SOME RESULTS ON v-MULTIPLICATION RINGS

MALCOLM GRIFFIN

1. Introduction. A family Ω of valuations of the field K is said to be of *finite character* if only a finite number of valuations are non-zero at any non-zero element of K. If $w \in \Omega$ has ring $\mathfrak{R}_w$ and maximal ideal $\mathfrak{P}_w$, then $A = \bigcap_{w\in\Omega} \mathfrak{R}_w$ is said to be *defined by* Ω and $\mathfrak{P}_w \cap A$ is a prime ideal called the *centre of* w *on* A and denoted by $Z(w)$. If $\mathfrak{R}_w = A_{Z(w)}$, then w is said to be an *essential valuation* for A. A domain defined by a family of finite character in which every valuation is essential is called a *ring of Krull type.*

By generalizing a proposition due to Lorenzen we extend some results known for Prufer rings to v-multiplication rings. We use these results to investigate the relationship between v-multiplication rings and rings of Krull type. Finally we investigate the preservation of rings of Krull type under various extensions. Further properties of rings of Krull type are investigated in **(2, 3)**.

We give a brief outline of results from the theory of systems of ideals. For further details the reader is referred to Jaffard's book **(4)**.

Let A be a ring with quotient field K. To each non-zero fractionary ideal M associate a fractionary ideal M_r so that this mapping has the following properties:

(1) $M \subseteq M_r$;

(2) $M \subseteq N_r$ implies that $M_r \subseteq N_r$;

(3) $A = A_r$;

(4) $aM_r = (aM)_r$ for all $a \in K$.

The ideals of the form M_r are said to form a *system of r-ideals* for A. An *r-ideal*, M_r, is a *finite* (an *integral*) *r-ideal* if M is a finitely generated (an integral) ideal. An integral r-ideal, M_r, is a *prime r-ideal*, if $ab \in M_r$, with $a, b \in A$, $a \notin M_r$, implies that $b \in M_r$. An integral *r-ideal*, M_r, is a *maximal r-ideal* if $M_r \subseteq N_r \subset A$ implies that $N_r = M_r$.

The product of r-ideals is defined by $M_r \times {}_r N_r = (M_r N_r)_r$. It is easily shown that $(MN)_r = M_r \times_r N_r = M \times_r N$. An r-ideal M is *r-invertible* if there is an r-ideal N such that $M \times_r N = A$.

We say that the r-system is *coarser* than the g-system if for every fractionary ideal M, $M_g \subseteq M_r$. The coarsest possible system of ideals is the *v-system* which is defined by $M_v = \bigcap_{(x)\supseteq M} (x)$. Any *r-system* of ideals gives

Received February 1, 1966. This constitutes part of the research done for a Ph.D. degree at Queen's University, Kingston, Ontario, under the direction of Dr. Paulo Ribenboim, with financial assistance from the National Research Council of Canada.

rise to a new system of ideals, the r_s-*system*, defined by $M_{r_s} = \bigcup_{N \in \mathfrak{E}} N_r$, where $\mathfrak{E}$ is the family of ideals generated by finite subsets of M. An r-system is said to be of *finite character* if it is equal to its own r_s-system. The v_s-system (which is of finite character) is called the *t-system.*

We use the following propositions, the proofs of which may be found in **(4)**.

I. Any r-invertible r-ideal is a v-ideal.

II. If the system of r-ideals has finite character, then (a) all r-invertible r-ideals are finite, and (b) if $M_r \subset A$, then M_r is contained in a maximal r-ideal; this maximal r-ideal is prime.

A *Prufer ring* is a ring in which every finitely generated ideal is invertible.

A ring in which the finite v-ideals, with the product defined above, form a group is called a *v-multiplication ring.*

Let A be a ring with quotient field K. Form a multiplicative pre-ordered group by setting $x \leqslant y$ when $x, y \in K$ and $y = ax$ for some $a \in A$. The corresponding ordered group is called the *divisibility group of* A.

Let Γ be a totally ordered group. A subgroup Δ of Γ is called an *isolated subgroup* if $\delta \in \Delta$ with $\delta \geqslant \gamma \geqslant 0$ implies that $\gamma \in \Delta$.

Let Γ be a totally ordered group; then a subset U of Γ is called an *upper class* provided that:

(1) if $\alpha \in U$ and $\beta \in \Gamma$ with $\beta > \alpha$, then $\beta \in U$;

(2) there exists $\gamma \in \Gamma$ such that $\gamma \notin U$.

If Γ is a totally ordered group it is possible to construct a totally ordered monoid Λ^* **(7)**, and a one-to-one map ϕ from the upper classes of Γ into Λ^* such that:

(1) $U \subset U' \Leftrightarrow \phi(U) > \phi(U')$;

(2) $\phi(U + U') = \phi(U) + \phi(U')$.

There is no loss of generality in assuming that the image Γ^* of the upper classes of Γ under ϕ contains Γ as an ordered group.

Let w, w' be valuations of K with groups Γ, Γ' and rings $\mathfrak{R}$, $\mathfrak{R}'$. Then w' *is coarser than* w, written $w' \leqslant w$ if $\mathfrak{R} \subseteq \mathfrak{R}'$.

