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Correspondences, von Neumann Algebras and Holomorphic L^2 Torsion

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Abstract. Given a holomorphic Hilbertian bundle on a compact complex manifold, we introduce the notion of holomorphic L^2 torsion, which lies in the determinant line of the twisted L^2 Dolbeault cohomology and represents a volume element there. Here we utilise the theory of determinant lines of Hilbertian modules over finite von Neumann algebras as developed in [CFM]. This specialises to the Ray-Singer-Quillen holomorphic torsion in the finite dimensional case. We compute a metric variation formula for the holomorphic L^2 torsion, which shows that it is *not* in general independent of the choice of Hermitian metrics on the complex manifold and on the holomorphic Hilbertian bundle, which are needed to define it. We therefore initiate the theory of correspondences of determinant lines, that enables us to define a relative holomorphic L^2 torsion for a pair of flat Hilbertian bundles, which we prove is independent of the choice of Hermitian metrics on the complex manifold and on the flat Hilbertian bundles.

0 Introduction

Ray and Singer (*cf.* [RS]) introduced the notion of holomorphic torsion of a holomorphic bundle over a compact complex manifold. In [Q], Quillen viewed the holomorphic torsion as an element in the real determinant line of the twisted Dolbeault cohomology, or equivalently, as a metric in the dual of the determinant line of the twisted Dolbeault cohomology. Since then there have been many generalisations in the finite dimensional case, particularly by Bismut, Freed, Gillet and Soule, [BF], [BGS].

In this paper, we investigate generalisations of aspects of this previous work to the case of infinite dimensional representations of the fundamental group. Our approach combines features from [BFKM], where the equality of L^2-analytic and L^2-RF torsion was proved, and from [CFM] where the notion of determinant line for certain infinite dimensional representations of the fundamental group was defined. The latter approach has as a corollary, a reformulation of the main theorem of [BFKM], placing it in its appropriate topological setting. The study of holomorphic analogues of these initial results is the logical next step.

Our approach is to introduce the concepts of holomorphic Hilbertian bundles and of connections compatible with the holomorphic structure. These bundles have fibres which are von Neumann algebra modules. We are able to define the *determinant line bundle* of a holomorphic Hilbertian bundle over a compact complex manifold, generalising the construction of the determinant line of a finitely generated Hilbertian module that was developed in our earlier paper [CFM]. A nonzero element of the determinant line bundle can be naturally viewed as a volume form on the Hilbertian bundle. This enables us to make sense of the notions of volume form and determinant line bundle in this infinite dimensional

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and non-commutative situation. Given an isomorphism of the determinant line bundles of holomorphic Hilbertian bundles, we introduce the concept of a *correspondence* between the determinant lines of the twisted L^2 Dolbeault cohomologies. This was previously studied in the finite dimensional situation in [F].

In this paper we restrict our attention to pairs consisting of manifolds and holomorphic vector bundles satisfying the so-called determinant or D-class property, whose real analogue was studied in [BFKM], [CFM]. Here we define the holomorphic L^2 torsion of a holomorphic Hilbertian bundle; it reduces to the classical constructions in the finite dimensional situation. This new torsion invariant lives in the determinant line of the twisted L^2 Dolbeault cohomology. Some key results in our paper are a metric variation formula for the holomorphic L^2 torsion, and the definition of a correspondence between the determinant lines of the twisted L^2 Dolbeault cohomologies for a pair of flat holomorphic Hilbertian bundles, and finally the definition of a metric independent relative holomorphic L^2 torsion associated to a correspondence between determinant line bundles of flat Hilbertian bundles. To prove that a correspondence between determinant line bundles of flat Hilbertian bundles is well defined, we need to prove a generalised local index theorem for almost Kähler manifolds, and as a consequence, we give an alternate proof of Bismut's local index theorem for almost Kähler manifolds [Bi], where we use instead the methods of Donnelly [D] and Getzler [Ge].

