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AN EXTENSION OF THE FORELLI–RUDIN PROJECTION THEOREM

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For a measurable function f on the unit ball B in $\mathbb{C}^n$ we define $(M_1 f)(w)$, $|w|<1$, to be the mean modulus of f over a hyperbolic ball with center at w and of a fixed radius. The space $L_1^p, 0<p<\infty$, is defined by the requirement that $M_1 f$ belongs to the Lebesgue space L^p. It is shown that the subspace of L^p spanned by holomorphic functions coincides with the corresponding subspace of L_1^p. It is proved that if $s>(n+1)(p^{-1}-1)$, $0<p<1$, then this subspace is complemented in L_1^p by the projection whose reproducing kernel is $(1-|w|^2)^s(1-\langle z,w\rangle)^{-(s+n+1)}$. As corollaries we get an extension of the Forelli–Rudin projection theorem and we show that a holomorphic function f is L^p-integrable, $0<p<\infty$, over the unit ball B iff $u=Re f$ is L^p-integrable over B. Finally, we sketch an alternative proof of the main result of this paper in the case $0<p<1$.

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0. Introduction

Throughout this paper n will denote a fixed positive integer. Let B be the unit ball in $\mathbb{C}^n$ and dv the normalized Lebesgue measure on B. Following Forelli and Rudin [4] we let

$$(T_s f)(z)=\binom{n+s}{n}\int_B f(w)\,\frac{(1-|w|^2)^s}{(1-\langle z,w\rangle)^{s+n+1}}\,dv(w),\quad z\in B, \tag{0.1}$$

where s is a real number >-1, and f is any complex-valued measurable function on B satisfying

$$\int_B |f(w)|(1-|w|^2)^s\,dv(w)<\infty. \tag{0.2}$$

The set of all f satisfying (0.2) will be denoted by $D(T_s)$. It is clear that (0.1) defines a linear operator which maps $D(T_s)$ into $H(B)$, the set of all functions holomorphic in B. The most important property of T_s is that

$$T_s f=f \quad\text{and}\quad T_s\bar{f}=\overline{f(0)} \quad\text{for}\quad f\in H(B)\cap D(T_s). \tag{0.3}$$

See [4].

In [4], Forelli and Rudin gave a necessary and sufficient condition for T_s to be a bounded operator on $L^p(v)$:

Forelli–Rudin Theorem. *For $1 \leqq p < \infty$, T_s is a bounded operator on $L^p(v)$ if and only if*

$$s > p^{-1} - 1. \tag{0.4}$$

If (0.4) *holds, then T_s projects $L^p(v)$ onto $L^p(v) \cap H(B)$.*

In this paper we extend the Forelli–Rudin theorem to a class of non-locally convex spaces. We are motivated by the fact that if $0 < p < 1$, then there is no bounded operator which maps $L^p(v)$ onto $L^p(v) \cap H(B)$. (The dual of $L^p(v)$ is trivial. On the other hand, for each $z \in B$, the functional $f \to f(z)$ is continuous on $L^p(v) \cap H(B)$; see [10, Theorem 7.2.5].) Our main result is that there is a scale of spaces, denoted by $L_1^p(v)$, satisfying the following:

(i) $L^p(v) \cap H(B) = L_1^p(v) \cap H(B)$ for $0 < p < \infty$;

(ii) $L_1^p(v) \subset L^p(v)$ for $p \leqq 1$, and $L^p(v) \subset L_1^p(v)$ for $1 \leqq p < \infty$;

(iii) for $0 < p < 1$ (resp. $1 \leqq p < \infty$), T_s is a bounded operator on $L_1^p(v)$ if and only if $s > (n+1)(p^{-1} - 1)$ (resp. $s > p^{-1} - 1$).

The definition of $L_1^p(v)$ and of some related spaces is in Section 2. In Section 1 we list some properties of the automorphisms of the unit ball and give a short proof of (0.3).

The proof of the assertion (iii) is in Section 3. We also extend a result of Forelli and Rudin by proving that $f \in H(B)$ and $Re f \in L^p(v)$, $p < 1$, imply $f \in L^p(v)$.

In Section 4, we consider a discrete version of L_1^p obtained by decomposing the disk into hyperbolically "equal"-sized pieces as in [2] and use it to sketch a proof of the part "if" of property (iii) (see above) in the case $0 < p < 1$.