If $w' \leqslant w$, then for some prime ideal $\mathfrak{P}$ of $\mathfrak{R}$, $\mathfrak{R}' \cong \mathfrak{R}_{\mathfrak{P}}$ and $\Gamma' \cong \Gamma/\Delta$, where Δ is the isolated subgroup of Γ generated by those elements of $\mathfrak{R}$ which do not belong to $\mathfrak{P}$.

2. Rings defined by well-centred valuations.

A valuation w with ring containing A is said to be *well centred* on A if for each positive element γ in the value group of w there is an element $a \in A$ such that $w(a) = \gamma$. It is easy to show that if w is essential for A then w is well centred on A.

Let w be well centred on A with value group Γ_w. For a fractionary ideal M, define $w(M) = \{w(x) | x \in M, x \neq 0\}$. Then if $\alpha \in w(M)$ and $\beta > \alpha$ there exists $a \in A$ such that $w(a) = \beta - \alpha > 0$, since $w(x) = \alpha$ for some $x \in M$, $ax \in M$ and $w(ax) = \alpha + \beta - \alpha = \beta$; it follows that $\beta \in w(M)$. Also, since M is fractionary, $w(M)$ is bounded below. It follows that $w(M)$ is an

https://doi.org/10.4153/CJM-1967-065-8 Published online by Cambridge University Press

upper class. For each such w we may define $M(w) \in \Gamma_w^*$ to be the element corresponding to the upper class $w(M)$.

Let Ω be a family defining A with every valuation of Ω well centred on A. We define a map Φ by associating to each non-zero fractionary ideal M of A an element of $\prod_{w\in\Omega} \Gamma_w^*$ given by $(\Phi(M))_w = M(w)$. We define addition and order on $\prod_{w\in\Omega} \Gamma_w^*$ componentwise using the structure on each Γ_w^*.

THEOREM 1. *Let Ω be a family of well-centred valuations of K defining A. Let $L, M, N, \{M_i, i \in I\}, \sum_{i\in I} M_i$ be non-zero fractionary ideals of A. Then*
(1) *$M \subseteq N$ implies that $\Phi(M) \geqslant \Phi(N)$;*
(2) $\Phi(MN) = \Phi(M) + \Phi(N)$;
(3) $\Phi(\sum_{i\in I} M_i) = \bigcap_{i\in I} \Phi(M_i)$;
(4) *if* $\Phi((x_1 \ldots x_n)M) = \Phi((x_1 \ldots x_n)N)$ *with* $x_i \in K$, $i = 1, \ldots, n$, *then* $\Phi(M) = \Phi(N)$; (5′) $\Phi(M \cap N) \geqslant \Phi(M) \vee \Phi(N)$.
If all the valuations of Ω are essential we have in addition:
(5) $\Phi(M \cap N) = \Phi(M) \vee \Phi(N)$;
(6) $\Phi(M(L \cap N)) = \Phi(ML \cap MN)$;
(7) $\Phi((M + N)(M \cap N)) = \Phi(MN)$;
(8) $\Phi(M \cap (L + N)) = \Phi(M \cap L + M \cap N)$.
If the valuations of Ω are essential and N and N' are finitely generated, then
(9) $\Phi(M: N) = \Phi(M) - \Phi(N)$;
(10) $\Phi((L + M): N) = \Phi(L: N + M: N)$;
(11) $\Phi(M: N \cap N') = \Phi(M: N + M: N')$.

Proof. (1) Trivial.
(2) If $x \in MN$, then

$$x = \sum_1^n a_i b_i$$

with $a_i \in M$, $b_i \in N$. Thus $w(x) \geqslant \min_{1\leqslant i\leqslant n} \{w(a_i) + w(b_i)\}$, so that

$$w(MN) \subseteq w(M) + w(N).$$

If $\alpha \in w(M)$, $\beta \in w(N)$, let $x \in M$, $y \in N$ be such that $w(x) = \alpha$, $w(y) = \beta$. Then $xy \in MN$, $w(xy) = \alpha + \beta$, so $w(M) + w(N) \subseteq w(MN)$ and $w(M) + w(N) = w(MN)$.

Since $(MN)(w) \in \Gamma_w^*$, $M(w) \in \Gamma_w^*$, $N(w) \in \Gamma_w^*$, correspond respectively to the upper classes $w(MN)$, $w(M)$, $w(N)$, it follows that $M(w) + N(w)$ corresponds to the upper class $w(M) + w(N) = w(MN)$; that is

$$(MN)(w) = M(w) + N(w).$$

This holds for all $w \in \Omega$ so that (2) is proved.

(3) Since $w(a + b) \geqslant \min \{w(a), w(b)\}$ with equality when $w(a) \neq w(b)$, it follows that $w(M_i + M_k) = w(M_i) \cup w(M_k)$ and

$$w(\textstyle\sum_{i\in I} M_i) = \bigcup_{i\in I} w(M_i),$$

so that $(\sum_{i\in I} M_i)(w) = \inf_{i\in I}\{M_i(w)\}$, and $\Phi(\sum_{i\in I} M) = \bigwedge_{i\in I}\Phi(M)$.