This paper also lays some groundwork for studying the torsion of a family of operators parametrised by a manifold. A fundamental difficulty in such a generalisation is the fact that the 'determinant class' condition seems not to be stable under perturbations. However in this paper we introduce some examples of families where this stability can be proved, suggesting that the holomorphic setting is one where a certain 'rigidity' of the determinant class condition occurs.

The paper is organized as follows. In the first section, we recall some preliminary material on Hilbertian modules over finite von Neumann algebras, the canonical trace on the commutant of a finitely generated Hilbertian module, the Fuglede-Kadison determinant on Hilbertian modules and the construction of determinant lines for finitely generated Hilbertian modules. Details of the material in this section can be found in [CFM]. In Section 2, we define Hilbertian bundles and connections on these. The definition of a connection is tricky in the infinite dimensional context, and we use some fundamental theorems in von Neumann algebras to make sense of our definition. Then we define holomorphic Hilbertian bundles and connections compatible with the holomorphic structure as well as Cauchy-Riemann operators on these. In Section 3, we study the properties of the zeta function associated to holomorphic Hilbertian bundles of D-class. In Section 4, we define the holomorphic L^2 torsion as an element in the determinant line of reduced L^2 Dolbeault cohomology. Here we also prove metric variation formulae and we deduce that holomorphic L^2 torsion *does* depend on the choices of Hermitian metrics on the compact complex manifold and on the holomorphic Hilbertian bundle. However, in Sections 5 and 6, we give situations when a relative version of the holomorphic L^2 torsion is indeed independent of the choice of metric. In Section 5, we are able to deduce the following theorem (Theorem 5.3 in the text) from the variation formula: let $\mathcal{E}$ and $\mathcal{F}$ be two flat Hilbert bundles of D-class over a compact Hermitian manifold X. Then one can define a relative holomorphic L^2 torsion

$$\rho^{p}_{\mathcal{E},\mathcal{F}} \in \det\big(H^{p,*}(X,\mathcal{E})\big) \otimes \det\big(H^{p,*}(X,\mathcal{F})\big)^{-1}$$

which is independent of the choice of Hermitian metric on X. In Section 6, we define the notion of the determinant line bundle of a Hilbertian bundle and also of correspondences between determinant lines. The proof that a correspondence is well defined, uses techniques of Bismut [Bi], Donnelly [D] and Getzler [Ge] in their proof of the local index theorem in different situations. Using the notion of a correspondence of determinant line bundles, we prove one of the main theorems in our paper (Theorem 6.9 in the text), which can be briefly stated as follows: let $\mathcal{E}$ and $\mathcal{F}$ be two flat Hilbertian bundles of D-class over a compact almost Kähler manifold X and $\varphi\colon \det(\mathcal{E}) \to \det(\mathcal{F})$ be an isomorphism of the corresponding determinant line bundles. Then one can define a relative holomorphic L^2 torsion

$$\rho_\varphi^p \in \det\big(H^{p,*}(X,\mathcal{E})\big) \otimes \det\big(H^{p,*}(X,\mathcal{F})\big)^{-1}.$$

Using the correspondence defined by the isomorphism φ, we show that the relative holomorphic L^2 torsion ρ_φ^p is independent of the choices of Hermitian metrics on $\mathcal{E}$ and $\mathcal{F}$ and the choice of almost Kähler metric on X which are needed to define it. Recall that an almost Kähler manifold is a Hermitian manifold whose "Kähler" 2-form ω is not necessarily closed, but satisfies the weaker condition $\bar\partial\partial\omega = 0$. A result of Gauduchon (*cf.* [Gau]) asserts that every compact complex surface is almost Kähler, whereas there are many examples of complex surfaces which are not Kähler. In Section 7, we give some examples of calculation of the holomorphic L^2 torsion for locally symmetric spaces and Riemann surfaces. We also give an example of a family of operators (parametrised by projective representations of the fundamental group of a Riemann surface) for which the holomorphic torsion may be calculated.

