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Poncelet's closure theorem and the embedded topology of conic-line arrangements

Shinzo Bannai, Ryosuke Masuya, Taketo Shirane,
Hiro-o Tokunaga, and Emiko Yorisaki

Abstract. In the study of plane curves, one of the problems is to classify the embedded topology of plane curves in the complex projective plane that have a given fixed combinatorial type, where the *combinatorial type* of a plane curve is data equivalent to the embedded topology in its tubular neighborhood. A pair of plane curves with the same combinatorial type but distinct embedded topology is called a *Zariski pair*. In this paper, we consider Zariski pairs consisting of conic-line arrangements that arise from Poncelet's closure theorem. We study unramified double covers of the union of two conics that are induced by a $2m$-sided Poncelet transverse. As an application, we show the existence of families of Zariski pairs of degree $2m+6$ for $m \geq 2$ that consist of reducible curves having two conics and $2m+2$ lines as irreducible components.

1 Introduction

There is a very famous theorem in projective geometry known as Poncelet's closure theorem or Poncelet's porism, first introduced in [15]. Let $\mathbb{P}^2(=\mathbb{CP}^2)$ be the complex projective plane and let $C_1, C_2 \subset \mathbb{P}^2$ be smooth conics that intersect transversely. Given a general point $P_1 \in C_1$, let L_1 be a line passing through P_1 and tangent to C_2. Then L_1 intersects C_1 at another point P_2, and we can choose a line L_2 passing through P_2 and tangent to C_2. By repeating this process, we obtain a sequence $(P_1, L_1), (P_2, L_2), \ldots$ of pairs of points $P_i \in C_1$ and tangent lines $L_i \in C_2^*$ such that $P_i \in L_i$, where C_2^* is the dual curve of C_2. Such a sequence is called a *Poncelet transverse originating at* P_1. Poncelet's closure theorem can be stated as follows:

Theorem 1.1 [15] *Let C_1, C_2 be as above. If there exists a point $P \in C_1$ such that the Poncelet transverse originating at P is periodic with period n, then for any $Q \in C_1$, the Poncelet transverse originating at $Q \in C_1$ is also periodic with period n.*

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A modern proof in terms of torsion points of elliptic curves was given by P. Griffiths and J. Harris in [10]. We call a Poncelet transverse $\sqcap_n$ with minimum period n an *n-sided* Poncelet transverse and denote it by a sequence $\sqcap_n : (P_1, L_1), \dots, (P_n, L_n)$ of n pairs. Sometimes, we omit "n-sided Poncelet" and simply use "transverse" when it is evident from the context that we are talking about an n-sided Poncelet transverse. For a general choice of origin $P_1 \in C_1$, an n-sided Poncelet transverse will consist of n distinct points $P_1, \dots, P_n$ and n distinct lines $L_1, \dots, L_n$ and can be viewed as an n-gon with these points and lines as vertices and edges that is inscribed in C_1 and circumscribed about C_2. Poncelet's theorem also holds for special choices of origins, where $\sqcap_n$ will involve intersection points of C_1 and C_2 and/or bitangent lines of C_1 and C_2 and can be viewed as a degenerated n-gon having double edges. In this paper, we consider curves arising from $2m$-sided Poncelet transverses ($m \geq 2$) and construct a new infinite sequence of curves that are interesting with regard to their *embedded topology*.

The embedded topology of an algebraic plane curve $\mathcal{C} \subset \mathbb{P}^2 (= \mathbb{CP}^2)$, which is defined over the field of complex numbers $\mathbb{C}$, is the homeomorphism class of the pair $(\mathbb{P}^2, \mathcal{C})$ of topological spaces. One of the main objectives in this area of research is to give a complete classification of the embedded topology of plane curves. It is known that for two plane curves $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{P}^2$, if $(\mathbb{P}^2, \mathcal{C}_1)$ and $(\mathbb{P}^2, \mathcal{C}_2)$ are homeomorphic as pairs, then $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same combinatorial type. Here, the *combinatorial type* of plane curves is data consisting of the number of irreducible components, the degrees and the topological types of singularities, and configuration of irreducible components (see [5], [2] for details). However, the converse is not true because there exist pairs of plane curves $\mathcal{C}_1, \mathcal{C}_2$ that have the same combinatorial type, but the homeomorphism classes of $(\mathbb{P}^2, \mathcal{C}_1)$ and $(\mathbb{P}^2, \mathcal{C}_2)$ are distinct. The first example of such pairs $(\mathcal{C}_1, \mathcal{C}_2)$ was given by Zariski in [17], and the name *Zariski pair* was coined by E. Artal in [1] (see [17], [1], [5] for precise definitions and details).

Remark 1.2 It is known that the combinatorial type of a plane curve is equivalent to the embedded topology in its regular neighborhood (cf. [5]). Furthermore, the study of Zariski pairs can be regarded as an algebraic analogue of the study of surface-knots.

Understanding what causes the differences in the embedded topology of Zariski pairs should lead to a better understanding of the embedded topology of plane curves in general and hence is important. Concerning Zariski pairs of reducible plane curves with irreducible components of small degree, the following results are known. In the case where $\mathcal{C}_i$ are line arrangements, it is known that Zariski pairs do not exist for $\deg \mathcal{C}_i \leq 9$ (see [13]). Also, it is known that there exists a Zariski pair of line arrangements of $\deg \mathcal{C}_i = 11$ (see [4]). However, the case of $\deg \mathcal{C}_i = 10$ is open. In the case of conic-line arrangements (i.e., reducible curves whose irreducible components are lines and smooth conics), the existence of a Zariski pair of degree 7 consisting of curves with two conics and three lines as irreducible components is known (see [16]). Also, a Zariski pair of degree 6 consisting of curves with three conics as irreducible components is known (see [14]). It would be interesting to find a Zariski pair of conic-line arrangements of low degree consisting of curves with a single conic and additional lines.

Now, we explain our main result. Note that in the following, we work over the field of complex numbers $\mathbb{C}$. Let C_1, C_2 be smooth conics intersecting transversely that admit a $2m$-sided Poncelet transverse. Let $L_1, \ldots, L_{2m}$ be distinct lines forming a nondegenerated $2m$-sided Poncelet transverse $\sqcap_{2m} : (P_1, L_1), \ldots, (P_{2m}, L_{2m})$, where $L_1, \ldots, L_{2m}$ are tangent lines of C_2 and the points $P_1 = L_{2m} \cap L_1$ and $P_i = L_{i-1} \cap L_i$ $(2 \leq i \leq 2m)$ lie on C_1. Let $\mathcal{P}_{2m} := \sum_{i=1}^{2m} L_i$ be the union of the lines and let T_1, T_2, T_3, T_4 be the four bitangent lines of C_1 and C_2. The curves that we are interested in are reducible curves of the form

$$\mathcal{C}_{ij} := C_1 + C_2 + \mathcal{P}_{2m} + T_i + T_j \quad (\{i, j\} \subset \{1, 2, 3, 4\})$$

having two conics and $2m + 2$ lines as irreducible components. We consider the double covers of $\mathbb{P}^2$ branched along the $2m + 2$ lines $\mathcal{P}_{2m} + T_i + T_j$ and see how $C_1 + C_2$ behaves under these double covers. Namely, we calculate the *splitting types* of $C_1 + C_2$ with respect to these double covers. (See Section 2 and [6] for details on splitting types.) This will be done by studying the invertible sheaves $\mathcal{F}$ of order 2 on $C_1 + C_2$, or equivalently, torsion points of order 2 of the Jacobian $J(C_1 + C_2)$ of the singular curve $C_1 + C_2$. As an application of these calculations, we obtain the following theorem.