(4) From (2) it follows that $((x_1 \ldots x_n)M)(w) = (x_1 \ldots x_n)(w) + M(w)$ and $((x_1 \ldots x_n)N)(w) = (x_1 \ldots x_n)(w) + N(w)$, but

$$(x_1 \ldots x_n)(w) = \inf_{1\leqslant i\leqslant n} w(x_i)$$

and consequently has an inverse in Γ_w^* (for it is an element of Γ_w). We deduce that $M(w) = N(w)$ and consequently that $\Phi(M) = \Phi(N)$.

(5′) This is trivial since for all $w \in \Omega$, $w(M \cap N) \subseteq w(M) \cap w(N)$.

(5) Using the fact that intersections of ideals are preserved on passing to quotient rings,

$$\mathfrak{R}_w(M \cap N) = A_P(M \cap N) = A_P M \cap A_P N = \mathfrak{R}_w M \cap \mathfrak{R}_w N,$$

so that

$$\begin{aligned} w(M \cap N) &= w(\mathfrak{R}_w(M \cap N)) = w(\mathfrak{R}_w M \cap \mathfrak{R}_w N) \\ &= w(\mathfrak{R}_w M) \cap w(\mathfrak{R}_w N) = w(M) \cap w(N). \end{aligned}$$

Hence $(M \cap N)(w) = M(w) \vee N(w)$ and $\Phi(M \cap N) = \Phi(M) \vee \Phi(N)$.

(6–8) Since the valuations in Ω are essential, $\mathfrak{R}_w = A_P$, but transition to quotient rings preserves sums, products, and intersections of ideals, so it is sufficient to prove (6), (7), and (8) componentwise. These relations follow at once from (2), (3), and (5) applied to each quotient ring $\mathfrak{R}_w$.

(9) Since quotients of ideals by finitely generated ideals are preserved by passage to quotient rings,

$$\begin{aligned} \mathfrak{R}_w(M: N) &= A_P(M: N) = A_P M: A_P N = \mathfrak{R}_w M: \mathfrak{R}_w N \\ &= \{x \in K \mid x\mathfrak{R}_w N \subseteq \mathfrak{R}_w M\} = \{x \in K \mid w(x) + N(w) \geqslant M(w)\} \\ &= \{x \in K \mid w(x) \geqslant M(w) - N(w)\}, \end{aligned}$$

for since N is finitely generated, $N(w) \in \Gamma_w$. Thus $(M: N)(w) = M(w) - N(w)$, i.e. $\Phi(M: N) = \Phi(M) - \Phi(N)$.

(10, 11) It is sufficient to prove (10) and (11) componentwise, i.e. we may assume that $A = \mathfrak{R}_w$, and hence, since the ideals of a valuation ring are totally ordered by inclusion, that $L \subseteq M$ and $N' \subseteq N$. Then

$$\begin{aligned} (L + M): N &= M: N = L: N + M: N, \\ M: (N \cap N') &= M: N' = M: N' + M: N. \end{aligned}$$

This completes the proof.

We define an equivalence relation on the non-zero fractionary ideals of A by setting $M \equiv N$ when $\Phi(M) = \Phi(N)$.

Let $M_c = \bigcup_{N\equiv M} N$. Then by Theorem 1 (3) we see that $\Phi(M_c) = \Phi(M)$ and M_c is the largest ideal which is equivalent to M.

We note that if $x \in K$, then $(x)_c = (x)$ and it follows as a consequence of Theorem 1 (2) that $(xM)_c = (x)M_c = xM_c$. Since it is obviously true that $M \subseteq M_c$ and that if $M \subseteq N_c$, then $M_c \subseteq N_c$, we see that we have a system of ideals, the *c-ideals*.

PROPOSITION 2. *Let Ω be a well-centred family of valuations defining the ring A. Let M be any fractionary ideal. Then $M_c = \bigcap_{w \in \Omega} M\mathfrak{R}_w$.*

Proof. Let $M' = \bigcap_{w \in \Omega} M\mathfrak{R}_w$.

From $M = MA = M(\bigcap_{w \in \Omega} \mathfrak{R}_w) \subseteq M\mathfrak{R}_w$ it follows that $M \subseteq M'$ so that $M_c \subseteq (M')_c$.

From $M' \subseteq M\mathfrak{R}_w$ we deduce that $M'(w) \geqslant (M\mathfrak{R}_w)(w) = M(w)$ so that $(M')_c \subseteq M_c$, and consequently $(M')_c = M_c$.

Now if $x \in (M')_c$, then from $w(x) \geqslant M'(w) \geqslant M(w)$ it follows that there exists $y \in M$ such that $w(x) \geqslant w(y)$, i.e. $x \in \mathfrak{R}_w y \subseteq \mathfrak{R}_w M$. Since this holds for every $w \in \Omega$, $x \in \bigcap_{w \in \Omega} \mathfrak{R}_w M = M'$, and $(M')_c \subseteq M'$.

Thus $M' = (M')_c = M_c$.