1. Preliminaries

The definitions and notation are for the most part those given in Rudin [10]. By $\mathbb{C}^n$ we denote the vector space of n-tuples $z = (z_1, \ldots, z_n)$ of complex numbers, with inner product and norm given by

$$\langle z, w \rangle = z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n, \quad |z| = \langle z, z \rangle^{1/2}.$$

Let $Aut(B)$ be the group of all (biholomorphic) automorphisms of the unit ball $B = \{z \in \mathbb{C}^n : |z| < 1\}$. Each $\psi \in Aut(B)$ can be written as $\psi = U \circ \phi_a$ $(a \in B)$, where U is a unitary transformation on $\mathbb{C}^n$, and $\phi_a \in Aut(B)$ satisfies

$$\phi_a(0) = a, \ \phi_a(a) = 0, \ \phi_a = \phi_a^{-1}.$$

The main property of ϕ_a is given by

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$$1-\langle\phi_a(z),\phi_a(w)\rangle=\frac{(1-|a|^2)(1-\langle z,w\rangle)}{(1-\langle a,w\rangle)(1-\langle z,a\rangle)} \tag{1.1}$$

for all, $a,z,w\in B$ (see [**10**, Theorem 2.2.2]). In particular

$$1-|\phi_a(w)|^2=\frac{(1-|a|^2)(1-|w|^2)}{|1-\langle a,w\rangle|^2} \tag{1.2}$$

and

$$1-\langle a,\phi_a(w)\rangle=1-\langle\phi_a(0),\phi_a(w)\rangle=\frac{1-|a|^2}{1-\langle a,w\rangle}. \tag{1.3}$$

Combining (1.2) and (1.3) yields

$$\frac{1-|\phi_a(w)|^2}{1-\langle a,\phi_a(w)\rangle}=\frac{1-|w|^2}{1-\langle w,a\rangle}. \tag{1.4}$$

For $a,w\in B$ let $d(w,a)=|\phi_a(w)|=|\phi_w(a)|$. It is well-known that d is an invariant metric on B satisfying

$$d(w,z)\leqq\frac{d(w,a)+d(a,z)}{1+d(w,a)d(a,z)} \tag{1.5}$$

for all $a,z,w\in B$. (Note that the Bergman metric on B is equal to $c_n\log((1+d)/(1-d))$ and that d is called the pseudo-hyperbolic metric.)

The measure $d\tau$ defined by

$$d\tau(w)=(1-|w|^2)^{-(n+1)}dv(w),\ w\in B,$$

(dv is the normalized Lebesgue measure on B) is invariant with respect to the group $Aut(B)$ [**10**, Theorem 2.2.6]. In particular, if we put

$$E(a,\varepsilon)=\{z\in B: d(a,z)<\varepsilon\}=\phi_a(\varepsilon B),$$

$$\varepsilon B=E(0,\varepsilon)=\{z:|z|<\varepsilon\};\ 0<\varepsilon<1,$$

then we have $\tau(E(a,\varepsilon))=\tau(\varepsilon B)$. By integration in polar coordinates we find that

$$\tau(E(a,\varepsilon))=\varepsilon^{2n}(1-\varepsilon^2)^{-n}=:\tau(\varepsilon),\quad a\in B;\ 0<\varepsilon<1. \tag{1.6}$$

We also note that the invariance property of d and the mean value property of M-harmonic functions imply the formula

$$g(a)\tau(\varepsilon)=\int_{E(a,\varepsilon)} g\,d\tau, \quad a\in B, 0<\varepsilon<1, \tag{1.7}$$

which is valid for every M-harmonic function g on B. (In particular, (1.7) holds for holomorphic and antiholomorphic functions.)

A proof of (0.3). That $T_s\bar{f}=\overline{f(0)}$ for $f\in H(B)\cap D(T_s)$ is easily deduced from the mean value property of antiholomorphic functions [**10**, Proposition 7.1.2]. Then the first equality in (0.3) is obtained by use of the formula

$$\overline{(T_sf)(a)}=(T_s(\overline{f\circ\phi_a}))(a)), \quad f\in D(T_s),\ a\in B. \tag{1.8}$$

To prove (1.8) we write T_s as

$$(T_sf)(a)=\int_B f(w)Q_s(a,w)\,d\tau(w), \tag{1.9}$$

where

$$Q_s(a,w)=\binom{n+s}{n}\left(\frac{1-|w|^2}{1-\langle a,w\rangle}\right)^{s+n+1}, \quad a, w\in B. \tag{1.10}$$

By using the invariance of $d\tau$ we get

$$(T_sf)(a)=\int_B f(\phi_a(w))Q_s(a,\phi_a(w))\,d\tau(w).$$

Combining this with the identity $Q_s(a,\phi_a(w))=\overline{Q_s(a,w)}$ (which follows from (1.4)) yields (1.8). (The proof shows that if f belongs to $D(T_s)$, then so does $f\circ\phi_a$.)

