1}^\sigma \beta_g^{\sigma U}\eta_{h,1}^\sigma \left(\beta_g^{\sigma U}\right)^{-1}\left(\eta_{gh,1}^\sigma\right)^{-1}\circ \left(K_{gh}^\sigma\right)^{-1}, \qquad (2.125)$$

we obtain

$$\pi_L \otimes \pi_R \circ \alpha_R \eta_{g,2}^R \beta_g^{RU}\eta_{h,2}^R \left(\beta_g^{RU}\right)^{-1}\left(\eta_{gh,2}^R\right)^{-1}\alpha_R^{-1}$$

$$= \pi_0 \circ \left(\mathrm{id}_L \otimes \alpha_R \circ K_g^R \circ \left(\eta_{g,1}^R \beta_g^{RU}K_h^R \left(\eta_{g,1}^R \beta_g^{RU}\right)^{-1}\right)\right.$$

$$\left. \circ \eta_{g,1}^R \beta_g^{RU}\eta_{h,1}^R \left(\beta_g^{RU}\right)^{-1}\left(\eta_{gh,1}^R\right)^{-1}\circ \left(K_{gh}^R\right)^{-1}\alpha_R^{-1}\right)$$

$$= \pi_L \otimes \mathrm{Ad}\left(V_g^R v_{g,h}^R u_R^{(1)}(g,h)\left(V_{gh}^R\right)^*\right)\pi_R, \qquad (2.126)$$

substituting equations (2.118), (2.124) and (2.122). This proves equation (2.123) for $\sigma = R$. An analogous result for $\sigma = L$ can be proven by the same argument. Hence we have proven the claim (2.124) and (2.123).

Setting

$$V_g := V_g^L \otimes V_g^R \in \mathcal{U}(\mathcal{H}_0),$$ (2.127)

we have

$$\begin{aligned}
\mathrm{Ad}\left(V_g W_{g,1}\right) \circ \pi_0 &= \pi_0 \circ \alpha_0 \circ K_g \circ \alpha_0^{-1} \circ \alpha_0 \circ \Theta \circ \eta_{g,1} \circ \beta_g^U \circ \Theta^{-1} \circ \alpha_0^{-1} \\
&= \pi_0 \circ \alpha_0 \circ \Theta \circ \eta_{g,2} \circ \beta_g^U \circ \Theta^{-1} \circ \alpha_0^{-1}.
\end{aligned}$$ (2.128)

In the last equality, we used the definition of $K_g$ and the commutativity of $\Theta$ and $K_g$ again. From equations (2.128) and (2.123), setting

$$u_\sigma^{(2)}(g,h) := V_g^\sigma v_{g,h}^\sigma u_\sigma^{(1)}(g,h) \left(V_{gh}^\sigma\right)^*,$$ (2.129)

we see that

$$\left(\left(V_g W_{g,1}\right), \left(u_R^{(2)}(g,h)\right)\right) \in \mathrm{IP}\left(\omega, \alpha, \theta, \left(\tilde{\beta}_g^{(2)}\right), \left(\eta_{g,2}^\sigma\right), (\alpha_L, \alpha_R, \Theta)\right)$$ (2.130)

and

$$\mathbb{I}_{\mathcal{H}_L} \otimes u_R^{(2)}(g,h) = \left(\mathbb{I}_{\mathcal{H}_L} \otimes V_g^R\right) W_{g,1} \left(\mathbb{I}_{\mathcal{H}_L} \otimes V_h^R\right) (W_{g,1})^* \left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R^{(1)}(g,h) \left(V_{gh}^R\right)^*\right).$$ (2.131)

Now we set

$$\begin{aligned}
c_{R,1} &:= c_R\left(\omega, \alpha, \theta, \left(\tilde{\beta}_g^{(1)}\right), \left(\eta_{g,1}^\sigma\right), (\alpha_L, \alpha_R, \Theta), \left((W_{g,1}), \left(u_\sigma^{(1)}(g,h)\right)\right)\right), \\
c_{R,2} &:= c_R\left(\omega, \alpha, \theta, \left(\tilde{\beta}_g^{(2)}\right), \left(\eta_{g,2}^\sigma\right), (\alpha_L, \alpha_R, \Theta), \left((V_g W_{g,1}), \left(u_R^{(2)}(g,h)\right)\right)\right).
\end{aligned}$$ (2.132)

To prove the Lemma, it suffices to show $c_{R,1} = c_{R,2}$. By equation (2.131), we have

$$\begin{aligned}
&\mathbb{I}_{\mathcal{H}_L} \otimes u_R^{(2)}(g,h) u_R^{(2)}(gh,k) \\
&= \left(\mathbb{I}_{\mathcal{H}_L} \otimes V_g^R\right) W_{g,1} \left(\mathbb{I}_{\mathcal{H}_L} \otimes V_h^R\right) (W_{g,1})^* \left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R^{(1)}(g,h) \left(V_{gh}^R\right)^*\right) \\
&\quad \cdot \left(\mathbb{I}_{\mathcal{H}_L} \otimes V_{gh}^R\right) W_{gh,1} \left(\mathbb{I}_{\mathcal{H}_L} \otimes V_k^R\right) (W_{gh,1})^* \left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R^{(1)}(gh,k) \left(V_{ghk}^R\right)^*\right) \\
&= \left(\mathbb{I}_{\mathcal{H}_L} \otimes V_g^R\right) W_{g,1} \left(\mathbb{I}_{\mathcal{H}_L} \otimes V_h^R\right) (W_{g,1})^* \left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R^{(1)}(g,h)\right) \\
&\quad \cdot W_{gh,1} \left(\mathbb{I}_{\mathcal{H}_L} \otimes V_k^R\right) (W_{gh,1})^* \left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R^{(1)}(gh,k) \left(V_{ghk}^R\right)^*\right) \\
&= \left(\mathbb{I}_{\mathcal{H}_L} \otimes V_g^R\right) W_{g,1} \left(\mathbb{I}_{\mathcal{H}_L} \otimes V_h^R\right) (W_{g,1})^* \left\{\mathrm{Ad}\left(\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R^{(1)}(g,h)\right) \cdot W_{gh,1}\right) \left(\mathbb{I}_{\mathcal{H}_L} \otimes V_k^R\right)\right\} \\
&\quad \cdot \left(\mathbb{I}_{\mathcal{H}_L} \otimes \left[u_R^{(1)}(g,h) u_R^{(1)}(gh,k)\right] \left(V_{ghk}^R\right)^*\right) \\
&= c_{R,1}(g,h,k) \left(\mathbb{I}_{\mathcal{H}_L} \otimes V_g^R\right) W_{g,1} \left(\mathbb{I}_{\mathcal{H}_L} \otimes V_h^R\right) (W_{g,1})^* \left\{\mathrm{Ad}\left(W_{g,1} W_{h,1}\right) \left(\mathbb{I}_{\mathcal{H}_L} \otimes V_k^R\right)\right\} \\
&\quad \cdot \left(W_{g,1} \left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R^{(1)}(h,k)\right) W_{g,1}^*\right) \left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R^{(1)}(g,hk) \left(V_{ghk}^R\right)^*\right).
\end{aligned}$$ (2.133)

We used equation (2.52) for the $[\cdot]$ part and Lemma 2.3(ii) and equation (2.122) for the $\{\cdot\}$ part in the fourth equality. Again using equation (2.131), we have

$$
\begin{aligned}
&\mathbb{I}_{\mathcal{H}_L} \otimes u_R^{(2)}(g,h) u_R^{(2)}(gh,k) = \text{equation (2.134)}\\
&= c_{R,1}(g,h,k)\left(\mathbb{I}_{\mathcal{H}_L} \otimes V_g^R\right) W_{g,1}\left(\mathbb{I}_{\mathcal{H}_L} \otimes V_h^R\right)\left\{\text{Ad}\left(W_{h,1}\right)\left(\mathbb{I}_{\mathcal{H}_L} \otimes V_k^R\right)\right\}\\
&\quad \cdot\left(W_{h,1}\left(\mathbb{I}_{\mathcal{H}_L} \otimes \left(V_k^R\right)^*\right)\right)\left(W_{h,1}\right)^*\left(\mathbb{I}_{\mathcal{H}_L} \otimes V_h^R\right)^*\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R^{(2)}(h,k)\right)\left(\mathbb{I}_{\mathcal{H}_L} \otimes \left(V_{hk}^R\right)\right)\left(W_{g,1}\right)^*\right)\\
&\quad W_{g,1}\left(\mathbb{I}_{\mathcal{H}_L} \otimes \left(V_{hk}^R\right)^*\right)\left(W_{g,1}\right)^*\left(\mathbb{I}_{\mathcal{H}_L} \otimes V_g^R\right)^*\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R^{(2)}(g,hk)\right)\left(\mathbb{I}_{\mathcal{H}_L} \otimes \left(V_{ghk}^R\right)\right)\left(\mathbb{I}_{\mathcal{H}_L} \otimes \left(V_{ghk}^R\right)^*\right)\\
&= c_{R,1}(g,h,k)\left(\mathbb{I}_{\mathcal{H}_L} \otimes V_g^R\right) W_{g,1}\cdot\left(\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R^{(2)}(h,k)\right)\right)\cdot\left(W_{g,1}\right)^*\left(\mathbb{I}_{\mathcal{H}_L} \otimes V_g^R\right)^*\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R^{(2)}(g,hk)\right)\\
&= c_{R,1}(g,h,k)\cdot\text{Ad}\left(\left(\mathbb{I}_{\mathcal{H}_L} \otimes V_g^R\right) W_{g,1}\right)\left(\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R^{(2)}(h,k)\right)\right)\cdot\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R^{(2)}(g,hk)\right)\\
&= c_{R,1}(g,h,k)\cdot\text{Ad}\left(\left(V_g^{L^*} \otimes \mathbb{I}_{\mathcal{H}_R}\right) V_g W_{g,1}\right)\left(\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R^{(2)}(h,k)\right)\right)\cdot\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R^{(2)}(g,hk)\right)\\
&= c_{R,1}(g,h,k)\cdot\text{Ad}\left(V_g W_{g,1}\right)\left(\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R^{(2)}(h,k)\right)\right)\cdot\left(\mathbb{I}_{\mathcal{H}_L} \otimes u_R^{(2)}(g,hk)\right) \qquad (2.134)
\end{aligned}
$$

In the last line we used formula (2.130) and Lemma 2.3(iii) to remove $V_g^{L^*}$. From this, we see that $c_{R,1} = c_{R,2}$, completing the proof. $\qquad\square$

**Lemma 2.17.** *Set*

$$\omega \in \mathcal{SL}, \quad 0 < \theta_1 < \theta_2 < \frac{\pi}{2}, \quad \text{with } \text{IG}(\omega,\theta_1),\ \text{IG}(\omega,\theta_2) \neq \emptyset. \qquad (2.135)$$

*Then we have*

$$h^{(5)}(\omega,\theta_1) = h^{(5)}(\omega,\theta_2). \qquad (2.136)$$

**Definition 2.18.** From this lemma, for any $\omega \in \mathcal{SL}$ with $\text{IG}(\omega) \neq \emptyset$, we may define

$$h(\omega) := h^{(5)}(\omega,\theta) \qquad (2.137)$$

independent of the choice of $\theta$. This is the index we associate to $\omega \in \mathcal{SL}$ with $\text{IG}(\omega) \neq \emptyset$.

*Proof.* By the assumption, there are some $(\tilde\beta_g) \in \text{IG}(\omega,\theta_1)$ and $(\eta_g^\sigma) \in \mathcal{T}((\theta_1,\tilde\beta_g))$. Because $\omega \in \mathcal{SL}$, there are $\alpha \in \text{EAut}(\omega)$ and $(\alpha_L,\alpha_R,\Theta) \in \mathcal{D}_\alpha^{\theta_2}$ by the definition. Setting $\alpha_0 := \alpha_L \otimes \alpha_R$, we have $\alpha = (\text{inner})\circ\alpha_0\circ\Theta$. Because $0 < \theta_1 < \theta_2 < \frac{\pi}{2}$, we also have $(\eta_g^\sigma) \in \mathcal{T}((\theta_2,\tilde\beta_g))$, and $(\tilde\beta_g) \in \text{IG}(\omega,\theta_2)$. For the same reason, we also have $(\alpha_L,\alpha_R,\Theta) \in \mathcal{D}_\alpha^{\theta_1}$.

By Lemma 2.1, there is some

$$((W_g),(u_\sigma(g,h))) \in \text{IP}\left(\omega,\alpha,\theta_1,(\tilde\beta_g),(\eta_g^\sigma),(\alpha_L,\alpha_R,\Theta)\right). \qquad (2.138)$$

However, we also have

$$((W_g),(u_\sigma(g,h))) \in \text{IP}\left(\omega,\alpha,\theta_2,(\tilde\beta_g),(\eta_g^\sigma),(\alpha_L,\alpha_R,\Theta)\right). \qquad (2.139)$$

Therefore, we obtain $h^{(5)}(\omega,\theta_1) = h^{(5)}(\omega,\theta_2)$. $\qquad\square$

This completes the proof of Theorem 2.7.

## 3. The existence of $\tilde{\beta}$ for SPT phases

In this section, we give a sufficient condition for IG($\omega$) to be nonempty. We consider the same setting as in Section 2.2.

**Theorem 3.1.** *For any $0 < \theta < \frac{\pi}{2}$ and $\alpha \in \mathrm{SQAut}(\mathcal{A})$ satisfying $\omega_0 \circ \alpha \circ \beta_g = \omega_0 \circ \alpha$ for all $g \in G$, IG($\omega_0 \circ \alpha, \theta$) is not empty.*

The strategy is as follows. Our infinite tensor product state $\omega_0$ can be written as $\omega_0 = \omega_U \otimes \omega_D$, with pure states $\omega_U, \omega_D$ on $\mathcal{A}_{H_U}, \mathcal{A}_{H_D}$. Using the factorisation property of $\alpha \in \mathrm{SQAut}(\mathcal{A})$, we can show that

$$\alpha \circ \beta_g^U \circ \alpha^{-1} = \left( id_{\mathcal{A}_{H_D}} \otimes \tilde{Y}_{g,U} \right) \circ \left(\text{an automorphism localised at } C_{\theta_2}\right) \circ (\text{inner}), \tag{3.1}$$

$$\alpha \circ \beta_g \circ \alpha^{-1} = \left( \tilde{Y}_{g,D} \otimes \tilde{Y}_{g,U} \right) \circ \left(\text{an automorphism localised at } C_{\theta_2}\right) \circ (\text{inner}), \tag{3.2}$$

with $\tilde{Y}_{g,U} := \beta_g^{C_{(\theta_{0.8}, \frac{\pi}{2}]}, U} \xi_U, \tilde{Y}_{g,D} := \beta_g^{C_{(\theta_{0.8}, \frac{\pi}{2}]}, U} \xi_D$ automorphisms on $\left(C_{\theta_{0.8}}\right)^c \cap H_U, \left(C_{\theta_{0.8}}\right)^c \cap H_D$, respectively. The 'automorphism localised at $C_{\theta_2}$' can be split into left and right parts. (See equation (3.28).) From the latter equation and $\omega_0 \circ \alpha \circ \beta_g = \omega_0 \circ \alpha$, one can show that $\omega_U \tilde{Y}_{g,U}$ is quasiequivalent to a state of the form $\varphi_L \otimes \varphi_R \otimes \omega_{C_{\theta_2}^c}$, where $\varphi_L, \varphi_R$ are states on $\mathcal{A}_{C_{\theta_2} \cap H_L}, \mathcal{A}_{C_{\theta_2} \cap H_R}$ and $\omega_{C_{\theta_2}^c}$ is the pure state given as the restriction of $\omega_0$ to $\mathcal{A}_{C_{\theta_2}^c}$ (with $\theta_{0.8} < \theta_2$). A general lemma proven in the following (Lemma 3.2), derived from the homogeneity of pure state spaces on UHF-algebras, then allows us to show the existence of automorphisms $Z_{g,L}, Z_{g,R}$ on $\mathcal{A}_{C_{\theta_2} \cap H_L}, \mathcal{A}_{C_{\theta_2} \cap H_R}$ satisfying $\omega_U \tilde{Y}_{g,U} \simeq \omega_U \circ \left( Z_{g,L} \otimes Z_{g,R} \otimes id_{C_{\theta_2}^c} \right)$. Combining this with equation (3.1) basically gives the Theorem.

Now let us start with a precise mathematical proof. We first prepare the general lemma just mentioned.

**Lemma 3.2.** *Let $\mathfrak{A}, \mathfrak{B}$ be UHF-algebras. Let $\omega$ be a pure state on $\mathfrak{A} \otimes \mathfrak{B}$ and $\varphi_{\mathfrak{A}}, \varphi_{\mathfrak{B}}$ states on $\mathfrak{A}, \mathfrak{B}$, respectively. Assume that $\omega$ is quasiequivalent to $\varphi_{\mathfrak{A}} \otimes \varphi_{\mathfrak{B}}$. Then for any pure states $\psi_{\mathfrak{A}}, \psi_{\mathfrak{B}}$ on $\mathfrak{A}, \mathfrak{B}$, there are automorphisms $\gamma_{\mathfrak{A}} \in \mathrm{Aut}\left(\mathfrak{A}\right), \gamma_{\mathfrak{B}} \in \mathrm{Aut}\left(\mathfrak{B}\right)$ and a unitary $u \in \mathcal{U}(\mathfrak{A} \otimes \mathfrak{B})$ such that*

$$\omega = \left( (\psi_{\mathfrak{A}} \circ \gamma_{\mathfrak{A}}) \otimes (\psi_{\mathfrak{B}} \circ \gamma_{\mathfrak{B}}) \right) \circ \mathrm{Ad}(u). \tag{3.3}$$

*If $\psi_{\mathfrak{A}}$ and $\varphi_{\mathfrak{A}}$ are quasiequivalent, then we may set $\gamma_{\mathfrak{A}} = id_{\mathfrak{A}}$.*

*Proof.* Let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega), (\mathcal{H}_{\varphi_{\mathfrak{A}}}, \pi_{\varphi_{\mathfrak{A}}}, \Omega_{\varphi_{\mathfrak{A}}}), (\mathcal{H}_{\varphi_{\mathfrak{B}}}, \pi_{\varphi_{\mathfrak{B}}}, \Omega_{\varphi_{\mathfrak{B}}})$ be GNS triples of $\omega, \varphi_{\mathfrak{A}}, \varphi_{\mathfrak{B}}$, respectively. Then $\left(\mathcal{H}_{\varphi_{\mathfrak{A}}} \otimes \mathcal{H}_{\varphi_{\mathfrak{B}}}, \pi_{\varphi_{\mathfrak{A}}} \otimes \pi_{\varphi_{\mathfrak{B}}}, \Omega_{\varphi_{\mathfrak{A}}} \otimes \Omega_{\varphi_{\mathfrak{B}}}\right)$ is a GNS triple of $\varphi_{\mathfrak{A}} \otimes \varphi_{\mathfrak{B}}$. As $\omega$ is quasiequivalent to $\varphi_{\mathfrak{A}} \otimes \varphi_{\mathfrak{B}}$, there is a $*$-isomorphism $\tau : \pi_\omega\left(\mathfrak{A} \otimes \mathfrak{B}\right)'' \to \pi_{\varphi_{\mathfrak{A}}}(\mathfrak{A})'' \otimes \pi_{\varphi_{\mathfrak{B}}}(\mathfrak{B})''$ such that $\tau \circ \pi_\omega = \pi_{\varphi_{\mathfrak{A}}} \otimes \pi_{\varphi_{\mathfrak{B}}}$. Because $\omega$ is pure, we have $\pi_\omega\left(\mathfrak{A} \otimes \mathfrak{B}\right)'' = \mathcal{B}(\mathcal{H}_\omega)$, and from the isomorphism $\tau$, we have that $\pi_{\varphi_{\mathfrak{A}}}(\mathfrak{A})'' \otimes \pi_{\varphi_{\mathfrak{B}}}(\mathfrak{B})''$ is also a type I factor. Then from [T, Theorem 2.30V], both $\pi_{\varphi_{\mathfrak{A}}}(\mathfrak{A})''$ and $\pi_{\varphi_{\mathfrak{B}}}(\mathfrak{B})''$ are type I factors. The restriction of $\tau$ to $\pi_\omega\left(\mathfrak{A} \otimes \mathbb{C}\mathbb{I}_{\mathfrak{B}}\right)''$ implies a $*$-isomorphism from $\pi_\omega\left(\mathfrak{A} \otimes \mathbb{C}\mathbb{I}_{\mathfrak{B}}\right)''$ onto the type I factor $\pi_{\varphi_{\mathfrak{A}}}(\mathfrak{A})''$. Hence we see that $\pi_\omega\left(\mathfrak{A} \otimes \mathbb{C}\mathbb{I}_{\mathfrak{B}}\right)''$ is a type I factor. Therefore, from [T, Theorem 1.31V], there are Hilbert spaces $\mathcal{K}_{\mathfrak{A}}, \mathcal{K}_{\mathfrak{B}}$ and a unitary $W : \mathcal{H}_\omega \to \mathcal{K}_{\mathfrak{A}} \otimes \mathcal{K}_{\mathfrak{B}}$ such that $\mathrm{Ad}\left(W\right)\left(\pi_\omega\left(\mathfrak{A} \otimes \mathbb{C}\mathbb{I}_{\mathfrak{B}}\right)''\right) = \mathcal{B}(\mathcal{K}_{\mathfrak{A}}) \otimes \mathbb{C}\mathbb{I}_{\mathcal{K}_{\mathfrak{B}}}$. Because $\omega$ is pure, we also have $\mathrm{Ad}\left(W\right)\left(\pi_\omega\left(\mathbb{C}\mathbb{I}_{\mathfrak{A}} \otimes \mathfrak{B}\right)''\right) = \mathbb{C}\mathbb{I}_{\mathcal{K}_{\mathfrak{A}}} \otimes \mathcal{B}(\mathcal{K}_{\mathfrak{B}})$. From this, we see that there are irreducible representations $\rho_{\mathfrak{A}}, \rho_{\mathfrak{B}}$ of $\mathfrak{A}$ and $\mathfrak{B}$ on $\mathcal{K}_{\mathfrak{A}}, \mathcal{K}_{\mathfrak{B}}$ such that $\mathrm{Ad}(W) \circ \pi_\omega = \rho_{\mathfrak{A}} \otimes \rho_{\mathfrak{B}}$. Fix some unit vectors $\xi_{\mathfrak{A}} \in \mathcal{K}_{\mathfrak{A}}, \xi_{\mathfrak{B}} \in \mathcal{K}_{\mathfrak{B}}$. Then because of the irreducibility of $\rho_{\mathfrak{B}}$ and $\rho_{\mathfrak{B}}$, we have that $\omega_{\mathfrak{A}} := \langle \xi_{\mathfrak{A}}, \rho_{\mathfrak{A}} (\cdot) \xi_{\mathfrak{A}} \rangle$ and $\omega_{\mathfrak{B}} := \langle \xi_{\mathfrak{B}}, \rho_{\mathfrak{B}} (\cdot) \xi_{\mathfrak{B}} \rangle$ are pure states on $\mathfrak{A}, \mathfrak{B}$. By [KOS, Theorem 1.1] (originally proved by Powers [P] for UHF-algebras), for any pure states $\psi_{\mathfrak{A}}, \psi_{\mathfrak{B}}$ on $\mathfrak{A}, \mathfrak{B}$, there exist automorphisms $\gamma_{\mathfrak{A}} \in \mathrm{Aut}(\mathfrak{A}), \gamma_{\mathfrak{B}} \in \mathrm{Aut}(\mathfrak{B})$ such that $\omega_{\mathfrak{A}} = \psi_{\mathfrak{A}} \circ \gamma_{\mathfrak{A}} \; \omega_{\mathfrak{B}} = \psi_{\mathfrak{B}} \circ \gamma_{\mathfrak{B}}$. Now for unit vectors $W^* \left( \xi_{\mathfrak{A}} \otimes \xi_{\mathfrak{B}} \right), \Omega_\omega \in \mathcal{H}_\omega$, by Kadison's transitivity theorem and the irreducibility of $\pi_\omega$ there

exists a unitary $u \in \mathcal{U}(\mathfrak{A} \otimes \mathfrak{B})$ such that $\pi_\omega(u)\Omega_\omega = W^* (\xi_\mathfrak{A} \otimes \xi_\mathfrak{B})$. Substituting this, we obtain

$$
\begin{aligned}
\omega &= \langle \Omega_\omega, \pi_\omega(\cdot)\Omega_\omega \rangle = \langle \pi_\omega(u^*)W^* (\xi_\mathfrak{A} \otimes \xi_\mathfrak{B}), \pi_\omega(\cdot)\pi_\omega(u^*)W^* (\xi_\mathfrak{A} \otimes \xi_\mathfrak{B}) \rangle \\
&= \langle W^* (\xi_\mathfrak{A} \otimes \xi_\mathfrak{B}), \pi_\omega \circ \mathrm{Ad}(u)(\cdot)W^* (\xi_\mathfrak{A} \otimes \xi_\mathfrak{B}) \rangle \\
&= \langle (\xi_\mathfrak{A} \otimes \xi_\mathfrak{B}), (\rho_\mathfrak{A} \otimes \rho_\mathfrak{B}) \circ \mathrm{Ad}(u)(\cdot)(\xi_\mathfrak{A} \otimes \xi_\mathfrak{B}) \rangle \\
&= (\omega_\mathfrak{A} \otimes \omega_\mathfrak{B}) \circ \mathrm{Ad}(u) = (\psi_\mathfrak{A} \circ \gamma_\mathfrak{A} \otimes \psi_\mathfrak{B} \circ \gamma_\mathfrak{B}) \circ \mathrm{Ad}(u).
\end{aligned}
\tag{3.4}
$$

Now assume that $\psi_\mathfrak{A}$ and $\varphi_\mathfrak{A}$ are quasiequivalent – that is, the GNS representations of $\psi_\mathfrak{A}, \varphi_\mathfrak{A}$, denoted by $\pi_{\psi_\mathfrak{A}}$ and $\pi_{\varphi_\mathfrak{A}}$ are quasiequivalent. From the foregoing argument, $\pi_\omega|_\mathfrak{A}$ and $\pi_\varphi|_\mathfrak{A}$ are quasiequivalent. At the same time, $\pi_\omega|_\mathfrak{A}$ and $\rho_\mathfrak{A}$ are quasiequivalent. Therefore, $\pi_{\psi_\mathfrak{A}}$ and $\rho_\mathfrak{A}$ are quasiequivalent. Because both of them are irreducible, we see that a pure state $\psi_\mathfrak{A}$ can be represented by a unit vector $\zeta \in \mathcal{K}_\mathfrak{A}$, as $\psi_\mathfrak{A} = \langle \zeta, \rho_\mathfrak{A}(\cdot)\zeta \rangle$. Because $\rho_\mathfrak{A}$ is irreducible, by Kadison's transitivity theorem there exists a unitary $w \in \mathcal{U}(\mathfrak{A})$ such that $\rho_\mathfrak{A}(w^*)\zeta = \xi_\mathfrak{A}$. Hence we obtain $\psi_\mathfrak{A} \circ \mathrm{Ad}(w) = \omega_\mathfrak{A}$. Substituting this instead of $\omega_\mathfrak{A} = \psi_\mathfrak{A} \circ \gamma_\mathfrak{A}$ in equation (3.4), we obtain

$$
\omega = (\psi_\mathfrak{A} \otimes \psi_\mathfrak{B} \circ \gamma_\mathfrak{B}) \circ \mathrm{Ad}\left((w \otimes \mathrm{id}_\mathfrak{B})u\right),
\tag{3.5}
$$

proving the last claim. □

**Lemma 3.3.** *Let* $\mathfrak{B}, \mathfrak{A}_{1,L}, \mathfrak{A}_{2,L}, \mathfrak{A}_{1,R}, \mathfrak{A}_{2,R}$ *be UHF-algebras. Set* $\mathfrak{A}_1 := \mathfrak{A}_{1,L} \otimes \mathfrak{A}_{1,R}, \mathfrak{A}_2 := \mathfrak{A}_{2,L} \otimes \mathfrak{A}_{2,R},$ $\mathfrak{A}_L := \mathfrak{A}_{1,L} \otimes \mathfrak{A}_{2,L}$ *and* $\mathfrak{A}_R := \mathfrak{A}_{1,R} \otimes \mathfrak{A}_{2,R}$. *Let* $\omega, \varphi_L^{(1,2)}, \varphi_R^{(1,2)}, \psi$ *be pure states on* $\mathfrak{B} \otimes \mathfrak{A}_1, \mathfrak{A}_L, \mathfrak{A}_R, \mathfrak{B},$ *respectively. Suppose that* $\omega$ *is quasiequivalent to* $\left(\psi \otimes \varphi_L^{(1,2)} \otimes \varphi_R^{(1,2)}\right)\Big|_{\mathfrak{B} \otimes \mathfrak{A}_1}$. *Then for any pure states* $\varphi_L^{(1)}, \varphi_R^{(1)}$ *on* $\mathfrak{A}_{1,L}, \mathfrak{A}_{1,R},$ *respectively, there are automorphisms* $\gamma_L^{(1)} \in \mathrm{Aut}\left(\mathfrak{A}_{1,L}\right), \gamma_R^{(1)} \in \mathrm{Aut}\left(\mathfrak{A}_{1,R}\right)$ *and a unitary* $u \in \mathcal{U}(\mathfrak{B} \otimes \mathfrak{A}_1)$ *such that*

$$
\omega = \left(\psi \otimes \left(\varphi_L^{(1)} \circ \gamma_L^{(1)}\right) \otimes \left(\varphi_R^{(1)} \circ \gamma_R^{(1)}\right)\right) \circ \mathrm{Ad}\, u.
\tag{3.6}
$$