3. Systems of ideals.

PROPOSITION 3. *The following conditions for an integral domain A are equivalent:*

(1) *the finite v-ideals of A are v-invertible;*

(2) *there is a system of ideals of A, the r-ideals, such that for any finite integral r-ideal M and family of integral r-ideals N_i, $i \in I$, with non-zero intersection*

$$M \times_r (\bigcap_{i \in I} N_i) = \bigcap_{i \in I} (M \times_r N_i);$$

(3) *there is a system of ideals of A, the r-ideals, such that for any finite integral r-ideals M and N, $(M + N) \times_r (M \cap N) = M \times_r N$.*

Proof. We first show that if the finite v-ideals are *v-invertible*, then (2) holds for the v-ideals. Let M be a finite integral v-ideal and N_i, $i \in I$, be integral v-ideals with non-zero intersection.

$$M(\bigcap_{i \in I} N_i) \subseteq MN_i \subseteq (MN_i)_v = M \times_v N_i,$$

so that

$$M \times_v (\bigcap_{i \in I} N_i) = (M(\bigcap_{i \in I} N_i))_v \subseteq \bigcap_{i \in I} (M \times_v N_i).$$

Since M is a finite v-ideal, it has an inverse M^{-1}, and

$$M^{-1} \times_v (\bigcap_{j \in I} (M \times_v N_i)) \subseteq M^{-1} \times_v M \times_v N_i = N_i,$$

so that

$$M^{-1} \times_v (\bigcap_{i \in I} (M \times_v N_i)) \subseteq \bigcap_{i \in I} N_i$$

and

$$\bigcap_{i \in I} (M \times_v N_i) \subseteq M \times_v (\bigcap_{i \in I} N_i);$$

hence

$$\bigcap_{i \in I} (M \times_v N_i) = M \times_v (\bigcap_{i \in I} N_i).$$

We show that if (2) holds for the r-ideals, then so does (3). For any integral ideals we have $(M + N)(M \cap N) \subseteq MN + NM = MN$ so that

$$(M + N) \times_r (M \cap N) \subseteq M \times_r N.$$

By (2) we deduce that

$$(M + N) \times_r (M \cap N) = ((M + N) \times_r M) \cap ((M + N) \times_r N) \supseteq M \times_r N$$

and hence $(M + N) \times_r (M \cap N) = M \times_r N$.

We prove by induction on the number of generators of the finite r-ideal $L = (m_1 \ldots m_n)_r$ that if (3) holds then every finitely generated r-ideal is r-invertible.

It is obvious that every principal ideal is r-invertible. We assume that every r-ideal with $n - 1$ or less generators is r-invertible. Let

$$M = (m_1 \ldots m_{n-1})_r, \ N = (m_n);$$

then by (3) $L \times_r (M \cap N) = (M + N) \times_r (M \cap N) = M \times_r N$ but M, N have inverses M^{-1}, N^{-1}, so that

$$L \times_r (M \cap N) \times_r N^{-1} \times_r M^{-1} = M \times_r N \times_r N^{-1} \times_r M^{-1} = A$$

and L is r-invertible.

Since every finite r-ideal is r-invertible, the finite r-ideals coincide with the finite v-ideals and (1) is proved.

PROPOSITION 4. *Let the r-ideals be a system of ideals of A having finite character. Let Π be the family of maximal r-ideals. Then $X_r = \bigcap_{P \in \Pi} A_P X_r$. If $X_r = A$, then $A = \bigcap_{P \in \Pi} A_P X$ and $A_P X = A_P$.*

Proof. If $a \in A_P X_r$, then for some $x \in X_r$ and $b \in A$, $b \notin P$, $a = x/b$, i.e. $a^{-1}x = b$ so that $a^{-1}X_r \cap A \nsubseteq P$. Consequently it follows that if

$$a \in \bigcap_{P \in \Pi} A_P X_r,$$

then for each prime $P \in \Pi$, $a^{-1}X_r \cap A \nsubseteq P$ and so

$$A = (a^{-1}X_r \cap A)_r = (a^{-1}X_r)_r \cap A = a^{-1}X_r \cap A.$$

It follows that $1 \in a^{-1}X_r$ so that $a \in X_r$ and $X_r \supseteq \bigcap_{P \in \Pi} A_P X_r$. The opposite inclusion follows immediately since $X_r \subseteq A_P X_r$.

Let $X_r = A$. Since A_P is a local ring, either $A_P X = A_P$ or $A_P X \subseteq A_P P$. In the latter case $P = A_P P \cap A \supseteq A_P X \cap A \supseteq X$ and consequently $X_r \subseteq P_r = P \subset A$, a contradiction. It follows that $A_P X = A_P$ and also that $X_r = A = \bigcap_{P \in \Pi} A_P = \bigcap_{P \in \Pi} A_P X$.

THEOREM 5. *Let A be a ring with a system of ideals having finite character, the r-ideals. If L, M, N represent any integral r-ideals, then the following six conditions are equivalent*:

(1) *the finite r-ideals form a group*;
(2) *A_P is a valuatiʼin ring for each maximal r-ideal*;
(3) $L \times_r (M \cap N) = (L \times_r M) \cap (L \times_r N)$;
(4) $(M + N) \times_r (M \cap N) = M \times_r N$.
(5) *every finite r-ideal is r-invertible*;
(6) $L \cap (M + N)_r = (L \cap M + L \cap N)_r$.

https://doi.org/10.4153/CJM-1967-065-8 Published online by Cambridge University Press

Proof. We show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1) and that (2) $\Leftrightarrow$ (6).

Let Π be the family of maximal r-ideals.