1 Preliminaries

This section contains some preliminary material from [CFM].

1.1 Hilbertian Modules over von Neumann Algebras

Throughout the paper $\mathcal{A}$ will denote a finite von Neumann algebra with a fixed finite, normal, and faithful trace $\tau\colon \mathcal{A} \to \mathbb{C}$. The involution in $\mathcal{A}$ will be denoted $*$ while $\ell^2(\mathcal{A})$ denotes the completion of $\mathcal{A}$ in the norm derived from the inner product $\tau(a^*b)$, $a, b \in \mathcal{A}$. A *Hilbert module* over $\mathcal{A}$ is a Hilbert space M together with a continuous left $\mathcal{A}$-module structure such that there exists an isometric $\mathcal{A}$-linear embedding of M into $\ell^2(\mathcal{A}) \otimes H$, for some Hilbert space H. (Note that this embedding is not part of the structure.) A Hilbert module M is *finitely generated* if it admits an imbedding as above with finite dimensional H. To introduce the notion of determinant line requires us to forget the scalar product on H but keep the topology and the $\mathcal{A}$-action.

Definition 1.1 A *Hilbertian module* is a topological vector space M with continuous left $\mathcal{A}$-action such that there exists a scalar product $\langle\ ,\ \rangle$ on M which generates the topology of M and such that M together with $\langle\ ,\ \rangle$ and with the $\mathcal{A}$-action is a Hilbert module. Any scalar product $\langle\ ,\ \rangle$ on M with the above properties will be called *admissible*.

1.2 Remarks and Further Definitions

The choice of any other admissible scalar product $\langle\ ,\ \rangle_1$ gives an isomorphic Hilbert module. In fact there exists an operator $A\colon M \to M$ such that

$$\langle v, w\rangle_1 = \langle Av, w\rangle \tag{1}$$

for any $v, w \in M$. The operator A must be a self-adjoint, positive linear homeomorphism (since the scalar products $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle_1$ define the same topology), which commutes with the $\mathcal{A}$-action. A *finitely generated Hilbertian module* is one for which the corresponding Hilbert module is finitely generated. Finally, a *morphism* of Hilbertian modules is a continuous linear map $f\colon M \to N$, commuting with the $\mathcal{A}$-action. Note that the kernel of any morphism f is again a Hilbertian module as is the closure of the image $\mathrm{cl}\big(\mathrm{im}(f)\big)$.

1.3 The Canonical Trace on the Commutant

Any choice of an admissible scalar product $\langle\ ,\ \rangle$ on M, defines obviously a $*$-operator on $\mathcal{B}$ (by assigning to an operator its adjoint) and turns $\mathcal{B}$ into a von Neumann algebra. If we choose another admissible scalar product $\langle\ ,\ \rangle_1$ on M then the new involution will be given by

$$f \mapsto A^{-1} f^* A \quad \text{for } f \in \mathcal{B}, \tag{2}$$

where $A \in \mathcal{B}$ satisfies $\langle v, w\rangle_1 = \langle Av, w\rangle$ for $v, w \in M$. The trace on the commutant may now be defined as in [Dix] and here will be denoted Tr_τ. It is finite, normal, and faithful. If M and N are two finitely generated modules over $\mathcal{A}$, then the canonical traces Tr_τ on $\mathcal{B}(M)$, $\mathcal{B}(N)$ and on $\mathcal{B}(M \oplus N)$ are compatible in the following sense:

$$\mathrm{Tr}_\tau \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{Tr}_\tau(A) + \mathrm{Tr}_\tau(D), \tag{3}$$

for all $A \in \mathcal{B}(M)$, $D \in \mathcal{B}(N)$ and any morphisms $B\colon M \to N$, and $C\colon N \to M$. Note that the *von Neumann dimension* of a Hilbertian submodule N of M is defined as $\dim_\tau(M) = \mathrm{Tr}_\tau(P_N)$ where P_N is the orthogonal projection onto N.