*Proof.* Because the pure state $\omega$ is quasiequivalent to $\left(\psi \otimes \varphi_L^{(1,2)} \otimes \varphi_R^{(1,2)}\right)\Big|_{\mathfrak{B} \otimes \mathfrak{A}_1} = \psi \otimes \left(\varphi_L^{(1,2)} \otimes \varphi_R^{(1,2)}\right)\Big|_{\mathfrak{A}_1}$, applying Lemma 3.2 means that for any pure states $\varphi_L^{(1)}, \varphi_R^{(1)}$ on $\mathfrak{A}_{1,L}, \mathfrak{A}_{1,R}$, there exist an automorphism $S \in \mathrm{Aut}\, \mathfrak{A}_1$ and a unitary $v \in \mathcal{U}(\mathfrak{B} \otimes \mathfrak{A}_1)$ such that

$$
\omega = \left(\psi \otimes \left(\left(\varphi_L^{(1)} \otimes \varphi_R^{(1)}\right) \circ S\right)\right) \circ \mathrm{Ad}\, v.
\tag{3.7}
$$

From equation (3.7) and $\omega \sim_{\mathrm{q.e.}} \left(\psi \otimes \varphi_L^{(1,2)} \otimes \varphi_R^{(1,2)}\right)\Big|_{\mathfrak{B} \otimes \mathfrak{A}_1}$, we get $\left(\psi \otimes \left(\left(\varphi_L^{(1)} \otimes \varphi_R^{(1)}\right) \circ S\right)\right) \sim_{\mathrm{q.e.}} \left(\psi \otimes \varphi_L^{(1,2)} \otimes \varphi_R^{(1,2)}\right)\Big|_{\mathfrak{B} \otimes \mathfrak{A}_1}$, which implies

$$
\left(\varphi_L^{(1)} \otimes \varphi_R^{(1)}\right) \circ S \sim_{\mathrm{q.e.}} \left(\varphi_L^{(1,2)} \otimes \varphi_R^{(1,2)}\right)\Big|_{\mathfrak{A}_1}.
\tag{3.8}
$$

Applying Lemma 3.2 to formula (3.8), there are automorphisms $\gamma_L^{(1)} \in \mathrm{Aut}\left(\mathfrak{A}_{1,L}\right), \gamma_R^{(1)} \in \mathrm{Aut}\left(\mathfrak{A}_{1,R}\right)$ and a unitary $w \in \mathcal{U}(\mathfrak{A}_1)$ such that

$$
\left(\varphi_L^{(1)} \otimes \varphi_R^{(1)}\right) \circ S = \left(\left(\varphi_L^{(1)} \circ \gamma_L^{(1)}\right) \otimes \left(\varphi_R^{(1)} \circ \gamma_R^{(1)}\right)\right) \circ \mathrm{Ad}\, w.
\tag{3.9}
$$

Substituting this into equation (3.7), we obtain equation (3.6). □

**Lemma 3.4.** *Let $\mathfrak{A}_L, \mathfrak{A}_R, \mathfrak{B}_{LU}, \mathfrak{B}_{LD}, \mathfrak{B}_{RU}, \mathfrak{B}_{RD}, \mathfrak{C}_U, \mathfrak{C}_D$ be UHF-algebras, and set*

$$
\begin{array}{llll}
\mathfrak{B}_U := \mathfrak{B}_{LU} \otimes \mathfrak{B}_{RU}, & \mathfrak{B}_D := \mathfrak{B}_{LD} \otimes \mathfrak{B}_{RD}, & \mathfrak{B}_L := \mathfrak{B}_{LD} \otimes \mathfrak{B}_{LU}, & \mathfrak{B}_R := \mathfrak{B}_{RD} \otimes \mathfrak{B}_{RU}, \\
\mathfrak{A} := \mathfrak{A}_L \otimes \mathfrak{A}_R, & \mathfrak{B} := \mathfrak{B}_D \otimes \mathfrak{B}_U = \mathfrak{B}_L \otimes \mathfrak{B}_R, & \mathfrak{C} := \mathfrak{C}_D \otimes \mathfrak{C}_U, & \mathfrak{D} := \mathfrak{A} \otimes \mathfrak{B} \otimes \mathfrak{C}.
\end{array}
\tag{3.10}
$$

*Let $\omega_X$ be a pure state on each $X = \mathfrak{A}_L, \mathfrak{A}_R, \mathfrak{B}_{LU}, \mathfrak{B}_{LD}, \mathfrak{B}_{RU}, \mathfrak{B}_{RD}, \mathfrak{C}_U, \mathfrak{C}_D$, and set*

$$
\begin{aligned}
\omega_{\mathfrak{B}\mathfrak{C}}^U &:= \omega_{\mathfrak{B}_{LU}} \otimes \omega_{\mathfrak{B}_{RU}} \otimes \omega_{\mathfrak{C}_U} && \text{on } \mathfrak{B}_U \otimes \mathfrak{C}_U, \\
\omega_{\mathfrak{B}\mathfrak{C}}^D &:= \omega_{\mathfrak{B}_{LD}} \otimes \omega_{\mathfrak{B}_{RD}} \otimes \omega_{\mathfrak{C}_D} && \text{on } \mathfrak{B}_D \otimes \mathfrak{C}_D, \\
\omega_{\mathfrak{A}} &:= \omega_{\mathfrak{A}_L} \otimes \omega_{\mathfrak{A}_R} && \text{on } \mathfrak{A}, \\
\omega_{\mathfrak{A}\mathfrak{B}}^L &:= \omega_{\mathfrak{A}_L} \otimes \omega_{\mathfrak{B}_{LU}} \otimes \omega_{\mathfrak{B}_{LD}} && \text{on } \mathfrak{A}_L \otimes \mathfrak{B}_L, \\
\omega_{\mathfrak{A}\mathfrak{B}}^R &:= \omega_{\mathfrak{A}_R} \otimes \omega_{\mathfrak{B}_{RU}} \otimes \omega_{\mathfrak{B}_{RD}} && \text{on } \mathfrak{A}_R \otimes \mathfrak{B}_R, \\
\omega_0 &:= \bigotimes_{\substack{X=\mathfrak{A}_L, \mathfrak{A}_R, \mathfrak{B}_{LU}, \mathfrak{B}_{LD}, \\ \mathfrak{B}_{RU}, \mathfrak{B}_{RD}, \mathfrak{C}_U, \mathfrak{C}_D}} \omega_X && \text{on } \mathfrak{D}.
\end{aligned}
\tag{3.11}
$$

*Let $\alpha, \hat{\alpha}$ be automorphisms on $\mathfrak{D}$ which allow the following decompositions:*

$$
\hat{\alpha} = \left( \rho_{\mathfrak{B}\mathfrak{C}}^U \otimes \mathrm{id}_{\mathfrak{A}} \otimes \rho_{\mathfrak{B}\mathfrak{C}}^D \right) \circ \left( \mathrm{id}_{\mathfrak{C}_U} \otimes \hat{\gamma}_{\mathfrak{A}\mathfrak{B}}^L \otimes \hat{\gamma}_{\mathfrak{A}\mathfrak{B}}^R \otimes \mathrm{id}_{\mathfrak{C}_D} \right) \circ (inner),
\tag{3.12}
$$

$$
\alpha = \left( \rho_{\mathfrak{B}\mathfrak{C}}^U \otimes \mathrm{id}_{\mathfrak{A}} \otimes \mathrm{id}_{\mathfrak{B}_D \otimes \mathfrak{C}_D} \right) \circ \left( \mathrm{id}_{\mathfrak{C}_U} \otimes \gamma_{\mathfrak{A}\mathfrak{B}}^L \otimes \gamma_{\mathfrak{A}\mathfrak{B}}^R \otimes \mathrm{id}_{\mathfrak{C}_D} \right) \circ (inner).
\tag{3.13}
$$

*Here, $\rho_{\mathfrak{B}\mathfrak{C}}^U, \rho_{\mathfrak{B}\mathfrak{C}}^D$ are automorphisms on $\mathfrak{B}_U \otimes \mathfrak{C}_U, \mathfrak{B}_D \otimes \mathfrak{C}_D$, respectively. For each $\sigma = L, R, \gamma_{\mathfrak{A}\mathfrak{B}}^\sigma, \hat{\gamma}_{\mathfrak{A}\mathfrak{B}}^\sigma$ are automorphisms on $\mathfrak{A}_\sigma \otimes \mathfrak{B}_{\sigma D} \otimes \mathfrak{B}_{\sigma U}$. Suppose that $\omega_0 \circ \hat{\alpha} = \omega_0$. Then there are automorphisms $\eta_L, \eta_R$ on $\mathfrak{A}_L \otimes \mathfrak{B}_{LD} \otimes \mathfrak{B}_{LU}, \mathfrak{A}_R \otimes \mathfrak{B}_{RD} \otimes \mathfrak{B}_{RU}$ such that $\omega_0 \circ \alpha$ is quasiequivalent to $\omega_0 \circ \left( \mathrm{id}_{\mathfrak{C}_U} \otimes \eta_L \otimes \eta_R \otimes \mathrm{id}_{\mathfrak{C}_D} \right)$.*

*Proof.* First we claim that there are automorphisms $\theta_{\mathfrak{B}}^{LU} \in \mathrm{Aut}\, \mathfrak{B}_{LU}, \theta_{\mathfrak{B}}^{RU} \in \mathrm{Aut}\, \mathfrak{B}_{RU}$ and a unitary $u \in \mathcal{U}\left( \mathfrak{B}^U \otimes \mathfrak{C}^U \right)$ such that

$$
\omega_{\mathfrak{B}\mathfrak{C}}^U \circ \rho_{\mathfrak{B}\mathfrak{C}}^U = \omega_{\mathfrak{B}\mathfrak{C}}^U \circ \left( \theta_{\mathfrak{B}}^{LU} \otimes \theta_{\mathfrak{B}}^{RU} \otimes \mathrm{id}_{\mathfrak{C}^U} \right) \circ \mathrm{Ad}\,(u).
\tag{3.14}
$$

To prove this, we first note that from $\omega_0 \circ \hat{\alpha} = \omega_0$ and the decomposition (3.12), we have

$$
\omega_{\mathfrak{B}\mathfrak{C}}^U \circ \rho_{\mathfrak{B}\mathfrak{C}}^U \otimes \omega_{\mathfrak{A}} \otimes \omega_{\mathfrak{B}\mathfrak{C}}^D \circ \rho_{\mathfrak{B}\mathfrak{C}}^D \sim_{\mathrm{q.e.}} \omega_{\mathfrak{C}_U} \otimes \omega_{\mathfrak{A}\mathfrak{B}}^L \circ \left( \widehat{\gamma_{\mathfrak{A}\mathfrak{B}}^L} \right)^{-1} \otimes \omega_{\mathfrak{A}\mathfrak{B}}^R \circ \left( \widehat{\gamma_{\mathfrak{A}\mathfrak{B}}^R} \right)^{-1} \otimes \omega_{\mathfrak{C}_D}.
\tag{3.15}
$$

From this, because both states are pure (hence the restrictions of their GNS representations onto $\mathfrak{C}_U \otimes \mathfrak{B}_U$ are factors), we have

$$
\begin{aligned}
\omega_{\mathfrak{B}\mathfrak{C}}^U \circ \rho_{\mathfrak{B}\mathfrak{C}}^U &= \left. \left( \omega_{\mathfrak{B}\mathfrak{C}}^U \circ \rho_{\mathfrak{B}\mathfrak{C}}^U \otimes \omega_{\mathfrak{A}} \otimes \omega_{\mathfrak{B}\mathfrak{C}}^D \circ \rho_{\mathfrak{B}\mathfrak{C}}^D \right) \right|_{\mathfrak{C}_U \otimes \mathfrak{B}_U} \\
&\sim_{\mathrm{q.e.}} \left. \omega_{\mathfrak{C}_U} \otimes \left( \omega_{\mathfrak{A}\mathfrak{B}}^L \circ \left( \widehat{\gamma_{\mathfrak{A}\mathfrak{B}}^L} \right)^{-1} \otimes \omega_{\mathfrak{A}\mathfrak{B}}^R \circ \left( \widehat{\gamma_{\mathfrak{A}\mathfrak{B}}^R} \right)^{-1} \right) \right|_{\mathfrak{B}_U}.
\end{aligned}
\tag{3.16}
$$

We apply Lemma 3.3 for $\mathfrak{B}, \mathfrak{A}_{1L}, \mathfrak{A}_{1R}, \mathfrak{A}_{2L}, \mathfrak{A}_{2R}, \omega, \varphi_L^{(1,2)}, \varphi_R^{(1,2)}, \psi$, replaced by $\mathfrak{C}_U, \mathfrak{B}_{LU}, \mathfrak{B}_{RU}, \mathfrak{A}_L \otimes \mathfrak{B}_{LD}, \mathfrak{A}_R \otimes \mathfrak{B}_{RD}, \omega_{\mathfrak{B}\mathfrak{C}}^U \circ \rho_{\mathfrak{B}\mathfrak{C}}^U, \omega_{\mathfrak{A}\mathfrak{B}}^L \circ \left( \widehat{\gamma_{\mathfrak{A}\mathfrak{B}}^L} \right)^{-1}, \omega_{\mathfrak{A}\mathfrak{B}}^R \circ \left( \widehat{\gamma_{\mathfrak{A}\mathfrak{B}}^R} \right)^{-1}, \omega_{\mathfrak{C}_U}$, respectively. From equation (3.16), they satisfy the conditions in Lemma 3.3. Applying Lemma 3.3 – for pure states $\varphi_L^{(1)} = \omega_{\mathfrak{B}_{LU}}$

and $\varphi_R^{(1)} = \omega_{\mathfrak{B}_{RU}}$ – we obtain automorphisms $\theta_{\mathfrak{B}}^{LU} \in \text{Aut}(\mathfrak{B}_{LU})$, $\theta_{\mathfrak{B}}^{RU} \in \text{Aut}(\mathfrak{B}_{RU})$ and a unitary $u \in \mathcal{U}(\mathfrak{B}_U \otimes \mathfrak{C}_U)$ satisfying equation (3.14).

We set

$$
\begin{aligned}
\eta_L &:= \left(\theta_{\mathfrak{B}}^{LU} \otimes \text{id}_{\mathfrak{A}_L} \otimes \text{id}_{\mathfrak{B}_{LD}}\right) \circ \gamma_{\mathfrak{A}\mathfrak{B}}^L \in \text{Aut}\left(\mathfrak{B}_{LU} \otimes \mathfrak{A}_L \otimes \mathfrak{B}_{LD}\right) \\
\eta_R &:= \left(\theta_{\mathfrak{B}}^{RU} \otimes \text{id}_{\mathfrak{A}_R} \otimes \text{id}_{\mathfrak{B}_{RD}}\right) \circ \gamma_{\mathfrak{A}\mathfrak{B}}^R \in \text{Aut}\left(\mathfrak{B}_{RU} \otimes \mathfrak{A}_R \otimes \mathfrak{B}_{RD}\right).
\end{aligned}
\tag{3.17}
$$

Then we have

$$
\begin{aligned}
\omega_0 \circ \alpha &= \left(\omega_{\mathfrak{A}_L} \otimes \omega_{\mathfrak{A}_R} \otimes \omega_{\mathfrak{B}\mathfrak{C}}^U \otimes \omega_{\mathfrak{B}\mathfrak{C}}^D\right) \circ \alpha \\
&\sim_{\text{q.e.}} \left(\omega_{\mathfrak{A}_L} \otimes \omega_{\mathfrak{A}_R} \otimes \omega_{\mathfrak{B}\mathfrak{C}}^U \circ \rho_{\mathfrak{B}\mathfrak{C}}^U \otimes \omega_{\mathfrak{B}\mathfrak{C}}^D\right) \circ \left(\text{id}_{\mathfrak{C}_U} \otimes \gamma_{\mathfrak{A}\mathfrak{B}}^L \otimes \gamma_{\mathfrak{A}\mathfrak{B}}^R \otimes \text{id}_{\mathfrak{C}_D}\right) \\
&\sim_{\text{q.e.}} \left(\omega_{\mathfrak{A}_L} \otimes \omega_{\mathfrak{A}_R} \otimes \omega_{\mathfrak{B}\mathfrak{C}}^U \otimes \omega_{\mathfrak{B}\mathfrak{C}}^D\right) \\
&\quad \circ \left(\text{id}_{\mathfrak{C}_U} \otimes \left(\left(\theta_{\mathfrak{B}}^{LU} \otimes \text{id}_{\mathfrak{A}_L} \otimes \text{id}_{\mathfrak{B}_{LD}}\right) \circ \gamma_{\mathfrak{A}\mathfrak{B}}^L\right) \otimes \left(\left(\theta_{\mathfrak{B}}^{RU} \otimes \text{id}_{\mathfrak{A}_R} \otimes \text{id}_{\mathfrak{B}_{RD}}\right) \circ \gamma_{\mathfrak{A}\mathfrak{B}}^R\right) \otimes \text{id}_{\mathfrak{C}_D}\right) \\
&= \omega_0 \circ \left(\text{id}_{\mathfrak{C}_U} \otimes \eta_L \otimes \eta_R \otimes \text{id}_{\mathfrak{C}_D}\right).
\end{aligned}
\tag{3.18}
$$

This completes the proof.    □

Now we are ready to prove Theorem 3.1.

*Proof of Theorem 3.1.* Set $0 < \theta < \frac{\pi}{2}$ and $\alpha \in \text{SQAut}(\mathcal{A})$ satisfying $\omega_0 \circ \alpha \circ \beta_g = \omega_0 \circ \alpha$ for all $g \in G$. We would like to show that $\text{IG}(\omega_0 \circ \alpha, \theta)$ is not empty.

Let us set $\theta_{2.2} := \theta$ and consider $\theta_{0.8}, \theta_1, \theta_{1.2}, \theta_{1.8}, \theta_2, \theta_{2.8}, \theta_3, \theta_{3.2}$ satisfying formula (2.11) for this $\theta_{2.2}$. Because $\alpha \in \text{SQAut}(\mathcal{A})$, there is a decomposition given by formulas (2.12), (2.13) and (2.14). Using this decomposition, set

$$
\begin{aligned}
\alpha_1 &:= \alpha_{1D} \otimes \alpha_{1U}, \quad \text{where} \\
\alpha_{1\zeta} &:= \left(\alpha_{(\theta_1,\theta_2],\zeta} \otimes \alpha_{(\theta_2,\theta_3],\zeta} \otimes \alpha_{(\theta_3,\frac{\pi}{2}],\zeta}\right) \\
&\quad \circ \left(\alpha_{(\theta_{0.8},\theta_{1.2}],\zeta} \otimes \alpha_{(\theta_{1.8},\theta_{2.2}],\zeta} \otimes \alpha_{(\theta_{2.8},\theta_{3.2}],\zeta}\right) \in \text{Aut}\left(\mathcal{A}_{\left((C_{\theta_{0.8}})^c\right)_\zeta}\right), \quad \zeta = U, D, \\
\alpha_2 &:= \alpha_{[0,\theta_1]} \in \text{Aut}\left(\mathcal{A}_{C_{\theta_1}}\right).
\end{aligned}
\tag{3.19}
$$

We have $\alpha = (\text{inner}) \circ \alpha_2 \circ \alpha_1$.

We would like to show that $\left(\alpha \circ \beta_g^U \circ \alpha^{-1}, \alpha \circ \beta_g \circ \alpha^{-1}\right)$ satisfy the conditions of $(\alpha, \hat{\alpha})$ in Lemma 3.4. We first show that they satisfy a decomposition corresponding to equations (3.12) and (3.13). For $\Gamma = \mathbb{Z}^2$, $H_U$, we have

$$
\left(\beta_g^\Gamma\right)^{-1} \alpha \circ \beta_g^\Gamma \circ \alpha^{-1} = (\text{inner}) \circ \left(\beta_g^\Gamma\right)^{-1} \circ \left(\alpha_1 \beta_g^\Gamma \alpha_1^{-1}\right) \left(\alpha_1 \beta_g^\Gamma \alpha_1^{-1}\right)^{-1} \alpha_2 \alpha_1 \beta_g^\Gamma \alpha_1^{-1} \alpha_2^{-1}.
\tag{3.20}
$$

The latter part, $\left(\alpha_1 \beta_g^\Gamma \alpha_1^{-1}\right)^{-1} \alpha_2 \alpha_1 \beta_g^\Gamma \alpha_1^{-1} \alpha_2^{-1}$, decomposes to left and right. To see this, first note that

$$
\alpha_1^{-1} \alpha_2 \alpha_1 = \alpha_{(\theta_{0.8},\theta_{1.2}]}^{-1} \alpha_{[0,\theta_1]} \alpha_{(\theta_{0.8},\theta_{1.2}]} \in \text{Aut}\left(\mathcal{A}_{C_{\theta_{1.2}}}\right).
\tag{3.21}
$$

Because the conjugation $\left(\beta_g^\Gamma\right)^{-1} \cdot \beta_g^\Gamma$ does not change the support of an automorphism, $\left(\beta_g^\Gamma\right)^{-1} \left(\alpha_1^{-1}\alpha_2\alpha_1\right)\beta_g^\Gamma$ is also supported on $C_{\theta_{1.2}}$. Therefore, we have

$$\alpha_1 \left(\left(\beta_g^\Gamma\right)^{-1}\left(\alpha_1^{-1}\alpha_2\alpha_1\right)\beta_g^\Gamma\right)\alpha_1^{-1}$$
$$= \alpha_{(\theta_1,\theta_2]}\alpha_{(\theta_{0.8},\theta_{1.2}]}\left(\beta_g^\Gamma\right)^{-1}\alpha_{(\theta_{0.8},\theta_{1.2}]}^{-1}\alpha_{[0,\theta_1]}\alpha_{(\theta_{0.8},\theta_{1.2}]}\beta_g^\Gamma\alpha_{(\theta_{0.8},\theta_{1.2}]}^{-1}\alpha_{(\theta_1,\theta_2]}^{-1}. \qquad (3.22)$$

Hence we get the left-right decomposition

$$\left(\alpha_1\beta_g^\Gamma\alpha_1^{-1}\right)^{-1}\alpha_2\alpha_1\beta_g^\Gamma\alpha_1^{-1}\alpha_2^{-1} = \alpha_1\left(\left(\beta_g^\Gamma\right)^{-1}\left(\alpha_1^{-1}\alpha_2\alpha_1\right)\beta_g^\Gamma\right)\alpha_1^{-1}\alpha_2^{-1}$$
$$= \alpha_{(\theta_1,\theta_2]}\alpha_{(\theta_{0.8},\theta_{1.2}]}\left(\beta_g^\Gamma\right)^{-1}\alpha_{(\theta_{0.8},\theta_{1.2}]}^{-1}\alpha_{[0,\theta_1]}\alpha_{(\theta_{0.8},\theta_{1.2}]}\beta_g^\Gamma\alpha_{(\theta_{0.8},\theta_{1.2}]}^{-1}\alpha_{(\theta_1,\theta_2]}^{-1} \circ \alpha_{[0,\theta_1]}^{-1}$$
$$=: \bigotimes_{\sigma=L,R} \Xi_{\Gamma,g,\sigma}. \qquad (3.23)$$

Here we set

$$\Xi_{\Gamma,g,\sigma} := \left(\alpha_{(\theta_1,\theta_2],\sigma}\alpha_{(\theta_{0.8},\theta_{1.2}],\sigma}\left(\beta_g^{\Gamma_\sigma}\right)^{-1}\right.$$
$$\left.\alpha_{(\theta_{0.8},\theta_{1.2}],\sigma}^{-1}\alpha_{[0,\theta_1],\sigma}\alpha_{(\theta_{0.8},\theta_{1.2}],\sigma}\beta_g^{\Gamma_\sigma}\alpha_{(\theta_{0.8},\theta_{1.2}],\sigma}^{-1}\alpha_{(\theta_1,\theta_2],\sigma}^{-1} \circ \alpha_{[0,\theta_1],\sigma}^{-1}\right) \quad \in \mathrm{Aut}\left(\mathcal{A}_{(C_{\theta_2})_\sigma}\right). \qquad (3.24)$$

On the other hand, the first part of equation (3.20) with $\Gamma = \mathbb{Z}^2$, $H_U$ satisfies

$$\beta_g^{-1}\alpha_1\beta_g\alpha_1^{-1} = \xi_D \otimes \xi_U, \qquad \left(\beta_g^U\right)^{-1}\alpha_1\beta_g^U\alpha_1^{-1} = \mathrm{id}_{\mathcal{A}_{H_D}}\otimes\xi_U, \qquad (3.25)$$

where

$$\xi_\zeta := \left(\beta_g^\zeta\right)^{-1}\alpha_{1,\zeta}\beta_g^\zeta\alpha_{1,\zeta}^{-1} \in \mathrm{Aut}\left(\mathcal{A}_{((C_{\theta_{0.8}})^c)_\zeta}\right), \quad \zeta = U, D. \qquad (3.26)$$

Hence we obtain decompositions

$$\left(\beta_g^U\right)^{-1} \circ \alpha \circ \beta_g^U \circ \alpha^{-1} = (\text{inner}) \circ \left(\mathrm{id}_{\mathcal{A}_{H_D}}\otimes\xi_U\right) \circ \left(\Xi_{H_U,g,L} \otimes \Xi_{H_U,g,R}\right),$$
$$\left(\beta_g\right)^{-1} \circ \alpha \circ \beta_g \circ \alpha^{-1} = (\text{inner}) \circ (\xi_D \otimes \xi_U) \circ \left(\Xi_{\mathbb{Z}^2,g,L} \otimes \Xi_{\mathbb{Z}^2,g,R}\right). \qquad (3.27)$$

Because $\xi_\zeta \in \mathrm{Aut}\left(\mathcal{A}_{((C_{\theta_{0.8}})^c)_\zeta}\right)$ commutes with $\beta_g^{C_{[0,\theta_{0.8}]}}$ and $\beta_g^{C_{[0,\theta_{0.8}]},U}$, we get

$$\alpha \circ \beta_g^U \circ \alpha^{-1} = (\text{inner}) \circ \left(\mathrm{id}_{\mathcal{A}_{H_D}}\otimes\beta_g^{C_{(\theta_{0.8},\frac{\pi}{2}]},U}\xi_U\right) \circ \left(\beta_g^{C_{[0,\theta_{0.8}]},L,U}\Xi_{H_U,g,L} \otimes \beta_g^{C_{[0,\theta_{0.8}]},R,U}\Xi_{H_U,g,R}\right),$$
$$\alpha \circ \beta_g \circ \alpha^{-1} = (\text{inner}) \circ \left(\beta_g^{C_{(\theta_{0.8},\frac{\pi}{2}]},D}\xi_D \otimes \beta_g^{C_{(\theta_{0.8},\frac{\pi}{2}]},U}\xi_U\right) \circ \left(\beta_g^{C_{[0,\theta_{0.8}]},L}\Xi_{\mathbb{Z}^2,g,L} \otimes \beta_g^{C_{[0,\theta_{0.8}]},R}\Xi_{\mathbb{Z}^2,g,R}\right). \qquad (3.28)$$