Theorem 1.3 *Under the above notation, it is possible to choose labels of $T_1, \ldots, T_4$ so that the pair $(\mathcal{C}_{ij}, \mathcal{C}_{kl})$ is a Zariski pair if $\{i, j\} = \{1, 2\}$ or $\{3, 4\}$ and $\{k, l\} \neq \{1, 2\}, \{3, 4\}$.*

We note that in the above setting, C_1, C_2 admits two degenerated $2m$-sided Poncelet transverses each having two bitangent lines as edges. The differences of the curves $\mathcal{C}_{ij}$ and $\mathcal{C}_{kl}$ in the theorem are whether the two bitangent lines lie in the same degenerated Poncelet transverse or not. Although the curves that are proven to be Zariski pairs in Theorem 1.3 are conic-line arrangements of $\deg \mathcal{C}_{ij} \geq 10$, we believe that our method of systematically constructing Zariski pairs from Poncelet transverses is in itself interesting and worth sharing. We hope this topological viewpoint is new and will add to the already rich literature on curves related to Poncelet's closure theorem.

Similar studies relating torsion elements of the Jacobian $J(C)$ and the embedded topology of reducible curves having C as an irreducible component have been done in [8], [2], [3] when C is a smooth curve. This paper can be considered as a variation of these works in the case where C is reducible and singular.

This paper is organized as follows: In Section 2, we review the definition of splitting types and state the proposition that is used in distinguishing the embedded topology. In Section 3, we give a discussion on unramified double covers of conic-line arrangements, especially in the case of two transversal conics. In Section 4, we study n-sided Poncelet transverses and double covers related to them. Finally, in Section 5, we give the proof of our main result, Theorem 1.3.

2 Splitting types

In this section, we review the notion of *splitting types* of plane algebraic curves with respect to a double cover, which will be used to distinguish the embedded topology of the curves that we are interested in. We refer the reader to [6] for details. Let $\pi_{\mathcal{B}} : S' \to \mathbb{P}^2$ be a double cover branched along a curve $\mathcal{B} \subset \mathbb{P}^2$ of even degree and let $C \subset \mathbb{P}^2$ be an irreducible plane curve. The preimage $\pi_{\mathcal{B}}^{-1}(C)$ can be either reducible or irreducible, depending on the relation between C and the branch locus $\mathcal{B}$. In the former case where $\pi_{\mathcal{B}}^{-1}(C)$ is reducible, $\pi_{\mathcal{B}}^{-1}(C)$ will have two irreducible components since $\pi_{\mathcal{B}}^{-1}$ is a double cover. In this case, we say that C *is a splitting curve with respect to* $\pi_{\mathcal{B}}$ *or* $\mathcal{B}$. Let C_1, C_2 be splitting curves with respect to $\mathcal{B}$ and let $\pi_{\mathcal{B}}^{-1}(C_i) = C_i^+ + C_i^-$, $(i = 1, 2)$. The relation between the components $C_1^\pm$, $C_2^\pm$ reflects how the curves $\mathcal{B}, C_1, C_2$ are "intertwined" in $\mathbb{P}^2$ and hence gives information about the embedded topology of the reducible curve $\mathcal{C} = \mathcal{B} + C_1 + C_2$. The information can be formulated as follows:

Definition 2.1 Let $\mathcal{B}, C_1, C_2$ be as above. For integers $m_1 \le m_2$, we say that the triple $(C_1, C_2; \mathcal{B})$ has splitting type (m_1, m_2), if $C_1^+ \cdot C_2^+ = m_1$ and $C_1^+ \cdot C_2^- = m_2$ for a suitable choice of labels.

The splitting types can be used to distinguish the embedded topology of reducible plane curves by the following proposition.

Proposition 2.2 [6, Proposition 2.5] *Let* $\mathcal{B}_1$, $\mathcal{B}_2$ *be plane curves of degree* $2d$ *and let* C_{i1}, C_{i2} *be splitting curves with respect to* $\mathcal{B}_i$, $(i = 1, 2)$. *Suppose that* $C_{i1} \cap C_{i2} \cap \mathcal{B}_i = \emptyset$, C_{i1} *and* C_{i2} *intersect transversely and that* $(C_{11}, C_{12}; \mathcal{B}_1)$ *and* $(C_{21}, C_{22}; \mathcal{B}_2)$ *have distinct spitting types. Then a homeomorphism* $h : \mathbb{P}^2 \to \mathbb{P}^2$ *such that* $h(\mathcal{B}_1) = \mathcal{B}_2$ *and* $\{h(C_{11}), h(C_{12})\} = \{C_{21}, C_{22}\}$ *does not exist.*

Remark 2.3 It is known that Definition 2.1 and Proposition 2.2 can be modified to a more general version (see [7]), but the above version is enough for our purposes.

Later, we will calculate the splitting types of pairs of conics C_1, C_2 with respect to various double covers in order to prove our main theorem.

3 Line bundles of order two and unramified double covers of conic-line arrangements with simple nodes

In this section, we briefly recall the theory of double covers. We will especially consider the case of conic-line arrangements with simple nodes for later use. We refer the reader to [11, Section 2, Section 3] for details and arguments in a more general setting.

First, we consider topological (unramified) double covers. Let $\mathcal{C} = C_1 + \cdots + C_k$ be a conic-line arrangement with simple nodes (i.e., each irreducible component C_i of $\mathcal{C}$ is either a line or a smooth conic, and all intersection points are ordinary double points). Let $\varphi : \mathcal{C}' \to \mathcal{C}$ be a topological double cover of $\mathcal{C}$. Then, since $C_i \cong \mathbb{P}^1$ and is simply connected, $\varphi^{-1}(C_i)$ splits into two disjoint sets $C_i^\pm \subset \mathcal{C}'$. We fix a labeling $C_i^\pm$ for the meantime. Let $Q \in \mathrm{Sing}(\mathcal{C})$, and let C_i, C_j be the irreducible components intersecting at Q. Then C_i^+ will intersect with either C_j^+ or C_j^- over Q. We say that $\mathcal{C}'$

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is glued by $+$ over Q if C_i^+ intersects C_j^+ and is glued by $-$ over Q if C_i^+ intersects C_j^-. We summarize this data in the form of a map defined as below.

Definition 3.1 A *gluing data of order two on* $\mathcal{C}$ is a map $\kappa : \mathrm{Sing}(\mathcal{C}) \to \{+,-\}$. If there is no confusion, we simply call it a *gluing data*. The *gluing data* κ_φ *of a topological double cover* $\varphi : \mathcal{C}' \to \mathcal{C}$ is a gluing data on $\mathcal{C}$ defined by $\kappa_\varphi(Q) = +$ if $\mathcal{C}'$ is glued by $+$ over Q and $\kappa_\varphi(Q) = -$ if $\mathcal{C}'$ is glued by $-$ over Q.

If we reverse the labeling of $C_i^\pm$ of $\pi^{-1}(C_i)$, then all of the signs for $Q \in \mathrm{Sing}(\mathcal{C}) \cap C_i$ will be reversed. Namely, for a gluing data $\kappa : \mathrm{Sing}(\mathcal{C}) \to \{+,-\}$ and each $i = 1, \dots, k$, a new gluing data κ_i is obtained by

$$\kappa_i(Q) := \begin{cases} -\kappa(Q) & \text{if } Q \in C_i, \\ \kappa(Q) & \text{otherwise} \end{cases} \tag{3.1}$$

for each $Q \in \mathrm{Sing}(\mathcal{C})$. We say that two gluing data κ and κ' are *equivalent*, and write $\kappa \sim \kappa'$, if κ' can be constructed from κ by a finite number of the above operations. In this way, we have a map from the set of topological double covers $\varphi : \mathcal{C}' \to \mathcal{C}$ to the set of equivalence classes of gluing data κ_φ on $\mathcal{C}$.