1.4 Fuglede-Kadison Determinant for Hilbertian Modules

Let $\mathrm{GL}(M)$ denote the group of all invertible elements of the algebra $\mathcal{B}(M)$ equipped with the norm topology. With this topology it is a Banach Lie group whose Lie algebra may be identified with the commutant $\mathcal{B}(M)$. The canonical trace Tr_τ on the commutant $\mathcal{B}(M)$ is a homomorphism of the Lie algebra $\mathcal{B}(M)$ into $\mathbb{C}$ and by standard theorems, it defines a group homomorphism of the universal covering group of $\mathrm{GL}(M)$ into $\mathbb{C}$. This approach leads to following construction of the Fuglede-Kadison determinant, compare [HS].

Theorem 1.2 *There exists a function* $\mathrm{Det}_\tau\colon \mathrm{GL}(M) \to \mathbb{R}^{>0}$ *(called the Fuglede-Kadison determinant) whose key properties are:*

(i) Det_τ *is a group homomorphism and is continuous if* $\mathrm{GL}(M)$ *is supplied with the norm topology;*

(ii) If A_t *for* $t \in [0,1]$ *is a continuous piecewise smooth path in* $\mathrm{GL}(M)$ *then*

$$\log\left[\frac{\mathrm{Det}_\tau(A_1)}{\mathrm{Det}_\tau(A_0)}\right] = \int_0^1 \Re\,\mathrm{Tr}_\tau[A_t^{-1}A_t']\,dt. \tag{4}$$

Here $\Re$ *denotes the real part and* A_t' *denotes the derivative of* A_t *with respect to* t*.*

Let M *and* N *be two finitely generated modules over* $\mathcal{A}$*, and* $A \in \mathrm{GL}(M)$ *and* $B \in \mathrm{GL}(N)$ *two automorphisms, and* $\gamma\colon N \to M$ *be a homomorphism. Then the map given by the matrix*

$$\begin{pmatrix} A & \gamma \\ 0 & B \end{pmatrix}$$

belongs to $\mathrm{GL}(M \oplus N)$ *and*

$$\mathrm{Det}_\tau \begin{pmatrix} A & \gamma \\ 0 & B \end{pmatrix} = \mathrm{Det}_\tau(A) \cdot \mathrm{Det}_\tau(B). \tag{5}$$

Given an operator $A \in \mathrm{GL}(M)$, there is a continuous piecewise smooth path $A_t \in \mathrm{GL}(M)$ with $t \in [0,1]$ such that $A_0 = I$ and $A_1 = A$ (it is well known that the group $\mathrm{GL}(M)$ is pathwise connected, *cf.* [Dix]). Then from (4) we have the formula:

$$\log \mathrm{Det}_\tau(A) = \int_0^1 \Re\,\mathrm{Tr}_\tau[A_t^{-1}A_t']\,dt. \tag{6}$$

This integral does not depend on the choice of the path. As an example consider the following situation. Suppose that a self-adjoint operator $A \in \mathrm{GL}(M)$ has spectral resolution

$$A = \int_0^\infty \lambda\, dE_\lambda \tag{7}$$

where dE_λ is the spectral measure. Then we can choose the path

$$A_t = t(A - I) + I, \quad t \in [0,1]$$

joining A with I inside $\mathrm{GL}(M)$. Applying (6) we obtain

$$\log \mathrm{Det}_\tau(A) = \int_0^\infty \ln\lambda\, d\phi_\lambda \tag{8}$$

where $\phi_\lambda = \mathrm{Tr}_\tau E_\lambda$ is the spectral density function.