We finish this section with two useful lemmas.

Lemma 1.1. *If* $d(a,w)\leqq\varepsilon<1$, *then*

$$\frac{1}{C}\leqq\frac{1-|a|^2}{1-|w|^2}\leqq C,$$

where $C=4/(1-\varepsilon^2)<\infty$.

Proof. Clearly we have to prove one of the required inequalities; the other will follow by symmetry. If $d(a,w)\leqq\varepsilon$, then, by (1.2),

$$\frac{1-|a|^2}{1-|w|^2}=\frac{(1-d(a,w)^2)|1-\langle a,w\rangle|^2}{(1-|w|^2)^2}$$

$$\geqq\frac{(1-\varepsilon^2)(1-|w|)^2}{(1-|w|^2)^2}\geqq\frac{1-\varepsilon^2}{4}.$$

Lemma 1.2. *If* $d(a,w) \leqq \varepsilon < 1$ *and* $z \in B$, *then*

$$\frac{1}{C} \leqq \left| \frac{1 - \langle z, w \rangle}{1 - \langle z, a \rangle} \right| \leqq C,$$

where $C = 2/(1-\varepsilon) < \infty$.

Proof. By (1.1),

$$\left| \frac{1 - \langle z, w \rangle}{1 - \langle z, a \rangle} \right| = \frac{|1 - \langle \phi_a(z), \phi_a(w) \rangle| \, |1 - \langle a, w \rangle|}{1 - |a|^2}$$

$$\geqq \frac{(1 - |\phi_a(w)|)(1 - |a|)}{1 - |a|^2}.$$

The result follows.

2. L^p_q-spaces

Unless specified otherwise, we shall assume that p, q, ε and δ are positive and satisfy $p < \infty$, $q \leqq \infty$, $\varepsilon < 1$ and $\delta < 1$. For a complex-valued measurable function f on B we define

$$(M_\infty f)(w) = (M_{\infty,\varepsilon} f)(w) = \operatorname{ess\,sup} \{ |f(a)| : a \in E(w, \varepsilon) \},$$

$$(M_q f)(w) = (M_{q,\varepsilon} f)(w) = \left\{ \frac{1}{\tau(\varepsilon)} \int_{E(w,\varepsilon)} |f|^q \, d\tau \right\}^{1/q}, \quad q < \infty,$$

where $\tau(\varepsilon) = \tau(E(w, \varepsilon))$, $w \in B$. (See (1.6).)

The simplest properties of M_q are collected in the following proposition.

Proposition 2.1. *Let* f *be a measurable function on* B. *Then*

$$M_\infty f \geqq M_q f \geqq M_p f \quad \text{for} \quad q \geqq p, \tag{2.1}$$

$$M_{q,\delta} f \leqq C M_{q,\varepsilon} f \quad \text{for} \quad 0 < \delta < \varepsilon, \tag{2.2}$$

$$M_{\infty,\delta}(M_{q,\delta} f) \leqq C M_{q,\varepsilon} f, \quad \text{where} \quad 2\delta/(1+\delta^2) = \varepsilon, \tag{2.3}$$

$$\int_B |f|^q \, d\tau = \int_B (M_q f)^q \, d\tau \quad \text{for} \quad q < \infty. \tag{2.4}$$

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Remark. Throughout this paper the letter "C" denotes a positive real constant which may vary from line to line. In (2.2) and (2.3), C is independent of f.

Proof. The proofs of (2.1) and (2.2) are simple. To prove (2.3) observe that, by (1.5),

$$E(a,\delta)\subset E(w,2\delta/(1+\delta^2)) \quad \text{if} \quad d(a,w)<\delta. \tag{2.5}$$

Hence, if $a\in E(w,\delta)$, then

$$\tau(\delta)(M_{q,\delta}f)^q(a)=\int_{E(a,\delta)}|f|^q\,d\tau$$

$$\leqq \int_{E(w,\varepsilon)}|f|^q\,d\tau=\tau(\varepsilon)(M_{q,\varepsilon}f)^q(w),$$

which gives (2.3) with $C=(\tau(\varepsilon)/\tau(\delta))^{1/q}$.