Furthermore, from the $\beta_g$-invariance of $\omega_0 \circ \alpha$, we have

$$\omega_0 \circ \alpha \circ \beta_g \circ \alpha^{-1} = \omega_0. \tag{3.29}$$

Now we apply Lemma 3.4 for $\mathfrak{A}_\sigma, \mathfrak{B}_{\sigma\zeta}, \mathfrak{C}_\zeta$ replaced by $\mathcal{A}_{\left(C_{[0,\theta_{0.8}]}\right)_\sigma}, \mathcal{A}_{\left(C_{(\theta_{0.8},\theta_2]}\right)_{\sigma,\zeta}}, \mathcal{A}_{\left(C_{(\theta_2,\frac{\pi}{2}]}\right)_\zeta}$,
for $\sigma = L, R, \zeta = D, U$. By equations (3.29) and (3.28), $\left(\alpha \circ \beta_g^U \circ \alpha^{-1}, \alpha \circ \beta_g \circ \alpha^{-1}\right)$ satisfy the
conditions of $(\alpha, \hat\alpha)$ in Lemma 3.4, for $\omega_0$ and its restrictions. Applying Lemma 3.4, there are $\tilde\eta_{\sigma,g} \in$
$\mathrm{Aut}\left(\mathcal{A}_{\left(C_{\theta_2}\right)_\sigma}\right), g \in G, \sigma = L, R$, such that

$$\omega_0 \circ \alpha \circ \beta_g^U \circ \alpha^{-1} \sim_{\text{q.e.}} \omega_0 \circ \left(\tilde\eta_{Lg} \otimes \tilde\eta_{Rg}\right), \quad g \in G. \tag{3.30}$$

Because both $\omega_0 \circ \alpha \circ \beta_g^U \circ \alpha^{-1}$ and $\omega_0 \circ \left(\tilde\eta_{Lg} \otimes \tilde\eta_{Rg}\right)$ are pure, by Kadison's transitivity theorem there
exists a unitary $\tilde v_g \in \mathcal{U}(\mathcal{A})$ such that

$$\omega_0 \circ \alpha \circ \beta_g^U \circ \alpha^{-1} = \omega_0 \circ \mathrm{Ad}_{\tilde v_g} \circ \left(\tilde\eta_{Lg} \otimes \tilde\eta_{Rg}\right), \quad g \in G. \tag{3.31}$$

We define

$$\tilde\beta_g := \mathrm{Ad}\left(\alpha^{-1}\left(\tilde v_{g^{-1}}\right)\right) \circ \alpha^{-1} \circ \left(\tilde\eta_{Lg^{-1}} \otimes \tilde\eta_{Rg^{-1}}\right) \circ \alpha \circ \beta_g^U, \quad g \in G. \tag{3.32}$$

It suffices to show that $(\tilde\beta_g) \in \mathrm{IG}(\omega_0 \circ \alpha, \theta) = \mathrm{IG}(\omega_0 \circ \alpha, \theta_{2.2})$. By equation (3.31), we have $\omega_0 \circ \alpha \circ \tilde\beta_g = \omega_0 \circ \alpha$. Therefore, what is left to be proven is that there are $\eta_g^\sigma \in \mathrm{Aut}\left((C_\theta)_\sigma\right), g \in G, \sigma = L, R$, such that

$$\tilde\beta_g = (\text{inner}) \circ \left(\eta_g^L \otimes \eta_g^R\right) \circ \beta_g^U, \quad \text{for all } g \in G. \tag{3.33}$$

By the decomposition (2.12) and the fact that $\tilde\eta_{Lg^{-1}} \otimes \tilde\eta_{Rg^{-1}}$ has support in $C_{\theta_2}$, we have

$$\begin{aligned}
\alpha^{-1} &\circ \left(\tilde\eta_{Lg^{-1}} \otimes \tilde\eta_{Rg^{-1}}\right) \circ \alpha \\
&= (\text{inner}) \circ \left(\alpha_{(\theta_{0.8},\theta_{1.2}]} \otimes \alpha_{(\theta_{1.8},\theta_{2.2}]}\right)^{-1} \left(\alpha_{[0,\theta_1]} \otimes \alpha_{(\theta_1,\theta_2]}\right)^{-1} \left(\tilde\eta_{Lg^{-1}} \otimes \tilde\eta_{Rg^{-1}}\right) \left(\alpha_{[0,\theta_1]} \otimes \alpha_{(\theta_1,\theta_2]}\right) \\
&\quad \circ \left(\alpha_{(\theta_{0.8},\theta_{1.2}]} \otimes \alpha_{(\theta_{1.8},\theta_{2.2}]}\right) \\
&= (\text{inner}) \circ \left(\eta_g^L \otimes \eta_g^R\right),
\end{aligned} \tag{3.34}$$

where

$$\begin{aligned}
\eta_g^\sigma &= \left(\alpha_{(\theta_{0.8},\theta_{1.2}],\sigma} \otimes \alpha_{(\theta_{1.8},\theta_{2.2}],\sigma}\right)^{-1} \left(\alpha_{[0,\theta_1],\sigma} \otimes \alpha_{(\theta_1,\theta_2],\sigma}\right)^{-1} \left(\tilde\eta_{\sigma g^{-1}}\right) \left(\alpha_{[0,\theta_1],\sigma} \otimes \alpha_{(\theta_1,\theta_2],\sigma}\right) \\
&\quad \circ \left(\alpha_{(\theta_{0.8},\theta_{1.2}],\sigma} \otimes \alpha_{(\theta_{1.8},\theta_{2.2}],\sigma}\right) \\
&\in \mathrm{Aut}\left((C_{\theta_{2.2}})_\sigma\right), \quad \sigma = L, R.
\end{aligned} \tag{3.35}$$

Substituting this into formula (3.32), we obtain equation (3.33). This completes the proof. □

## 4. The stability of the index $h(\omega)$

In this section we prove the stability of the index $h(\omega)$ with respect to $\gamma \in \mathrm{GUQAut}(\mathcal{A})$.

**Theorem 4.1.** *Set* $\omega \in \mathcal{SL}$ *with* $\mathrm{IG}(\omega) \neq \emptyset$. *Set* $\gamma \in \mathrm{GUQAut}(\mathcal{A})$. *Then we have* $\omega \circ \gamma \in \mathcal{SL}$ *with* $\mathrm{IG}(\omega \circ \gamma) \neq \emptyset$ *and*

$$h(\omega \circ \gamma) = h(\omega). \tag{4.1}$$

*Proof.* The point of the proof is that we can derive $\left(\hat{\alpha}_L, \hat{\alpha}_R, \hat{\Theta}\right) \in \mathcal{D}_{\alpha\gamma}^{\theta_{1.2}}$ (formulas (4.10) and (4.11)) and $\left(\gamma^{-1}\tilde{\beta}_g\gamma\right) \in \mathrm{IG}(\omega \circ \gamma, \theta_{1.2}), \left(\hat{\eta}_g^\sigma\right) \in \mathcal{T}(\theta_{1.2}, (\gamma^{-1}\tilde{\beta}_g\gamma))$ (formula (4.16)) from the corresponding objects for $\alpha$, using the factorisation property of $\alpha, \gamma$. And it is straightforward to see that the $\beta_g^U$-invariance of $\gamma_C$ results in $\mathrm{IP}\left(\omega, \alpha, \theta_2, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_L, \alpha_R, \Theta)\right) = \mathrm{IP}\left(\omega \circ \gamma, \alpha \circ \gamma, \theta_{1.2}, (\gamma^{-1}\tilde{\beta}_g\gamma), (\hat{\eta}_g^\sigma), (\hat{\alpha}_L, \hat{\alpha}_R, \hat{\Theta})\right)$, which immediately implies the Theorem.

*Step 1.* From $\omega \in \mathcal{SL}$, there is an $\alpha \in \mathrm{EAut}(\omega)$. For any $0 < \theta < \frac{\pi}{2}$ fixed, we show that $\mathcal{D}_{\alpha\gamma}^\theta \neq \emptyset$, hence $\alpha \circ \gamma \in \mathrm{QAut}(\mathcal{A})$ and $\omega \circ \gamma = \omega_0 \circ \alpha\gamma \in \mathcal{SL}$. Set $\theta_{1.2} := \theta$ and choose

$$0 < \theta_0 < \theta_{0.8} < \theta_1 < \theta_{1.2} := \theta < \theta_{1.8} < \theta_2 < \theta_{2.2} < \theta_{2.8} < \theta_3 < \theta_{3.2} < \frac{\pi}{2}. \tag{4.2}$$

Because $\alpha \in \mathrm{QAut}(\mathcal{A})$, there exists some $(\alpha_L, \alpha_R, \Theta) \in \mathcal{D}_\alpha^{\theta_2}$. Setting $\alpha_0 := \alpha_L \otimes \alpha_R$, we have $\alpha = (\text{inner}) \circ \alpha_0 \circ \Theta$. Because $\gamma \in \mathrm{GUQAut}(\mathcal{A})$, there are $\gamma_H \in \mathrm{HAut}(\mathcal{A})$ and $\gamma_C \in \mathrm{GSQAut}(\mathcal{A})$ such that

$$\gamma = \gamma_C \circ \gamma_H. \tag{4.3}$$

Because $\gamma_H \in \mathrm{HAut}(\mathcal{A})$, we may decompose $\gamma_H$ as

$$\gamma_H = (\text{inner}) \circ \left(\gamma_{H,L} \otimes \gamma_{H,R}\right) = (\text{inner}) \circ \gamma_0, \tag{4.4}$$

with some $\gamma_{H,\sigma} \in \mathrm{Aut}\left(\mathcal{A}_{(C_{\theta_0})_\sigma}\right), \sigma = L, R$. We set $\gamma_0 := \gamma_{H,L} \otimes \gamma_{H,R} \in \mathrm{Aut}\left(\mathcal{A}_{C_{\theta_0}}\right)$. By definition, $\gamma_C \in \mathrm{GSQAut}(\mathcal{A})$ allows a decomposition

$$\gamma_C = (\text{inner}) \circ \gamma_{CS},$$
$$\gamma_{CS} = \left(\gamma_{[0,\theta_1]} \otimes \gamma_{(\theta_1,\theta_2]} \otimes \gamma_{(\theta_2,\theta_3]} \otimes \gamma_{(\theta_3,\frac{\pi}{2}]}\right) \circ \left(\gamma_{(\theta_{0.8},\theta_{1.2}]} \otimes \gamma_{(\theta_{1.8},\theta_{2.2}]} \otimes \gamma_{(\theta_{2.8},\theta_{3.2}]}\right), \tag{4.5}$$

with

$$\gamma_X := \bigotimes_{\sigma=L,R,\,\zeta=D,U} \gamma_{X,\sigma,\zeta}, \qquad \gamma_{[0,\theta_1]} := \bigotimes_{\sigma=L,R} \gamma_{[0,\theta_1],\sigma}, \qquad \gamma_{(\theta_3,\frac{\pi}{2}]} := \bigotimes_{\zeta=D,U} \gamma_{(\theta_3,\frac{\pi}{2}],\zeta},$$

$$\gamma_{X,\sigma,\zeta} \in \mathrm{Aut}\left(\mathcal{A}_{C_{X,\sigma,\zeta}}\right), \qquad \gamma_{X,\sigma} := \bigotimes_{\zeta=U,D} \gamma_{X,\sigma,\zeta}, \qquad \gamma_{X,\zeta} := \bigotimes_{\sigma=L,R} \gamma_{X,\sigma,\zeta}, \tag{4.6}$$

$$\gamma_{[0,\theta_1],\sigma} \in \mathrm{Aut}\left(\mathcal{A}_{C_{[0,\theta_1],\sigma}}\right), \qquad \gamma_{(\theta_3,\frac{\pi}{2}],\zeta} \in \mathrm{Aut}\left(\mathcal{A}_{C_{(\theta_3,\frac{\pi}{2}],\zeta}}\right),$$

for

$$X = (\theta_1,\theta_2], (\theta_2,\theta_3], (\theta_{0.8},\theta_{1.2}], (\theta_{1.8},\theta_{2.2}], (\theta_{2.8},\theta_{3.2}], \quad \sigma = L, R, \ \zeta = D, U. \tag{4.7}$$

Here we have

$$\gamma_I \circ \beta_g^U = \beta_g^U \circ \gamma_I \quad \text{for all } g \in G, \tag{4.8}$$

for any

$$I = [0, \theta_1], (\theta_1, \theta_2], (\theta_2, \theta_3], \left(\theta_3, \frac{\pi}{2}\right], (\theta_{0.8}, \theta_{1.2}], (\theta_{1.8}, \theta_{2.2}], (\theta_{2.8}, \theta_{3.2}]. \tag{4.9}$$

Set

$$\hat{\Theta} := \Theta \circ \left(\gamma_{(\theta_2, \theta_3]} \otimes \gamma_{\left(\theta_3, \frac{\pi}{2}\right]}\right) \circ \left(\gamma_{(\theta_{1.8}, \theta_{2.2}]} \otimes \gamma_{(\theta_{2.8}, \theta_{3.2}]}\right) \in \mathrm{Aut}\left(\mathcal{A}_{C^c_{\theta_{1.8}}}\right) \subset \mathrm{Aut}\left(\mathcal{A}_{C^c_{\theta_{1.2}}}\right) \tag{4.10}$$

and

$$\hat{\alpha}_\sigma := \alpha_\sigma \circ \left(\gamma_{[0,\theta_1],\sigma} \otimes \gamma_{(\theta_1,\theta_2],\sigma}\right) \circ \gamma_{(\theta_{0.8},\theta_{1.2}],\sigma} \circ \gamma_{H,\sigma} \in \mathrm{Aut}\left(\mathcal{A}_{H_\sigma}\right), \quad \sigma = L, R. \tag{4.11}$$

We claim

$$\alpha \circ \gamma = (\mathrm{inner}) \circ (\hat{\alpha}_L \otimes \hat{\alpha}_R) \circ \hat{\Theta}. \tag{4.12}$$

This means $(\hat{\alpha}_L, \hat{\alpha}_R, \hat{\Theta}) \in \mathcal{D}^{\theta_{1.2}}_{\alpha\gamma}$, hence $\mathcal{D}^{\theta}_{\alpha\gamma} = \mathcal{D}^{\theta_{1.2}}_{\alpha\gamma} \neq \emptyset$. The claim (4.12) can be checked as follows. Note that $\gamma_{(\theta_2,\theta_3]} \otimes \gamma_{\left(\theta_3, \frac{\pi}{2}\right]}$ and $\gamma_{(\theta_{0.8},\theta_{1.2}]}$ commute because of their disjoint supports. Because $\Theta \in \mathrm{Aut}(\mathcal{A}_{C^c_{\theta_2}})$, it commutes with $\gamma_{[0,\theta_1]} \otimes \gamma_{(\theta_1,\theta_2]}$ and $\gamma_{(\theta_{0.8},\theta_{1.2}]}$. Therefore, we have

$$\begin{aligned}
\alpha \circ \gamma &= (\mathrm{inner}) \circ \alpha_0 \circ \Theta \circ \left(\gamma_{[0,\theta_1]} \otimes \gamma_{(\theta_1,\theta_2]} \otimes \gamma_{(\theta_2,\theta_3]} \otimes \gamma_{\left(\theta_3, \frac{\pi}{2}\right]}\right) \\
&\quad \circ \left(\gamma_{(\theta_{0.8},\theta_{1.2}]} \otimes \gamma_{(\theta_{1.8},\theta_{2.2}]} \otimes \gamma_{(\theta_{2.8},\theta_{3.2}]}\right) \circ \gamma_0 \\
&= (\mathrm{inner}) \circ \alpha_0 \circ \left(\gamma_{[0,\theta_1]} \otimes \gamma_{(\theta_1,\theta_2]}\right) \circ \gamma_{(\theta_{0.8},\theta_{1.2}]} \circ \Theta \circ \left(\gamma_{(\theta_2,\theta_3]} \otimes \gamma_{\left(\theta_3, \frac{\pi}{2}\right]}\right) \\
&\quad \circ \left(\gamma_{(\theta_{1.8},\theta_{2.2}]} \otimes \gamma_{(\theta_{2.8},\theta_{3.2}]}\right) \circ \gamma_0 \\
&= (\mathrm{inner}) \circ \alpha_0 \circ \left(\gamma_{[0,\theta_1]} \otimes \gamma_{(\theta_1,\theta_2]}\right) \circ \gamma_{(\theta_{0.8},\theta_{1.2}]} \circ \hat{\Theta} \circ \gamma_0. 
\end{aligned} \tag{4.13}$$

Because $\gamma_0 \in \mathrm{Aut}\left(\mathcal{A}_{C_{\theta_0}}\right)$ and $\hat{\Theta} \in \mathrm{Aut}\left(\mathcal{A}_{C^c_{\theta_{1.8}}}\right)$ commute, we have

$$\begin{aligned}
\alpha \circ \gamma &= \text{equation (4.13)} = (\mathrm{inner}) \circ \alpha_0 \circ \left(\gamma_{[0,\theta_1]} \otimes \gamma_{(\theta_1,\theta_2]}\right) \circ \gamma_{(\theta_{0.8},\theta_{1.2}]} \circ \gamma_0 \circ \hat{\Theta} \\
&= (\mathrm{inner}) \circ (\hat{\alpha}_L \otimes \hat{\alpha}_R) \circ \hat{\Theta},
\end{aligned} \tag{4.14}$$

proving equation (4.12).

*Step 2.* From $\mathrm{IG}(\omega) \neq \emptyset$, we fix a $0 < \theta_0 < \frac{\pi}{2}$ such that $\mathrm{IG}(\omega, \theta_0) \neq \emptyset$. We choose $\theta_{0.8}, \theta_1, \theta_{1.2}, \theta_{1.8}, \theta_2, \theta_{2.2}, \theta_{2.8}, \theta_3, \theta_{3.2}$ such that

$$0 < \theta_0 < \theta_{0.8} < \theta_1 < \theta_{1.2} < \theta_{1.8} < \theta_2 < \theta_{2.2} < \theta_{2.8} < \theta_3 < \theta_{3.2} < \frac{\pi}{2}. \tag{4.15}$$

For these $\theta$s, we associate the decomposition of $\gamma$ in step 1. Fix $(\tilde{\beta}_g) \in \mathrm{IG}(\omega, \theta_0)$ and $\left(\eta_g^\sigma\right) \in \mathcal{T}(\theta_0, (\tilde{\beta}_g))$. Set $\eta_g := \eta_g^L \otimes \eta_g^R$. Note that $\left(\eta_g^\sigma\right)$ also belongs to $\mathcal{T}(\theta_2, (\tilde{\beta}_g))$. Set

$$\begin{aligned}
\hat{\eta}_g^\sigma &:= \left(\gamma_{[0,\theta_1],\sigma} \gamma_{(\theta_{0.8},\theta_{1.2}],\sigma} \gamma_{H,\sigma}\right)^{-1} \eta_g^\sigma \left(\beta_g^{\sigma U} \gamma_{[0,\theta_1],\sigma} \gamma_{(\theta_{0.8},\theta_{1.2}],\sigma} \gamma_{H,\sigma} \left(\beta_g^{\sigma U}\right)^{-1}\right) \\
&\in \mathrm{Aut}\left(\mathcal{A}_{\left(C_{\theta_{1.2}}\right)_\sigma}\right),
\end{aligned} \tag{4.16}$$

for $\sigma = L, R$. We also set $\hat{\eta}_g := \hat{\eta}_g^L \otimes \hat{\eta}_g^R$. We claim that $(\gamma^{-1}\tilde{\beta}_g\gamma) \in \mathrm{IG}(\omega \circ \gamma, \theta_{1.2})$ with $(\hat{\eta}_g^\sigma) \in \mathcal{T}(\theta_{1.2}, (\gamma^{-1}\tilde{\beta}_g\gamma))$. Clearly we have

$$\omega \circ \gamma \circ \left(\gamma^{-1}\tilde{\beta}_g\gamma\right) = \omega \circ \tilde{\beta}_g \circ \gamma = \omega \circ \gamma. \tag{4.17}$$

Therefore, what remains to be shown is

$$\gamma^{-1}\tilde{\beta}_g\gamma = (\text{inner}) \circ \left(\hat{\eta}_g^L \otimes \hat{\eta}_g^R\right) \circ \beta_g^U. \tag{4.18}$$

To see this, we first have

$$
\begin{aligned}
\gamma^{-1} \circ \eta_g \circ \gamma = {} & (\text{inner}) \circ \gamma_0^{-1} \circ \left(\gamma_{(\theta_{0.8}, \theta_{1.2}]} \otimes \gamma_{(\theta_{1.8}, \theta_{2.2}]} \otimes \gamma_{(\theta_{2.8}, \theta_{3.2}]}\right)^{-1} \\
& \circ \left(\gamma_{[0, \theta_1]} \otimes \gamma_{(\theta_1, \theta_2]} \otimes \gamma_{(\theta_2, \theta_3]} \otimes \gamma_{(\theta_3, \frac{\pi}{2}]}\right)^{-1} \\
& \circ \eta_g \circ \left(\gamma_{[0, \theta_1]} \otimes \gamma_{(\theta_1, \theta_2]} \otimes \gamma_{(\theta_2, \theta_3]} \otimes \gamma_{(\theta_3, \frac{\pi}{2}]}\right) \circ \left(\gamma_{(\theta_{0.8}, \theta_{1.2}]} \otimes \gamma_{(\theta_{1.8}, \theta_{2.2}]} \otimes \gamma_{(\theta_{2.8}, \theta_{3.2}]}\right)\gamma_0 \quad (4.19)
\end{aligned}
$$

from the decomposition of equations (4.3), (4.4) and (4.5). Because $\gamma_{(\theta_1, \theta_2]} \otimes \gamma_{(\theta_2, \theta_3]} \otimes \gamma_{(\theta_3, \frac{\pi}{2}]}$ commutes with $\eta_g \in \mathrm{Aut}(\mathcal{A}_{C_{\theta_0}})$ and $\gamma_{(\theta_{1.8}, \theta_{2.2}]} \otimes \gamma_{(\theta_{2.8}, \theta_{3.2}]}$ commutes with $\left(\gamma_{[0, \theta_1]}\right)^{-1}\eta_g\gamma_{[0, \theta_1]} \in \mathrm{Aut}(\mathcal{A}_{C_{\theta_1}})$, we have

$$
\begin{aligned}
& \gamma^{-1} \circ \eta_g \circ \gamma \\
& = \text{equation (4.19)} = (\text{inner}) \circ \gamma_0^{-1} \circ \left(\gamma_{(\theta_{0.8}, \theta_{1.2}]} \otimes \gamma_{(\theta_{1.8}, \theta_{2.2}]} \otimes \gamma_{(\theta_{2.8}, \theta_{3.2}]}\right)^{-1} \\
& \quad \circ \left(\gamma_{[0, \theta_1]}\right)^{-1} \circ \eta_g \circ \left(\gamma_{[0, \theta_1]}\right) \circ \left(\gamma_{(\theta_{0.8}, \theta_{1.2}]} \otimes \gamma_{(\theta_{1.8}, \theta_{2.2}]} \otimes \gamma_{(\theta_{2.8}, \theta_{3.2}]}\right)\gamma_0 \\
& = (\text{inner}) \circ \gamma_0^{-1} \circ \left(\gamma_{(\theta_{0.8}, \theta_{1.2}]}\right)^{-1} \circ \left(\gamma_{[0, \theta_1]}\right)^{-1} \circ \eta_g \circ \left(\gamma_{[0, \theta_1]}\right) \circ \left(\gamma_{(\theta_{0.8}, \theta_{1.2}]}\right)\gamma_0. \tag{4.20}
\end{aligned}
$$

On the other hand, because $\gamma_{CS}$ and $\beta_g^U$ commute, we have

$$\gamma^{-1} \circ \beta_g^U \circ \gamma = (\text{inner})\gamma_0^{-1} \circ \gamma_{CS}^{-1}\beta_g^U\gamma_{CS}\gamma_0 = (\text{inner})\gamma_0^{-1} \circ \beta_g^U\gamma_0. \tag{4.21}$$

Combining equations (4.20) and (4.21), we obtain

$$
\begin{aligned}
\gamma^{-1}\tilde{\beta}_g\gamma & = (\text{inner}) \circ \gamma_0^{-1}\left(\gamma_{(\theta_{0.8}, \theta_{1.2}]}\right)^{-1} \circ \left(\gamma_{[0, \theta_1]}\right)^{-1} \circ \eta_g \circ \left(\gamma_{[0, \theta_1]}\right) \circ \left(\gamma_{(\theta_{0.8}, \theta_{1.2}]}\right)\gamma_0 \circ \gamma_0^{-1} \circ \beta_g^U\gamma_0 \\
& = (\text{inner}) \circ \gamma_0^{-1}\left(\gamma_{(\theta_{0.8}, \theta_{1.2}]}\right)^{-1} \circ \left(\gamma_{[0, \theta_1]}\right)^{-1} \circ \eta_g\beta_g^U \circ \left(\gamma_{[0, \theta_1]}\right) \circ \left(\gamma_{(\theta_{0.8}, \theta_{1.2}]}\right) \circ \gamma_0 \\
& = (\text{inner}) \circ \left(\hat{\eta}_g^L \otimes \hat{\eta}_g^R\right) \circ \beta_g^U. \tag{4.22}
\end{aligned}
$$

In the second equality, we used the fact that $\gamma_{[0, \theta_1]}\gamma_{(\theta_{0.8}, \theta_{1.2}]}$ and $\beta_g^U$ commute. This completes the proof of the claim.