Lemma 3.2 *Let $\mathcal{C} = C_1 + \cdots + C_k$ be a conic-line arrangement with simple nodes. The following map Ψ from the set of homeomorphism classes of topological double covers of $\mathcal{C}$ to the set of equivalence classes of gluing data on $\mathcal{C}$ is well-defined and one-to-one:*

$$\begin{array}{cccc} \Psi : & \{\varphi : \textit{a topological double cover of } \mathcal{C}\}/\cong & \to & \{\kappa : \textit{a gluing data on } \mathcal{C}\}/\sim \\ & \Psi & & \Psi \\ & [\varphi] & \mapsto & [\kappa_\varphi] \end{array},$$

where κ_φ is the gluing data of the topological double cover φ of $\mathcal{C}$. In particular, any continuous deformation $\varphi_t : \mathcal{C}'_t \to \mathcal{C}$ ($t \in \Delta$) of topological double covers of $\mathcal{C}$ is constant, where $\Delta \subset \mathbb{C}$ is a small neighborhood of the origin.

Proof Suppose that $h : \mathcal{C}' \to \mathcal{C}''$ is a homeomorphism over $\mathcal{C}$ of topological double covers $\varphi' : \mathcal{C}' \to \mathcal{C}$ and $\varphi'' : \mathcal{C}'' \to \mathcal{C}$. Put $\varphi'^{-1}(C_i) = C'^+_i + C'^-_i$ and $\varphi''^{-1}(C_i) = C''^+_i + C''^-_i$. Note that h satisfies $h(\varphi'^{-1}(C_i)) = \varphi''^{-1}(C_i)$, but there may be $1 \le i \le k$ such that $h(C'^+_i) = C''^-_i$ depending on how the components are labeled by $\pm$. Since h is a homeomorphism, $\kappa_{\varphi''}$ can be obtained from $\kappa_{\varphi'}$ by applying the operations (3.1) to $\kappa_{\varphi'}$ for $1 \le i \le k$ with $h(C'^+_i) = C''^-_i$, which corresponds to exchanging C'^+_i and C'^-_i. Hence, the gluing data $\kappa_{\varphi''}$ is equivalent to $\kappa_{\varphi'}$, and the map Ψ is well-defined.

Let $\kappa : \mathrm{Sing}(\mathcal{C}) \to \{+,-\}$ be a gluing data on $\mathcal{C}$. For each $i = 1, \dots, k$, let $C_i^+ \sqcup C_i^-$ be the disjoint union of two copies $C_i^\pm$ of C_i, and let $\varphi_i : C_i^+ \sqcup C_i^- \to C_i$ be the projection, which is the topological double cover of C_i. We construct a topological space $\mathcal{C}'_\kappa$ by gluing $C_i^\pm \cap \varphi_i^{-1}(Q)$ to $C_j^\pm \cap \varphi_j^{-1}(Q)$ if $\kappa(Q) = +$, and to $C_j^\mp \cap \varphi_j^{-1}(Q)$ if $\kappa(Q) = -$, for each $1 \le i < j \le k$ and $Q \in C_i \cap C_j$. Then the topological double covers φ_i induce a topological double cover $\varphi_\kappa : \mathcal{C}'_\kappa \to \mathcal{C}$. Since the operation (3.1) corresponds to replacing C_i^+ and C_i^-, the map $[\kappa] \mapsto [\varphi_\kappa]$ is well-defined, and is the inverse map of Ψ. Hence, Ψ is one-to-one.

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Let $\varphi_t : \mathcal{C}'_t \to \mathcal{C}$ $(t \in \Delta)$ be a continuous deformation of topological double covers, and let $\Phi : \overline{\mathcal{C}}' \to \Delta \times \mathcal{C}$ be the continuous family of the topological double covers φ_t, where $\overline{\mathcal{C}}' := \{(t, P') \mid t \in \Delta,\ P' \in \mathcal{C}'_t\}$ and $\Phi(t, P') := (t, \varphi_t(P'))$. Then Φ is a topological double cover of $\Delta \times \mathcal{C}$. Since $\Delta \times C_i$ is simply connected for each irreducible component $C_i \subset \mathcal{C}$, the preimage $\Phi^{-1}(\Delta \times C_i)$ consists of two connected components $\overline{C}_i^{\pm}$. For each $Q \in \mathrm{Sing}(\mathcal{C})$, the preimage $\Phi^{-1}(\Delta \times \{Q\})$ also consists of two components $\Delta_Q^{\pm}$. For each $t \in \Delta$, we define a gluing data $\kappa_t : \mathrm{Sing}(\mathcal{C}) \to \{+, -\}$ by, for each $Q \in C_i \cap C_j$ $(i \neq j)$, $\kappa_t(Q) = +$ if $\Delta_Q^+ \subset \overline{C}_i^+ \cap \overline{C}_j^+$, and $\kappa_t(Q) = -$ otherwise. This κ_t coincides with the gluing data κ_{φ_t} of φ_t for any $t \in \Delta$. Since κ_t is constant under t, all topological double covers φ_t are homeomorphic. ■

Next, we consider the relation between unramified double covers of $\mathcal{C}$ and invertible sheaves of order 2 on $\mathcal{C}$ following [11]. Let $\mathcal{C}$ be a reduced curve and let $\mathcal{F}$ be an invertible sheaf of order 2 on $\mathcal{C}$ (i.e., $\mathcal{F} \otimes \mathcal{F} \cong \mathcal{O}_{\mathcal{C}}$, where $\mathcal{O}_{\mathcal{C}}$ is the structure sheaf of, and, by abuse of terminology, the structure sheaf is considered as order 2 $\mathcal{C}$). Let $p_{\mathcal{F}} : \boldsymbol{L}_{\mathcal{F}} \to \mathcal{C}$ be the line bundle corresponding to $\mathcal{F}$ and let $t \in \Gamma(\boldsymbol{L}_{\mathcal{F}}, p_{\mathcal{F}}^*\mathcal{F})$ be the tautological section. Then, since we have assumed that $\mathcal{F} \otimes \mathcal{F} \cong \mathcal{O}_{\mathcal{C}}$, the zero divisor of $t^2 - 1$ in $\boldsymbol{L}_{\mathcal{F}}$ gives an unramified double cover $\varpi_{\mathcal{F}} : \mathcal{C}'_{\mathcal{F}} \to \mathcal{C}$ of $\mathcal{C}$. Note that the construction is algebraic, but since it is unramified, $\varpi_{\mathcal{F}}$ is also a topological double cover. It is known that this relation induces a one-to-one correspondence between isomorphism classes of invertible sheaves $\mathcal{F}$ of order 2, and isomorphism classes of unramified double covers $\varpi_{\mathcal{F}} : \mathcal{C}'_{\mathcal{F}} \to \mathcal{C}$ of $\mathcal{C}$. In the case where $\mathcal{C}$ is a conic-line arrangement with simple nodes, the relation between invertible sheaves of order 2 and unramified double covers can be described using the gluing data as follows: Let $\mathcal{F}$ be an invertible sheaf of order 2 on $\mathcal{C}$. Then, since the components of $\mathcal{C}$ are isomorphic to $\mathbb{P}^1$ and $\mathrm{Pic}(\mathbb{P}^1)$ is torsion free, the restrictions $\mathcal{F}|_{C_i}$ are isomorphic to the trivial sheaf $\mathcal{O}_{C_i}$. Given a node $Q \in C_i \cap C_j$, the line bundle $\mathcal{F}$ and its transition functions give the data of gluing of the trivial sheaves $\mathcal{F}|_{C_i}$ and $\mathcal{F}|_{C_j}$ over Q which is given by multiplication with 1 or -1, since $\mathcal{F}$ is of order 2. Conversely, an invertible sheaf of order 2 can be constructed by assigning this gluing data ± 1 of the trivial sheaves at each node Q. In this way, we can associate a gluing data $\kappa_{\mathcal{F}}$ to $\mathcal{F}$. Again, changing the signs of the gluing data at every node in an irreducible component C_i will result in an isomorphic sheaf. Hence, we have the following Lemma:

Lemma 3.3 *Let $\mathcal{C} = C_1 + \cdots + C_k$ be a conic-line arrangement with simple nodes. Then, there is a one-to-one correspondence between the set of invertible sheaves $\mathcal{F}$ of order 2 on $\mathcal{C}$ and the set of equivalence classes of gluing data on $\mathcal{C}$. Furthermore, this correspondence is compatible with the correspondence between the unramified double cover $\varpi_{\mathcal{F}} : \mathcal{C}'_{\mathcal{F}} \to \mathcal{C}$ of $\mathcal{C}$ associated to $\mathcal{F}$ and its gluing data. Namely, if $\kappa_{\mathcal{F}}$ is the gluing data of $\mathcal{F}$ and $\kappa_{\varpi_{\mathcal{F}}}$ is the gluing data of $\varpi_{\mathcal{F}} : \mathcal{C}'_{\mathcal{F}} \to \mathcal{C}$, then $\kappa_{\mathcal{F}} \sim \kappa_{\varpi_{\mathcal{F}}}$.*

Proof The first part follows from the discussion stated before the lemma. See also [11, Section 2b]. For the second part, let $\mathcal{F}$ be an invertible sheaf of order 2 and let $\mathcal{C}'_{\mathcal{F}} \to \mathcal{C}$ be the associated unramified double cover. The preimage of C_i in $\mathcal{C}'_{\mathcal{F}}$ consists of disjoint copies C_i^+ and C_i^- of C_i corresponding to the decomposition $t_i^2 - 1 = (t_i - 1)(t_i + 1)$, where t_i is the tautological section of $p^*(\mathcal{F}|_{C_i})$. The gluing

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data of $\mathcal{F}$ at a node $Q \in C_i \cap C_j$ tells us how the tautological sections t_i, t_j are related and in turn how $C_i^\pm$ and $C_j^\pm$ intersect as curves in $\boldsymbol{L}_\mathcal{F}$. If the sheaves $\mathcal{F}|_{C_i}$ and $\mathcal{F}|_{C_j}$ are glued by multiplication by 1 over Q, then $t_i = t_j$ over Q and C_i^+ intersects C_j^+ over Q. If they are glued by -1, then $t_i = -t_j$ over Q and C_i^+ intersects C_j^- over Q. Hence, the gluing data $\kappa_\mathcal{F}$ of $\mathcal{F}$ as a sheaf and the gluing data $\kappa_{\varpi_\mathcal{F}}$ of the topological unramified double cover $\varpi_\mathcal{F} : \mathcal{C}'_\mathcal{F} \to \mathcal{C}$ associated to $\mathcal{F}$ coincide. ■

Furthermore, given two invertible sheaves $\mathcal{F}_1, \mathcal{F}_2$ that are each of order 2, the product $\mathcal{F}_1 \otimes \mathcal{F}_2$ is again of order 2. The gluing data of $\mathcal{F}_1 \otimes \mathcal{F}_2$ is given by simply taking the products of the gluing data of $\mathcal{F}_1$ and $\mathcal{F}_2$ at each $Q \in \operatorname{Sing}(\mathcal{C})$, as the transition functions of $\mathcal{F}_1 \otimes \mathcal{F}_2$ are given by products of the transition function of $\mathcal{F}_1$ and $\mathcal{F}_2$. The gluing data of the unramified double cover associated to $\mathcal{F}_1 \otimes \mathcal{F}_2$ can also be obtained likewise.

Understanding and calculating the structure of the unramified double covers through this gluing data is useful and will be used in the proof of the main theorem. Also, in some cases where $\mathcal{C} \subset \mathbb{P}^2$, and the unramified double cover of $\mathcal{C}$ is induced by a (possibly ramified) double cover of $\mathbb{P}^2$, the structure of the former can be deduced from the latter as follows: Let $\pi : S' \to \mathbb{P}^2$ be a double cover branched along a plane curve $\mathcal{B} \subset \mathbb{P}^2$ of degree $2d$, and let $F \in \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2d))$ be a defining polynomial of $\mathcal{B}$. Assume that there is an effective divisor D on the curve $\mathcal{C}$ such that $\mathcal{B}|_\mathcal{C} = 2D$ and $\operatorname{Supp} D \cap \operatorname{Sing} \mathcal{C} = \varnothing$ (i.e., $\mathcal{C}$ intersects with $\mathcal{B}$ at smooth points of $\mathcal{C}$ with even multiplicities). Put $\mathcal{L} := \mathcal{O}_{\mathbb{P}^2}(d)$, and let $p_\mathcal{L} : \boldsymbol{L}_\mathcal{L} \to \mathbb{P}^2$ be the line bundle corresponding to $\mathcal{L}$, where $\boldsymbol{L}_\mathcal{L} := \operatorname{Spec} S(\mathcal{L}^{-1})$ is the spectrum of the symmetric algebra $S(\mathcal{L}^{-1})$ of $\mathcal{L}^{-1}$. Let $t \in \Gamma(\boldsymbol{L}_\mathcal{L}, p_\mathcal{L}^*\mathcal{L})$ be the tautological section. Then S' can be regarded as the subvariety of $\boldsymbol{L}_\mathcal{L}$ defined by $t^2 - F = 0$, and $\pi = p_\mathcal{L}|_{S'}$. Since $\operatorname{Supp} D$ is contained in the smooth part of $\mathcal{C}$, D corresponds to a Cartier divisor on $\mathcal{C}$, and there is a section $s_D \in \Gamma(\mathcal{C}, \mathcal{O}_\mathcal{C}(D))$ defining D and satisfying $s_D^2 = F|_\mathcal{C}$. Put $\mathcal{F} := \mathcal{L}|_\mathcal{C} \otimes \mathcal{O}_\mathcal{C}(-D)$ and let $p_\mathcal{F} : \boldsymbol{L}_\mathcal{F} \to \mathcal{C}$ be the line bundle corresponding to $\mathcal{F}$. Note that $\mathcal{F}$ is of order 2, and $\frac{t|_\mathcal{C}}{s_D}$ can be regarded as a section of $\Gamma(\boldsymbol{L}_\mathcal{F}, p_\mathcal{F}^*\mathcal{F})$. We say that the unramified double cover $\varpi : \mathcal{C}'_\mathcal{F} \to \mathcal{C}$ given by

$$\left(\frac{t|_\mathcal{C}}{s_D}\right)^2 - 1 = 0$$

in $\boldsymbol{L}_\mathcal{F}$ and $\varpi := p_\mathcal{F}|_{\mathcal{C}'}$ is induced by π. The morphism $\mathcal{F} \to \mathcal{L}|_\mathcal{C}$ given by multiplication of s_D induces the morphism $\boldsymbol{L}_\mathcal{F} \to \boldsymbol{L}_\mathcal{L}|_\mathcal{C}$ over $\mathcal{C}$, which is given by multiplication of the value of s_D to each fiber coordinate of $\boldsymbol{L}_\mathcal{F}$. This morphism induces $\mathcal{C}'_\mathcal{F} \to \pi^{-1}(\mathcal{C})$, which is isomorphic over $\mathcal{C} \setminus \operatorname{Supp} D$. Hence, we can deduce the structure of $\mathcal{C}'_\mathcal{F}$ by studying $\pi^{-1}(\mathcal{C})$. In the above cases, explicit calculations of transition functions may be avoided when calculating the gluing data, which we see in the following example.