(1) $\Rightarrow$ (2). Let $P \in \Pi$. Since P is a prime ideal, there exists a valuation w with valuation ring $\mathfrak{R} \supseteq A$ and with centre P on A (**9**, p. 12). It follows that $A_P \subseteq \mathfrak{R}$. Let $x \in \mathfrak{R}$. Let $(y_1, \ldots, y_n)_r$ be the inverse of $(1, x)_r$ in the group of finite r-ideals, so that $(y_1, \ldots, y_n, xy_1, \ldots, xy_n)_r = A$.

By Proposition 4,

$$\begin{aligned} A &= \bigcap_{P\in\Pi} A_P\{y_1, \ldots, y_n, xy_1, \ldots, xy_n\} \\ &\subseteq A_P\{y_1, \ldots, y_n, xy_1, \ldots, xy_n\} \\ &\subseteq \mathfrak{R}\{y_1, \ldots, y_n, xy_1, \ldots, xy_n\} \\ &= \mathfrak{R}y_i, \end{aligned}$$

where $w(y_i) = \min_{1\leqslant j\leqslant n} \{w(y_j), w(xy_j)\}$.

Hence $1 \in \mathfrak{R}y_i$, and since this means that $w(y_i) \leqslant 0$, we have $y_i \notin P$. But $y_i, xy_i \in A$ so $x = xy_i/y_i \in A_P$ and $A_P = \mathfrak{R}$.

(2) $\Rightarrow$ (3). It follows from Propositions 4 and 2 that

$$M_r = \bigcap_{P\in\Pi} M_r A_P = (M_r)_c,$$

i.e. that $M_c \subseteq M_r$.

By Proposition 4, A_P, $P \in \Pi$ is a defining family of (essential) valuations for A, and so using Theorem 1 (6),

$$L \times_c (M_c \cap N_c) = (L \times_c M) \cap (L \times_c N). \tag{3.1}$$

Consequently it follows by Proposition 3 that every finite c-ideal is c-invertible and hence, by the result quoted in the Introduction, that the finite c-ideals and finite v-ideals coincide. Now for any finite set $X \subseteq A$,

$$X_v = X_c \subseteq X_r \subseteq X_v$$

and it follows that the finite ideals of all three systems coincide. Since a family of finite character is the finest having specified finite ideals, for any set $X \subseteq A$, $X_r \subseteq X_c$, and since $X_c \subseteq X_r$, it follows that the c-system and r-system are identical. (3) follows immediately from (3.1).

(3) $\Rightarrow$ (4) $\Rightarrow$ (5). This follows from Proposition 3.

(5) $\Rightarrow$ (1). If M is finite, then it has an inverse M^{-1}, but M^{-1} has an inverse M, and so must be finite by the result quoted in the Introduction; consequently, the finite ideals form a group.

(2) $\Rightarrow$ (6). Now that we have shown that the c-ideals and the r-ideals coincide, this follows from Theorem 1 (8).

(6) $\Rightarrow$ (2). If (6) holds, then, for $a, b \in A$

$$(a) = (a) \cap ((b) + (a - b))_r = ((a) \cap (b) + (a) \cap (a - b))_r,$$

so that

$$A = a^{-1}((a) \cap (b) + (a) \cap (a - b))_r = (a^{-1}((a) \cap (b) + (a) \cap (a - b))_r.$$

https://doi.org/10.4153/CJM-1967-065-8 Published online by Cambridge University Press

By Proposition 4 it follows that for each $P \in \Pi$

$$A_P a^{-1}((a) \cap (b) + (a) \cap (a - b)) = A_P,$$

that is

$$((a) \cap (b))A_P + ((a) \cap (a - b))A_P = aA_P$$

or $(a)^* \cap (b)^* + (a)^* \cap (a - b)^* = (a)^*$, where $(a)^*$ denotes the ideal aA_P. Thus $a = t + (a - b)c$ with $t \in (a)^* \cap (b)^*$ and $bc \in (a)^*$.

If c is a unit of A_P, then $b \in (a)^*$, i.e. $a|b$.

If c is not a unit of A_P, then $1 - c$ is, since A_P is a local ring; so

$$(a)^* = (a(1 - c))^* = (t - bc)^* \subseteq (b)^*, \text{ i.e. } b|a.$$

We conclude that A_P is a valuation ring, since its divisibility group is totally ordered. This completes the proof.

Because the r-ideals of Theorem 5 have finite character and the finite r-ideals are the v-ideals, it follows that the r-ideals coincide with the t-ideals. Thus any ring satisfying Theorem 5 is a v-multiplication ring. A more particular case is given by the following corollary.

COROLLARY (Krull **(6)**, Jensen **(5)**). *Let A be an integral domain with integral ideals L, M, N. Then the following conditions are equivalent:*

(1) *the finitely generated ideals form a group;*
(2) *A_P is a valuation ring for every maximal ideal P of A;*
(3) $L(M \cap N) = LM \cap LN$;
(4) $(M + N)(M \cap N) = MN$;
(5) $L \cap (M + N) = L \cap M + L \cap N$;
(6) *A is a Prufer ring.*

Proof. This follows by noting that the ideals have finite character.

We note that the finite v-ideals of a v-multiplication ring form a lattice-ordered group with the order relation $X_v \leqslant Y_v$ when $Y_v \subseteq X_v$.