*Step 3.* We use the setting and notation of steps 1 and 2 (with $\theta_0$ chosen in step 2). By Lemma 2.1, there exists

$$((W_g), (u_\sigma(g, h))) \in \mathrm{IP}\left(\omega, \alpha, \theta_2, (\tilde{\beta}_g), (\eta_g^\sigma), (\alpha_L, \alpha_R, \Theta)\right). \tag{4.23}$$

Now we have

$$
\omega \circ \gamma \in \mathcal{SL}, \qquad \alpha \circ \gamma \in \mathrm{EAut}(\omega \circ \gamma), \qquad \left( \gamma^{-1} \circ \tilde{\beta}_g \circ \gamma \right) \in \mathrm{IG}(\omega \circ \gamma, \theta_{1.2}),
$$
$$
\left( \hat{\eta}_g^\sigma \right) \in \mathcal{T}\left( \theta_{1.2}, \left( \gamma^{-1} \tilde{\beta}_g \gamma \right) \right), \qquad \left( \hat{\alpha}_L, \hat{\alpha}_R, \hat{\Theta} \right) \in \mathcal{D}_{\alpha\gamma}^{\theta_{1.2}}. \tag{4.24}
$$

We claim

$$
((W_g), (u_\sigma(g, h))) \in \mathrm{IP}\left( \omega \circ \gamma, \alpha \circ \gamma, \theta_{1.2}, \left( \gamma^{-1} \tilde{\beta}_g \gamma \right), \left( \hat{\eta}_g^\sigma \right), \left( \hat{\alpha}_L, \hat{\alpha}_R, \hat{\Theta} \right) \right). \tag{4.25}
$$

This immediately implies $h(\omega) = h(\omega \circ \gamma)$. To prove the claim, we first see from formulas (4.10) and (4.11) that

$$
\begin{aligned}
&(\hat{\alpha}_L \otimes \hat{\alpha}_R) \circ \hat{\Theta} \circ \gamma_0^{-1} \left( \gamma_{(\theta_{0.8}, \theta_{1.2}]} \right)^{-1} \circ \left( \gamma_{[0,\theta_1]} \right)^{-1} \\
&= \alpha_0 \circ \left( \gamma_{[0,\theta_1]} \otimes \gamma_{(\theta_1,\theta_2]} \right) \circ \gamma_{(\theta_{0.8},\theta_{1.2}]} \circ \gamma_0 \circ \Theta \circ \left( \gamma_{(\theta_2,\theta_3]} \otimes \gamma_{(\theta_3,\frac{\pi}{2}]} \right) \circ \left( \gamma_{(\theta_{1.8},\theta_{2.2}]} \otimes \gamma_{(\theta_{2.8},\theta_{3.2}]} \right) \\
&\quad \circ \gamma_0^{-1} \left( \gamma_{(\theta_{0.8},\theta_{1.2}]} \right)^{-1} \circ \left( \gamma_{[0,\theta_1]} \right)^{-1} \\
&= \alpha_0 \circ \left( \gamma_{[0,\theta_1]} \otimes \gamma_{(\theta_1,\theta_2]} \right) \circ \Theta \circ \left( \gamma_{(\theta_2,\theta_3]} \otimes \gamma_{(\theta_3,\frac{\pi}{2}]} \right) \circ \left( \gamma_{(\theta_{1.8},\theta_{2.2}]} \otimes \gamma_{(\theta_{2.8},\theta_{3.2}]} \right) \circ \left( \gamma_{[0,\theta_1]} \right)^{-1},
\end{aligned} \tag{4.26}
$$

because $\gamma_{(\theta_{0.8},\theta_{1.2}]} \circ \gamma_0 \in \mathrm{Aut}(\mathcal{A}_{C_{\theta_{1.2}}})$ and $\Theta \circ (\gamma_{(\theta_2,\theta_3]} \otimes \gamma_{(\theta_3,\frac{\pi}{2}]}) \circ (\gamma_{(\theta_{1.8},\theta_{2.2}]} \otimes \gamma_{(\theta_{2.8},\theta_{3.2}]}) \in \mathrm{Aut}(\mathcal{A}_{C_{\theta_{1.8}}{}^c})$ commute. Furthermore, because $\gamma_{[0,\theta_1]}$ and $\Theta \circ (\gamma_{(\theta_2,\theta_3]} \otimes \gamma_{(\theta_3,\frac{\pi}{2}]}) \circ (\gamma_{(\theta_{1.8},\theta_{2.2}]} \otimes \gamma_{(\theta_{2.8},\theta_{3.2}]}) \in \mathrm{Aut}(\mathcal{A}_{C_{\theta_{1.8}}{}^c})$ commute and $\gamma_{(\theta_1,\theta_2]}$ and $\Theta \in \mathrm{Aut}(\mathcal{A}_{C_{\theta_2}{}^c})$ commute, we have

$$
\begin{aligned}
&(\hat{\alpha}_L \otimes \hat{\alpha}_R) \circ \hat{\Theta} \circ \gamma_0^{-1} \left( \gamma_{(\theta_{0.8},\theta_{1.2}]} \right)^{-1} \circ \left( \gamma_{[0,\theta_1]} \right)^{-1} = \text{equation (4.26)} \\
&\quad = \alpha_0 \circ \gamma_{(\theta_1,\theta_2]} \circ \Theta \circ \left( \gamma_{(\theta_2,\theta_3]} \otimes \gamma_{(\theta_3,\frac{\pi}{2}]} \right) \circ \left( \gamma_{(\theta_{1.8},\theta_{2.2}]} \otimes \gamma_{(\theta_{2.8},\theta_{3.2}]} \right) \\
&\quad = \alpha_0 \circ \Theta \circ \gamma_{(\theta_1,\theta_2]} \circ \left( \gamma_{(\theta_2,\theta_3]} \otimes \gamma_{(\theta_3,\frac{\pi}{2}]} \right) \circ \left( \gamma_{(\theta_{1.8},\theta_{2.2}]} \otimes \gamma_{(\theta_{2.8},\theta_{3.2}]} \right) = \alpha_0 \circ \Theta \circ \hat{\gamma}.
\end{aligned} \tag{4.27}
$$

Here $\hat{\gamma} := \gamma_{(\theta_1,\theta_2]} \circ (\gamma_{(\theta_2,\theta_3]} \otimes \gamma_{(\theta_3,\frac{\pi}{2}]}) \circ (\gamma_{(\theta_{1.8},\theta_{2.2}]} \otimes \gamma_{(\theta_{2.8},\theta_{3.2}]}) \in \mathrm{Aut}(\mathcal{A}_{C_{\theta_1}{}^c})$ commutes with $\beta_g^U$. Combining this and

$$
\hat{\eta}_g \beta_g^U = \left( \gamma_{[0,\theta_1]} \gamma_{(\theta_{0.8},\theta_{1.2}]} \gamma_0 \right)^{-1} \eta_g \beta_g^U \gamma_{[0,\theta_1]} \gamma_{(\theta_{0.8},\theta_{1.2}]} \gamma_0, \tag{4.28}
$$

we obtain

$$
\pi_0 \circ (\hat{\alpha}_L \otimes \hat{\alpha}_R) \circ \hat{\Theta} \circ \hat{\eta}_g \beta_g^U (\hat{\Theta})^{-1} (\hat{\alpha}_L \otimes \hat{\alpha}_R)^{-1} = \pi_0 \circ \alpha_0 \circ \Theta \circ \hat{\gamma} \circ \eta_g \beta_g^U \circ \hat{\gamma}^{-1} \circ \Theta^{-1} \circ \alpha_0^{-1}. \tag{4.29}
$$

Because $\hat{\gamma}$ commutes with $\beta_g^U$ and $\eta_g \in \mathrm{Aut}(\mathcal{A}_{C_{\theta_0}})$ commutes with $\hat{\gamma} \in \mathrm{Aut}(\mathcal{A}_{C_{\theta_1}{}^c})$, we have

$$
\begin{aligned}
&\pi_0 \circ (\hat{\alpha}_L \otimes \hat{\alpha}_R) \circ \hat{\Theta} \circ \hat{\eta}_g \beta_g^U (\hat{\Theta})^{-1} (\hat{\alpha}_L \otimes \hat{\alpha}_R)^{-1} \\
&\quad = \text{equation (4.29)} = \pi_0 \circ \alpha_0 \circ \Theta \circ \eta_g \beta_g^U \circ \Theta^{-1} \circ \alpha_0^{-1} = \mathrm{Ad}(W_g) \circ \pi_0. \tag{4.30}
\end{aligned}
$$

Hence the condition for $W_g$ in formula (4.25) is checked. On the other hand, substituting formulas (4.11) and (4.16), we get

$$\pi_R \circ \hat{\alpha}_R \circ \hat{\eta}_g^R \beta_g^{RU} \hat{\eta}_h^R \left(\beta_g^{RU}\right)^{-1} \left(\hat{\eta}_{gh}^R\right)^{-1} \hat{\alpha}_R^{-1}$$

$$= \pi_R \circ \alpha_R \circ \left(\gamma_{[0,\theta_1],R} \otimes \gamma_{(\theta_1,\theta_2],R}\right) \circ \gamma_{(\theta_{0.8},\theta_{1.2}],R} \circ \gamma_{H,R} \circ \left(\gamma_{[0,\theta_1],R} \circ \gamma_{(\theta_{0.8},\theta_{1.2}],R} \circ \gamma_{H,R}\right)^{-1}$$

$$\eta_g^R \beta_g^{RU} \eta_h^R \left(\beta_g^{RU}\right)^{-1} \left(\eta_{gh}^R\right)^{-1} \circ \gamma_{[0,\theta_1],R} \circ \gamma_{(\theta_{0.8},\theta_{1.2}],R}$$

$$\circ \gamma_{H,R} \circ \left(\left(\gamma_{[0,\theta_1],R} \otimes \gamma_{(\theta_1,\theta_2],R}\right) \circ \gamma_{(\theta_{0.8},\theta_{1.2}],R} \circ \gamma_{H,R}\right)^{-1} \alpha_R^{-1}$$

$$= \pi_R \circ \alpha_R \circ \gamma_{(\theta_1,\theta_2],R} \circ \eta_g^R \beta_g^{RU} \eta_h^R \left(\beta_g^{RU}\right)^{-1} \left(\eta_{gh}^R\right)^{-1} \circ \left(\gamma_{(\theta_1,\theta_2],R}\right)^{-1} \circ \alpha_R^{-1}. \tag{4.31}$$

Because $\eta_g^R \beta_g^{RU} \eta_h^R (\beta_g^{RU})^{-1} (\eta_{gh}^R)^{-1} \in \mathrm{Aut}(\mathcal{A}_{C_{\theta_0}})$ commutes with $\gamma_{(\theta_1,\theta_2],R}$, we obtain

$$\pi_R \circ \hat{\alpha}_R \circ \hat{\eta}_g^R \beta_g^{RU} \hat{\eta}_h^R \left(\beta_g^{RU}\right)^{-1} \left(\hat{\eta}_{gh}^R\right)^{-1} \hat{\alpha}_R^{-1} = \text{equation (4.31)}$$

$$= \pi_R \circ \alpha_R \circ \eta_g^R \beta_g^{RU} \eta_h^R \left(\beta_g^{RU}\right)^{-1} \left(\eta_{gh}^R\right)^{-1} \alpha_R^{-1} = \mathrm{Ad}\left(u_R(g,h)\right) \circ \pi_R. \tag{4.32}$$

An analogous statement for $\sigma = L$ also holds. This completes the proof of formula (4.25). Hence the statement of the theorem is proven. □

## 5. Proof of Theorem 1.5

In this section, we prove Theorem 1.5. The proof relies heavily on the machinery of quasilocal automorphisms developed in [BMNS, NSY, MO]. (A summary is given in Appendix D.) We use terminology and facts from Appendixes C and D freely. We introduce a set of $F$-functions with fast decay, $\mathcal{F}_a$, as Definition C.2. A crucial point for us is the following:

**Theorem 5.1.** *Set* $\Phi_0, \Phi_1 \in \mathcal{P}_{UG}$ *and let* $\omega_{\Phi_0}, \omega_{\Phi_1}$ *be their unique gapped ground states. Suppose that* $\Phi_0 \sim \Phi_1$ *holds, via a path* $\Phi : [0,1] \to \mathcal{P}_{UG}$. *Then there exists some* $\Psi \in \hat{\mathcal{B}}_F([0,1])$ *with* $\Psi_1 \in \hat{\mathcal{B}}_F([0,1])$ *for some* $F \in \mathcal{F}_a$ *of the form* $F(r) = \frac{\exp(-r^\theta)}{(1+r)^4}$ *with a constant* $0 < \theta < 1$, *such that* $\omega_{\Phi_1} = \omega_{\Phi_0} \circ \tau_{1,0}^\Psi$. *If* $\Phi_0, \Phi_1 \in \mathcal{P}_{UG\beta}$ *and* $\Phi \sim_\beta \Phi_0$, *we may take* $\Psi$ *to be* $\beta$-*invariant.*

For the proof, see Appendix D.

From this and Theorems 3.1 and 4.1, in order to show Theorem 1.5 it suffices to show the following, which says that the automorphism $\tau_{1,0}^\Psi$ in Theorem 5.1 satisfies all the good factorisation properties which we assumed in previous sections:

**Theorem 5.2.** *Let* $F \in \mathcal{F}_a$ *be an F-function of the form* $F(r) = \frac{\exp(-r^\theta)}{(1+r)^4}$ *with a constant* $0 < \theta < 1$. *Let* $\Psi \in \hat{\mathcal{B}}_F([0,1])$ *be a path of interactions satisfying* $\Psi_1 \in \hat{\mathcal{B}}_F([0,1])$. *Then we have* $\tau_{1,0}^\Psi \in \mathrm{SQAut}(\mathcal{A})$. *Furthermore, if* $\Psi$ *is* $\beta_g^U$-*invariant – that is,* $\beta_g^U\left(\Psi(X;t)\right) = \Psi(X;t)$ *for any* $X \in \mathfrak{S}_{\mathbb{Z}^2}$, $t \in [0,1]$ *and* $g \in G$ – *then we have* $\tau_{1,0}^\Psi \in \mathrm{GSQAut}(\mathcal{A})$.

*Proof.* Fix arbitrary

$$0 < \theta_{0.8} < \theta_1 < \theta_{1.2} < \theta_{1.8} < \theta_2 < \theta_{2.2} < \theta_{2.8} < \theta_3 < \theta_{3.2} < \frac{\pi}{2}. \tag{5.1}$$

We show the existence of the decomposition

$$\tau_{1,0}^\Psi = \mathrm{Ad}(u) \circ \left(\alpha_{(0,\theta_1]} \otimes \alpha_{(\theta_1,\theta_2]} \otimes \alpha_{(\theta_2,\theta_3]} \otimes \alpha_{(\theta_3,\frac{\pi}{2}]}\right)$$
$$\circ \left(\alpha_{(\theta_{0.8},\theta_{1.2}]} \alpha_{(\theta_{1.8},\theta_{2.2}]} \otimes \alpha_{(\theta_{2.8},\theta_{3.2}]}\right), \tag{5.2}$$

with $\alpha$s of the forms in formulas (2.13) and (2.14). We follow the strategy of [NO].

*Step 1.* Fix some $0 < \theta' < \theta$ and set

$$\tilde{F}(r) := \frac{\exp\left(-r^{\theta'}\right)}{(1+r)^4}. \tag{5.3}$$

With a suitably chosen constant $c_1 > 0$, we have

$$\max\left\{F\left(\frac{r}{3}\right), \left(F\left(\left[\frac{r}{3}\right]\right)\right)^{\frac{1}{2}}\right\} \le c_1 \tilde{F}(r), \quad r \ge 0. \tag{5.4}$$

Namely, $c_1 \tilde{F}(r)$ satisfy the condition on $\tilde{F}_\theta$ in Definition C.2(ii) for our $F = \frac{\exp\left(-cr^\theta\right)}{(1+r)^4}$ and $\theta = \frac{1}{2}$. Set

$$\mathcal{C}_0 := \left\{ \begin{matrix} C_{[0,\theta_1],\sigma}, \ C_{(\theta_1,\theta_2],\sigma,\zeta}, \ C_{(\theta_2,\theta_3],\sigma,\zeta}, \ C_{\left(\theta_3,\frac{\pi}{2}\right],\zeta}, \\ \sigma = L, R, \ \zeta = D, U \end{matrix} \right\}, \tag{5.5}$$

$$\mathcal{C}_1 := \left\{ \begin{matrix} C_{(\theta_{0.8},\theta_{1.2}),\sigma,\zeta}, \ C_{(\theta_{1.8},\theta_{2.2}),\sigma,\zeta}, \ C_{(\theta_{2.8},\theta_{3.2}),\sigma,\zeta}, \\ \sigma = L, R, \ \zeta = D, U \end{matrix} \right\}. \tag{5.6}$$

Define $\Psi^{(0)}, \Psi^{(1)} \in \hat{\mathcal{B}}_F([0,1])$ by

$$\Psi^{(0)}(X;t) := \begin{cases} \Psi(X;t) & \text{if there exists a } C \in \mathcal{C}_0 \text{ such that } X \subset C, \\ 0 & \text{otherwise}, \end{cases} \tag{5.7}$$

$$\Psi^{(1)}(X;t) := \Psi^{(0)}(X;t) - \Psi(X;t),$$

for each $X \in \mathfrak{S}_{\mathbb{Z}^2}$, $t \in [0,1]$.

First we would like to represent $(\tau_{1,0}^{\Psi^{(0)}})^{-1} \circ \tau_{1,0}^\Psi$ as some quasilocal automorphism. Set $t, s \in [0,1]$. We apply Proposition D.6 for $\Psi$ replaced by $\Psi^{(1)}$ and $\tilde{\Psi}$ by $\Psi$. Hence we set

$$\Xi^{(s)}(Z,t) := \sum_{m \ge 0} \sum_{X \subset Z, \ X(m)=Z} \Delta_{X(m)}\left(\tau_{t,s}^\Psi\left(\Psi^{(1)}(X;t)\right)\right) \tag{5.8}$$

and

$$\Xi^{(n)(s)}(Z,t) := \sum_{m \ge 0} \sum_{X \subset Z, \ X(m)\cap\Lambda_n=Z} \Delta_{X(m)}\left(\tau_{t,s}^{(\Lambda_n)\Psi}\left(\Psi^{(1)}(X;t)\right)\right). \tag{5.9}$$

Corresponding to equation (D.31), we obtain

$$\tau_{t,s}^{(\Lambda_n),\Psi}\left(H_{\Lambda_n,\Psi^{(1)}}(t)\right) = H_{\Lambda_n,\Xi^{(n)(s)}}(t). \tag{5.10}$$

Applying Proposition D.6. we have $\Xi^{(n)(s)}, \Xi^{(s)} \in \hat{\mathcal{B}}_{\tilde{F}}([0,1])$, and

$$\lim_{n\to\infty} \left\| \tau_{t,u}^{\Xi^{(n)(s)}}(A) - \tau_{t,u}^{\Xi^{(s)}}(A) \right\| = 0, \quad A \in \mathcal{A}, \ t, u \in [0,1], \tag{5.11}$$

holds. Two functions $\hat{\tau}_{t,s}^{(\Lambda_n),\Xi^{(n)(s)}}(A)$ and $\tau_{t,s}^{(\Lambda_n),\Psi} \circ (\tau_{t,s}^{(\Lambda_n),\Psi^{(0)}})^{-1}(A)$ satisfy the same differential equation and initial condition. Therefore we obtain

$$\hat{\tau}_{t,s}^{(\Lambda_n),\Xi^{(n)(s)}}(A) = \tau_{t,s}^{(\Lambda_n),\Psi} \circ \left(\tau_{t,s}^{(\Lambda_n),\Psi^{(0)}}\right)^{-1}(A), \quad t \in [0,1], \ A \in \mathcal{A}. \tag{5.12}$$

From the fact that $\hat{\tau}_{t,u}^{(\Lambda_n),\Xi^{(n)(s)}} = \tau_{u,t}^{(\Lambda_n),\Xi^{(n)(s)}} = \tau_{u,t}^{\Xi^{(n)(s)}}$ converges strongly to an automorphism $\tau_{u,t}^{\Xi^{(s)}}$ on $\mathcal{A}$ (equation (5.11)), we have

$$\lim_{n\to\infty} \left\| \hat{\tau}_{t,s}^{(\Lambda_n)\Xi^{(n)(s)}}(A) - \tau_{s,t}^{\Xi^{(s)}}(A) \right\| = 0, \quad A \in \mathcal{A}. \tag{5.13}$$

On the other hand, by Theorem D.3 we have, for $t \in [0,1]$ and $A \in \mathcal{A}$,

$$\lim_{n\to\infty} \left\| \tau_{t,s}^{(\Lambda_n),\Psi} \circ \left( \tau_{t,s}^{(\Lambda_n),\Psi^{(0)}} \right)^{-1}(A) - \tau_{t,s}^{\Psi} \circ \left( \tau_{t,s}^{\Psi^{(0)}} \right)^{-1}(A) \right\| = 0. \tag{5.14}$$

Therefore, taking the $n \to \infty$ limit in equation (5.12), we obtain

$$\tau_{s,t}^{\Xi^{(s)}}(A) = \tau_{t,s}^{\Psi} \circ \left( \tau_{t,s}^{\Psi^{(0)}} \right)^{-1}(A), \quad t,s \in [0,1], \ A \in \mathcal{A}. \tag{5.15}$$

Hence we have

$$\tau_{s,t}^{\Psi} = \left( \tau_{t,s}^{\Psi} \right)^{-1} = \left( \tau_{t,s}^{\Psi^{(0)}} \right)^{-1} \left( \tau_{s,t}^{\Xi^{(s)}} \right)^{-1} = \tau_{s,t}^{\Psi^{(0)}} \tau_{t,s}^{\Xi^{(s)}}. \tag{5.16}$$

In particular, we get

$$\tau_{1,0}^{\Psi} = \tau_{1,0}^{\Psi^{(0)}} \tau_{0,1}^{\Xi^{(1)}}. \tag{5.17}$$

*Step 2.* We show

$$\sum_{\substack{Z \in \mathfrak{S}(\mathbb{Z}^2), \\ \nexists C \in \mathcal{C}_1 \text{ s.t. } Z \subset C}} \sup_{t \in [0,1]} \left\| \Xi^{(1)}(Z,t) \right\| < \infty. \tag{5.18}$$

From this,

$$V(t) := \sum_{\substack{Z \in \mathfrak{S}(\mathbb{Z}^2), \\ \nexists C \in \mathcal{C}_1 \text{ s.t. } Z \subset C}} \Xi^{(1)}(Z,t) \in \mathcal{A} \tag{5.19}$$

converges absolutely in the norm topology and defines an element in $\mathcal{A}$. Furthermore, for

$$V_n(t) := \sum_{\substack{Z \in \mathfrak{S}(\mathbb{Z}^2), Z \subset \Lambda_n \\ \nexists C \in \mathcal{C}_1 \text{ s.t. } Z \subset C}} \Xi^{(1)}(Z,t) \in \mathcal{A}_{\Lambda_n}, \quad n \in \mathbb{N}, \tag{5.20}$$

we get

$$\lim_{n\to\infty} \sup_{t \in [0,1]} \| V_n(t) - V(t) \| = 0 \tag{5.21}$$

from formula (5.18).

To prove formula (5.18), we first bound

$$
\sum_{\substack{Z \in \mathfrak{S}(\mathbb{Z}^2) \\ \nexists C \in \mathcal{C}_1 \text{ s.t. } Z \subset C}} \sup_{t \in [0,1]} \left\| \Xi^{(1)}(Z,t) \right\|
$$

$$
\leq \sum_{\substack{Z \in \mathfrak{S}(\mathbb{Z}^2), \\ \nexists C \in \mathcal{C}_1 \text{ s.t. } Z \subset C}} \sum_{m \geq 0} \sum_{\substack{X : X \subset Z \\ X(m)=Z}} \left[ \sup_{t \in [0,1]} \left\| \Delta_{X(m)} \left( \tau_{t,1}^{\Psi} \left( \Psi^{(1)}(X;t) \right) \right) \right\| \right]
$$

$$
\leq \sum_{m \geq 0} \sum_{\substack{X : \\ \nexists C \in \mathcal{C}_1 \text{ s.t. } X(m) \subset C}} \sup_{t \in [0,1]} \left\| \Delta_{X(m)} \left( \tau_{t,1}^{\Psi} \left( \Psi^{(1)}(X;t) \right) \right) \right\| \tag{5.22}
$$

$$
\leq \sum_{m \geq 0} \sum_{\substack{X : \\ \nexists C \in \mathcal{C}_1 \text{ s.t. } X(m) \subset C}} \left[ \sup_{t \in [0,1]} \frac{8 \left\| \Psi^{(1)}(X;t) \right\|}{C_F} \left( e^{2 I_F(\Psi)} - 1 \right) |X| \, G_F(m) \right]
$$

$$
= \frac{8}{C_F} \left( e^{2 I_F(\Psi)} - 1 \right) \sum_{m \geq 0} \sum_{\substack{X : \\ \nexists C \in \mathcal{C}_1 \text{ s.t. } X(m) \subset C}} \left[ \sup_{t \in [0,1]} \left( \left\| \Psi^{(1)}(X;t) \right\| \right) |X| \, G_F(m) \right].
$$

For the third inequality, we used Theorem D.3 3. For any cone $C_1, C_2$ of $\mathbb{Z}^2$ with its apex at the origin, we set

$$
M(C_1, C_2) := \sum_{m \geq 0} \sum_{\substack{X : \\ \forall C \in \mathcal{C}_1, \, X \cap ((C^c)(m)) \neq \emptyset, \\ X \cap C_1 \neq \emptyset, \, X \cap C_2 \neq \emptyset}} \left[ \sup_{t \in [0,1]} \left( \left\| \Psi^{(1)}(X;t) \right\| \right) |X| \, G_F(m) \right]. \tag{5.23}
$$

From the definition of $\Psi^{(1)}$, we have $\Psi^{(1)}(X;t) = 0$, unless $X$ has a nonempty intersection with at least two elements in $\mathcal{C}_0$. Therefore, if $X$ gives a nonzero contribution in formula (5.22), then it has to satisfy

$$
X \cap ((C^c)(m)) \neq \emptyset, \quad \text{for all } C \in \mathcal{C}_1,
$$
$$
\exists C_1, C_2 \in \mathcal{C}_0 \text{ such that } C_1 \neq C_2, \, X \cap C_1 \neq \emptyset, \, X \cap C_2 \neq \emptyset.
$$

Hence we have

$$
\text{formula (5.22)} \leq \frac{8}{C_F} \left( e^{2 I_F(\Psi)} - 1 \right) \sum_{\substack{C_1, C_2 \in \mathcal{C}_0 \\ C_1 \neq C_2}} M(C_1, C_2). \tag{5.24}
$$

Hence it suffices to show that $M(C_1, C_2) < \infty$ for all $C_1, C_2 \in \mathcal{C}_0$ with $C_1 \neq C_2$.