Example 3.4 Let $C_1, C_2 \subset \mathbb{P}^2$ be smooth conics intersecting transversely, T_1, T_2, T_3, T_4 be the four bitangent lines to C_1, C_2, and $P_{ik} = T_i \cap C_k$ be the tangent points. Since $T_i + T_j$ has degree 2 and $(T_i + T_j)|_\mathcal{C} = 2(P_{i1} + P_{i2} + P_{j1} + P_{j2})$, the ramified double cover $\pi_{ij} : S'_{ij} \to \mathbb{P}^2$ of $\mathbb{P}^2$ branched along $T_i + T_j$ induces an unramified double cover $\varpi_{ij} : \mathcal{C}'_{ij} \to \mathcal{C}$ of $\mathcal{C} = C_1 + C_2$ as explained above by taking $\mathcal{B} = T_i + T_j$

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and $D = P_{i1} + P_{i2} + P_{j1} + P_{j2}$. Note again that the covers ϖ_{ij} and $\pi_{ij}|_{\pi_{ij}^{-1}(\mathcal{C})}$ of $\mathcal{C}$ are isomorphic outside the points $\{P_{i1}, P_{i2}, P_{j1}, P_{j2}\}$. Now let $S_{ij} \to S'_{ij}$ be the canonical resolution of π_{ij}. Then $S_{ij} \cong \Sigma_2$, where Σ_2 is the Hirzebruch surface of degree 2, and we have the following diagram:

$$\begin{tikzcd} S'_{ij} \arrow[d, "\pi_{ij}"'] & S_{ij} \cong \Sigma_2 \arrow[l] \arrow[d] \\ \mathbb{P}^2 & \widehat{\mathbb{P}^2}, \arrow[l, "\sigma_{ij}"] \end{tikzcd}$$

where σ_{ij} is the blow-up at the intersection point $T_i \cap T_j$. The pencil of lines through the intersection point induces the ruling of Σ_2. It can be readily checked that the preimages $C_1^\pm, C_2^\pm$ of C_1, C_2 in S_{ij} are all linearly equivalent to $2F + \Delta_0$, since $C_i^+ + C_i^- \sim 4F + 2\Delta_0$ and $C_i^\pm \cdot \Delta_0 = 0$, where F is the divisor class of fibers and Δ_0 is the unique negative section with $\Delta_0^2 = -2$. Hence, $C_1^+ \cdot C_2^+ = C_1^+ \cdot C_2^- = 2$ and the splitting type of $(C_1, C_2; \mathcal{B})$ is $(2, 2)$ since $C_i^\pm \cdot \Delta_0 = 0$. Also, since ϖ_{ij} and $\pi_{ij}|_{\pi_{ij}^{-1}(\mathcal{C})}$ are isomorphic outside P_{ik}, this implies that the gluing data of $\varpi_{ij} : \mathcal{C}'_{ij} \to \mathcal{C}$ and $\mathcal{F}_{ij} := \mathcal{O}_{\mathbb{P}^2}(1)|_{\mathcal{C}} \otimes \mathcal{O}_{\mathcal{C}}(-P_{i1} - P_{i2} - P_{j1} - P_{j2})$ is $(+,+,-,-)$ for a suitable choice of labels on the nodes $C_1 \cap C_2$. Note again that changing all of the signs in the gluing data gives isomorphic covers/sheaves, so the data $(+,+,-,-)$ and $(-,-,+,+)$ give equivalent covers/sheaves. Let Q_1, Q_2, Q_3, Q_4 be the nodes of $C_1 + C_2$ and suppose that the gluing data of $\mathcal{F}_{12}$ is $(+,+,-,-)$ for (Q_1, Q_2, Q_3, Q_4) in this order. Since $\mathcal{F}_{12} \otimes \mathcal{F}_{13} \cong \mathcal{F}_{23}$ and all of these sheaves have two "+"s and two "−"s in the gluing data and are nontrivial, we can assume that $\mathcal{F}_{12}, \mathcal{F}_{13}, \mathcal{F}_{23}$ are all distinct and the gluing data of $\mathcal{F}_{13}$ is $(+,-,+,-)$ and the gluing data of $\mathcal{F}_{23}$ is $(-,+,+,-)$, after changing the labels of Q_3, Q_4 if necessary. By the same argument, each triple $\mathcal{F}_{ij}, \mathcal{F}_{jk}, \mathcal{F}_{ik}$, $\{i, j, k\} \subset \{1, 2, 3, 4\}$ gives all three possible distinct invertible sheaves with two "+"s and two "−"s in the gluing data. Moreover, $\mathcal{F}_{ij} \ncong \mathcal{F}_{ik}$ if $j \neq k$. Furthermore, this implies that $\mathcal{F}_{ij} \cong \mathcal{F}_{kl}$ for $\{i, j, k, l\} = \{1, 2, 3, 4\}$, as $\mathcal{F}_{kl}$ cannot be isomorphic to $\mathcal{F}_{ik}$ or $\mathcal{F}_{jk}$ and must be isomorphic to the remaining $\mathcal{F}_{ij}$ in the triple $\mathcal{F}_{ij}, \mathcal{F}_{jk}, \mathcal{F}_{ik}$.

Remark 3.5 In Example 3.4, $\mathcal{O}_{\mathcal{C}}(2P_{i1} + 2P_{i2}) \cong \mathcal{L}|_{\mathcal{C}} \cong \omega_{\mathcal{C}}$, where $\omega_{\mathcal{C}}$ is the dualizing sheaf of $\mathcal{C}$. Hence, $\mathcal{O}_{\mathcal{C}}(P_{i1} + P_{i2})$ is a *theta characteristic* of $\mathcal{C}$ (see [11]). Now, $\mathcal{F}_{ij} = \mathcal{L}|_{\mathcal{C}} \otimes \mathcal{O}_{\mathcal{C}}(-P_{i1} - P_{i2} - P_{j1} - P_{j2}) \cong \mathcal{O}_{\mathcal{C}}((P_{i1} + P_{i2}) - (P_{j1} + P_{j2}))$ and is nothing but the difference between the odd theta characteristics $\mathcal{O}_{\mathcal{C}}(P_{i1} + P_{i2})$ and $\mathcal{O}_{\mathcal{C}}(P_{j1} + P_{j2})$.

Remark 3.6 Since $\mathcal{F}_{ij} \cong \mathcal{F}_{kl}$, we have $\mathcal{F}_{ij} \otimes \mathcal{F}_{kl} \cong \mathcal{O}_{\mathcal{C}}$, which gives

$$\mathcal{O}_{\mathcal{C}}\left(\sum_{i=1}^{4}(P_{i1} + P_{i2})\right) \cong \mathcal{O}_{\mathbb{P}^2}(2)|_{\mathcal{C}}.$$

This implies Salmon's theorem – namely, the eight points of tangency $\{P_{11}, \ldots, P_{42}\}$ lie on a conic. See [11, Theorem 3.3] for details. Also, another different proof can be found in [12, Corollary 1.5].