To prove (2.4) write M_qf as

$$(M_qf)^q(w)=\frac{1}{\tau(\varepsilon)}\int_B|f(a)|^q k_\varepsilon(w,a)\,d\tau(a),$$

where

$$k_\varepsilon(w,a)=\begin{cases}1 & \text{if } d(w,a)<\varepsilon,\\ 0 & \text{if } d(w,a)\geqq\varepsilon.\end{cases} \tag{2.6}$$

Then, by Fubini's theorem,

$$\int_B (M_qf)^q(w)\,d\tau(w)=\frac{1}{\tau(\varepsilon)}\int_B|f(a)|^q\,d\tau(a)\int_B k_\varepsilon(w,a)\,d\tau(w)$$

$$=\frac{1}{\tau(\varepsilon)}\int_B|f(a)|^q\tau(E(a,\varepsilon))\,d\tau(a),$$

and this concludes the proof because $\tau(E(a,\varepsilon))=\tau(\varepsilon)$.

Definition. Let μ be one of the measures v or τ. We define $L^p_{q,\varepsilon}(\mu)=L^p_q(\mu)$ to be the space of all measurable functions f on B for which

$$\|f\|_{L^p_q(\mu)}:=\|M_qf\|_{L^p(\mu)}<\infty.$$

Proposition 2.2. (i) *The operator S_p defined by $(S_pf)(w)=(1-|w|^2)^{(n+1)/p}f(w)$ acts as an isomorphism of $L^p_q(v)$ onto $L^p_q(\tau)$.*

(ii) $L^p_p(\mu)=L^p(\mu)$; $L^p_q(\mu)\subset L^p(\mu)\,(q\geqq p)$; $L^p_q(\mu)\supset L^p(\mu)\,(q\leqq p)$.

(iii) *The spaces L^p_q are complete.*

Proof. The assertion (i) is an immediate consequence of Lemma 1.1. If $\mu=\tau$, then $L_p^p(\mu)=L^p(\mu)$ because of (2.4). Combining this with (i) gives the first relation in (ii). On the other hand, it follows from (2.1) that $L_q^p(\mu)\subset L_p^p(\mu)$ for $q\geqq p$, and $L_q^p(\mu)\supset L_p^p(\mu)$ for $q\leqq p$, and this completes the proof of the assertion (ii).

The completeness of $L_q^p(\tau)$ reduces to the completeness of $L_q^p(v)$, by (i). The completeness of $L_q^p(v)$ is deduced from the completeness of $L^r(v)$, $r=\min(p,q)$, by using Fatou's lemma and the (continuous) inclusion $L_q^p(v)\subset L^r(v)$. The proof is standard and is omitted here.

The main difference between L^p and L_1^p is given by the following proposition.

Proposition 2.3. *If* $0<p<1$, *then* $L_1^p(\tau)\subset L^1(\tau)$, *and the inclusion map is continuous.*

Remark. This shows that, in contrast to the case of L^p, the dual of L_1^p separates points.

Proof of the proposition. Let f be a norm-one element of $L_1^p(\tau)=L_{1,\varepsilon}^p(\tau)$, $p<1$. Let $g=M_{1,\delta}f$, where $2\delta/(1+\delta^2)=\varepsilon$, $\delta<\varepsilon$. We have, by (2.4) and (2.2),

$$\int_B |f|\,d\tau=\int_B g\,d\tau=\int_B g^{1-p}g^p\,d\tau\leqq\|g\|_\infty^{1-p}\int_B g^p\,d\tau$$

$$\leqq C\|g\|_\infty^{1-p}\int_B (M_{1,\varepsilon}f)^p\,d\tau$$

$$=C\|g\|_\infty^1,$$

where C is independent of f. On the other hand,

$$\int_B (M_{\infty,\delta}g)^p\,d\tau\leqq C$$

because $M_{\infty,\delta}g\leqq CM_{1,\varepsilon}f$ (by (2.3)). Hence, by using the equality

$$(M_{\infty,\delta}g)^p(w)=\operatorname*{ess\,sup}_{a\in B} g(a)^p k_\delta(w,a)$$

(see (2.6)) we obtain

$$\operatorname*{ess\,sup}_{a\in B}\int_B g(a)^p k_\delta(w,a)\,d\tau(w)\leqq C,$$

which yields

$$\|g\|_\infty^p\tau(\delta)\leqq C.$$

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Combining the above inequalities concludes the proof.

As a consequence of Propositions 2.3 and 2.2 (i) we have

Proposition 2.3′. *If* $f \in L_1^p(v)$, $0<p<1$, *then*

$$\int_B |f(w)|(1-|w|^2)^{(n+1)(p^{-1}-1)}\,dv(w)<\infty. \tag{2.7}$$

For $p>1$, the above arguments show that $L^1(\tau)\subset L_1^p(\tau)$. In other words, if f satisfies (2.7), $p>1$, then $f\in L_1^p(v)$. By using this remark we prove the following.