In order to proceed, we prepare two estimates. We will freely identify $\mathbb{C}$ and $\mathbb{R}^2$ in an obvious manner. In particular, $\arg z$ of $z \in \mathbb{Z}^2 \subset \mathbb{R}^2$ in the following definition is considered with this identification: For $\varphi_1 < \varphi_2$, we set

$$
\check{C}_{[\varphi_1, \varphi_2]} := \left\{ z \in \mathbb{Z}^2 \mid \arg z \in [\varphi_1, \varphi_2] \right\}. \tag{5.25}
$$

We define $\check{C}_{(\varphi_1, \varphi_2)}$ and so on analogously. Set

$$
c^{(0)}_{\zeta_1, \zeta_2, \zeta_3, \zeta_4} := \sqrt{1 - \max \left\{ \cos(\zeta_3 - \zeta_2), \cos(\zeta_4 - \zeta_1), 0 \right\}}, \quad \zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \mathbb{R}. \tag{5.26}
$$

**Lemma 5.3.** *Set* $\varphi_1 < \varphi_2 < \varphi_3 < \varphi_4$ *with* $\varphi_4 - \varphi_1 < 2\pi$. *Then*

$$b_0(\varphi_1, \varphi_2, \varphi_3, \varphi_4) := \sum_{m \geq 0} \sum_{\substack{X: \\ X \cap \check{C}_{[\varphi_1, \varphi_2]} \neq \emptyset \\ X \cap \check{C}_{[\varphi_3, \varphi_4]} \neq \emptyset}} \left[ \sup_{t \in [0,1]} \left( \|\Psi(X; t)\| \right) |X| \, G_F(m) \right]$$

$$\leq (64)^3 \frac{3^4 \kappa_{1,4,F}}{\left( c^{(0)}_{\varphi_1, \varphi_2, \varphi_3, \varphi_4} \right)^4} \left( \||\Psi_1|\|_F \right) \left( \sum_{m \geq 0} G_F(m) \right) < \infty.$$

*Proof.* Substituting Lemma C.4, we obtain

$$b_0(\varphi_1, \varphi_2, \varphi_3, \varphi_4) := \sum_{m \geq 0} \sum_{\substack{X: \\ X \cap \check{C}_{[\varphi_1, \varphi_2]} \neq \emptyset \\ X \cap \check{C}_{[\varphi_3, \varphi_4]} \neq \emptyset}} \left[ \sup_{t \in [0,1]} \left( \|\Psi(X; t)\| \right) |X| \, G_F(m) \right]$$

$$\leq \sum_{m \geq 0} \sum_{\substack{x \in \check{C}_{[\varphi_1, \varphi_2]} \\ y \in \check{C}_{[\varphi_3, \varphi_4]}}} \sum_{X \ni x, y} \left[ \sup_{t \in [0,1]} \left( \|\Psi(X; t)\| \right) |X| \, G_F(m) \right]$$

$$\leq \left( \||\Psi_1|\|_F \right) \sum_{\substack{x \in \check{C}_{[\varphi_1, \varphi_2]} \\ y \in \check{C}_{[\varphi_3, \varphi_4]}}} F\left( d(x, y) \right) \left( \sum_{m \geq 0} G_F(m) \right)$$

$$\leq (64)^3 \frac{3^4 \kappa_{1,4,F}}{\left( c^{(0)}_{\varphi_1, \varphi_2, \varphi_3, \varphi_4} \right)^4} \left( \||\Psi_1|\|_F \right) \left( \sum_{m \geq 0} G_F(m) \right) < \infty. \tag{5.27}$$

We used Lemma C.4 in the last inequality. The last value is finite by equation (C.14) for our $F \in \mathcal{F}_a$. □

Set

$$c^{(1)}_{\zeta_1, \zeta_2, \zeta_3} := \sqrt{1 - \max\{\cos(\zeta_1 - \zeta_2), \cos(\zeta_1 - \zeta_3)\}}, \quad \zeta_1, \zeta_2, \zeta_3 \in [0, 2\pi). \tag{5.28}$$

**Lemma 5.4.** *For* $\varphi_1 < \varphi_2 < \varphi_3$ *with* $\varphi_3 - \varphi_1 < \frac{\pi}{2}$, *we have*

$$b_1(\varphi_1, \varphi_2, \varphi_3) := \sum_{m \geq 0} \sum_{\substack{X: \\ X \subset \check{C}_{[\varphi_1, \varphi_3]} \\ X \cap \check{C}_{[\varphi_1, \varphi_2]} \neq \emptyset \\ X \cap \check{C}_{[\varphi_2, \varphi_3]} \neq \emptyset \\ X \cap \left( \left( \left( \check{C}_{(\varphi_1, \varphi_3)} \right)^c \right)(m) \right) \neq \emptyset}} \left[ \sup_{t \in [0,1]} \left( \|\Psi(X; t)\| \right) |X| \, G_F(m) \right] \tag{5.29}$$

$$\leq 64 \cdot 144 \cdot 24 \cdot \left( \pi \kappa_{1,2,F} + F(0) \right) \left( \||\Psi_1|\|_F \right) \left( \sum_{m \geq 0} (m+1)^4 G_F(m) \right)$$

$$\left( \left( c^{(1)}_{\varphi_1, \varphi_2, \varphi_3} \right)^{-4} + \left( c^{(1)}_{\varphi_3, \varphi_1, \varphi_2} \right)^{-4} \right) < \infty.$$

*Proof.* Set

$$L_\varphi := \left\{ z \in \mathbb{R}^2 \mid \arg z = \varphi \right\}, \quad \varphi \in [0, 2\pi). \tag{5.30}$$

Note that if $X \in \mathfrak{S}_{\mathbb{Z}^2}$ satisfies $X \subset \check{C}_{[\varphi_1, \varphi_3]}$ and $X \cap (((\check{C}_{(\varphi_1, \varphi_3)})^c)(m)) \neq \emptyset$, then we have

$$d(X, L_{\varphi_1}) \leq m \quad \text{or} \quad d(X, L_{\varphi_3}) \leq m. \tag{5.31}$$

Therefore, we have

$$\sum_{m \geq 0} \sum_{\substack{X: \\ X \subset \check{C}_{[\varphi_1, \varphi_3]} \\ X \cap \check{C}_{[\varphi_1, \varphi_2]} \neq \emptyset \\ X \cap \check{C}_{[\varphi_2, \varphi_3]} \neq \emptyset \\ X \cap \left( \left( \left( \check{C}_{(\varphi_1, \varphi_3)} \right)^c \right)(m) \right) \neq \emptyset}} \left[ \sup_{t \in [0,1]} (\|\Psi(X; t)\|) \, |X| \, G_F(m) \right]$$

$$\leq \sum_{m \geq 0} G_F(m) \left( \sum_{\substack{X: \\ X \cap \check{C}_{[\varphi_2, \varphi_3]} \neq \emptyset \\ d(X, L_{\varphi_1}) \leq m}} + \sum_{\substack{X: \\ X \cap \check{C}_{[\varphi_1, \varphi_2]} \neq \emptyset \\ d(X, L_{\varphi_3}) \leq m}} \right) \left[ \sup_{t \in [0,1]} (\|\Psi(X; t)\|) \, |X| \right]$$

$$\leq \sum_{m \geq 0} G_F(m) \left( \sum_{\substack{x \in \check{C}_{[\varphi_2, \varphi_3]} \\ y \in L_{\varphi_1}(m)}} + \sum_{\substack{x \in \check{C}_{[\varphi_1, \varphi_2]} \\ y \in L_{\varphi_3}(m)}} \right) \sum_{X: X \ni x, y} \left[ \sup_{t \in [0,1]} (\|\Psi(X; t)\|) \, |X| \right]$$

$$\leq (\|\|\Psi_1\|\|_F) \sum_{m \geq 0} G_F(m) \left( \sum_{\substack{x \in \check{C}_{[\varphi_2, \varphi_3]} \\ y \in L_{\varphi_1}(m)}} + \sum_{\substack{x \in \check{C}_{[\varphi_1, \varphi_2]} \\ y \in L_{\varphi_3}(m)}} \right) F(d(x, y))$$

$$\leq 64 \cdot 144 \cdot 24 \cdot \left( \pi \kappa_{1,2,F} + F(0) \right) (\|\|\Psi_1\|\|_F)$$
$$\left( \sum_{m \geq 0} (m+1)^4 G_F(m) \right) \left( \left( c^{(1)}_{\varphi_1, \varphi_2, \varphi_3} \right)^{-4} + \left( c^{(1)}_{\varphi_3, \varphi_1, \varphi_2} \right)^{-4} \right). \tag{5.32}$$

In the last inequality, we used Lemma C.5 with $\varphi_3 - \varphi_1 < \frac{\pi}{2}$. Because $\varphi_3 - \varphi_1 < \frac{\pi}{2}$ and because of formula (C.14), the last value is finite. $\quad \square$

Now let us go back to the estimate of formula (5.23). If $C_1, C_2 \in \mathcal{C}_0$ are $C_1 = \check{C}_{[\varphi_1, \varphi_2]}, C_2 = \check{C}_{[\varphi_3, \varphi_4]}$ with $\varphi_1 < \varphi_2 < \varphi_3 < \varphi_4, \varphi_4 - \varphi_1 < 2\pi$, then from Lemma 5.3, we have

$$M(C_1, C_2) \leq b_0(\varphi_1, \varphi_2, \varphi_3, \varphi_4) < \infty. \tag{5.33}$$

Now suppose that $C_1, C_2 \in \mathcal{C}_0$ are $C_1 = \check{C}_{[\varphi_1, \varphi_2]}, C_2 = \check{C}_{[\varphi_2, \varphi_3]}$ with $\varphi_1 < \varphi_2 < \varphi_3, \varphi_3 - \varphi_1 < 2\pi$. (Recall definition (5.5).) By the definition of $\mathcal{C}_0$ and $\mathcal{C}_1$, there is some $C = C_{(\zeta_1, \zeta_2)} \in \mathcal{C}_1$ such that

$\varphi_1 < \zeta_1 < \varphi_2 < \zeta_2 < \varphi_3$ and $\zeta_2 - \zeta_1 < \frac{\pi}{2}$. For $X \in \mathfrak{S}_{\mathbb{Z}^2}$ to give a nonzero contribution in formula (5.23), it has to satisfy

$$X(m) \cap (\check{C}_{[\zeta_1,\zeta_2]})^c \neq \emptyset, \qquad X \cap \check{C}_{[\varphi_1,\varphi_2]} \neq \emptyset, \qquad X \cap \check{C}_{[\varphi_2,\varphi_3]} \neq \emptyset. \tag{5.34}$$

For such an $X$, one of the following occurs:

(i) $X \cap \check{C}_{[\zeta_2,\varphi_3]} \neq \emptyset$ and $X \cap \check{C}_{[\varphi_1,\varphi_2]} \neq \emptyset$.
(ii) $X \cap \check{C}_{[\varphi_1,\zeta_1]} \neq \emptyset$ and $X \cap \check{C}_{[\varphi_2,\varphi_3]} \neq \emptyset$.
(iii) $X \cap \check{C}_{[\varphi_2,\zeta_2]} \neq \emptyset$ (and $X \cap \check{C}_{[\zeta_1,\varphi_2]} \neq \emptyset$) and $X \cap \check{C}_{[\varphi_3,\varphi_1+2\pi]} \neq \emptyset$.
(iv) $X \subset \check{C}_{\zeta_1,\zeta_2}$, $X \cap ((\check{C}_{\zeta_1,\zeta_2})^c)(m) \neq \emptyset$, $X \cap \check{C}_{[\varphi_2,\zeta_2]} \neq \emptyset$ and $X \cap \check{C}_{[\zeta_1,\varphi_2]} \neq \emptyset$.

Hence we get

$$M(C_1, C_2) \leq b_0(\varphi_1, \varphi_2, \zeta_2, \varphi_3) + b_0(\varphi_1, \zeta_1, \varphi_2, \varphi_3) + b_0(\varphi_2, \zeta_2, \varphi_3, \varphi_1 + 2\pi) + b_1(\zeta_1, \varphi_2, \zeta_2)$$
$$< \infty. \tag{5.35}$$

Hence we have proven the claim of step 2.

*Step 3.* Next we set

$$\tilde{\Xi}(Z, t) := \begin{cases} \Xi^{(1)}(Z, t) & \text{if } \exists C \in C_1 \text{ s.t. } Z \subset C, \\ 0 & \text{otherwise.} \end{cases} \tag{5.36}$$

Clearly, we have $\tilde{\Xi} \in \hat{\mathcal{B}}_{\bar{F}}([0, 1])$. Note that

$$H_{\Lambda_n, \tilde{\Xi}}(t) + V_n(t) = H_{\Lambda_n, \Xi^{(1)}}(t). \tag{5.37}$$

As a uniform limit of $[0, 1] \ni t \mapsto V_n(t) \in \mathcal{A}$ (equation (5.21)), $[0, 1] \ni t \mapsto V(t) \in \mathcal{A}$ is norm-continuous. Because $\tilde{\Xi} \in \hat{\mathcal{B}}_{\bar{F}}([0, 1])$, $[0, 1] \ni t \mapsto \tau_{t,s}^{\tilde{\Xi}}(V(t)) \in \mathcal{A}$ is also norm-continuous, for each $s \in [0, 1]$. Therefore, for each $s \in [0, 1]$, there is a unique norm-differentiable map $[0, 1] \ni t \mapsto W^{(s)}(t) \in \mathcal{U}(\mathcal{A})$ such that

$$\frac{d}{dt} W^{(s)}(t) = -i\tau_{t,s}^{\tilde{\Xi}}(V(t)) W^{(s)}(t), \quad W^{(s)}(s) = \mathbb{I}. \tag{5.38}$$

It is given by

$$W^{(s)}(t) := \sum_{k=0}^{\infty} (-i)^k \int_s^t ds_1 \int_s^{s_1} ds_2 \cdots \int_s^{s_{k-1}} ds_k \tau_{s_1,s}^{\tilde{\Xi}}(V(s_1)) \cdots \tau_{s_k,s}^{\tilde{\Xi}}(V(s_k)). \tag{5.39}$$

Analogously, for each $s \in [0, 1]$ and $n \in \mathbb{N}$, we define a unique norm-differentiable map from $[0, 1]$ to $\mathcal{U}(\mathcal{A})$ such that

$$\frac{d}{dt} W_n^{(s)}(t) = -i\tau_{t,s}^{(\Lambda_n)\tilde{\Xi}}(V_n(t)) W_n^{(s)}(t), \quad W_n^{(s)}(s) = \mathbb{I}. \tag{5.40}$$

It is given by

$$W_n^{(s)}(t) := \sum_{k=0}^{\infty} (-i)^k \int_s^t ds_1 \int_s^{s_1} ds_2 \cdots \int_s^{s_{k-1}} ds_k \tau_{s_1,s}^{(\Lambda_n)\tilde{\Xi}}(V_n(s_1)) \cdots \tau_{s_k,s}^{(\Lambda_n)\tilde{\Xi}}(V_n(s_k)). \tag{5.41}$$

By the uniform convergence (5.21) and Lemma D.3, we have

$$\lim_{n \to \infty} \sup_{t \in [0,1]} \left\| \tau_{t,s}^{(\Lambda_n)\tilde{\Xi}}(V_n(t)) - \tau_{t,s}^{\tilde{\Xi}}(V(t)) \right\| = 0. \tag{5.42}$$

From this and formulas (5.39) and (5.41), we obtain

$$\lim_{n \to \infty} \sup_{t \in [0,1]} \left\| W_n^{(s)}(t) - W^{(s)}(t) \right\| = 0. \tag{5.43}$$

This and Theorem D.3 4 for $\Xi^{(1)}, \tilde{\Xi} \in \mathcal{B}_{\tilde{F}}([0,1])$ imply

$$\lim_{n \to \infty} \tau_{s,t}^{(\Lambda_n),\tilde{\Xi}} \circ \mathrm{Ad}\left( W_n^{(s)}(t) \right)(A) = \tau_{s,t}^{\tilde{\Xi}} \circ \mathrm{Ad}\left( W^{(s)}(t) \right)(A),$$
$$\lim_{n \to \infty} \tau_{s,t}^{(\Lambda_n),\Xi^{(1)}}(A) = \tau_{s,t}^{\Xi^{(1)}}(A), \tag{5.44}$$

for any $A \in \mathcal{A}$.

Note that for any $A \in \mathcal{A}$,

$$\frac{d}{dt} \tau_{s,t}^{(\Lambda_n),\tilde{\Xi}} \circ \mathrm{Ad}\left( W_n^{(s)}(t) \right)(A) = -i \left[ H_{\Lambda_n,\tilde{\Xi}}(t), \tau_{s,t}^{(\Lambda_n),\tilde{\Xi}} \circ \mathrm{Ad}\left( W_n^{(s)}(t) \right)(A) \right]$$
$$- i \tau_{s,t}^{(\Lambda_n),\tilde{\Xi}} \left( \left[ \tau_{t,s}^{(\Lambda_n),\tilde{\Xi}}(V_n(t)), \mathrm{Ad}\left( W_n^{(s)}(t) \right)(A) \right] \right)$$
$$= -i \left[ H_{\Lambda_n,\tilde{\Xi}}(t) + V_n(t), \tau_{s,t}^{(\Lambda_n),\tilde{\Xi}} \circ \mathrm{Ad}\left( W_n^{(s)}(t) \right)(A) \right]$$
$$= -i \left[ H_{\Lambda_n,\Xi^{(1)}}(t), \tau_{s,t}^{(\Lambda_n),\tilde{\Xi}} \circ \mathrm{Ad}\left( W_n^{(s)}(t) \right)(A) \right].$$

We used equation (D.10) for the second equality and equation (5.37) for the third. On the other hand, for any $A \in \mathcal{A}$, we have

$$\frac{d}{dt} \tau_{s,t}^{(\Lambda_n),\Xi^{(1)}}(A) = -i \left[ H_{\Lambda_n,\Xi^{(1)}}(t), \tau_{s,t}^{(\Lambda_n),\Xi^{(1)}}(A) \right]. \tag{5.45}$$

Therefore, $\tau_{s,t}^{(\Lambda_n),\tilde{\Xi}} \circ \mathrm{Ad}(W_n^{(s)}(t))(A)$ and $\tau_{s,t}^{(\Lambda_n),\Xi^{(1)}}(A)$ satisfy the same differential equation. Also note that we have $\tau_{s,s}^{(\Lambda_n),\tilde{\Xi}} \circ \mathrm{Ad}(W_n^{(s)}(s))(A) = \tau_{s,s}^{(\Lambda_n),\Xi^{(1)}}(A) = A$. Therefore, we get

$$\tau_{s,t}^{(\Lambda_n),\tilde{\Xi}} \circ \mathrm{Ad}\left( W_n^{(s)}(t) \right)(A) = \tau_{s,t}^{(\Lambda_n),\Xi^{(1)}}(A). \tag{5.46}$$

By equation (5.44), we obtain

$$\tau_{s,t}^{\tilde{\Xi}} \circ \mathrm{Ad}\left( W^{(s)}(t) \right)(A) = \tau_{s,t}^{\Xi^{(1)}}(A), \quad A \in \mathcal{A}, \ t, s \in [0,1]. \tag{5.47}$$

Taking the inverse, we get

$$\mathrm{Ad}\left( W^{(s)*}(t) \right) \circ \tau_{t,s}^{\tilde{\Xi}} = \tau_{t,s}^{\Xi^{(1)}}, \quad t, s \in [0,1]. \tag{5.48}$$

*Step 4.* Combining equations (5.17) and (5.48), we have

$$\tau_{1,0}^{\Psi} = \tau_{1,0}^{\Psi^{(0)}} \tau_{0,1}^{\Xi^{(1)}} = \tau_{1,0}^{\Psi^{(0)}} \circ \mathrm{Ad}\left( \left( W^{(1)}(0) \right)^* \right) \circ \tau_{0,1}^{\tilde{\Xi}}. \tag{5.49}$$

By the definitions of $\Psi^{(0)}$ and $\tilde{\Xi}$, we obtain decompositions

$$\tau_{1,0}^{\Psi^{(0)}} = \alpha_{[0,\theta_1]} \otimes \alpha_{(\theta_1,\theta_2]} \otimes \alpha_{(\theta_2,\theta_3]} \otimes \alpha_{(\theta_3,\frac{\pi}{2}]},$$
$$\tau_{0,1}^{\tilde{\Xi}} = \alpha_{(\theta_{0.8},\theta_{1.2}]} \otimes \alpha_{(\theta_{1.8},\theta_{2.2}]} \otimes \alpha_{(\theta_{2.8},\theta_{3.2}]}, \tag{5.50}$$

with $\alpha$s in the form of formulas (2.13) and (2.14). This completes the proof of the first part.

*Step 5.* Suppose that $\beta_g^U \left( \Psi(X;t) \right) = \Psi(X;t)$ for any $X \in \mathfrak{S}_{\mathbb{Z}^2}$, $t \in [0,1]$ and $g \in G$. Then clearly we have $\beta_g^U \left( \Psi^{(0)}(X;t) \right) = \Psi^{(0)}(X;t)$ for any $X \in \mathfrak{S}_{\mathbb{Z}^2}$, $t \in [0,1]$ and $g \in G$. By Theorem D.3 5, this implies $\tau_{1,0}^{\Psi^{(0)}} \beta_g^U = \beta_g^U \tau_{1,0}^{\Psi^{(0)}}$. From the decomposition (5.50), this means that all of $\alpha_{[0,\theta_1],\sigma}, \alpha_{(\theta_1,\theta_2],\sigma,\zeta}$, $\alpha_{(\theta_2,\theta_3],\sigma,\zeta}, \alpha_{(\theta_3,\frac{\pi}{2}],\zeta}$, $\sigma = L, R$, $\zeta = U, D$, commute with $\beta_g^U$. Because $\Pi_X$ commutes with $\beta_g^U$, $\tau_{t,s}^{\Psi}$ commutes with $\beta_g^U$ (Theorem D.3 5), and $\Psi^{(1)}$ and $\Xi^{(s)}$ are $\beta_g^U$-invariant from the definition (5.8). Therefore, from the definition (5.36), $\tilde{\Xi}$ is also $\beta_g^U$-invariant. Hence by Theorem D.3 5, $\tau_{0,1}^{\tilde{\Xi}}$ commutes with $\beta_g^U$. The decomposition (5.50) then implies that $\alpha_{(\theta_{0.8},\theta_{1.2}],\sigma,\zeta}, \alpha_{(\theta_{1.8},\theta_{2.2}],\sigma,\zeta}, \alpha_{(\theta_{2.8},\theta_{3.2}],\sigma,\zeta}$, $\sigma = L, R$, $\zeta = U, D$, commute with $\beta_g^U$. $\qquad\square$

An analogous proof shows the following:

**Proposition 5.5.** *Let $F \in \mathcal{F}_a$ be an F-function of the form $F(r) = \frac{\exp(-r^\theta)}{(1+r)^4}$ with a constant $0 < \theta < 1$. Let $\Psi \in \hat{\mathcal{B}}_F([0,1])$ be a path of interactions satisfying $\Psi_1 \in \hat{\mathcal{B}}_F([0,1])$. Define $\Psi^{(0)} \in \hat{\mathcal{B}}_F([0,1])$ by*

$$\Psi^{(0)}(X;t) := \begin{cases} \Psi(X;t) & \text{if } X \subset H_U \text{ or } X \subset H_D, \\ 0 & \text{otherwise,} \end{cases} \tag{5.51}$$

*for each $X \in \mathfrak{S}_{\mathbb{Z}^2}, t \in [0,1]$. Then $\left( \tau_{1,0}^{\Psi^{(0)}} \right)^{-1} \tau_{1,0}^{\Psi}$ belongs to $\mathrm{HAut}(\mathcal{A})$.*

*Proof.* Define $\tilde{F}$ as in formula (5.3) with some $0 < \theta' < \theta$. The same argument as in Theorem 5.2, step 2, implies that there exists $\Xi^{(1)} \in \hat{\mathcal{B}}_{\tilde{F}}[0,1]$ with $\tilde{F} \in \mathcal{F}_a$, such that

$$\tau_{1,0}^{\Psi} = \tau_{1,0}^{\Psi^{(0)}} \tau_{0,1}^{\Xi^{(1)}}. \tag{5.52}$$

This $\Xi^{(1)}$ is given by formula (5.8) for current $\Psi$ and $\Psi^{(1)}(X;t) := \Psi^{(0)}(X;t) - \Psi(X;t)$. To prove the theorem, it suffices to show that $\tau_{0,1}^{\Xi^{(1)}}$ belongs to $\mathrm{HAut}(\mathcal{A})$. Indeed, for any $0 < \theta_0 < \frac{\pi}{4}$, as in Theorem 5.2, step 2, we have

$$\sum_{\substack{Z: Z \not\subseteq C_{[0,\theta_0],L} \\ Z \not\subseteq C_{[0,\theta_0],R}}} \sup_{t \in [0,1]} \left\| \Xi^{(1)}(Z,t) \right\|$$

$$\leq \frac{8}{C_F} \left( e^{2I_F(\Psi)} - 1 \right) \sum_{m \geq 0} \sum_{\substack{X: X(m) \not\subseteq C_{[0,\theta_0],L} \\ X(m) \not\subseteq C_{[0,\theta_0],R}}} \left[ \sup_{t \in [0,1]} \left( \left\| \Psi^{(1)}(X;t) \right\| \right) |X| G_F(m) \right] < \infty. \tag{5.53}$$

To see this, note that if $X$ in the last line has a nonzero contribution to the sum, then at least one of the following occurs:

(i) $X \cap C_{\left[\theta_0, \frac{\pi}{2}\right], U} \neq \emptyset$ and $X \cap H_D \neq \emptyset$.

(ii) $X \cap C_{\left[\theta_0, \frac{\pi}{2}\right], D} \neq \emptyset$ and $X \cap H_U \neq \emptyset$.

(iii) $X \subset C_{[0,\theta_0]}$ and

    (1) $X \cap C_{[0,\theta_0],L} \neq \emptyset$ and $X \cap C_{[0,\theta_0],R} \neq \emptyset$, or

    (2) $X \subset C_{[0,\theta_0],R}, X \cap \check{C}_{[0,\theta_0]} \neq \emptyset, X \cap \check{C}_{[-\theta_0,0]} \neq \emptyset$ and $X(m) \cap \left( C_{[0,\theta_0],R} \right)^c \neq \emptyset$, or

    (3) $X \subset C_{[0,\theta_0],L}, X \cap \check{C}_{[\pi-\theta_0,\pi]} \neq \emptyset, X \cap \check{C}_{[\pi,\pi+\theta_0]} \neq \emptyset$ and $X(m) \cap \left( C_{[0,\theta_0],L} \right)^c \neq \emptyset$.

Therefore, the summation in the second line of formula (5.53) is bounded by

$$\frac{8}{C_F} \left( e^{2I_F(\Psi)} - 1 \right) \begin{pmatrix} b_0(\theta_0, \pi - \theta_0, \pi, 2\pi) + b_0(0, \pi, \pi + \theta_0, 2\pi - \theta_0) + b_0(-\theta_0, \theta_0, \pi - \theta_0, \pi + \theta_0) \\ + b_1(-\theta_0, 0, \theta_0) + b_1(\pi - \theta_0, \pi, \pi + \theta_0) \end{pmatrix} < \infty,$$

from Lemmas 5.3 and 5.4, proving formula (5.53).

Therefore, as in step 3 of Theorem 5.2, setting

$$\tilde{\Xi}(Z, t) := \begin{cases} \Xi^{(1)}(Z, t) & \text{if } Z \subseteq C_{[0,\theta_0],L} \text{ or } Z \subseteq C_{[0,\theta_0],R}, \\ 0 & \text{otherwise,} \end{cases} \tag{5.54}$$

we obtain $\tau_{0,1}^{\Xi^{(1)}} = (\text{inner}) \circ \tau_{0,1}^{\tilde{\Xi}}$. By the definition, $\tau_{0,1}^{\tilde{\Xi}}$ decomposes as $\tau_{0,1}^{\tilde{\Xi}} = \zeta_L \otimes \zeta_R$, with some $\zeta_\sigma \in \text{Aut}(\mathcal{A}_{C_{[0,\theta_0],\sigma}})$, $\sigma = L, R$. As this holds for any $0 < \theta_0 < \frac{\pi}{4}$, we conclude $\tau_{0,1}^{\Xi^{(1)}} \in \text{HAut}(\mathcal{A})$. □

**Theorem 5.6.** *Let $F \in \mathcal{F}_a$ be an F-function of the form $F(r) = \frac{\exp(-r^\theta)}{(1+r)^4}$ with a constant $0 < \theta < 1$. Let $\Psi \in \hat{\mathcal{B}}_F([0,1])$ be a path of interactions satisfying $\Psi_1 \in \hat{\mathcal{B}}_F([0,1])$. If $\Psi$ is $\beta$-invariant, then $\tau_{1,0}^\Psi$ belongs to* GUQAut$(\mathcal{A})$.

*Proof.* Define $\Psi^{(0)}$ as in formula (5.51) for our $\Psi$. By Proposition 5.5, we have $(\tau_{1,0}^{\Psi^{(0)}})^{-1} \tau_{1,0}^\Psi \in \text{HAut}(\mathcal{A})$. On the other hand, applying Theorem 5.2 to $\Psi^{(0)} \in \hat{\mathcal{B}}_F([0,1])$, we see that $\tau_{1,0}^{\Psi^{(0)}}$ belongs to SQAut$(\mathcal{A})$. Note that $\Psi^{(0)}(X; t)$ is nonzero only if $X \subset H_U$ or $X \subset H_D$, and it coincides with $\Psi(X; t)$ when it is nonzero. Therefore, if $\Psi$ is $\beta$-invariant, $\Psi^{(0)}$ is $\beta_g^U$-invariant. Therefore, by Theorem 5.2, we have $\tau_{1,0}^{\Psi^{(0)}} \in \text{GSQAut}(\mathcal{A})$. Hence we have $\tau_{1,0}^\Psi \in \text{GUQAut}(\mathcal{A})$. □

*Proof of Theorem 1.5.* Let $\Phi_0 \in \mathcal{P}_{UG}$ be the fixed trivial interaction with a unique gapped ground state. Its ground state $\omega_0 := \omega_{\Phi_0}$ is of a product form (formula (2.18)). For any $\Phi \in \mathcal{P}_{SL\beta}$, we have $\Phi_0 \sim \Phi$. Then by Theorem 5.1, there exists some $\Psi \in \hat{\mathcal{B}}_F([0,1])$ with $\Psi_1 \in \hat{\mathcal{B}}_F([0,1])$ for some $F \in \mathcal{F}_a$ of the form $F(r) = \frac{\exp(-r^\theta)}{(1+r)^4}$ with $0 < \theta < 1$, such that $\omega_\Phi = \omega_{\Phi_0} \circ \tau_{1,0}^\Psi$. From Theorem 5.2, $\tau_{1,0}^\Psi$ belongs to SQAut$(\mathcal{A})$. Because $\Phi \in \mathcal{P}_{SL\beta}$, $\omega_\Phi = \omega_{\Phi_0} \circ \tau_{1,0}^\Psi$ is $\beta$-invariant. Then, by Theorem 3.1, IG$(\omega_\Phi)$ is not empty. Therefore, we may define $h_\Phi := h(\omega_\Phi)$ by Definition 2.18.

To see that $h_\Phi$ is an invariant of $\sim_\beta$, set $\Phi_1, \Phi_2 \in \mathcal{P}_{SL\beta}$ with $\Phi_1 \sim_\beta \Phi_2$. Then by Theorem 5.1, there exists some $\beta$-invariant $\Psi \in \hat{\mathcal{B}}_F([0,1])$ with $\Psi_1 \in \hat{\mathcal{B}}_F([0,1])$ for some $F \in \mathcal{F}_a$ of the form $F(r) = \frac{\exp(-r^\theta)}{(1+r)^4}$ with a constant $0 < \theta < 1$ such that $\omega_{\Phi_2} = \omega_{\Phi_1} \circ \tau_{1,0}^\Psi$. Applying Theorem 5.6 to this $\Psi$, we see that $\tau_{1,0}^\Psi$ belongs to GUQAut$(\mathcal{A})$. Then Theorem 4.1 implies

$$h_{\Phi_2} = h(\omega_{\Phi_2}) = h\left(\omega_{\Phi_1} \circ \tau_{1,0}^\Psi\right) = h(\omega_{\Phi_1}) = h_{\Phi_1}, \tag{5.55}$$

proving the stability. □

# 6. Automorphisms with factorised $d_{H_U}^0 \alpha$

When $\alpha \in \text{EAut}(\omega)$ has some good factorisation property with respect to the action of $\beta_g^U$, the index $h(\omega)$ can be calculated without going through GNS representations.