4 Unramified double covers of two conics induced by Poncelet transverses

Let C_1, C_2 be smooth conics intersecting transversely as before. In this section, we consider the unramified double covers of $\mathcal{C} = C_1 + C_2$ induced by n-sided Poncelet transverses. We note that it is known that there exist C_1, C_2 intersecting transversely that admit an n-sided Poncelet transverse for any $n \geq 3$ (see [9]).

4.1 Degenerated Poncelet transverses

First, we study degenerated n-sided Poncelet transverses. Let C_1, C_2 be smooth conics intersecting transversely with an n-sided Poncelet transverse $\sqcap_n : (P_1, L_1), \ldots, (P_n, L_n)$. If $\sqcap_n$ is degenerated, there exists a pair (P_i, L_i), (P_j, L_j) $(i \neq j)$ such that either $P_i = P_j$ or $L_i = L_j$. We can assume $i < j$ without loss of generality. Note that $(P_i, L_i) \neq (P_j, L_j)$ by the minimality of the period n.

- Suppose $P_i = P_j$. If $P_i \in C_2$, then there is only one unique line L passing through P_i and tangent to C_2. Then $L_i = L_j$, which contradicts the minimality of the period. Hence, we can assume $P_i \notin C_2$ and that there exist two distinct tangent lines L_i', L_i'', of C_2 passing through P_i. Since $L_i \neq L_j$ by the minimality of the period, we have $\{L_i, L_j\} = \{L_i', L_i''\}$. This implies that (P_i, L_i), (P_j, L_j) are consecutive in the sequence and are of the form $(P_i, L_i), (P_i, L_{i+1})$; this can only occur if L_i is a tangent line of C_1 and hence a bitangent line to $C_1 + C_2$.
- Suppose $L_i = L_j$. Then since P_i and P_j lie on the same line, again we can assume that (P_i, L_i), (P_j, L_j) are consecutive in the sequence and are of the form (P_i, L_i), (P_{i+1}, L_i). If $P_{i+1} \notin C_2$, then there exist two distinct lines through P_{i+1} tangent to C_2 and $L_i \neq L_{i+1}$, which contradicts $L_i = L_j$. Hence, $P_{i+1} \in C_2$, and $L_i = L_j$ is the unique tangent line of C_2 passing through P_{i+1}.

In both cases, the sequence is "reflected" at (P_i, L_i), (P_{i+1}, L_{i+1}), and the points and lines leading up to this position appear in reverse order leading away. In order to be periodic, the sequence must be "reflected" once more to come back to (P_i, L_i), (P_{i+1}, L_{i+1}). A "reflection" only occurs in the above two cases; hence, if $\sqcap_n$ is degenerated, then it must contain exactly two lines that are either a bitangent line or a line tangent to C_2 at a point of $C_1 \cap C_2$. A bitangent line will appear in the whole sequence only once, and the other lines will appear exactly twice. Therefore, we have the following:

- If $n = 2m$, two cases can occur. In the first case, $\sqcap_{2m}$ has two bitangent lines. The set of vertices consists of m distinct points, and the set of edges consists of two bitangents and $m-1$ general lines. In this case, the transverse is of the form

$$(P_1, L_1), \ldots, (P_m, L_m), (P_m, L_{m-1}), (P_{m-1}, L_{m-2}), \ldots, (P_1, L_0),$$

  where L_0 and L_m are the bitangent lines, under a suitable choice of labels (see Figure 1 (a)). In the second case, $\sqcap_{2m}$ has two lines each tangent to C_2 at a point of $C_1 \cap C_2$. In this case, the set of vertices consists of $m+1$ distinct points, and the

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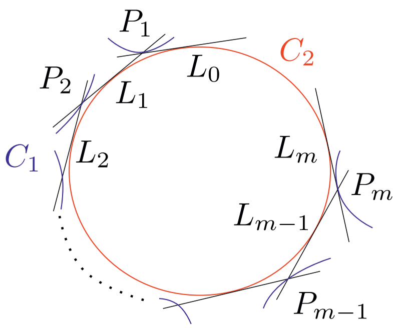
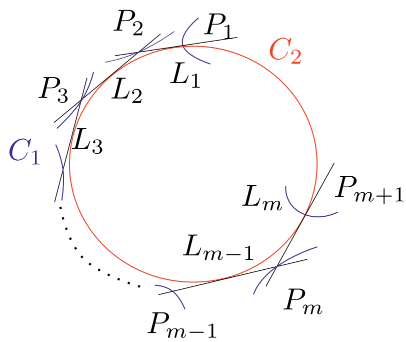
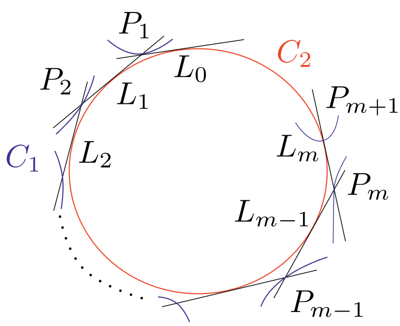


(a) Case of $n = 2m$. L_0 and L_m are bitangent lines.



(b) Case of $n = 2m$. L_1 and L_m are tangent to C_2 at points $P_1, P_{m+1} \in C_1 \cap C_2$.



(c) Case of $n = 2m + 1$. L_0 is a bitangent line and L_m is a line tangent to C_2 at a point $P_{m+1} \in C_1 \cap C_2$.

Figure 1: Three types of degenerated n-sided Poncelet transverses.

set of edges consists of the two lines each tangent to C_2 at a point of $C_1 \cap C_2$ and $m - 2$ general lines. In this case, the transverse is of the form

$$(P_1, L_1), \dots, (P_m, L_m), (P_{m+1}, L_m), (P_m, L_{m-1}), \dots, (P_3, L_2), (P_2, L_1),$$

where $P_1, P_{m+1} \in C_1 \cap C_2$ and L_1 and L_m are the lines tangent to C_2 at a point of $C_1 \cap C_2$, under a suitable choice of labels (see Figure 1 (b)). There exist two degenerated $2m$-sided transverses of each kind.

- If $n = 2m + 1$, $\sqcap_{2m+1}$ will have one bitangent line and one line tangent to C_2 at a point of $C_1 \cap C_2$. The set of vertices consists of $m + 1$ distinct points, and the set of edges consists of the bitangent line, the line tangent to C_2 at a point in $C_1 \cap C_2$, and $m - 1$ general lines. The transverse is of the form

$$(P_1, L_1), \dots, (P_m, L_m), (P_{m+1}, L_m), (P_m, L_{m-1}) \dots, (P_1, L_0),$$

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where L_0 is the bitangent line, $P_{m+1} \in C_1 \cap C_2$, and L_m is the line tangent to C_2 at P_{m+1}, under a suitable choice of labels (see Figure 1 (c)). There exist four degenerated $2m + 1$-sided transverses of this kind.

4.2 Deformation and degeneration of Poncelet transverses and line bundles of order two

Let C_1, C_2 be smooth conics intersecting transversely admitting a $2m$-sided Poncelet transverse $\sqcap_{2m}$. In this subsection, we consider double covers of $\mathbb{P}^2$ branched along the lines of $\sqcap_{2m}$ and its relation with the induced unramified double covers of $\mathcal{C} = C_1 + C_2$. We study the unramified double cover through a degeneration argument, where we deform general $2m$-sided Poncelet transverses to a degenerated transverse with two bitangent lines. Note that we do not consider the other type of degeneration, as it will not induce an unramified double cover of $\mathcal{C}$.