Proposition 2.4. *The inclusions* $L_1^p(v)\subset L_1^p(v)$ $(p<1)$ *and* $L^p(v)\subset L_1^p(v)$ $(p>1)$, *which occur in Proposition* 2.2(ii), *are proper.*

Proof. Let $\{A_j\}_{j=1}^{\infty}$ be a sequence of pairwise disjoint subsets of B such that $\tau(A_j)=2^{-j}$. Let

$$f(w)=(1-|w|^2)^{-(n+1)/p}\sum_{j=1}^{\infty} c_j K_j(w), \quad w\in B,$$

where K_j is the characteristic function of A_j. We put $c_j=2^{j/p}$ if $p>1$, and $c_j=2^j$ if $p<1$. If $p>1$, then f satisfies (2.7) and therefore $f\in L_1^p(v)$. On the other hand, $f\notin L^p(v)$, and this shows that $L_1^p(v)\neq L^p(v)$ for $p>1$. The case $p<1$ is considered similarly.

Although the spaces L_1^p and L^p are different, their restrictions to some important classes coincide. Here we consider the case of holomorphic functions.

Proposition 2.5. *We have* $L_q^p(v)\cap H(B)=L^p(v)\cap H(B)$, *and the corresponding "norms" are equivalent.*

Proof. Let $f\in H(B)$ and $a\in B$. Then $f\circ\phi_a\in H(B)$ and therefore the function $|f\circ\phi_a|^q$ $(q<\infty)$ is subharmonic, whence

$$|f(a)|^q=|f(\phi_a(0))|^q\leqq \varepsilon^{-2n}\int_{\varepsilon B}|f\circ\phi_a|^q\,dv\leqq \varepsilon^{-2n}\int_{\varepsilon B}|f\circ\phi_a|^q\,d\tau=\varepsilon^{-2n}\int_{E(a,\varepsilon)}|f|^q\,d\tau.$$

Hence

$$|f|\leqq CM_{q,\varepsilon}f, \tag{2.8}$$

where C is independent of f. This proves that $L_q^p(v)\cap H(B)\subset L^p(v)\cap H(B)$.

(Observe that the case $q=\infty$ is trivial.) To conclude the proof we have to prove that $L^p(v)\cap H(B)\subset L_\infty^p(B)\cap H(B)$.

From (2.8) and the obvious modification of (2.3) we have $M_{\infty,\varepsilon}f \leqq CM_{\infty,\varepsilon}(M_{p,\varepsilon}f) \leqq CM_{p,\delta}f$, where $\delta = 2\varepsilon/(1+\varepsilon^2)$. Hence $L^p(v) \cap H(B) = L^p_{p,\delta}(v) \cap H(B) \subset L^p_{\infty,\varepsilon}(v) \cap H(B)$, which was to be proved.

3. Projections

Our main result is the following.

Theorem 3.1. *For* $0<p<1$, T_s *is a bounded operator on* $L^p_1(v) = L^p_{1,\varepsilon}(v)$ *if and only if*

$$s > (n+1)(p^{-1}-1). \tag{3.1}$$

If (3.1) *holds, then* T_s *projects* $L^p_1(v)$ *onto* $L^p(v) \cap H(B)$.

The second assertion is easily deduced from the first, (0.3), and Proposition 2.5. To prove the first assertion we need the following lemma which can be found in [**10**, Proposition 1.4.10].

Lemma 3.1. *For a real number* α *let*

$$J_\alpha(w) = \int_B \frac{dv(z)}{|1-\langle z, w\rangle|^{\alpha+n+1}}, \quad w \in B.$$

Then

$$J_\alpha(w) \doteq 1 \quad \text{if } \alpha < 0,$$

$$\doteq \log\frac{1}{1-|w|^2} \quad \text{if } \alpha = 0,$$

$$\doteq (1-|w|^2)^{-\alpha} \quad \text{if } \alpha > 0.$$

Remark. For two nonnegative functions F and G defined on a set S we write $F(w) \doteq G(w)$, $w \in S$, if there is a positive constant C such that $G(w)/C \leqq F(w) \leqq CG(w)$ for all $w \in S$.