**Definition 6.1.** For $\alpha \in \text{Aut}(\mathcal{A})$, we set

$$\left( d_{H_U}^0 \alpha \right)(g) := \alpha^{-1} \beta_g^U \circ \alpha \circ \left( \beta_g^U \right)^{-1}, \quad g \in G. \tag{6.1}$$

We say that $d^0_{H_U}\alpha$ is factorised into left and right if there are automorphisms $\gamma_{g,\sigma} \in \mathrm{Aut}\left(\mathcal{A}_{H_\sigma}\right)$, $g \in G, \sigma = L, R$, such that

$$\left(d^0_{H_U}\alpha\right)(g) = (\text{inner}) \circ \left(\gamma_{g,L} \otimes \gamma_{g,R}\right), \quad g \in G. \tag{6.2}$$

For known examples of 2-dimensional SPT phases like [CGLW, MM, Y, DW] or injective projected entangled-pair states [MGSC], this property holds. Namely, with a bit of effort, states in these models can be written in the form $\omega_0\alpha$, where $\omega_0$ is a pure infinite tensor product state and $\alpha$ is an automorphism satisfying the property in Definition 6.1. From such an automorphism, we can derive an outer action of $G$.

**Lemma 6.2.** *Let $\alpha \in \mathrm{Aut}(\mathcal{A})$ be an automorphism. Suppose that $d^0_{H_U}\alpha$ is factorised into left and right – that is, there are automorphisms $\gamma_{g,\sigma} \in \mathrm{Aut}\left(\mathcal{A}_{H_\sigma}\right)$, $g \in G, \sigma = L, R$, such that*

$$\left(d^0_{H_U}\alpha\right)(g) = (\text{inner}) \circ \left(\gamma_{g,L} \otimes \gamma_{g,R}\right), \quad g \in G. \tag{6.3}$$

*Then there are unitaries $v_\sigma(g, h) \in \mathcal{U}\left(\mathcal{A}_{H_\sigma}\right)$, $g, h \in G, \sigma = L, R$, such that*

$$\gamma_{g,\sigma}\beta^{\sigma U}_g \gamma_{h,\sigma}\beta^{\sigma U}_h \left(\gamma_{gh,\sigma}\beta^{\sigma U}_{gh}\right)^{-1} = \mathrm{Ad}(v_\sigma(g, h)). \tag{6.4}$$

*Proof.* Because $\beta^U_g$ is a group action, substituting equation (6.3) we get

$$\mathrm{id}_\mathcal{A} = \alpha^{-1}\beta^U_g\alpha \circ \alpha^{-1}\beta^U_h\alpha \circ \left(\alpha^{-1}\beta^U_{gh}\alpha\right)^{-1}$$

$$= (\text{inner}) \circ \left(\gamma_{g,L}\beta^{LU}_g \otimes \gamma_{g,R}\beta^{RU}_g\right) \circ \left(\gamma_{h,L}\beta^{LU}_h \otimes \gamma_{h,R}\beta^{RU}_h\right) \circ \left(\gamma_{gh,L}\beta^{LU}_{gh} \otimes \gamma_{gh,R}\beta^{RU}_{gh}\right)^{-1}$$

$$= (\text{inner}) \circ \left(\gamma_{g,L}\beta^{LU}_g\gamma_{h,L}\beta^{LU}_h \left(\gamma_{gh,L}\beta^{LU}_{gh}\right)^{-1} \otimes \gamma_{g,R}\beta^{RU}_g\gamma_{h,R}\beta^{RU}_h \left(\gamma_{gh,R}\beta^{RU}_{gh}\right)^{-1}\right). \tag{6.5}$$

By Lemma B.1, we then see that there are unitaries $v_\sigma(g, h) \in \mathrm{Aut}\left(\mathcal{A}_{H_\sigma}\right)$, $g \in G, \sigma = L, R$, satisfying equation (6.4). □

It is well known that a third cohomology class can be associated to cocycle actions [C, J].

**Lemma 6.3.** *Let $\alpha \in \mathrm{Aut}(\mathcal{A})$ be an automorphism such that $d^0_{H_U}\alpha$ is factorised into left and right as in equation (6.3). Let $v_\sigma(g, h) \in \mathcal{U}\left(\mathcal{A}_{H_\sigma}\right)$, $g, h \in G, \sigma = L, R$, be unitaries satisfying equation (6.4). Then there is some $c_\sigma \in C^3(G, \mathbb{T})$, $\sigma = L, R$, such that*

$$v_\sigma(g, h)v_\sigma(gh, k) = c_\sigma(g, h, k)\left(\gamma_{g,\sigma} \circ \beta^{\sigma U}_g (v_\sigma(h, k))\right)v_\sigma(g, hk), \quad g, h, k \in G. \tag{6.6}$$

*Proof.* By equation (6.4), we have

$$\hat{\gamma}_{g,\sigma}\hat{\gamma}_{h,\sigma} = \mathrm{Ad}(v_\sigma(g, h)) \circ \hat{\gamma}_{gh,\sigma} \tag{6.7}$$

for $\hat{\gamma}_{g,\sigma} := \gamma_{g,\sigma}\beta^{\sigma U}_g$. Using this, we have

$$\mathrm{Ad}(v_\sigma(g, h)) \circ \mathrm{Ad}\left(v_\sigma(gh, k)\right) \circ \hat{\gamma}_{ghk,\sigma}$$
$$= \mathrm{Ad}(v_\sigma(g, h)) \circ \hat{\gamma}_{gh,\sigma} \circ \hat{\gamma}_{k,\sigma} = \hat{\gamma}_{g,\sigma}\hat{\gamma}_{h,\sigma}\hat{\gamma}_{k,\sigma} = \hat{\gamma}_{g,\sigma} \circ \mathrm{Ad}\left(v_\sigma(h, k)\right) \circ \hat{\gamma}_{hk,\sigma}$$
$$= \mathrm{Ad}\left(\hat{\gamma}_{g,\sigma}\left(v_\sigma(h, k)\right)\right)\hat{\gamma}_{g,\sigma} \circ \hat{\gamma}_{hk,\sigma} = \mathrm{Ad}\left(\hat{\gamma}_{g,\sigma}\left(v_\sigma(h, k)\right)v_\sigma(g, hk)\right) \circ \hat{\gamma}_{ghk,\sigma}. \tag{6.8}$$

Because $\mathcal{A}' \cap \mathcal{A} = \mathbb{I}_\mathcal{A}$, it must be the case that $\hat{\gamma}_{g,\sigma}\left(v_\sigma(h, k)\right)v_\sigma(g, hk)$ and $v_\sigma(g, h)v_\sigma(gh, k)$ are proportional to each other, proving the lemma. □

By the same argument as in Lemma 2.4, we can show that this $c_R$ is actually a 3-cocycle. If $\omega \in \mathcal{SL}$ is given by an automorphism $\alpha \in \mathrm{EAut}(\omega)$ with factorised $d^0_{H_U}\alpha$, and if $\omega_0$ is invariant under $\beta^U_g$, then we have $h(\omega) = [c_R]_{H^3(G,\mathbb{T})}$, for $c_R$ given in Lemma 6.3.

**Theorem 6.4.** *Let $\omega_0$ be a reference state of the form in formula (2.18), and assume that $\omega_0 \circ \beta^U_g = \omega_0$ for any $g \in G$. Let $\alpha \in \mathrm{QAut}(\mathcal{A})$ be an automorphism. Suppose that $d^0_{H_U}\alpha$ is factorised into left and right as in equation (6.3) with some $\gamma_{g,\sigma} \in \mathrm{Aut}(\mathcal{A}_{C_{\theta_0},\sigma})$ and $0 < \theta_0 < \frac{\pi}{2}$, for $\sigma = L, R$. Let $v_\sigma(g,h) \in \mathcal{U}(\mathcal{A}_{H_\sigma})$, $g,h \in G, \sigma = L, R$, be unitaries satisfying equation (6.4) and $c_R \in C^3(G,\mathbb{T})$ satisfying equation (6.6) for these $v_R(g,h)$ which are given in Lemma 6.2 and Lemma 6.3. Then we have $\omega_0 \circ \alpha \in \mathcal{SL}$ with $\mathrm{IG}(\omega_0 \circ \alpha) \neq \emptyset$, $c_R \in Z^3(G,\mathbb{T})$, and $h(\omega_0 \circ \alpha) = [c_R]_{H^3(G,\mathbb{T})}$.*

**Remark 6.5.** The situation of this theorem is special. We do not expect that it always occurs.

*Proof.* That $\omega_0 \circ \alpha \in \mathcal{SL}$ is by definition. Because

$$\mathrm{Ad}(v_\sigma(g,h)) = \gamma_{g,\sigma}\beta^{\sigma U}_g \gamma_{h,\sigma}\beta^{\sigma U}_h \left(\gamma_{gh,\sigma}\beta^{\sigma U}_{gh}\right)^{-1} \in \mathrm{Aut}\left(\mathcal{A}_{C_{\theta_0},\sigma}\right), \qquad (6.9)$$

our $v_\sigma(g,h)$ belongs to $\mathcal{U}(\mathcal{A}_{C_{\theta_0},\sigma})$. Because

$$\omega_0\alpha \circ \alpha^{-1}\beta^U_g\alpha = \omega_0\beta^U_g\alpha = \omega_0\alpha \qquad (6.10)$$

and

$$\alpha^{-1}\beta^U_g\alpha = (\text{inner}) \circ \left(\gamma_{g,L} \otimes \gamma_{g,R}\right) \circ \beta^U_g, \qquad (6.11)$$

with $\gamma_{g,\sigma} \in \mathrm{Aut}(\mathcal{A}_{C_{\theta_0},\sigma})$, we have $(\alpha^{-1}\beta^U_g\alpha) \in \mathrm{IG}(\omega_0\alpha, \theta_0)$, and $(\gamma_{g,\sigma}) \in \mathcal{T}(\theta_0, \alpha^{-1}\beta^U_g\alpha)$. Clearly $\alpha \in \mathrm{EAut}(\omega_0 \circ \alpha)$, and there is $(\alpha_L, \alpha_R, \Theta) \in \mathcal{D}^{\theta_0}_\alpha$ because $\alpha \in \mathrm{QAut}(\mathcal{A})$. Set $\gamma_g := \gamma_{g,L} \otimes \gamma_{g,R}$. From Lemma 2.1, there is some $W_g \in \mathcal{U}(\mathcal{H}_0)g \in G$ satisfying

$$\mathrm{Ad}(W_g) \circ \pi_0 = \pi_0 \circ (\alpha_L \otimes \alpha_R) \circ \Theta \circ \gamma_g\beta^U_g \circ \Theta^{-1} \circ (\alpha_L \otimes \alpha_R)^{-1}, \quad g \in G. \qquad (6.12)$$

In particular, because $v_R(h,k)$ belongs to $\mathcal{U}(\mathcal{A}_{(C_{\theta_0})_R})$, $\Theta \in \mathrm{Aut}(\mathcal{A}_{C^c_{\theta_0}})$, and $\gamma_g\beta^U_g$ preserves $\mathcal{A}_{(C_{\theta_0})_R}$, we have

$$\begin{aligned}
\mathrm{Ad}(W_g) &\circ \pi_0 \circ (\alpha_L \otimes \alpha_R) \left(\mathrm{id}_{\mathcal{A}_L} \otimes (v_R(h,k))\right) \\
&= \pi_0 \circ (\alpha_L \otimes \alpha_R) \circ \Theta \circ \gamma_g\beta^U_g \circ \Theta^{-1} \circ \left(\mathrm{id}_{\mathcal{A}_L} \otimes (v_R(h,k))\right) \\
&= \pi_0 \circ (\alpha_L \otimes \alpha_R) \left(\mathrm{id}_{\mathcal{A}_L} \otimes\gamma_{g,R}\beta^{RU}_g (v_R(h,k))\right) = \mathbb{I}_{\mathcal{H}_L} \otimes \pi_R \circ \alpha_R \circ \gamma_{g,R}\beta^{RU}_g (v_R(h,k)).
\end{aligned} \qquad (6.13)$$

On the other hand, equation (6.4) means

$$\mathrm{Ad}\left(\pi_\sigma \circ \alpha_\sigma(v_\sigma(g,h))\right)\pi_\sigma = \pi_\sigma \circ \alpha_\sigma \circ \gamma_{g,\sigma}\beta^{\sigma U}_g\gamma_{h,\sigma}\left(\beta^{\sigma U}_g\right)^{-1}(\gamma_{gh,\sigma})^{-1} \circ \alpha^{-1}_\sigma. \qquad (6.14)$$

From equations (6.12) and (6.14), we have

$$\left((W_g), (\pi_\sigma \circ \alpha_\sigma(v_\sigma(g,h)))\right) \in \mathrm{IP}\left(\omega_0 \circ \alpha, \alpha, \theta_0, \left(\alpha^{-1}\beta^U_g\alpha\right), (\gamma_{g,\sigma}), (\alpha_L, \alpha_R, \Theta)\right). \qquad (6.15)$$

Now from equations (6.6) and then (6.13), we obtain

$$
\begin{aligned}
\mathbb{I}_{\mathcal{H}_L} &\otimes \pi_R \circ \alpha_R \left( v_R(g,h) v_R(gh,k) \right) \\
&= c_R(g,h,k) \mathbb{I}_{\mathcal{H}_L} \otimes \pi_R \circ \alpha_R \left( \left( \gamma_{g,R} \circ \beta_g^{RU} \left( v_R(h,k) \right) \right) v_R(g,hk) \right) \\
&= c_R(g,h,k) \left( \mathrm{Ad}(W_g) \left( \mathrm{id}_{\mathcal{H}_L} \otimes \pi_R \alpha_R \left( v_R(h,k) \right) \right) \right) \cdot \left( \mathbb{I}_{\mathcal{H}_L} \otimes \pi_R \circ \alpha_R \left( v_R(g,hk) \right) \right).
\end{aligned}
\tag{6.16}
$$

This means

$$
c_R = c_R \left( \omega_0 \circ \alpha, \alpha, \theta_0, \left( \alpha^{-1} \beta_g^U \alpha \right), \left( \gamma_{g,\sigma} \right), \left( \alpha_L, \alpha_R, \Theta \right), \left( (W_g), \left( (\pi_\sigma \circ \alpha_\sigma (v_\sigma(g,h))) \right) \right) \right)
\tag{6.17}
$$

in Definition 2.5. Hence we get $c_R \in Z^3(G, \mathbb{T})$, and $h(\omega_0 \circ \alpha) = [c_R]_{H^3(G, \mathbb{T})}$.                     □

## A.  Basic notation

For a finite set $S$, $\#S$ indicates the number of elements in $S$. For $t \in \mathbb{R}$, $[t]$ denotes the smallest integer less than or equal to $t$.

For a Hilbert space $\mathcal{H}$, $B(\mathcal{H})$ denotes the set of all bounded operators on $\mathcal{H}$. If $V : \mathcal{H}_1 \to \mathcal{H}_2$ is a linear map from a Hilbert space $\mathcal{H}_1$ to another Hilbert space $\mathcal{H}_2$, then $\mathrm{Ad}(V) : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$ denotes the map $\mathrm{Ad}(V)(x) := V x V^*$, $x \in B(\mathcal{H}_1)$. Occasionally we write $\mathrm{Ad}_V$ instead of $\mathrm{Ad}(V)$. For a $C^*$-algebra $\mathcal{B}$ and $v \in \mathcal{B}$, we set $\mathrm{Ad}(v)(x) := \mathrm{Ad}_v(x) := v x v^*$, $x \in \mathcal{B}$.

For a state $\omega, \varphi$ on a $C^*$-algebra $\mathcal{B}$, we write $\omega \sim_{\mathrm{q.e.}} \varphi$ when they are quasiequivalent (see [BR1]). We also write $\omega \simeq \varphi$ when they are equivalent. We denote by $\mathrm{Aut}\,\mathcal{B}$ the group of automorphisms on a $C^*$-algebra $\mathcal{B}$. The group of inner automorphisms on a unital $C^*$-algebra $\mathcal{B}$ is denoted by $\mathrm{Inn}\,\mathcal{B}$. For $\gamma_1, \gamma_2 \in \mathrm{Aut}(\mathcal{B})$, $\gamma_1 = (\text{inner}) \circ \gamma_2$ means there is some unitary $u$ in $\mathcal{B}$ such that $\gamma_1 = \mathrm{Ad}(u) \circ \gamma_2$. For a unital $C^*$-algebra $\mathcal{B}$, the unit of $\mathcal{B}$ is denoted by $\mathbb{I}_\mathcal{B}$. For a Hilbert space we write $\mathbb{I}_\mathcal{H} := \mathbb{I}_{B(\mathcal{H})}$. For a unital $C^*$-algebra $\mathcal{B}$, by $\mathcal{U}(\mathcal{B})$ we mean the set of all unitary elements in $\mathcal{B}$. For a Hilbert space we write $\mathcal{U}(\mathcal{H})$ for $\mathcal{U}(B(\mathcal{H}))$.

For a state $\varphi$ on $\mathcal{B}$ and a $C^*$-subalgebra $\mathcal{C}$ of $\mathcal{B}$, $\varphi|_\mathcal{C}$ indicates the restriction of $\varphi$ to $\mathcal{C}$.

To denote the composition of automorphisms $\alpha_1, \alpha_2$, all of $\alpha_1 \circ \alpha_2$, $\alpha_1 \alpha_2$, $\alpha_1 \cdot \alpha_2$ are used. Frequently, the first one serves as a bracket to visually separate a group of operators.

## B.  Automorphisms on UHF-algebras

**Lemma B.1.** *Let* $\mathfrak{A}, \mathfrak{B}$ *be UHF-algebras. If automorphisms* $\gamma_\mathfrak{A} \in \mathrm{Aut}(\mathfrak{A})$, $\gamma_\mathfrak{B} \in \mathrm{Aut}(\mathfrak{B})$ *and a unitary* $W \in \mathcal{U}(\mathfrak{A} \otimes \mathfrak{B})$ *satisfy*

$$
\left( \gamma_\mathfrak{A} \otimes \gamma_\mathfrak{B} \right)(X) = \mathrm{Ad}_W(X), \quad X \in \mathfrak{A} \otimes \mathfrak{B},
\tag{B.1}
$$

*then there are unitaries* $u_\mathfrak{A} \in \mathcal{U}(\mathfrak{A})$ *and* $u_\mathfrak{B} \in \mathcal{U}(\mathfrak{B})$ *such that*

$$
\begin{aligned}
\gamma_\mathfrak{A}(X_\mathfrak{A}) &= \mathrm{Ad}_{u_\mathfrak{A}}(X_\mathfrak{A}), \quad X_\mathfrak{A} \in \mathfrak{A}, \\
\gamma_\mathfrak{B}(X_\mathfrak{B}) &= \mathrm{Ad}_{u_\mathfrak{B}}(X_\mathfrak{B}), \quad X_\mathfrak{B} \in \mathfrak{B}.
\end{aligned}
\tag{B.2}
$$

*Proof.* Fix some irreducible representations $(\mathcal{H}_\mathfrak{A}, \pi_\mathfrak{A})$, $(\mathcal{H}_\mathfrak{B}, \pi_\mathfrak{B})$, of $\mathfrak{A}, \mathfrak{B}$. We claim that there are unitaries $v_\mathfrak{A} \in \mathcal{U}(\mathcal{H}_\mathfrak{A})$ and $v_\mathfrak{B} \in \mathcal{U}(\mathcal{H}_\mathfrak{B})$ such that

$$
\begin{aligned}
\mathrm{Ad}_{v_\mathfrak{A}}(\pi_\mathfrak{A}(X_\mathfrak{A})) &= \pi_\mathfrak{A} \circ \gamma_\mathfrak{A}(X_\mathfrak{A}), \quad X_\mathfrak{A} \in \mathfrak{A}, \\
\mathrm{Ad}_{v_\mathfrak{B}}(\pi_\mathfrak{B}(X_\mathfrak{B})) &= \pi_\mathfrak{B} \circ \gamma_\mathfrak{B}(X_\mathfrak{B}), \quad X_\mathfrak{B} \in \mathfrak{B}.
\end{aligned}
\tag{B.3}
$$

To see this, note that

$$\left(\pi_{\mathfrak{A}} \circ \gamma_{\mathfrak{A}} \otimes \pi_{\mathfrak{B}} \circ \gamma_{\mathfrak{B}}\right) = \mathrm{Ad}_{(\pi_{\mathfrak{A}} \otimes \pi_{\mathfrak{B}})(W)} \circ \left(\pi_{\mathfrak{A}} \otimes \pi_{\mathfrak{B}}\right). \tag{B.4}$$

From this, $\pi_{\mathfrak{A}} \circ \gamma_{\mathfrak{A}}$ (resp., $\pi_{\mathfrak{B}} \circ \gamma_{\mathfrak{B}}$) is quasiequivalent to $\pi_{\mathfrak{A}}$ (resp., $\pi_{\mathfrak{B}}$). Because $\pi_{\mathfrak{A}}$ and $\pi_{\mathfrak{B}}$ are irreducible, by the Wigner theorem there are unitaries $v_{\mathfrak{A}} \in \mathcal{U}(\mathcal{H}_{\mathfrak{A}})$ and $v_{\mathfrak{B}} \in \mathcal{U}(\mathcal{H}_{\mathfrak{B}})$ satisfying equation (B.3).

We then have

$$\begin{aligned}
\mathrm{Ad}_{(\pi_{\mathfrak{A}} \otimes \pi_{\mathfrak{B}})(W)} \circ \left(\pi_{\mathfrak{A}} \otimes \pi_{\mathfrak{B}}\right) &= \left(\pi_{\mathfrak{A}} \circ \gamma_{\mathfrak{A}}\right) \otimes \left(\pi_{\mathfrak{B}} \circ \gamma_{\mathfrak{B}}\right) \\
&= \left(\mathrm{Ad}_{v_{\mathfrak{A}}} \circ \pi_{\mathfrak{A}}\right) \otimes \left(\mathrm{Ad}_{v_{\mathfrak{B}}} \circ \pi_{\mathfrak{B}}\right) = \mathrm{Ad}_{v_{\mathfrak{A}} \otimes v_{\mathfrak{B}}} \circ \left(\pi_{\mathfrak{A}} \otimes \pi_{\mathfrak{B}}\right).
\end{aligned} \tag{B.5}$$

Because $\pi_{\mathfrak{A}} \otimes \pi_{\mathfrak{B}}$ is irreducible, there is a $c \in \mathbb{T}$ such that

$$\left(\pi_{\mathfrak{A}} \otimes \pi_{\mathfrak{B}}\right)(W) = c \left(v_{\mathfrak{A}} \otimes v_{\mathfrak{B}}\right). \tag{B.6}$$

We claim there is a unitary $u_{\mathfrak{B}} \in \mathcal{U}(\mathfrak{B})$ such that

$$\pi_{\mathfrak{B}}\left(u_{\mathfrak{B}}\right) = v_{\mathfrak{B}}. \tag{B.7}$$

Choose a unit vector $\xi \in \mathcal{H}_{\mathfrak{A}}$ with $\langle \xi, v_{\mathfrak{A}} \xi \rangle \neq 0$. For each $x \in \mathcal{B}(\mathcal{H}_{\mathfrak{A}} \otimes \mathcal{H}_{\mathfrak{B}})$, the map

$$\mathcal{H}_{\mathfrak{B}} \times \mathcal{H}_{\mathfrak{B}} \ni (\eta_1, \eta_2) \mapsto \langle (\xi \otimes \eta_1), x (\xi \otimes \eta_2) \rangle \tag{B.8}$$

is a bounded sesquilinear form. Therefore, there is a unique $\Phi_\xi(x) \in \mathcal{B}(\mathcal{H}_{\mathfrak{B}})$ such that

$$\left\langle \eta_1, \Phi_\xi(x)\eta_2 \right\rangle = \langle (\xi \otimes \eta_1), x (\xi \otimes \eta_2) \rangle, \quad (\eta_1, \eta_2) \in \mathcal{H}_{\mathfrak{B}} \times \mathcal{H}_{\mathfrak{B}}. \tag{B.9}$$

The map $\Phi_\xi : \mathcal{B}(\mathcal{H}_{\mathfrak{A}} \otimes \mathcal{H}_{\mathfrak{B}}) \to \mathcal{B}(\mathcal{H}_{\mathfrak{B}})$ is linear and

$$\left\|\Phi_\xi(x)\right\| \leq \|x\|, \quad x \in \mathcal{B}(\mathcal{H}). \tag{B.10}$$

Because $W$ belongs to $\mathfrak{A} \otimes \mathfrak{B}$, there are sequences

$$z_N = \sum_{i=1}^{n_N} a_i^{(N)} \otimes b_i^{(N)}, \quad \text{with } a_i^{(N)} \in \mathfrak{A}, \ b_i^{(N)} \in \mathfrak{B}, \tag{B.11}$$

such that

$$\|W - z_N\| < \frac{1}{N}. \tag{B.12}$$

Because of formula (B.10), we have

$$\left\|\Phi_\xi \left((\pi_{\mathfrak{A}} \otimes \pi_{\mathfrak{B}})(W - z_N)\right)\right\| < \frac{1}{N}. \tag{B.13}$$

Note that

$$\Phi_\xi \left((\pi_{\mathfrak{A}} \otimes \pi_{\mathfrak{B}})(z_N)\right) = \sum_{i=1}^{n_N} \left\langle \xi, \pi_{\mathfrak{A}}\left(a_i^{(N)}\right) \xi \right\rangle \pi_{\mathfrak{B}}\left(b_i^{(N)}\right) \in \pi_{\mathfrak{B}}(\mathfrak{B}). \tag{B.14}$$

Therefore, we have

$$c \langle \xi, v_{\mathfrak{A}} \xi \rangle v_{\mathfrak{B}} = \Phi_\xi \left( c \left( v_{\mathfrak{A}} \otimes v_{\mathfrak{B}} \right) \right) = \Phi_\xi \left( \left( \pi_{\mathfrak{A}} \otimes \pi_{\mathfrak{B}} \right) (W) \right) \in \overline{\pi_{\mathfrak{B}} (\mathfrak{B})}^n, \tag{B.15}$$

where $\cdot^n$ denotes the norm closure. Because $\pi_{\mathfrak{B}} (\mathfrak{B})$ is norm-closed, we have $\overline{\pi_{\mathfrak{B}} (\mathfrak{B})}^n = \pi_{\mathfrak{B}} (\mathfrak{B})$. Hence we have $v_{\mathfrak{B}} \in \pi_{\mathfrak{B}} (\mathfrak{B})$ – that is, there is a unitary $u_{\mathfrak{B}} \in \mathfrak{B}$ such that $v_{\mathfrak{B}} = \pi_{\mathfrak{B}} (u_{\mathfrak{B}})$.

We then have

$$\pi_{\mathfrak{B}} \circ \mathrm{Ad}_{u_{\mathfrak{B}}} (X) = \mathrm{Ad}_{\pi_{\mathfrak{B}} (u_{\mathfrak{B}})} \circ \pi_{\mathfrak{B}} (X) = \mathrm{Ad}_{v_{\mathfrak{B}}} \circ \pi_{\mathfrak{B}} (X) = \pi_{\mathfrak{B}} \circ \gamma_{\mathfrak{B}} (X), \quad X \in \mathfrak{B}. \tag{B.16}$$

As $\mathfrak{B}$ is simple, $\mathrm{Ad}_{u_{\mathfrak{B}}} (X) = \gamma_{\mathfrak{B}} (X)$ for all $X \in \mathfrak{B}$.

The proof for $\mathfrak{A}$ is the same. □

## C.  *F*-functions

In this section, we collect various estimates about *F*-functions. These estimates are useful for the proof of the factorisation property. Let us first start from the definition:

**Definition C.1.** An *F*-function *F* on $\left( \mathbb{Z}^2, \mathrm{d} \right)$ is a nonincreasing function $F : [0, \infty) \to (0, \infty)$ such that

(i)  $\|F\| := \sup_{x \in \mathbb{Z}^2} \left( \sum_{y \in \mathbb{Z}^2} F \left( \mathrm{d}(x, y) \right) \right) < \infty$ and

(ii)  $C_F := \sup_{x, y \in \mathbb{Z}^2} \left( \sum_{z \in \mathbb{Z}^2} \frac{F(\mathrm{d}(x,z)) F(\mathrm{d}(z,y))}{F(\mathrm{d}(x,y))} \right) < \infty.$

These properties are called *uniform integrability* and the *convolution identity*, respectively.