Let $\sqcap_{2m} : (P_1, L_1), \dots, (P_{2m}, L_{2m})$ be a general nondegenerated Poncelet transverse and let $\mathcal{P}_{2m} := \sum_{i=1}^{2m} L_i$. Let $Q_i = C_2 \cap L_i$ be the tangent points of L_i and C_2 $(i = 1, \dots, 2m)$. Let $\pi_{\mathcal{P}} : S' \to \mathbb{P}^2$ be the double cover branched along $\mathcal{P}_{2m}$. Then, since $\mathcal{P}_{2m}$ has degree $2m$ and $\mathcal{P}_{2m}|_{\mathcal{C}} = 2(\sum_{i=1}^{2m} P_i + \sum_{j=1}^{2m} Q_j)$, $\pi_{\mathcal{P}}$ induces an unramified double cover $\varpi_{\mathcal{P}} : \mathcal{C}'_{\mathcal{P}} \to \mathcal{C}$ as in Section 3. The line bundle of order 2 on $\mathcal{C}$ defining $\mathcal{C}'_{\mathcal{P}}$ is $\mathcal{F}_{\mathcal{P}} := \mathcal{O}_{\mathbb{P}^2}(m)|_{\mathcal{C}} \otimes \mathcal{O}_{\mathcal{C}}(-\sum_{i=1}^{2m} P_i - \sum_{j=1}^{2m} Q_j)$. We are interested in the structure of this double cover $\mathcal{C}'_{\mathcal{P}}$. By Theorem 1.1 (Poncelet's closure theorem), when we continuously move P_1 on C_1 to a point P' that is a tangent point of a bitangent line of $\mathcal{C} = C_1 + C_2$, the $2m$-sided Poncelet transverse originating at P_1 continuously deforms along with P_1 to the degenerated Poncelet transverse originating at P'. Let $P'_1 = P'$ and let

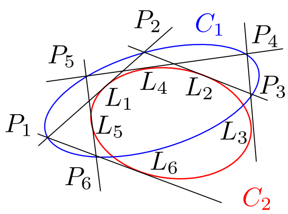
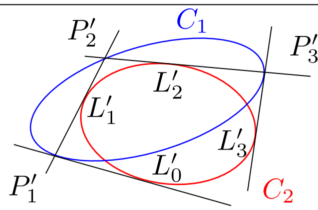
$$(P'_1, L'_1), \dots, (P'_m, L'_m), (P'_m, L'_{m-1}), (P'_{m-1}, L'_{m-2}), \dots, (P'_1, L'_0)$$

be the degenerated transverse originating at P'_1, where L'_0 and L'_m are bitangent lines of $\mathcal{C} = C_1 + C_2$. Let $Q'_i = L'_i \cap C_2$ $(i = 0, \dots, m)$ be the tangent points of L'_i and C_2. The correspondence between the lines, vertices, and tangent points under the degeneration are as follows:

$$\begin{aligned} P_i, P_{2m+1-i} &\to P'_i \quad (i = 1, \dots, m), \\ L_i, L_{2m-i} &\to L'_i \quad (i = 1, \dots, m-1), \quad L_m \to L'_m, \quad L_{2m} \to L'_0 \\ Q_i, Q_{2m-i} &\to Q'_i \quad (i = 1, \dots, m-1), \quad Q_m \to Q'_m, \quad Q_{2m} \to Q'_0 \end{aligned}$$

Here, the points $P_1, \dots, P_{2m}$ and $Q_1, \dots, Q_{2m}$ are continuously deformed on the smooth part of $\mathcal{C}$, while preserving the condition that $\mathcal{O}_{\mathbb{P}^2}(m)|_{\mathcal{C}} \otimes \mathcal{O}_{\mathcal{C}}(-\sum_{i=1}^{2m} P_i - \sum_{j=1}^{2m} Q_j)$ is of order 2. However, the set of invertible sheaves of order 2 of $\mathcal{C}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$ (see the discussions in Section 3 or [11, Section 3, 3a]) and is finite and discrete. By Lemma 3.2, the gluing data of the induced unramified double cover is constant under the deformation; hence, by Lemma 3.3, the gluing data of the

Table 1: $C_1 + C_2$ and the degeneration of a 6-sided Poncelet transverse.

<table>
<tr>
<td>Non-degenerated</td>
<td>
</td>
<td>$(P_1, L_1) \to (P_2, L_2)$
$\to (P_3, L_3) \to (P_4, L_4)$
$\to (P_5, L_5) \to (P_6, L_6)$</td>
</tr>
<tr>
<td>Degenerated</td>
<td>
</td>
<td>$(P'_1, L'_1) \to (P'_2, L'_2)$
$\to (P'_3, L'_3) \to (P'_3, L'_2)$
$\to (P'_2, L'_1) \to (P'_1, L'_0)$</td>
</tr>
</table>

invertible sheaves of order 2 must also be constant under the deformation, and the isomorphism classes of the sheaves $\mathcal{F}_{\mathcal{P}}$ must be constant. Hence, we have

$$\begin{aligned}\mathcal{O}_{\mathbb{P}^2}(m)|_{\mathcal{C}} \otimes \mathcal{O}_{\mathcal{C}}&\left(-\sum_{i=1}^{2m} P_i - \sum_{j=1}^{2m} Q_j\right)\\ &\cong \mathcal{O}_{\mathbb{P}^2}(m)|_{\mathcal{C}} \otimes \mathcal{O}_{\mathcal{C}}\left(-\sum_{i=1}^{m} 2P'_i - \sum_{j=1}^{m-1} 2Q'_j - Q'_0 - Q'_m\right).\end{aligned}$$

Furthermore, since $(L'_1 + \cdots + L'_{m-1})|_{\mathcal{C}} = P'_1 + \sum_{i=2}^{m-1} 2P'_i + P'_m + \sum_{j=1}^{m-1} 2Q'_j$ and $\mathcal{O}_{\mathbb{P}_2}(m-1)|_{\mathcal{C}} \cong \mathcal{O}_{\mathcal{C}}(P'_1 + \sum_{i=2}^{m-1} 2P'_i + P'_m + \sum_{j=1}^{m-1} 2Q'_j)$, we have

$$\begin{aligned}\mathcal{O}_{\mathbb{P}^2}(m)|_{\mathcal{C}} \otimes \mathcal{O}_{\mathcal{C}}&\left(-2\sum_{i=1}^{m} P'_i - \sum_{j=1}^{m-1} 2Q'_j - Q'_0 - Q'_m\right)\\ &\cong \mathcal{O}_{\mathbb{P}^2}(1)|_{\mathcal{C}} \otimes \mathcal{O}_{\mathcal{C}}(-P'_1 - P'_m - Q'_0 - Q'_m).\end{aligned}$$

The points P'_1, Q'_0 are the tangent points of the bitangent line L'_0, and the points P'_m, Q'_m are the tangent points of the bitangent line L'_m. Hence, we see that the structures of $\mathcal{F}_{\mathcal{P}}$ and the associated unramified double cover $\varpi_{\mathcal{P}} : \mathcal{C}'_{\mathcal{P}} \to \mathcal{C}$ induced by $\pi_{\mathcal{P}}$ are identical to that of the double cover in Example 3.4 associated to the bitangent lines $L'_0 + L'_m$. Summing up these arguments, we have the following Lemma, where ϖ_{ij} is the unramified double cover of $\mathcal{C}$ induced by the double cover of $\mathbb{P}^2$ branched along $T_i + T_j$ as defined in Example 3.4.