Proof of Theorem 3.1. Assuming (3.1) we have $L^p_1(v) \subset D(T_s)$, by Proposition 2.3′. Let $f \in L^p_1(v)$. For a fixed $z \in B$ let $h(w) = Q_s(z, w)$, $w \in B$, where Q_s is defined by (1.10). Then, by (1.9) and Proposition 2.3,

$$|(T_s f)(z)|^p \leqq C \int_B (M_1(fh))^p \, d\tau,$$

where C is independent of f, z. Since $M_1(fh) \leqq (M_1 f)(M_\infty h)$ and $M_\infty h \leqq C|h|$ (by Lemmas 1.1 and 1.2), we get

$$|(T_s f)(z)|^p \leqq C \int_B (M_1 f)^p(w) |Q_s(z,w)|^p \, d\tau(w), \tag{3.2}$$

where C is independent of f, z. Now integration yields

$$\int_B |T_s f)(z)|^p \, dv(z) \leqq C \int_B (M_1 f)^p(w) \, d\tau(w) \int_B |Q_s(z,w)|^p \, dv(z)$$

$$= C \int_B (M_1 f)^p(w) (1-|w|^2)^\alpha J_\alpha(w) \, dv(w),$$

where $\alpha = (s+n+1)p-(n+1)$. If (3.1) holds, then $\alpha > 0$, so the function $(1-|w|^2)^\alpha J_\alpha(w)$ is bounded on B (by Lemma 3.1). Hence we conclude that if (3.1) holds, then T_s is a bounded operator from $L_1^p(v)$ into $L^p(v) \cap H(B) = L_1^p(v) \cap H(B)$.

Assuming that $s \leqq (n+1)(p^{-1}-1)$, we have to prove that T_s is not bounded. Consider the functions f_b, $b \in B$, defined by

$$f_b(w) = \begin{cases} (1-|w|^2)^{-(s+n+1)} & \text{if } w \in E(b,\varepsilon) \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$(T_s f_b)(z) = \binom{n+s}{n} \int_{E(b,\varepsilon)} (1-\langle z,w \rangle)^{-(s+n+1)} \, d\tau(w).$$

For each $z \in B$ the function $w \to (1-\langle z,w \rangle)^{-(s+n+1)}$ is antiholomorphic. Hence, by (1.7),

$$(T_s f_b)(z) = \binom{n+s}{n} \tau(1/2)(1-\langle z,b \rangle)^{-(s+n+1)}.$$

Hence

$$\|T_s f_b\|_{L_1^p(v)} \doteq \|T_s f_b\|_{L^p(v)} \doteq J_\alpha(b)^{1/p}, \quad b \in B, \tag{3.3}$$

where $\alpha = sp-(n+1)(1-p) \leqq 0$. On the other hand, applying Lemma 1.1 and (2.5) gives

$$(M_\infty f_b)(w) \leqq C(1-|w|^2)^{-(s+n+1)} \quad \text{if } w \in E(b,\delta),$$

$$= 0 \quad \text{if } w \notin E(b,\delta),$$

where $\delta = 2\varepsilon/(1+\varepsilon^2)$. From this we find that

$$\|f_b\|_{L_1^p(v)} \leqq C(1-|b|^2)^{-\alpha/p}, \quad b \in B, \tag{3.4}$$

where α is the same as in (3.3). From (3.3), (3.4) and Lemma 3.1 we conclude that T_s is unbounded on the bounded set $\{(1-|b|^2)^{\alpha/p} f_b : b \in B\}$, and this completes the proof of the theorem.

Remark. The above proof shows that Theorem 3.1 remains true if we replace $L_1^p(v)$ by any of the spaces $L_q^p(v)$, $1 \leqq q \leqq \infty$.

Theorem 3.2. *The Forelli–Rudin theorem remains true if we replace $L^p(v)$ by $L_1^p(v)$.*

Proof. If T_s is bounded on $L_1^p(v)$, $p \geqq 1$, then T_s acts as a bounded operator from $L^p(v)$ to itself because of the continuous inclusions $L^p(v) \subset L_1^p(v)$ and $L^p(v) \supset L_1^p(v) \cap H(B)$ (= the image of $L_1^p(v)$). Hence, that the boundedness of T_s on $L_1^p(v)$ implies (0.4) is a consequence of the Forelli–Rudin theorem.

In view of Propositions 2.4 and 2.5, if $s > p^{-1} - 1$, then Theorem 3.2 states somewhat more than the Forelli–Rudin theorem. Nevertheless, a slight modification of Forelli and Rudin's proof proves Theorem 3.2. Namely, it follows from [**4**] that if $s > p^{-1} - 1$, $p \geqq 1$, then the equality

$$(U_s f)(z) = \int_B f(w) |Q_s(z, w)| \, d\tau(w), \quad z \in B,$$

defines a bounded linear operator on $L^p(v)$. This implies that if $f \in L_1^p(v)$, then

$$\|U_s M_1 f\|_{L^p(v)} \leqq C \|M_1 f\|_{L^p(v)} = C \|f\|_{L_1^p(v)}$$

(because $M_1 f \in L^p(v)$). On the other hand, by using (2.4) (applied to fQ_s) and Lemmas 1.1 and 1.2 we see that if $U_s M_1 f$ is defined, then so is $T_s f$ and $|T_s f| \leqq C U_s M_1 f$, where C is independent of f. Combining these estimates shows that T_s is a bounded operator from $L_1^p(v)$ to $L^p(v) \subset L_1^p(v)$, which completes the proof.