We denote by $\mathcal{F}_a$ a class of *F*-functions which decay quickly.

**Definition C.2.** We say an *F*-function *F* belongs to $\mathcal{F}_a$ if

(i)  for any $k \in \mathbb{N} \cup \{0\}$ and $0 < \vartheta \le 1$, we have

$$\kappa_{\vartheta, k, F} := \sum_{n=0}^{\infty} (n + 1)^k \left( F(n) \right)^{\vartheta} < \infty \tag{C.1}$$

and
(ii)  for any $0 < \vartheta < 1$, there is an *F*-function $\tilde{F}_\vartheta$ such that

$$\max \left\{ F \left( \frac{r}{3} \right), \left( F \left( \left[ \frac{r}{3} \right] \right) \right)^{\vartheta} \right\} \le \tilde{F}_\vartheta (r), \quad r \ge 0. \tag{C.2}$$

For example, a function $F(r) = \frac{\exp(-r^\theta)}{(1+r)^4}$ with a constant $0 < \theta < 1$ belongs to $\mathcal{F}_a$. (See [NSY, Appendix] for (i). The proof of (ii) is rather standard.)

In this appendix, we derive inequalities about $F \in \mathcal{F}_a$. In order for that, the following lemma is useful. We will freely identify $\mathbb{C}$ and $\mathbb{R}^2$ in an obvious manner.

**Lemma C.3.** *For $0 \le \theta_1 < \theta_2 \le 2\pi$, $c > 0$, and $r \ge 0$, set*

$$S_{r,c}^{[\theta_1, \theta_2]} := \left\{ s e^{i\theta} \in \mathbb{R}^2 \mid r \le s < r + c, \quad \theta \in [\theta_1, \theta_2] \right\}. \tag{C.3}$$

*Then we have*

$$\#\left(S_{r,c}^{[\theta_1,\theta_2]} \cap \mathbb{Z}^2\right) \le \pi\left(2\sqrt{2}+c\right)^2(r+1). \tag{C.4}$$

*In particular, we have*

$$\#\left(S_{r,1}^{[\theta_1,\theta_2]} \cap \mathbb{Z}^2\right) \le 64(r+1). \tag{C.5}$$

*Proof.* Because the diameter of a 2-dimensional unit square is $\sqrt{2}$, any unit square $B$ of $\mathbb{Z}^2$ with $B \cap S_{r,c}^{[\theta_1,\theta_2]} \cap \mathbb{Z}^2 \ne \emptyset$ satisfies $B \subset \hat{S}_{r,c}^{[\theta_1,\theta_2]}(\sqrt{2})$. Therefore, we have

$$\#\left\{B \mid \text{unit square of } \mathbb{Z}^2 \text{ with } B \cap S_{r,c}^{[\theta_1,\theta_2]} \cap \mathbb{Z}^2 \ne \emptyset\right\} = \sum_{B:B\cap S_{r,c}^{[\theta_1,\theta_2]}\cap\mathbb{Z}^2\ne\emptyset} 1 \le \left|\hat{S}_{r,c}^{[\theta_1,\theta_2]}(\sqrt{2})\right|. \tag{C.6}$$

Note that the area of $\hat{S}_{r,c}^{[\theta_1,\theta_2]}\left(\sqrt{2}\right)$, denoted by $\left|\hat{S}_{r,c}^{[\theta_1,\theta_2]}\left(\sqrt{2}\right)\right|$, is less than

$$\left|\hat{S}_{r,c}^{[\theta_1,\theta_2]}(\sqrt{2})\right| \le \pi\left(\left(r+c+\sqrt{2}\right)^2 - \left(r-\sqrt{2}\right)^2\right) \le \pi(2r+c)\left(2\sqrt{2}+c\right) \le \pi\left(2\sqrt{2}+c\right)^2(r+1) \tag{C.7}$$

if $r > \sqrt{2}$. For $r \le \sqrt{2}$, we have

$$\left|\hat{S}_{r,c}^{[\theta_1,\theta_2]}(\sqrt{2})\right| \le \pi\left(\left(r+c+\sqrt{2}\right)^2\right) \le \pi \cdot \left(2\sqrt{2}+c\right)^2 \le \pi\left(2\sqrt{2}+c\right)^2(r+1). \tag{C.8}$$

Hence, in any case we have

$$\left|\hat{S}_{r,c}^{[\theta_1,\theta_2]}(\sqrt{2})\right| \le \pi\left(2\sqrt{2}+c\right)^2(r+1). \tag{C.9}$$

Substituting this into equation (C.6), we obtain

$$\#\left\{B \mid \text{unit square of } \mathbb{Z}^2 \text{ with } B \cap S_{r,c}^{[\theta_1,\theta_2]} \cap \mathbb{Z}^2 \ne \emptyset\right\} \le \pi\left(2\sqrt{2}+c\right)^2(r+1). \tag{C.10}$$

On the other hand, we have

$$\#\left\{S_{r,c}^{[\theta_1,\theta_2]} \cap \mathbb{Z}^2\right\} = \sum_{z\in S_{r,c}^{[\theta_1,\theta_2]}\cap\mathbb{Z}^2} 1 = \sum_{z\in S_{r,c}^{[\theta_1,\theta_2]}\cap\mathbb{Z}^2} \sum_{B:\text{unit square of } \mathbb{Z}^2} \frac{1}{4}\mathbb{I}_{z\in B}$$

$$= \sum_{B:\text{unit square of } \mathbb{Z}^2} \sum_{z\in S_{r,c}^{[\theta_1,\theta_2]}\cap\mathbb{Z}^2} \frac{1}{4}\mathbb{I}_{z\in B} \le \sum_{\substack{B:\\ \text{unit square of } \mathbb{Z}^2 \\ B\cap S_{r,c}^{[\theta_1,\theta_2]}\cap\mathbb{Z}^2\ne\emptyset}} 1$$

$$= \#\left\{B \mid \text{unit square of } \mathbb{Z}^2 \text{ with } B \cap S_{r,c}^{[\theta_1,\theta_2]} \cap \mathbb{Z}^2 \ne \emptyset\right\} \le \pi\left(2\sqrt{2}+c\right)^2(r+1). \tag{C.11}$$

$\square$

For an $F$-function $F \in \mathcal{F}_a$, define a function $G_F$ on $t \geq 0$ by

$$G_F(t) := \sup_{x \in \mathbb{Z}^2} \left( \sum_{y \in \mathbb{Z}^2, d(x,y) \geq t} F(d(x,y)) \right), \quad t \geq 0. \tag{C.12}$$

Note that by uniform integrability, the supremum is finite for all $t$. In particular, for any $0 < \theta < 1$ we have

$$G_F(t) \leq \sum_{r=[t]}^{\infty} \sum_{\substack{y \in \mathbb{Z}^2: \\ r \leq d(0,y) < r+1}} F(d(0,y)) \leq \sum_{r=[t]}^{\infty} \sum_{y \in S_{r,1}^{[0,2\pi]} \cap \mathbb{Z}^2} F(r) \leq \sum_{r=[t]}^{\infty} \# \left( S_{r,1}^{[0,2\pi]} \cap \mathbb{Z}^2 \right) F(r)$$

$$\leq 64 \sum_{r=[t]}^{\infty} (r+1) F(r) = 64 \sum_{r=[t]}^{\infty} (r+1) F(r)^{\theta} F(r)^{1-\theta} \leq 64 \left( \sum_{r=0}^{\infty} (r+1) F(r)^{\theta} \right) F([t])^{1-\theta}$$

$$\leq 64 \cdot \kappa_{\theta,1,F} \cdot F([t])^{1-\theta} < \infty. \tag{C.13}$$

Substituting this, for any $0 < \alpha \leq 1$, $0 < \theta, \varphi < 1$, and $k \in \mathbb{N} \cup \{0\}$, we have

$$\sum_{n=0}^{\infty} (1+n)^k (G_F(n))^{\alpha} \leq \left( 64 \cdot \kappa_{\theta,1,F} \right)^{\alpha} \sum_{n=0}^{\infty} (1+n)^k \cdot F(n)^{\alpha(1-\theta)} = \left( 64 \cdot \kappa_{\theta,1,F} \right)^{\alpha} \kappa_{\alpha(1-\theta),k,F} < \infty,$$

$$\sum_{n=\left[\frac{r}{3}\right]}^{\infty} (1+n)^k (G_F(n))^{\alpha} \leq \left( 64 \cdot \kappa_{\theta,1,F} \right)^{\alpha} \sum_{n=\left[\frac{r}{3}\right]}^{\infty} (1+n)^k \cdot \left( F(n)^{\alpha(1-\theta)} \right)^{(1-\varphi)} \left( F(n)^{\alpha(1-\theta)} \right)^{\varphi}$$

$$\leq \left( 64 \cdot \kappa_{\theta,1,F} \right)^{\alpha} \kappa_{\alpha(1-\theta)(1-\varphi),k,F} F\left( \left[ \frac{r}{3} \right] \right)^{\alpha(1-\theta)\varphi}. \tag{C.14}$$

For any $0 < c \leq 1$, we have

$$\sum_{r=0}^{\infty} F(cr)(r+2)^3 = \sum_{l=0}^{\infty} \sum_{\substack{r \in \mathbb{Z}_{\geq 0} \\ l \leq cr < l+1}} F(cr)(r+2)^3 \leq \sum_{l=0}^{\infty} \sum_{\substack{r \in \mathbb{Z}_{\geq 0} \\ \frac{l}{c} \leq r < \frac{l+1}{c}}} F(l) \left( \frac{l+1}{c} + 2 \right)^3$$

$$\leq \sum_{l=0}^{\infty} F(l) \left( \frac{l+1}{c} + 2 \right)^3 \left( \frac{l+1}{c} - \left( \frac{l}{c} - 1 \right) + 1 \right) \leq \sum_{l=0}^{\infty} F(l) \left( \frac{l+1}{c} + 2 \right)^4 \tag{C.15}$$

$$\leq \frac{1}{c^4} \sum_{l=0}^{\infty} F(l) (l+3)^4 \leq \frac{3^4 \kappa_{1,4,F}}{c^4} < \infty.$$

We also have, for $m \in \mathbb{Z}_{\geq 0}$ and $0 < c \leq 1$,

$$\sum_{r_1=0}^{\infty} \sum_{\substack{r \in \mathbb{Z}_{\geq 0}: \\ \sqrt{r^2+r_1^2}c \geq (m+1)}} (r_1+1) F\left( \sqrt{r^2 + r_1^2}c - (m+1) \right)$$

$$\leq \sum_{l=0}^{\infty} \sum_{\substack{r_1, r \in \mathbb{Z}_{\geq 0} \\ l \leq \sqrt{r^2+r_1^2}c-(m+1) < l+1}} (r_1+1) F\left( \sqrt{r^2 + r_1^2}c - (m+1) \right)$$

$$\leq \sum_{l=0}^{\infty} \sum_{\substack{r_1,r \in \mathbb{Z}_{\geq 0} \\ l \leq \sqrt{r^2 + r_1^2} c - (m+1) < l+1}} \left( \frac{l+m+2}{c} + 1 \right) \cdot F(l)$$

$$\leq \sum_{l=0}^{\infty} \# \left\{ \mathbb{Z}^2 \cap S_{\frac{l+m+1}{c}, \frac{1}{c}}^{[0,2\pi]} \right\} \left( \frac{l+m+2}{c} + 1 \right) \cdot F(l)$$

$$\leq \sum_{l=0}^{\infty} \pi \left( 2\sqrt{2} + \frac{1}{c} \right)^2 \left( \frac{l+m+1}{c} + 1 \right) \cdot \left( \frac{l+m+2}{c} + 1 \right) \cdot F(l) \tag{C.16}$$

$$\leq \sum_{l=0}^{\infty} \pi \left( 2\sqrt{2} + \frac{1}{c} \right)^2 \left( \frac{l+m+3}{c} \right)^2 \cdot F(l)$$

$$\leq \pi \left( 2\sqrt{2} + \frac{1}{c} \right)^2 \frac{(m+3)^2}{c^2} \sum_{l=0}^{\infty} (l+1)^2 F(l)$$

$$\leq \pi \left( 2\sqrt{2} + \frac{1}{c} \right)^2 \frac{(m+3)^2}{c^2} \kappa_{1,2,F} \leq \left( \frac{3}{c} \right)^2 \left( 2\sqrt{2} + \frac{1}{c} \right)^2 \pi (m+1)^2 \kappa_{1,2,F}.$$

Recall formulas (5.25) and (5.26).

**Lemma C.4.** *Let $\varphi_1 < \varphi_2 < \varphi_3 < \varphi_4$ with $\varphi_4 - \varphi_1 < 2\pi$. Then we have*

$$\sum_{\substack{x \in \check{C}_{[\varphi_1,\varphi_2]} \\ y \in \check{C}_{[\varphi_3,\varphi_4]}}} F\left( \mathrm{d}(x,y) \right) \leq (64)^3 \frac{3^4 \kappa_{1,4,F}}{\left( c_{\varphi_1,\varphi_2,\varphi_3,\varphi_4}^{(0)} \right)^4}. \tag{C.17}$$

*Proof.* Let $x = s_1 e^{i\phi_1} \in \check{C}_{[\varphi_1,\varphi_2]}$ and $y = s_2 e^{i\phi_2} \in \check{C}_{[\varphi_3,\varphi_4]}$, with $s_1, s_2 \geq 0$. If $\cos(\phi_2 - \phi_1) \geq 0$, then we have

$$\mathrm{d}(x,y) = \sqrt{s_1^2 + s_2^2 - 2s_1 s_2 \cos(\phi_2 - \phi_1)} \geq \sqrt{s_1^2 + s_2^2} \sqrt{1 - \cos(\phi_2 - \phi_1)}$$

$$\geq \sqrt{1 - \max\{\cos(\varphi_3 - \varphi_2), \cos(\varphi_4 - \varphi_1), 0\}} \sqrt{s_1^2 + s_2^2}. \tag{C.18}$$

If $\cos(\phi_2 - \phi_1) < 0$, then we have

$$\mathrm{d}(x,y) = \sqrt{s_1^2 + s_2^2 - 2s_1 s_2 \cos(\phi_2 - \phi_1)} \geq \sqrt{s_1^2 + s_2^2}. \tag{C.19}$$

Hence for any $x = s_1 e^{i\phi_1} \in \check{C}_{[\varphi_1,\varphi_2]}$ and $y = s_2 e^{i\phi_2} \in \check{C}_{[\varphi_3,\varphi_4]}$ with $s_1, s_2 \geq 0$, we have

$$\mathrm{d}(x,y) \geq \sqrt{1 - \max\{\cos(\varphi_3 - \varphi_2), \cos(\varphi_4 - \varphi_1), 0\}} \sqrt{s_1^2 + s_2^2} = c_{\varphi_1,\varphi_2,\varphi_3,\varphi_4}^{(0)} \sqrt{s_1^2 + s_2^2}. \tag{C.20}$$

Substituting this estimate, we obtain

$$
\sum_{\substack{x \in \check{C}_{[\varphi_1, \varphi_2]} \\ y \in \check{C}_{[\varphi_3, \varphi_4]}}} F\left(\mathrm{d}(x, y)\right) \leq \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \sum_{\substack{x \in S_{r_1,1}^{[\varphi_1, \varphi_2]} \cap \mathbb{Z}^2 \\ y \in S_{r_2,1}^{[\varphi_3, \varphi_4]} \cap \mathbb{Z}^2}} F\left(\mathrm{d}(x, y)\right)
$$

$$
\leq \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} F\left(c_{\varphi_1, \varphi_2, \varphi_3, \varphi_4}^{(0)} \sqrt{r_1^2 + r_2^2}\right) \#\left(S_{r_1,1}^{[\varphi_1, \varphi_2]} \cap \mathbb{Z}^2\right) \#\left(S_{r_2,1}^{[\varphi_3, \varphi_4]} \cap \mathbb{Z}^2\right)
$$

$$
\leq (64)^2 \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} F\left(c_{\varphi_1, \varphi_2, \varphi_3, \varphi_4}^{(0)} \sqrt{r_1^2 + r_2^2}\right) (r_1 + 1)(r_2 + 1)
$$

$$
\leq (64)^2 \sum_{r=0}^{\infty} \sum_{\substack{r_1, r_2 \in \mathbb{Z}_{\geq 0} \\ (r_1, r_2) \in S_{r,1}^{[0, \frac{\pi}{2}]} \cap \mathbb{Z}^2}} F\left(c_{\varphi_1, \varphi_2, \varphi_3, \varphi_4}^{(0)} \sqrt{r_1^2 + r_2^2}\right) (r_1 + 1)(r_2 + 1) \quad \text{(C.21)}
$$

$$
\leq (64)^2 \sum_{r=0}^{\infty} F\left(c_{\varphi_1, \varphi_2, \varphi_3, \varphi_4}^{(0)} r\right) (r + 2)^2 \cdot \#\left(S_r^{[0, \frac{\pi}{2}]} \cap \mathbb{Z}^2\right)
$$

$$
\leq (64)^3 \sum_{r=0}^{\infty} F\left(c_{\varphi_1, \varphi_2, \varphi_3, \varphi_4}^{(0)} r\right) (r + 2)^3
$$

$$
\leq (64)^3 \frac{3^4 \kappa_{1,4,F}}{\left(c_{\varphi_1, \varphi_2, \varphi_3, \varphi_4}^{(0)}\right)^4}.
$$

We used Lemma C.3 to bound $\#\left(S_{r,1}^{[0, \frac{\pi}{2}]} \cap \mathbb{Z}^2\right)$ and so on, and in the last inequality we used equation (C.15). □

Set

$$
L_\varphi := \left\{z \in \mathbb{R}^2 \mid \arg z = \varphi\right\}, \quad \varphi \in [0, 2\pi), \quad \text{(C.22)}
$$

and

$$
c_{\zeta_1, \zeta_2, \zeta_3}^{(1)} := \sqrt{1 - \max\left\{\cos(\zeta_1 - \zeta_2), \cos(\zeta_1 - \zeta_3)\right\}}, \quad \zeta_1, \zeta_2, \zeta_3 \in [0, 2\pi). \quad \text{(C.23)}
$$

**Lemma C.5.** *Set $\varphi, \theta_1, \theta_2 \in \mathbb{R}$ with $\theta_1 < \theta_2$ and $0 < |\varphi - \theta_0| < \frac{\pi}{2}$ for all $\theta_0 \in [\theta_1, \theta_2]$. Then we have*

$$
\sum_{x \in \check{C}_{[\theta_1, \theta_2]}} \sum_{y \in L_\varphi(m)} F\left(\mathrm{d}(x, y)\right) \leq 64 \cdot 144 \cdot 24 \cdot \left(c^{(1)}{}_{\varphi, \theta_1, \theta_2}\right)^{-4} \left(\pi \kappa_{1,2,F} + F(0)\right) (m + 1)^4, \quad \text{(C.24)}
$$

*for any $m \in \mathbb{N} \cup \{0\}$.*

*Proof.* For each $r \in \mathbb{Z}$, set

$$
T_{\varphi, r, m} := \left\{s e^{i\theta} \in \mathbb{R}^2 \mid r \leq s \cos(\theta - \varphi) \leq r + 1, \ -m \leq s \sin(\theta - \varphi) \leq m\right\}. \quad \text{(C.25)}
$$

Note that $s \cos(\theta - \varphi)$ is a projection of $s e^{i\theta}$ onto $L_\varphi$ and $|s \sin(\theta - \varphi)|$ is the distance of $s e^{i\theta}$ from the line including $L_\varphi$. Then we have

$$
L_\varphi(m) \subset \cup_{r=-m}^{\infty} T_{\varphi, r, m} \cap \mathbb{Z}^2 \quad \text{and} \quad \left|\hat{T}_{\varphi, r, m}(\sqrt{2})\right| \leq \left(2\sqrt{2} + 1\right)\left(2m + 2\sqrt{2}\right) \leq 12(m + 1). \quad \text{(C.26)}
$$

Because the diameter of a 2-dimensional unit square is $\sqrt{2}$, any unit square $B$ of $\mathbb{Z}^2$ with $B \cap T_{\varphi,r,m} \cap \mathbb{Z}^2 \neq \emptyset$ satisfies $B \subset \hat{T}_{\varphi,r,m}\left(\sqrt{2}\right)$. Therefore, using formula (C.26) we have

$$
\begin{aligned}
\# \left\{B \mid \text{unit square of } \mathbb{Z}^2 \text{ with } B \cap T_{\varphi,r,m} \cap \mathbb{Z}^2 \neq \emptyset\right\} &= \sum_{B : B \cap T_{\varphi,r,m} \cap \mathbb{Z}^2 \neq \emptyset} 1 \\
&\leq \left|\hat{T}_{\varphi,r,m}(\sqrt{2})\right| \leq 12(m+1).
\end{aligned}
\tag{C.27}
$$

On the other hand, we have

$$
\begin{aligned}
\# \left\{T_{\varphi,r,m} \cap \mathbb{Z}^2\right\} &= \sum_{z \in T_{\varphi,r,m} \cap \mathbb{Z}^2} 1 = \sum_{z \in T_{\varphi,r,m} \cap \mathbb{Z}^2} \sum_{B : \text{unit square of } \mathbb{Z}^2} \frac{1}{4} \mathbb{I}_{z \in B} \\
&= \sum_{B : \text{unit square of } \mathbb{Z}^2} \sum_{z \in T_{\varphi,r,m} \cap \mathbb{Z}^2} \frac{1}{4} \mathbb{I}_{z \in B} \leq \sum_{\substack{B : \\ \text{unit square of } \mathbb{Z}^2 \\ B \cap T_{\varphi,r,m} \cap \mathbb{Z}^2 \neq \emptyset}} 1 \\
&= \# \left\{B \mid \text{unit square of } \mathbb{Z}^2 \text{ with } B \cap T_{\varphi,r,m} \cap \mathbb{Z}^2 \neq \emptyset\right\} \leq 12(m+1).
\end{aligned}
\tag{C.28}
$$

If $x \in \check{C}_{[\theta_1, \theta_2]}$, we have $x = r_0 e^{i\theta_0}$ for some $r_0 \geq 0$ and $\theta_0 \in [\theta_1, \theta_2]$. By the assumption, we have $0 < |\theta_0 - \varphi| < \frac{\pi}{2}$, hence $0 < \cos(\varphi - \theta_0) < 1$. Therefore, for any $r \in \mathbb{R}$, we have

$$
\begin{aligned}
\mathrm{d}\left(x, r e^{i\varphi}\right) &= \sqrt{r^2 + r_0^2 - 2 r_0 r \cos(\theta_0 - \varphi)} \geq \sqrt{r^2 + r_0^2} \sqrt{1 - \cos(\theta_0 - \varphi)} \\
&\geq \sqrt{r^2 + r_0^2} \sqrt{1 - \max\left\{\cos(\theta_1 - \varphi), \cos(\theta_2 - \varphi)\right\}}.
\end{aligned}
\tag{C.29}
$$

Therefore, for any $x \in \check{C}_{[\theta_1, \theta_2]}$ and $y \in T_{\varphi,r,m}$, we have

$$
\mathrm{d}(x, y) \geq \mathrm{d}\left(x, r e^{i\varphi}\right) - (m+1) = \sqrt{r^2 + r_0^2} \, c^{(1)}_{\varphi, \theta_1, \theta_2} - (m+1).
\tag{C.30}
$$

From this and formulas (C.26) and (C.28), for any $x = r_0 e^{i\theta_0} \in C_{[\theta_1, \theta_2]}, r_0 \geq 0$, we have

$$
\begin{aligned}
\sum_{y \in L_\varphi(m)} F\left(\mathrm{d}(x, y)\right) &\leq \sum_{r=-m}^{\infty} \sum_{y \in \left(T_{\varphi,r,m} \cap \mathbb{Z}^2\right)} F\left(\mathrm{d}(x, y)\right) \leq \sum_{r=-\infty}^{\infty} \sum_{y \in \left(T_{\varphi,r,m} \cap \mathbb{Z}^2\right)} F\left(\mathrm{d}(x, y)\right) \\
&\leq \sum_{\substack{r \in \mathbb{Z} : \\ \sqrt{r^2 + r_0^2}\, c^{(1)}_{\varphi, \theta_1, \theta_2} \geq (m+1)}} \sum_{y \in \left(T_{\varphi,r,m} \cap \mathbb{Z}^2\right)} F\left(\sqrt{r^2 + r_0^2}\, c^{(1)}_{\varphi, \theta_1, \theta_2} - (m+1)\right) \\
&\quad + \sum_{\substack{r \in \mathbb{Z} : \\ \sqrt{r^2 + r_0^2}\, c^{(1)}_{\varphi, \theta_1, \theta_2} < (m+1)}} \sum_{y \in \left(T_{\varphi,r,m} \cap \mathbb{Z}^2\right)} F(0)
\end{aligned}
$$

$$\leq \sum_{\substack{r \in \mathbb{Z}: \\ \sqrt{r^2 + r_0^2}\, c^{(1)}{}_{\varphi, \theta_1, \theta_2} \geq (m+1)}} 12(m+1) F\left(\sqrt{r^2 + r_0^2}\, c^{(1)}{}_{\varphi, \theta_1, \theta_2} - (m+1)\right) \qquad \text{(C.31)}$$

$$+ \sum_{\substack{r \in \mathbb{Z}: \\ \sqrt{r^2 + r_0^2}\, c^{(1)}{}_{\varphi, \theta_1, \theta_2} < (m+1)}} 12(m+1) F(0)$$

$$\leq \sum_{\substack{r \in \mathbb{Z}_{\geq 0}: \\ \sqrt{r^2 + r_0^2}\, c^{(1)}{}_{\varphi, \theta_1, \theta_2} \geq (m+1)}} 24(m+1) F\left(\sqrt{r^2 + r_0^2}\, c^{(1)}{}_{\varphi, \theta_1, \theta_2} - (m+1)\right)$$

$$+ 36 \frac{(m+1)^2}{c^{(1)}{}_{\varphi, \theta_1, \theta_2}} F(0) \mathbb{I}_{r_0 \leq \frac{m+1}{c^{(1)}{}_{\varphi, \theta_1, \theta_2}}}.$$

We then get

$$\sum_{x \in \check{C}_{[\theta_1, \theta_2]}} \sum_{y \in L_\varphi(m)} F\left(\mathrm{d}(x, y)\right)$$

$$\leq \sum_{r_1 = 0}^{\infty} \sum_{x \in S_{r_1, 1}^{[\theta_1, \theta_2]} \cap \mathbb{Z}^2} \left( \sum_{\substack{r \in \mathbb{Z}_{\geq 0}: \\ \sqrt{r^2 + r_1^2}\, c^{(1)}{}_{\varphi, \theta_1, \theta_2} \geq (m+1)}} 24(m+1) F\left(\sqrt{r^2 + r_1^2}\, c^{(1)}{}_{\varphi, \theta_1, \theta_2} - (m+1)\right) \right.$$

$$\left. + 36 \frac{(m+1)^2}{c^{(1)}{}_{\varphi, \theta_1, \theta_2}} F(0) \mathbb{I}_{r_1 \leq \frac{m+1}{c^{(1)}{}_{\varphi, \theta_1, \theta_2}}} \right)$$

$$\leq \sum_{r_1 = 0}^{\infty} 64(r_1 + 1) \left( \sum_{\substack{r \in \mathbb{Z}_{\geq 0}: \\ \sqrt{r^2 + r_1^2}\, c^{(1)}{}_{\varphi, \theta_1, \theta_2} \geq (m+1)}} 24(m+1) F\left(\sqrt{r^2 + r_1^2}\, c^{(1)}{}_{\varphi, \theta_1, \theta_2} - (m+1)\right) \right.$$

$$\left. + 36 \frac{(m+1)^2}{c^{(1)}{}_{\varphi, \theta_1, \theta_2}} F(0) \mathbb{I}_{r_1 \leq \frac{m+1}{c^{(1)}{}_{\varphi, \theta_1, \theta_2}}} \right)$$

$$\leq 64 \cdot 24 \cdot \left( \frac{3}{c^{(1)}{}_{\varphi, \theta_1, \theta_2}} \right)^2 \left( 2\sqrt{2} + \frac{1}{c^{(1)}{}_{\varphi, \theta_1, \theta_2}} \right)^2 \pi (m+1)^3 \kappa_{1,2,F}$$

$$+ 64 \cdot 36 \cdot \frac{(m+1)^2}{c^{(1)}{}_{\varphi, \theta_1, \theta_2}} F(0) \left( \frac{m+1}{c^{(1)}{}_{\varphi, \theta_1, \theta_2}} + 1 \right)^2$$

$$\leq 64 \cdot 144 \cdot 24 \cdot \left( c^{(1)}{}_{\varphi, \theta_1, \theta_2} \right)^{-4} \left( \pi \kappa_{1,2,F} + F(0) \right) (m+1)^4. \qquad \text{(C.32)}$$

We used formula (C.16). □

## D. Quasilocal automorphisms

In this appendix we collect some results from [NSY] and prove Theorem 5.1.