Jaffard **(4**, p. 55**)** has shown that a v-multiplication ring A may be characterized in terms of its divisibility group G as follows: A is a v-multiplication ring if and only if there exists a lattice-ordered group G' containing G as a subgroup in such a way that every element of G' is the infimum of a finite number of elements of G.

G' is lattice-order isomorphic to the lattice-ordered group of finite v-ideals.

A ring A is said to be an *essential ring* if it is equal to the intersection of its essential valuations.

Theorem 5 and the remarks following it show that every v-multiplication ring is an essential ring. We conjecture that the converse is false but have no counterexample.

4. Rings of Krull type and v-multiplication rings.

LEMMA 6. *Let G be a lattice-ordered group. Let $g_i \in G^+$, $i = 1, \ldots, n$, $n \geqslant 2$, be such that* $g_1 \vee g_2 \vee \ldots \vee g_{n-1} \ngeqslant g_n$ and $g_n \ngeqslant g_i$ *for* $i = 1, \ldots, n-1$. *Then* $g'_n = g_n - g_n \wedge (g_1 \vee \ldots \vee g_{n-1}) > 0$ *and* $g'_i = g_i - g_i \wedge g_n > 0$, $i = 1, \ldots, n-1$; *but* $g'_n \wedge g'_i = 0$, $\quad i = 1, \ldots, n-1$.

Proof. The first two conclusions follow easily, since from the hypothesis $g_n > g_n \wedge (g_1 \vee \ldots \vee g_{n-1})$ and $g_i > g_i \wedge g_n$. If $n = 2$ and

$$g_i - g_1 \wedge g_2 \geqslant h \geqslant 0, \qquad i = 1, 2,$$

then

$$g_i \geqslant h + g_1 \wedge g_2 \geqslant 0, \qquad i = 1, 2,$$

so that $g_1 \wedge g_2 \geqslant h + g_1 \wedge g_2$; thus $h = 0$, i.e. $g'_1 \wedge g'_2 = 0$.

Now we treat the general case:

$$\begin{aligned}(g_1 \vee \ldots \vee g_{n-1}) + g_n \wedge g_i &= (g_1 \vee \ldots \vee g_{n-1} + g_n) \wedge (g_1 \vee \ldots \vee g_{n-1} + g_i)\\ &\geqslant (g_i + g_n) \wedge (g_1 \vee \ldots \vee g_{n-1} + g_i)\\ &= g_i + (g_1 \vee \ldots \vee g_{n-1}) \wedge g_n.\end{aligned}$$

Thus $(g_1 \vee \ldots \vee g_{n-1}) - (g_1 \vee \ldots \vee g_{n-1}) \wedge g_n \geqslant g_i - g_n \wedge g_i = g'_i$. But by the lemma in the case $n = 2$,

$$((g_1 \vee \ldots \vee g_{n-1}) - (g_1 \vee \ldots \vee g_{n-1}) \wedge g_n) \wedge (g_n - (g_1 \vee \ldots \vee g_{n-1}) \wedge g_n) = 0,$$

so that certainly $g'_i \wedge g'_n = 0$.

A lattice-ordered group is said to satisfy *Conrad's* (F)*-condition* **(1)**, if each positive element is greater than only a finite number of pairwise disjoint elements.

THEOREM 7. *The following three conditions on a ring A are equivalent*:

(1) *A is a v-multiplication ring in which the lattice-ordered group of finite v-ideals satisfies Conrad's (F)-condition*;

(2) *A is a v-multiplication ring in which no non-zero element belongs to an infinite number of maximal t-ideals*;

(3) *A is defined by a family of essential valuations having finite character, i.e. A is a ring of Krull type.*

(1) $\Rightarrow$ (2). Let $x \in A$. Let $X_1, \ldots, X_n$ be a family of finite sets each containing x such that $(X_1)_v, \ldots, (X_n)_v$ are a maximal set of pairwise disjoint v-ideals in the corresponding lattice. Let $P_1, \ldots, P_n$ be maximal t-ideals containing $X_1, \ldots, X_n$ respectively. We show that x is contained in no other maximal t-ideal.

Suppose that $x \in P$, $P \neq P_i$, $i = 1, \ldots, n$. Let $b_i \in P$, $b_i \notin P_i$ and $a_i \in P_i$, $a_i \notin P$, for $i = 1, \ldots, n$.

For each $i = 1, \ldots, n$, $(X_i)_v \nsubseteq P$. For, suppose that $(X_i)_v \subseteq P$, then, since $(X_i, a_i)_v \nsubseteq (X_i, b_i)_v \nsubseteq (X_i, a_i)_v$ we may apply Lemma 6 with $n = 2$

to show that the finite v-ideals $(X_i, a_i)_v \times_v (X_i, a_i, b_i)_v^{-1}$, $(X_i, b_i)_v \times_v (X_i, a_i, b_i)_v^{-1}$ are disjoint and obtain a contradiction to maximality by substituting them for $(X_i)_v$ in the set $\{(X_1)_v, \ldots, (X_n)_v\}$.