At the end we use Theorem 3.1 to extend another result of Forelli and Rudin [**4**].

Theorem 3.3. *If $f \in H(B)$ and the real part of f belongs to $L^p(v)$ $(0 < p < \infty)$, then $f \in L^p(v)$.*

Forelli and Rudin considered the case there $p \geqq 1$.

Proof. Let $0 < p < 1$, $u = Re f$. The implication $u \in L_1^p(v) \Rightarrow f \in L^p(v)$ is a direct consequence of Theorem 3.1. and the identity

$$f = 2T_s u - \overline{f(0)}, \; u \in L_1^p(v), \; s > (n+1)(p^{-1} - 1). \tag{3.5}$$

Thus we have to prove (3.5) and the implication

$$u \in L^p(v) \Rightarrow u \in L_1^p(v). \tag{3.6}$$

For $0<r<1$ define f_r and u_r by $f_r(z)=f(rz)$ and $u_r(z)=u(rz)$. Since $f_r \in D(T_s)$ we have, by (0.3), $2T_s u_r = T_s f_r + T_s \bar{f}_r = f_r + \overline{f(0)}$. Hence

$$f(rz)+\overline{f(0)}=2\int_B u(rw)K_s(z,w)\,dv(w)=2r^{-2n}\int_{rB} u(w)K_s(z,w/r)\,dv(w), \quad z\in B,$$

where

$$K_s(z,w)=\binom{n+s}{n}\frac{(1-|w|^2)^s}{(1-\langle z,w\rangle)^{s+n+1}}.$$

Now (3.5) is proved by using the Lebesgue dominated convergence theorem and the inclusion $L_1^p(v)\subset D(T_s)$.

The implication (3.6) is proved (in the same way as Proposition 2.5) by using the following result of Hardy and Littlewood [**5**].

Theorem HL. *If u is a pluri-harmonic function on B, then for $0<p<1$*

$$|u(0)|^p \leqq C\varepsilon^{-2n}\int_{\varepsilon B}|u|^p\,dv, \quad 0<\varepsilon<1, \tag{3.7}$$

where C is a constant depending only on n, p.

In fact, (3.7) holds for any harmonic function, and a proof can be found in Fefferman and Stein [**3**]. An elementary proof, using only the mean-value property over balls, is given in [**8**].

4. New view to Theorem 3.1

After we wrote the first three sections of this manuscript and we had discussions mentioned in the acknowledgement (see below) we discovered a new approach to Theorem 3.1.

In this section we will only sketch a proof of Theorem 4.1 (below) in the setting of the unit disk. A proof with an obvious modification works in the case of the unit ball. A detailed proof with further results will appear in a later paper.

Let D denote the unit disk in $\mathbb{C}$ and let δ be a fixed positive number less than 1.

Let $P=\{D_k : k\geqq 1\}$ be a partition of D (i.e. $\bigcup_{k=1}^{\infty} D_k = D$, and $D_k\cap D_j=\phi$ for $k\neq j$) so that each D_k is a measurable set and the (pseudo-hyperbolic) diameter of each D_k is not greater than δ.

We denote by B^p, $0<p<\infty$, the space consisting of all analytic functions f such that

$$\|f\|_{B^p}^p=\int_D |f(z)|^p\,dx\,dy<+\infty.$$

These spaces are known as Bergman spaces. We refer the reader to Axler's survey paper [1] for properties of these spaces.

Recall that

$$K_s(z,w)=\binom{n+s}{n}(1-|w|^2)^s(1-\langle z,w\rangle)^{-(s+2)}.$$

Let $0<p<1$ and let φ be a measurable function on D. In order to define the space on which T_s is a bounded operator we first write formally

$$(T_s\varphi)(z)=\sum_{k=1}^{\infty}\int_{D_k} K_s(z,w)\varphi(w)\,du\,dv.$$

Next, let $\{a_k\}$ be a fixed sequence such that $a_k\in D_k$ for each $k\geqq 1$, and let $A_k(z)=(1-|a_k|^2)^{s+2-2/p}(1-z\bar{a}_k)^{s+2}$. If $s+2-2/p>0$, i.e. $(s+2)p>2$, the power of $(1-|a_k|^2)$ in the previous expression is exactly what we need to insure that such terms have an B^p norm which is bounded by a constant. The functions of the form as A_k are building blocks in the Coifman–Rochberg decomposition of Bergman's space and we are motivated by their approach [2].