Lemma 4.1 *Under the above settings and notation, let T_1, T_2, T_3, T_4 be the bitangent lines to C_1, C_2 labeled so that the pairs T_1, T_2 and T_3, T_4 each lie in the same degenerated $2m$-sided transverse. Then the unramified double covers $\varpi_{\mathcal{P}}, \varpi_{12}, \varpi_{34}$ of $\mathcal{C} = C_1 + C_2$ are all isomorphic.*

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Remark 4.2 The isomorphism between ϖ_{12} and ϖ_{34} has already been observed in Example 3.4, regardless of the existence of a $2m$-sided Poncelet transverse.

5 Proof of Main Theorem

In this section, we prove Theorem 1.3. Let C_1, C_2 be smooth conics intersecting transversely that admit a $2m$-sided Poncelet transverse. Let T_1, T_2, T_3, T_4 be bitangent lines of $\mathcal{C} = C_1 + C_2$ labeled so that the pairs T_1, T_2 and T_3, T_4 each lie in the same degenerated $2m$-sided transverse. Let, $\sqcap_{2m} : (P_1, L_1), \ldots, (P_{2m}, L_{2m})$ be a nondegenerated transverse, $\mathcal{P}_{2m} := \sum_{i=1}^{2m} L_i$ and let

$$\mathcal{C}_{ij} := C_1 + C_2 + \mathcal{P}_{2m} + T_i + T_j \quad (\{i, j\} \subset \{1, 2, 3, 4\})$$

as in the Introduction.

Lemma 5.1 *The combinatorial types* $\mathrm{Comb}(\mathcal{C}_{ij})$ *of* $\mathcal{C}_{ij}$ *are the same for all* $\{i, j\} \subset \{1, 2, 3, 4\}$ *and any choice of nondegenerated transverse* $\sqcap_{2m}$.

Proof Let $\sqcap_{2m} : (P_1, L_1), \ldots, (P_{2m}, L_{2m})$ be a nondegenerated transverse. Since all of the lines $L_1, \ldots, L_{2m}$ and T_1, T_2, T_3, T_4 are tangent lines of $\mathcal{C}_2$, no three are concurrent. A line L_i and a bitangent T_j cannot intersect on C_1, as we have assumed that L_i lies in a nondegenerated transverse and T_j lies in a degenerated transverse. They cannot intersect on C_2 as well because they are distinct tangent lines of C_2. Hence, the combinatoral types are the same. ■

Let $\mathcal{B}_{ij} := \mathcal{P}_{2m} + T_i + T_j \quad (\{i, j\} \subset \{1, 2, 3, 4\})$ and let $\pi_{\mathcal{B}_{ij}} : S' \to \mathbb{P}^2$ be the double cover of $\mathbb{P}^2$ branched along $\mathcal{B}_{ij}$.

Lemma 5.2 *Under the labeling above, the splitting type of* $(C_1, C_2; \mathcal{B}_{ij})$ *is* $(0, 4)$ *if* $\{i, j\} = \{1, 2\}$ *or* $\{3, 4\}$ *and is* $(2, 2)$ *otherwise.*

Proof Since $\mathcal{B}_{ij}$ can be viewed as a sum of $\mathcal{P}_{2m}$ and $(T_i + T_j)$, by the discussions in Section 3, the cover $\pi_{\mathcal{B}_{ij}}$ induces an unramified double cover of $\mathcal{C} = C_1 + C_2$, whose structure is given by the product of the covers $\varpi_{\mathcal{P}}$ of Section 4.2 and ϖ_{ij} of Example 3.4. Then since $\varpi_{\mathcal{P}}$, ϖ_{12}, ϖ_{34} are isomorphic by Lemma 4.1, $\mathcal{B}_{ij}$ induces the trivial unramified double cover if $\{i, j\} = \{1, 2\}$ or $\{3, 4\}$, and otherwise induces a nontrivial unramified double cover with gluing data $(+, +, -, -)$ for a suitable choice of labels of the nodes. ■

Now, Lemma 5.1 and 5.2 together with Proposition 2.2 give the proof of Theorem 1.3.

Example 5.3 We give explicit equations of conics C_1 and C_2 admitting a 4-sided Poncelet transverse, and also an example of a Zariski pair of degree 10 arising from the conics. We use $(x : y : z)$ as a system of homogeneous coordinates of $\mathbb{P}^2$. Let C_1 and C_2 be two conics defined by the following equations:

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$$C_1 : 4x^2 - y^2 - z^2 = 0,$$
$$C_2 : x^2 - yz = 0.$$

The bitangent lines of $C_1 + C_2$ are given as follows:

$$T_1 : \left(-\sqrt{-1} + \sqrt{3}\right) x - y - \left(\frac{1}{2} - \frac{\sqrt{-3}}{2}\right) z = 0,$$
$$T_2 : \left(-\sqrt{-1} - \sqrt{3}\right) x - y - \left(\frac{1}{2} + \frac{\sqrt{-3}}{2}\right) z = 0,$$
$$T_3 : \left(\sqrt{-1} - \sqrt{3}\right) x - y - \left(\frac{1}{2} - \frac{\sqrt{-3}}{2}\right) z = 0,$$
$$T_4 : \left(\sqrt{-1} + \sqrt{3}\right) x - y - \left(\frac{1}{2} + \frac{\sqrt{-3}}{2}\right) z = 0.$$

The tangent point of C_1 and T_1 is $P_1' := (\sqrt{-1} + \sqrt{3} : 2 + 2\sqrt{-3} : 4)$. Let L_1' be the line defined by $2\sqrt{-1}x - y + z = 0$, which passes through P_1' and is tangent to C_2. The intersection point of L_1' and C_1 is $P_2' = (\sqrt{-1} - \sqrt{3} : 2 - 2\sqrt{-3} : 4)$, which coincides with the intersection point of T_2 and C_1. Now it is evident that the following sequence is a degenerated 4-sided Poncelet transverse:

$$(P_1', T_1), (P_1', L_1'), (P_2', T_2), (P_2', L_1').$$

Hence, we have a 4-sided Poncelet transverse originating from any pair (P_1, L_0) of a point $P_1 \in C_1$ and a tangent line L_0 of C_2 passing through P_1 by Poncelet's closure theorem. Note that since T_1 and T_2 lie in a degenerated transverse, T_3 and T_4 must also lie on the other degenerated transverse. By the above argument, if $(i, j) \in \{(1, 2), (3, 4)\}$ and $(k, l) \notin \{(1, 2), (3, 4)\}$, then the following pair of plane curves is a Zariski pair for the square $\mathcal{P}_4$ of any nondegenerated 4-sided Poncelet transverse:

$$(C_1 + C_2 + \mathcal{P}_4 + T_i + T_j, C_1 + C_2 + \mathcal{P}_4 + T_k + T_l).$$

For example, the following $L_0, \ldots, L_3$ form a square $\mathcal{P}_4$ of a nondegenerated 4-sided Poncelet transverse:

$$L_0 : z = 0, \qquad L_1 : 2x - y - z = 0,$$
$$L_2 : y = 0, \qquad L_3 : 2x + y + z = 0.$$

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Department of Applied Mathematics, Faculty of Science, Okayama University of Science,
1-1 Ridai-cho, Kita-ku, Okayama 700-0005, Japan
e-mail: bannai@ous.ac.jp

Department of Mathematical Sciences, Graduate School of Science, Tokyo Metropolitan University, 1-1 Minami-Ohsawa, Hachiohji 192-0397, Japan
e-mail: tokunaga@tmu.ac.jp masuya-ryosuke@ed.tmu.ac.jp sagawa-emiko@ed.tmu.ac.jp

Department of Mathematical Science, Faculty of Science and Technology, Tokushima University, 2-1 Minamijyousanjima-cho, Tokushima 770-8506, Japan
e-mail: shirane@tokushima-u.ac.jp