**Definition D.1.** A norm-continuous interaction on $\mathcal{A}$ defined on an interval $[0, 1]$ is a map $\Phi : \mathfrak{S}_{\mathbb{Z}^2} \times [0, 1] \to \mathcal{A}_{\mathrm{loc}}$ such that

(i) for any $t \in [0, 1]$, $\Phi(\cdot, t) : \mathfrak{S}_{\mathbb{Z}^2} \to \mathcal{A}_{\mathrm{loc}}$ is an interaction and
(ii) for any $Z \in \mathfrak{S}_{\mathbb{Z}^2}$, the map $\Phi(Z, \cdot) : [0, 1] \to \mathcal{A}_Z$ is norm-continuous.

To ensure that the interactions induce quasilocal automorphisms we need to impose sufficient decay properties on the interaction strength.

**Definition D.2.** Let $F$ be an $F$-function on $(\mathbb{Z}^2, d)$. We denote by $\mathcal{B}_F([0,1])$ the set of all norm-continuous interactions $\Phi$ on $\mathcal{A}$ defined on an interval $[0,1]$ such that the function $\|\Phi\|_F : [0,1] \to \mathbb{R}$ defined by

$$\|\Phi\|_F(t) := \sup_{x,y \in \mathbb{Z}^2} \frac{1}{F(d(x,y))} \sum_{Z \in \mathfrak{S}_{\mathbb{Z}^2}, Z \ni x,y} \|\Phi(Z;t)\|, \quad t \in [0,1], \tag{D.1}$$

is uniformly bounded – that is, $\sup_{t \in [0,1]} \|\Phi\|(t) < \infty$. It follows that $t \mapsto \|\Phi\|_F(t)$ is integrable, and we set

$$I_F(\Phi) := I_{1,0}(\Phi) := C_F \int_0^1 dt \, \|\Phi\|_F(t), \tag{D.2}$$

with $C_F$ given in Definition C.1. We also set

$$\|\|\Phi\|\|_F := \sup_{x,y \in \mathbb{Z}^2} \frac{1}{F(d(x,y))} \sum_{Z \in \mathfrak{S}_{\mathbb{Z}^2}, Z \ni x,y} \sup_{t \in [0,1]} (\|\Phi(Z;t)\|) \tag{D.3}$$

and denote by $\hat{\mathcal{B}}_F([0,1])$ the set of all $\Phi \in \mathcal{B}_F([0,1])$ with $\|\|\Phi\|\|_F < \infty$.

We will need some more notation. For $\Phi \in \mathcal{B}_F([0,1])$ and $0 \le m \in \mathbb{R}$, we introduce a path of interactions $\Phi_m$ by

$$\Phi_m(X;t) := |X|^m \Phi(X;t), \quad X \in \mathfrak{S}(\mathbb{Z}^2), \, t \in [0,1]. \tag{D.4}$$

An interaction gives rise to local (and here, time-dependent) Hamiltonians via

$$H_{\Lambda,\Phi}(t) := \sum_{Z \in \Lambda} \Phi(Z,t), \quad t \in [0,1], \, \Lambda \in \mathfrak{S}_{\mathbb{Z}^2}. \tag{D.5}$$

We denote by $U_{\Lambda,\Phi}(t;s)$, the solution of

$$\frac{d}{dt} U_{\Lambda,\Phi}(t;s) = -i H_{\Lambda,\Phi}(t) U_{\Lambda,\Phi}(t;s), \quad s,t \in [0,1], \tag{D.6}$$

$$U_{\Lambda,\Phi}(s;s) = \mathbb{I}. \tag{D.7}$$

We define corresponding automorphisms $\tau_{t,s}^{(\Lambda),\Phi}, \hat{\tau}_{t,s}^{(\Lambda),\Phi}$ on $\mathcal{A}$ by

$$\tau_{t,s}^{(\Lambda),\Phi}(A) := U_{\Lambda,\Phi}(t;s)^* A U_{\Lambda,\Phi}(t;s), \tag{D.8}$$

$$\hat{\tau}_{t,s}^{(\Lambda),\Phi}(A) := U_{\Lambda,\Phi}(t;s) A U_{\Lambda,\Phi}(t;s)^*, \tag{D.9}$$

with $A \in \mathcal{A}$. Note that

$$\hat{\tau}_{t,s}^{(\Lambda),\Phi} = \tau_{s,t}^{(\Lambda),\Phi}, \tag{D.10}$$

by the uniqueness of the solution of the differential equation.

**Theorem D.3** ([NSY]). *Let $F$ be an $F$-function on $(\mathbb{Z}^2, d)$. Suppose that $\Phi \in \mathcal{B}_F([0,1])$. Then the following hold:*

1. *The limits*

$$\tau_{t,s}^{\Phi}(A) := \lim_{\Lambda \nearrow \mathbb{Z}^2} \tau_{t,s}^{(\Lambda),\Phi}(A), \qquad \hat{\tau}_{t,s}^{\Phi}(A) := \lim_{\Lambda \nearrow \mathbb{Z}^2} \hat{\tau}_{t,s}^{(\Lambda),\Phi}(A), \quad A \in \mathcal{A}, \, t, s \in [0, 1], \qquad \text{(D.11)}$$

*exist and define strongly continuous families of automorphisms on $\mathcal{A}$ such that $\hat{\tau}_{t,s}^{\Phi} = \tau_{s,t}^{\Phi} = {\tau_{t,s}^{\Phi}}^{-1}$ and*

$$\hat{\tau}_{t,s}^{\Phi} \circ \hat{\tau}_{s,u}^{\Phi} = \hat{\tau}_{t,u}^{\Phi}, \qquad \tau_{t,t}^{\Phi} = \mathrm{id}_{\mathcal{A}}, \quad t, s, u \in [0, 1]. \qquad \text{(D.12)}$$

2. *For any $X, Y \in \mathfrak{S}_{\mathbb{Z}^2}$ with $X \cap Y = \emptyset$, the bound*

$$\left\| \left[ \tau_{t,s}^{\Phi}(A), B \right] \right\| \leq \frac{2 \|A\| \|B\|}{C_F} \left( e^{2I_F(\Phi)} - 1 \right) |X| \, G_F \left( d(X, Y) \right) \qquad \text{(D.13)}$$

*holds for all $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, and $t, s \in [0, 1]$.*
    *If $\Lambda \in \mathfrak{S}_{\mathbb{Z}^2}$ and $X \cup Y \subset \Lambda$, a similar bound holds for $\tau_{t,s}^{(\Lambda),\Phi}$.*
3. *For any $X \in \mathfrak{S}_{\mathbb{Z}^2}$, we have*

$$\left\| \Delta_{X(m)} \left( \tau_{t,s}^{\Phi}(A) \right) \right\| \leq \frac{8 \|A\|}{C_F} \left( e^{2I_F(\Phi)} - 1 \right) |X| \, G_F(m), \qquad \text{(D.14)}$$

*for $A \in \mathcal{A}_X$. Here we set $\Delta_{X(0)} := \Pi_X$ and $\Delta_{X(m)} := \Pi_{X(m)} - \Pi_{X(m-1)}$ for $m \in \mathbb{N}$. A similar bound holds for $\tau_{t,s}^{(\Lambda),\Phi}$. (See formula (C.12) for the definition of $G_F$.)*
4. *For any $X, \Lambda \in \mathfrak{S}(\mathbb{Z}^2)$, with $X \subset \Lambda$, and $A \in \mathcal{A}_X$, we have*

$$\left\| \tau_{t,s}^{(\Lambda),\Phi}(A) - \tau_{t,s}^{\Phi}(A) \right\| \leq \frac{2}{C_F} \|A\| \, e^{2I_F(\Phi)} I_F(\Phi) |X| \, G_F \left( d \left( X, \mathbb{Z}^2 \setminus \Lambda \right) \right). \qquad \text{(D.15)}$$

5. *If $\beta_g^U(\Phi(X; t)) = \Phi(X; t)$ for any $X \in \mathfrak{S}_{\mathbb{Z}^2}$, $t \in [0, 1]$, and $g \in G$, then we have $\beta_g^U \circ \tau_{t,s}^{\Phi} = \tau_{t,s}^{\Phi} \circ \beta_g^U$ for any $t, s \in [0, 1]$ and $g \in G$,*

*Proof.* Item 1 is [NSY, Theorem 3.5], and 2 and 4 follow from Corollary 3.6 of the same paper by, respectively, a straightforward bounding of $D(X, Y)$ and the summation in [NSY, equation (3.80)]. Item 3 can be obtained using 2 and [NSY, Corollary 4.4].
    Suppose that $\beta_g^U(\Phi(X; t)) = \Phi(X; t)$ for any $X \in \mathfrak{S}_{\mathbb{Z}^2}$, $t \in [0, 1]$, and $g \in G$. Then we have

$$\frac{d}{dt} \beta_g^U \left( U_{\Lambda,\Phi}(t; s) \right) = -\beta_g^U \left( i H_{\Lambda,\Phi}(t) \right) \beta_g^U \left( U_{\Lambda,\Phi}(t; s) \right) = -i H_{\Lambda,\Phi}(t) \beta_g^U \left( U_{\Lambda,\Phi}(t; s) \right), \quad t \in [0, 1], \qquad \text{(D.16)}$$

and $\beta_g^U \left( U_{\Lambda,\Phi}(s; s) \right) = \mathbb{I}$. Hence $\beta_g^U \left( U_{\Lambda,\Phi}(t; s) \right)$ and $U_{\Lambda,\Phi}(t; s)$ satisfy the same differential equation and initial condition. Therefore we get $\beta_g^U \left( U_{\Lambda,\Phi}(t; s) \right) = U_{\Lambda,\Phi}(t; s)$. From this, we obtain $\beta_g^U \tau_{t,s}^{(\Lambda),\Phi} = \tau_{t,s}^{(\Lambda),\Phi} \beta_g^U$, and taking $\Lambda \uparrow \mathbb{Z}^2$, we obtain $\beta_g^U \circ \tau_{t,s}^{\Phi} = \tau_{t,s}^{\Phi} \circ \beta_g^U$. □

The following is slightly strengthened version of [NSY, Assumption 5.15]:

**Assumption D.4** ([NSY])**.** We assume that the family of linear maps $\{\mathcal{K}_t : \mathcal{A}_{\mathrm{loc}} \to \mathcal{A}\}_{t \in [0,1]}$ is norm-continuous and satisfies the following: There is a family of linear maps $\left\{ \mathcal{K}_t^{(n)} : \mathcal{A}_{\Lambda_n} \to \mathcal{A}_{\Lambda_n} \right\}_{t \in [0,1]}$ for each $n \geq 1$ such that the following are true:

(i) For each $n \geq 1$, the family $\left\{ \mathcal{K}_t^{(n)} : \mathcal{A}_{\Lambda_n} \to \mathcal{A}_{\Lambda_n} \right\}_{t \in [0,1]}$ satisfies the following conditions:

    (a) For each $t \in [0, 1]$, $\left( \mathcal{K}_t^{(n)}(A) \right)^* = \mathcal{K}_t^{(n)}(A^*)$ for all $\mathcal{A}_{\Lambda_n}$.

(b) For each $A \in \mathcal{A}_{\Lambda_n}$, the function $[0, 1] \ni t \to \mathcal{K}_t^{(n)}(A)$ is norm-continuous.

(c) For each $t \in [0, 1]$, the map $\mathcal{K}_t^{(n)} : \mathcal{A}_{\Lambda_n} \to \mathcal{A}_{\Lambda_n}$ is norm-continuous, and moreover, this continuity is uniform on $[0, 1]$.

(ii) There is some $p \geq 0$ and a constant $B_1 > 0$ for which, given any $X \in \mathfrak{S}_{\mathbb{Z}^2}$ and $n \geq 1$ large enough so that $X \subset \Lambda_n$,

$$\left\| \mathcal{K}_t^{(n)}(A) \right\| \leq B_1 |X|^p \|A\|, \quad \text{for all } A \in \mathcal{A}_X \quad \text{and} \quad t \in [0, 1].$$

(iii) There is some $q \geq 0$, a nonnegative, nonincreasing function $G$ with $G(x) \to 0$ as $x \to \infty$, and a constant $C_1 > 0$ for which, given any sets $X, Y \in \mathfrak{S}_{\mathbb{Z}^2}$ and $n \geq 1$ large enough so that $X \cup Y \subset \Lambda_n$,

$$\left\| \left[ \mathcal{K}_t^{(n)}(A), B \right] \right\| \leq C_1 |X|^q \|A\| \|B\| G(\mathrm{d}(X, Y)), \quad \text{for all } A \in \mathcal{A}_X, \, B \in \mathcal{A}_Y, \, t \in [0, 1].$$

(iv) There is some $r \geq 0$, a nonnegative, nonincreasing function $H$ with $H(x) \to 0$ as $x \to \infty$, and a constant $D_1 > 0$ for which, given any $X \in \mathfrak{S}_{\mathbb{Z}^2}$, there exists $N \geq 1$ such that for $n \geq N$,

$$\left\| \mathcal{K}_t^{(n)}(A) - \mathcal{K}_t(A) \right\| \leq D_1 |X|^r \|A\| H\left( \mathrm{d}\left( X, \mathbb{Z}^2 \setminus \Lambda_n \right) \right)$$

for all $A \in \mathcal{A}_X$ and $t \in [0, 1]$.

The following theorem is a slight modification of [NSY, Theorem 5.17]:

**Theorem D.5.** *Set $F \in \mathcal{F}_a$, with $\tilde{F}_\theta$ in formula (C.2) for each $0 < \theta < 1$. Assume that $\{\mathcal{K}_t\}_{t \in [0,1]}$ is a family of linear maps satisfying Assumption D.4, with $G = G_F$ in part (iii). (Recall Definition C.2 and formula (C.12)). Let $\Phi \in \mathcal{B}_F([0, 1])$ be an interaction satisfying $\Phi_m \in \mathcal{B}_F([0, 1])$ for $m = \max\{p, q, r\}$, where $p, q, r$ are numbers in Assumption D.4. Then the right-hand side of the sum*

$$\Psi(Z, t) := \sum_{m \geq 0} \sum_{X \subset Z, \, X(m)=Z} \Delta_{X(m)}(\mathcal{K}_t(\Phi(X; t))), \quad Z \in \mathfrak{S}_{\mathbb{Z}^2}, \, t \in [0, 1] \tag{D.17}$$

*defines a path of interaction such that $\Psi \in \mathcal{B}_{\tilde{F}_\theta}([0, 1])$, for any $0 < \theta < 1$. Furthermore, the formula*

$$\Psi^{(n)}(Z, t) := \sum_{m \geq 0} \sum_{X \subset Z, \, X(m) \cap \Lambda_n = Z} \Delta_{X(m)}\left( \mathcal{K}_t^{(n)}(\Phi(X; t)) \right) \tag{D.18}$$

*defines $\Psi^{(n)} \in \mathcal{B}_{\tilde{F}_\theta}([0, 1])$, for any $0 < \theta < 1$, such that $\Psi^{(n)}(Z, t) = 0$ unless $Z \subset \Lambda_n$, and satisfies*

$$\mathcal{K}_t^{(n)}\left( H_{\Lambda_n, \Phi}(t) \right) = H_{\Lambda_n, \Psi^{(n)}}(t). \tag{D.19}$$

*For any $t, u \in [0, 1]$, we have*

$$\lim_{n \to \infty} \left\| \tau_{t,u}^{\Psi^{(n)}}(A) - \tau_{t,u}^{\Psi}(A) \right\| = 0, \quad A \in \mathcal{A}. \tag{D.20}$$

*Furthermore, if $\Phi_{m+k} \in \hat{\mathcal{B}}_F([0, 1])$ for $k \in \mathbb{N} \cup \{0\}$, then we have $\Psi_k^{(n)}, \Psi_k \in \hat{\mathcal{B}}_{\tilde{F}_\theta}([0, 1])$ for any $0 < \theta < 1$.*

*Proof.* Because of $F \in \mathcal{F}_a$, we see from formula (C.14) that for any $0 < \alpha < 1$ and $k \in \mathbb{N}$, $G_F^\alpha$ has a finite $k$-moment. We also recall formulas (C.2) and (C.14) to see that

$$\max \left\{ F\left( \frac{r}{3} \right), \sum_{n=[\frac{r}{3}]}^{\infty} (1+n)^5 G_F(n)^\alpha \right\} \leq \tilde{C} \tilde{F}_{\alpha(1-\theta')\varphi}(r), \quad r \geq 0, \tag{D.21}$$

for any $0 < \alpha, \theta', \varphi < 1$. As this holds for any $0 < \alpha, \theta', \varphi < 1$, the condition in [NSY, Theorem 5.17(ii)] holds for any $\tilde{F}_\theta$. Therefore, from [NSY, Theorem 5.17(ii)], we get $\Psi, \Psi^{(n)} \in \mathcal{B}_{\tilde{F}_\theta}([0,1])$ and $\Psi^{(n)}$ converges locally in $F$-norm to $\Psi$ with respect to $\tilde{F}_\theta$, for any $0 < \theta < 1$.

From [NSY, Theorem 5.13] we have the implication

$$\sup_n \int_0^1 \left\| \Psi^{(n)} \right\|_{\tilde{F}_\theta} (t)dt < \infty \tag{D.22}$$

(see also [NSY, equation (5.101)]). Therefore, from [NSY, Theorem 3.8], we obtain equation (D.20).

By the proofs of [NSY, Theorems 5.17 and 5.13, equation (5.87)], if $\Phi_{k+m} \in \hat{\mathcal{B}}_F([0,1])$ for some $k \in \mathbb{N}$, then we have $\Psi_k^{(n)(s)}, \Psi_k^{(s)} \in \hat{\mathcal{B}}_{\tilde{F}}([0,1])$. More precisely, instead of [NSY, equation (5.88)], we obtain

$$\sum_{\substack{Z \in \mathfrak{S}_{\mathbb{Z}^2} \\ Z \ni x,y}} |Z|^k \sup_{t \in [0,1]} \|\Psi(Z;t)\|$$

$$\leq B_1 \sum_{\substack{Z \in \mathfrak{S}_{\mathbb{Z}^2} \\ Z \ni x,y}} |Z|^{k+p} \sup_{t \in [0,1]} \|\Phi(Z;t)\| + 4C_1 \sum_{n=0}^\infty G_F(n) \sum_{X: X(n+1) \ni x,y} |X|^q |X(n+1)|^k \sup_{t \in [0,1]} \|\Phi(X;t)\|$$

$$\leq B_1 \left\|\|\Phi_{k+p}\|\right\|_F F(\mathrm{d}(x,y)) + 4C_1 \sum_{n=0}^\infty G_F(n)(2n+3)^{2k} \sum_{X: X(n+1) \ni x,y} |X|^{q+k} \sup_{t \in [0,1]} \|\Phi(X;t)\|$$

$$\leq B_1 \left\|\|\Phi_{k+p}\|\right\|_F F(\mathrm{d}(x,y)) + \tilde{C}_\theta \tilde{F}_\theta(\mathrm{d}(x,y)) \left\|\|\Phi_{q+k}\|\right\|_F < \infty, \tag{D.23}$$

with some constant $\tilde{C}_\theta$, for each $0 < \theta < 1$. In the last line we used formula (C.14) and [NSY, Lemma 8.9]. Hence we get $\Psi_k^{(n)}, \Psi_k \in \hat{\mathcal{B}}_{\tilde{F}_\theta}([0,1])$. □

*Proof of Theorem 5.1.* Suppose $\Phi_0 \sim \Phi_1$ via a path $\Phi$. Our definition of $\Phi_0 \sim \Phi_1$ means the existence of a path of interactions satisfying [MO, Assumption 1.2]. Therefore, [MO, Theorem 1.3] guarantees the existence of a path of quasilocal automorphisms $\alpha_t$ satisfying $\omega_{\Phi_1} = \omega_{\Phi_0} \circ \alpha_1$. From the proof in [MO], the automorphism $\alpha_t$ is given by a family of interactions

$$\Psi(Z,t) := \sum_{m \geq 0} \sum_{X \subset Z, X(m)=Z} \Delta_{X(m)} \left( \mathcal{K}_t \left( \dot{\Phi}(X;t) \right) \right), \quad Z \in \mathfrak{S}_{\mathbb{Z}^2}, \ t \in [0,1], \tag{D.24}$$

with

$$\mathcal{K}_t(A) := -\int du W_\gamma(u) \tau_u^{\Phi(t)}(A), \tag{D.25}$$

as $\alpha_t = \tau_{t,0}^\Psi$. (Note that by the partial integral of [MO, equation (1.19)], we obtain [NSY, equation (6.103)] with function $W_\gamma$ in [NSY, equation (6.35)]).) The interaction $\Psi$ actually belongs to $\hat{\mathcal{B}}_{F_3}([0,1])$ for some $F_3 \in \mathcal{F}_a$. To see this, note that the path $\Phi$ in Definition 1.2 satisfies [NSY, Assumption 6.12] for any $F$-function, because

$$\sum_{\substack{X \in \mathfrak{S}_{\mathbb{Z}^2} \\ X \ni x,y}} \left( \|\Phi(X;s)\| + |X| \left\| \dot{\Phi}(X;s) \right\| \right) \leq \frac{2^{(2R+1)^2} C_b^\Phi}{F(R)} F(\mathrm{d}(x,y)), \tag{D.26}$$

with $C_b^\Phi$ and $R$ given in Definition 1.2 3 and 4. In particular, it satisfies [NSY, Assumption 6.12] with respect to the $F$-function (see [NSY, Section 8]) $F_1(r) := \frac{e^{-r}}{(1+r)^4}$. By [NSY, Section 8], $F_1$ belongs to $\mathcal{F}_a$.

Fix any $0 < \alpha < 1$. Then by [NSY, Proposition 6.13] and its proof, the family of maps given by formula (D.25) [NSY, equation (6.102)] satisfies Assumption D.4, with $p = 0$, $q = 1$, $r = 1$ and $G = G_{F_2}$, where $F_2(r) = (1+r)^{-4} \exp(-r^{\alpha})$. Furthermore, we have $\dot{\Phi}_m \in \hat{\mathcal{B}}_{F_2}([0,1])$ for any $m \in \mathbb{N}$, because

$$\left\| \left| \dot{\Phi}_m \right| \right\|_{F_2} := \sup_{x,y \in \mathbb{Z}^2} \frac{1}{F_2(d(x,y))} \sum_{Z \in \mathfrak{S}_{\mathbb{Z}^2}, \, Z \ni x,y} \sup_{t \in [0,1]} |Z|^m \left( \left\| \dot{\Phi}(Z;t) \right\| \right) \le \frac{2^{(2R+1)^2}(2R+1)^{2m} C_b^{\Phi}}{F_2(R)} < \infty. \tag{D.27}$$

We have $F_2 \in \mathcal{F}_a$, and fixing any $0 < \alpha' < \alpha$, $\tilde{F}_2(r) := (1+r)^{-4} \exp(-r^{\alpha'})$ satisfies

$$\max\left\{ F_2\left(\frac{r}{3}\right), \left(F_2\left(\left[\frac{r}{3}\right]\right)\right)^{\theta} \right\} \le C_{2,\theta,\alpha'} \tilde{F}_2(r), \quad r \ge 0, \tag{D.28}$$

for a suitable constant $C_{2,\theta,\alpha'}$.

Therefore, by Theorem D.5, $\Psi$ given by formula (D.24) for this $\mathcal{K}_t$ and $\dot{\Phi}$ satisfy $\Psi_1, \Psi \in \hat{\mathcal{B}}_{\tilde{F}_2}([0,1])$ for $\tilde{F}_2 \in \mathcal{F}_a$.

If $\Phi$ is $\beta_g$-invariant, then $\tau^{\Phi(t)}$ commutes with $\beta_g$, hence $\mathcal{K}_t$ commutes with $\beta_g$. As $\Pi_X$ commutes with $\beta_g$ and $\dot{\Phi}$ is $\beta_g$-invariant, we see that $\Psi$ is $\beta_g$-invariant. □

**Proposition D.6.** *Let $F, \tilde{F} \in \mathcal{F}_a$ be F-functions of the form $F(r) = (1+r)^{-4} \exp(-r^{\theta})$, $\tilde{F}(r) := (1+r)^{-4} \exp(-r^{\theta'})$ with some constants $0 < \theta' < \theta < 1$. Let $\Psi, \tilde{\Psi} \in \mathcal{B}_F([0,1])$ be a path of interactions such that $\Psi_1 \in \mathcal{B}_F([0,1])$. Finally, let $\tau_{t,s}^{\tilde{\Psi}}$ and $\tau_{t,s}^{(\Lambda_n),\tilde{\Psi}}$ be automorphisms given by $\Psi, \tilde{\Psi}$ from Theorem D.3.*

*Then, with $s \in [0,1]$, the right-hand side of the sum*

$$\Xi^{(s)}(Z,t) := \sum_{m \ge 0} \sum_{X \subset Z, \, X(m) = Z} \Delta_{X(m)}\left( \tau_{t,s}^{\tilde{\Psi}}(\Psi(X;t)) \right), \quad Z \in \mathfrak{S}_{\mathbb{Z}^2}, \, t \in [0,1], \tag{D.29}$$

*defines a path of interaction such that $\Xi^{(s)} \in \mathcal{B}_{\tilde{F}}([0,1])$. Furthermore, the formula*

$$\Xi^{(n)(s)}(Z,t) := \sum_{m \ge 0} \sum_{X \subset Z, \, X(m) \cap \Lambda_n = Z} \Delta_{X(m)}\left( \tau_{t,s}^{(\Lambda_n),\tilde{\Psi}}(\Psi(X;t)) \right) \tag{D.30}$$

*defines $\Xi^{(n)(s)} \in \mathcal{B}_{\tilde{F}}([0,1])$ such that $\Xi^{(n)}(Z,t) = 0$ unless $Z \subset \Lambda_n$, and satisfies*

$$\tau_{t,s}^{(\Lambda_n),\tilde{\Psi}}\left( H_{\Lambda_n,\Psi}(t) \right) = H_{\Lambda_n,\Xi^{(n)(s)}}(t). \tag{D.31}$$

*For any $t, u \in [0,1]$, we have*

$$\lim_{n \to \infty} \left\| \tau_{t,u}^{\Xi^{(n)(s)}}(A) - \tau_{t,u}^{\Xi^{(s)}}(A) \right\| = 0, \quad A \in \mathcal{A}. \tag{D.32}$$

*Furthermore, if $\Psi_1 \in \hat{\mathcal{B}}_F([0,1])$, then we have $\Xi^{(n)(s)}, \Xi^{(s)} \in \hat{\mathcal{B}}_{\tilde{F}}([0,1])$.*

*Proof.* From Theorem D.5, it suffices to show that the family $\left\{ \mathcal{K}_t := \tau_{t,u}^{\tilde{\Psi}} \right\}$ satisfies Assumption D.4. This follows from Theorem D.3. □

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[1]The present result and the main idea of the proof were announced publicly on 15 December 2020 at the IAMP One World Mathematical Physics Seminar (see YouTube video) [O2-1, O2-2, O2-3], the Theoretical Studies of Topological Phases of Matter international meeting on 17 December 2020, and in *Current Developments in Mathematics* on 4 January 2021 via Zoom with a lecture note [O3]. Our approach is operator-algebraic. Just after this paper was posted to arXiv, a paper reporting a similar result, based on quantum information [EN], was posted [S].

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