By Lemmas 1.1 and 1.2, there exist the functions $c_k(z,w)$ and an absolute constant c such that $(s+1)\pi^{-1}(1-|w|^2)^{s+2-2/p}(1-z\bar{w})^{s+2}=c_k(z,w)A_k(z)$ and $c^{-1}\leqq|c_k(z,w)|\leqq c$ for every $z\in D$ and every $w\in D_k$.

Hence

$$T_s\varphi(z)=\sum_{k=1}^{\infty}\lambda_k(z)A_k(z)$$

where

$$\lambda_k=\lambda_k(z)\equiv(s+1)\pi^{-1}\int_{D_k} c_k(z,w)(1-|w|^2)^{2(p^{-1}-1)}\varphi(w)\,du\,dv,$$

and consequently

$$|(T_s\varphi)(z)|\leqq c\sum_{k=1}^{\infty}\varphi_k|A_k(z)|, \tag{4.1}$$

where

$$\varphi_k=\int_{D_k}(1-|w|^2)^{2(p^{-1}-1)}|\varphi(w)|\,du\,dv. \tag{4.2}$$

Now, we are motivated for the following definition. Let $M^p, 0<p<\infty$, denote the space of all complex measurable functions φ on D for which

$$\|\varphi\|_{M^p}=\left\{\sum_{k=1}^{\infty}|\varphi_k|^p\right\}^{1/p}<+\infty,$$

where φ_k is defined by (4.2).

Theorem 4.1. *Let $0<p<1$ and $s>2(p^{-1}-1)$. Then T_s is a bounded operator from M^p into B^p.*

Proof. Let $\varphi\in M^p$. Then as above we have (4.1). Now, the desired conclusion follows from (4.1) and the inequality

$$|T_s\varphi)(z)|^p\leqq c\sum|\varphi_k|^p|A_k(z)|^p$$

by integration.

The following lemma shows that Theorem 4.1 contains the main part of Theorem 3.1. Recall that the space $L^p_{1,\delta}(v)$ was defined in Section 2 and that v denotes the normalized Lebesgue measure on D.

Lemma 4.1. *Let $P=\{D_k:k\geqq 1\}$ be a partition of the unit disk described above. In addition, if there is an absolute constant c such that $v(D_k)\geqq c(1-|a_k|^2)^2$, $k\geqq 1$, and $0<p<1$ then $L^p_1\subset M^p$, where $L^p_1=L^p_{1,\delta}(v)$.*

Proof. Let $\varphi\in L^p_1$, $a_k=\int_{D_k}|\varphi(z)|\,d\tau(z)$, $k\geqq 1$, and $z\in D_k$. Since the pseudo-hyperbolic diameter of D_k is less than δ, we have $D_k\subset E(z,\delta)$ and consequently $|\alpha_k|^p\leqq c[M_1\varphi(z)]^p$ for every $z\in D_k$. By integration over D_k, $k\geqq 1$,

$$v(D_k)|\alpha_k|^p\leqq c\int_{D_k}|M_1\varphi(z)|^p\,dx\,dy$$

and consequently

$$\sum_{k=1}^{\infty}|\alpha_k|^p v(D_k)\leqq c\sum_{k=1}^{\infty}\int_{D_k}|M_1\varphi(z)|^p\,dx\,dy. \tag{4.3}$$

Since $\varphi\in L^p_1$ it follows from (4.3) that

$$\sum_{k=1}^{\infty}|\alpha_k|^p v(D_k)<\infty. \tag{4.4}$$

Recall that by the hypothesis there is a constant c such that $v(D_k)\geqq c(1-|a_k|^2)^2$. Hence, by (4.4) we get

$$\{\alpha_k(1-|a_k|^2)^{2/p}\}_{k=1}^{\infty} \in l^p.$$

Thus by Lemma 1.1, $\varphi \in M^p$.

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Addendum. The dual of L_q^p, $0<p<\infty$, $1\leqq q<\infty$, is $L_{q'}^{p'}$, where $q'=q/(q-1)$, $p'=p/(p-1)$ for $p>1$ and $p'=\infty$ for $p\leqq 1$ and the pairing is given by $\int_B f(z)g(z)\,d\mu(z)$.

This answers a question posed by the referee.

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