From $(X_i)_v \nsubseteq P$ it follows that $\bigcap_1^n (X_i)_v \nsubseteq P$ (**8**, p. 210), i.e. that $(X_1)_v \vee (X_2)_v \vee \ldots \vee (X_n)_v \ngeqslant (b_1, \ldots, b_n, x)_v \geqslant P$. Since $(b_1, \ldots, b_n, x)_v \nsubseteq (X_i)_v$, i.e. $(b_1, \ldots, b_n, x)_v \ngeqslant (X_i)_v$ for $i = 1, \ldots, n$, we may apply Lemma 6 to construct a set of $n + 1$ finite disjoint v-ideals containing x, contradicting the maximality of $\{(X_1)_v, \ldots, (X_n)_v\}$. It follows that the maximal t-ideals containing x are a subset of $\{P_1, \ldots, P_n\}$.

(2) $\Rightarrow$ (3). Let Π be the family of maximal t-ideals. By Theorem 5, for each $P \in \Pi$, A_P is a valuation ring. Since each element of K belongs to only a finite number of maximal ideals, the family $\{A_P, P \in \Pi\}$ has finite character. Finally, by Proposition 4, $A = \bigcap_{P \in \Pi} A_P$. We conclude that A is a ring of Krull type.

(3) $\Rightarrow$ (1). We show first that the c-ideals have finite character; we need to show that if $a \in M_c$, then for some finite set of elements of M

$$m_i, i = 1, \ldots, n, \; a \in (m_1, \ldots, m_n)_c.$$

Let $m_1 \in M$; then $\Phi(a) \geqslant \Phi(m_1)$ except at a finite number of valuations, say $w_2 \ldots w_n$. By definition of $M(w_i)$ there exists $m_i \in M$ such that

$$w_i(a) \geqslant w_i(m_i) \geqslant M(w_i), \qquad i = 2, \ldots, n.$$

Now since $w(a) \geqslant \min \{w(m_1), \ldots, w(m_n)\}$ for all $w \in \Omega$ we conclude that the c-ideals have finite character.

Now by Theorem 1, since the valuations of the defining family Ω are essential,

$$(M + N) \times_c (M_c \cap N_c) = M \times_c N.$$

Thus by Theorem 5, A is a v-multiplication ring and the c-ideals are the t-ideals.

Let X, Y be finite sets containing a given element $x \in A$ and such that X_v, Y_v are disjoint. Then $\Phi((X, Y)_v) = \Phi(X, Y) = 0$. It follows that each $w \in \Omega$ is zero on some element of $X \cup Y$. We conclude that the number of disjoint finite v-ideals containing x is no greater than the number of non-zero valuations at x.

5. Preservation of rings of Krull type under extension.

PROPOSITION 8. *Let Ω be a family of valuations defining the ring A. Let K' be an algebraic extension of the quotient field K, of A. Let Ω' be the family of all the extensions of the valuations in Ω to the field K'. Then*

(a) *Ω' defines the integral closure A' of A in K';*
(b) *all extensions of essential valuations for A are essential for A';*
(c) *if Ω is of finite character and K' is a finite extension of K, then Ω' is of finite character.*

Proof. This proposition is well known for discrete rank-one valuations (9, Theorem 30, p. 87). The generalization to arbitrary valuations requires only minor changes in detail and will not be given.

COROLLARY. *Let A be an essential ring with quotient field K. Let A' be the integral closure of A in an algebraic extension of K. Then A' is an essential ring.*

COROLLARY. *Let A be a ring of Krull type with quotient field K. Let A' be the integral closure of A in a finite algebraic extension of K. Then A' is a ring of Krull type.*

PROPOSITION 9. *Let Ω be a family of valuations defining A. Let Ω' be the family of canonical extensions of valuations of Ω to $K(X_i)_{i\in I}$, where K is the quotient field of A. Let Ψ be the family of valuations of $K(X_i)_{i\in I}$ defined by the irreducible polynomials of $K[X_i]_{i\in I}$. Then:*

(1) *$A[X_i]_{i\in I}$ is defined by $\Omega' \cup \Psi$;*
(2) *each valuation of Ψ is essential for $A[X_i]_{i\in I}$;*
(3) *the canonical extension of an essential valuation is essential;*
(4) *if Ω is of finite character, so is $\Omega' \cup \Psi$.*

Proof. This proposition is well known for discrete rank-one valuations (9, Theorem 39, p. 111). The generalization to arbitrary valuations requires only minor changes in detail and will not be given.

COROLLARY. *If A is an essential ring, so is $A[X_i]_{i\in I}$.*

COROLLARY. *If A is a ring of Krull type, so is $A[X_i]_{i\in I}$.*

Let Ω be a family of valuations. We say that a family of valuations Ω' *is coarser than* Ω if there is a map f, from a subfamily Ω_1 of Ω onto Ω', such that $w \geqslant f(w)$ for all $w \in \Omega_1$.

Obviously, if Ω is of finite character then Ω' is of finite character.

PROPOSITION 10. *Let Ω be a defining family for the ring A; let $w \in \Omega$ be essential for A. Let Ω' be a family of valuations which is coarser than Ω and defines the ring A'. Let $w' \in \Omega'$ with $w' \leqslant w$. Then w' is essential for A'.*

Proof. Let P be the centre of w on A and P' be the centre of w' on A'. Let S be the complement of P in A, and S' the complement of P' in A'.

Since Ω' is coarser than Ω, $A' \supseteq A$, and since $w' \leqslant w$, $S \subseteq S'$. Thus

$$\mathfrak{R}_w = A_P = A_S \subseteq A'_{S'} = (A')_{P'}.$$

Hence $(A')_{P'}$ is a valuation ring and w' is essential.

COROLLARY. *Let Ω be a defining family (a defining family of finite character) of an essential ring (a ring of Krull type) A. Let Ω' be coarser than Ω. Then Ω' defines an essential ring (a ring of Krull type) A'.*

If $w' \in \Omega'$, then $w' \leqslant w \in \Omega$ and w is essential for A, so that w' is essential for A', i.e. A' is an essential ring.

If Ω is of finite character so is Ω'.

LEMMA 11. *Let Ω be a defining family of finite character for A. Let S be a multiplicatively closed subset of A with $0 \notin S$. Then there exists Ω' coarser than Ω such that*

$$A_S = \bigcap_{w \in \Omega'} \mathfrak{R}_w.$$

Proof. If $w \in \Omega$, then there exists $w' \leqslant w$ which is the finest of the valuations $v \leqslant w$ such that $Z(v) \cap S = \emptyset$.

Indeed, let

$$\mathfrak{E}_w = \{v \leqslant w | \, Z(v) \cap S = \emptyset\}.$$

Then $\mathfrak{E}_w$ is non-empty since it contains the trivial valuation. Let

$$\mathfrak{R}_{w'} = \bigcap_{v \in \mathfrak{E}_w} \mathfrak{R}_v.$$

If $a \in Z(w')$ and $a \neq 0$, then $a^{-1} \notin \mathfrak{R}_{w'}$ so that there exists $v \in \mathfrak{E}_w$ such that $a^{-1} \notin \mathfrak{R}_v$, i.e. $a \in \mathfrak{P}_v$; thus, $a \in Z(v)$ and $a \notin S$. It follows since $w' \leqslant w$ that $w' \in \mathfrak{E}_w$ and hence is the finest valuation of $\mathfrak{E}_w$.

Let w run through Ω; the w''s define a coarser family Ω'.

Since $S \cap Z(w') = \emptyset$, S is contained in the group of units of $\mathfrak{R}_{w'}$, so that if $a/s \in A_S$ then $a/s \in \mathfrak{R}_{w'}$, hence

$$A_S \subseteq \bigcap_{w' \in \Omega'} \mathfrak{R}_{w'}.$$

The converse inclusion consists of showing that if

$$x \in \bigcap_{w \in \Omega'} \mathfrak{R}_w,$$

then there exists $a \in S$ such that $ax \in A$.

Let $a \in S$ be such that the set Ω_1 of valuations $w \in \Omega$ with $w(ax) < 0$ is minimal (such an element exists since Ω has finite character).

Suppose that $w' \in \Omega_1$. Let Δ be the largest isolated subgroup of the value group Γ of w' not containing $w'(ax)$. Let $v \leqslant w'$ be the valuation with group Γ/Δ, so that $v(ax) \leqslant 0$. Suppose that $Z(v) \cap S = \emptyset$, then there exists $w \in \Omega'$ with $w \geqslant v$, i.e. $w(ax) \geqslant 0$ so that $v(ax) \geqslant 0$, a contradiction. From $Z(v) \cap S \neq \emptyset$ it follows that $v(b) > 0$ for some $b \in S$. Suppose that $w'(b^n ax) \leqslant 0$ for all n; then $nw'(b) \leqslant -w'(ax)$ for all n, and $w'(b)$ generates an isolated subgroup not containing $w'(ax)$, so that $w'(b) \in \Delta$ and $v(b) = 0$, a contradiction. It follows that for some n, $w'(b^n ax) > 0$. Since $b^n a \in S$ and since any $w \in \Omega$ is positive at $b^n ax$ if it is positive at ax, we conclude that Ω_1 is not a minimal family. This contradiction implies that $\Omega_1 = \emptyset$, so that $ax \in A$ and the lemma is proved.

PROPOSITION 12. *Let A be a ring of Krull type. Let S be a multiplicatively closed set $0 \notin S$. Then A_S is a ring of Krull type.*

Proof. By the preceding lemma, A_S has a defining family which is coarser than the family of finite character defining A; hence, by the Corollary to Proposition 8, A_S is a ring of Krull type.

REFERENCES

1. Paul Conrad, *Some structure theorems for lattice ordered groups*, Trans. Amer. Math. Soc., *99* (1961), 212–240.
2. Malcolm Griffin, *Families of finite character and essential valuations* (to appear in Trans. Amer. Math. Soc.).
3. ——— *Rings of Krull type* (to appear in J. Reine Angew. Math.).
4. Paul Jaffard, *Les systèmes d'idéaux* (Paris, 1960).
5. Chr. U. Jensen, *On characterisations of Prufer rings*, Math. Scand., *13* (1963), 90–98.
6. Wolfgang Krull, *Idealtheorie*, Ergebnisse der Math., vol. 4 (Berlin, 1935).
7. Paulo Ribenboim, *Sur les groupes totalement ordonnés et l'arithmétique des anneaux de valuation*. Summa Brasil. Math., *4* (1958), 1–64.
8. Oscar Zariski and Pierre Samuel, *Commutative algebra*, Vol. I (Princeton, 1958).
9. ——— *Commutative algebra*, Vol. II (Princeton, 1960).

Applied Mathematics Division,
D.S.I.R.,
Wellington, New Zealand