1.5 Operators of Determinant Class

Following [BFKM] and [CFM] we extend the previous ideas to a wider class of operators. An operator A as in (7) is said to be *D-class* (D for determinant) if

$$\int_0^\infty \ln\lambda\, d\phi_\lambda > -\infty. \tag{9}$$

A scalar product $\langle v, w\rangle = \langle Av, w\rangle_1$ is said to be *D-admissible* if A is D-class and $\langle\ ,\ \rangle_1$ is any admissible scalar product. The Fuglede-Kadison determinant extends to such operators via the formula:

$$\mathrm{Det}_\tau(A) = \exp\Big[\int_0^\infty \ln\lambda\, d\phi_\lambda\Big]. \tag{10}$$

1.6 Determinant Line of a Hilbertian Module

For a Hilbertian module M we defined in [CFM] the determinant line $\det(M)$ as a real vector space generated by symbols $\langle\ ,\ \rangle$, one for any admissible scalar product on M, subject to the following relations: for any pair $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ of admissible scalar products on M we require

$$\langle\ ,\ \rangle_2 = \sqrt{\mathrm{Det}_\tau(A)}^{-1}\cdot\langle\ ,\ \rangle_1, \tag{11}$$

where $A \in \mathrm{GL}(M)\cap\mathcal{B}(M)$ is such that $\langle v, w\rangle_2 = \langle Av, w\rangle_1$ for all $v, w \in M$. It is not difficult to see that $\det(M)$ *is one-dimensional generated by the symbol* $\langle\ ,\ \rangle$ *of any admissible scalar product on* M. Note also, that the real line has *the canonical orientation*, since the transition coefficient $\sqrt{\mathrm{Det}_\tau(A)}$ is always positive. Thus we may speak of *positive and negative* elements of $\det(M)$. We think of elements of $\det(M)$ as "volume forms" on M. If M is trivial module, $M = 0$, then we set $\det(M) = \mathbb{R}$, by definition.

Given two finitely generated Hilbertian modules M and N over $\mathcal{A}$, with admissible scalar products $\langle\ ,\ \rangle_M$ and $\langle\ ,\ \rangle_N$ respectively, we may obviously define the scalar product $\langle\ ,\ \rangle_M \oplus \langle\ ,\ \rangle_N$ on the direct sum. This defines the isomorphism

$$\det(M)\otimes\det(N) \to \det(M\oplus N). \tag{12}$$

By property (5) of the Fuglede-Kadison determinant it is easy to show that this homomorphism does not depend on the choice of the metrics $\langle\ ,\ \rangle_M$ and $\langle\ ,\ \rangle_N$ and preserves the orientations. Note that, any isomorphism $f\colon M \to N$ between finitely generated Hilbertian modules induces canonically an orientation preserving isomorphism of the determinant lines $f^*\colon \det(M) \to \det(N)$. Indeed, if $\langle\ ,\ \rangle_M$ is an admissible scalar product on M then set

$$f^*(\langle\ ,\ \rangle_M) = \langle\ ,\ \rangle_N, \tag{13}$$

where $\langle\ ,\ \rangle_N$ is the scalar product on N given by $\langle v, w\rangle_N = \langle f^{-1}(v), f^{-1}(w)\rangle_M$ for $v, w \in N$. This definition does not depend on the choice of the scalar product $\langle\ ,\ \rangle_M$ on M: if we

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have a different admissible scalar product $\langle\ ,\ \rangle'_M$ on M, where $\langle v, w\rangle'_M = \langle A(v), w\rangle_M$ with $A \in \mathrm{GL}(M)$ then the induced scalar product on N will be

$$\langle v, w\rangle'_N = \langle (f^{-1}Af)v, w\rangle_N$$

and our statement follows from property (5) of the Fuglede-Kadison determinant. Finally we note the *functorial* property: if $f\colon M \to N$ and $g\colon N \to L$ are two isomorphisms between finitely generated Hilbertian modules then $(g \circ f)^* = g^* \circ f^*$.

Proposition 1.3 *If $f\colon M \to M$ is an automorphism of a finitely generated Hilbertian module M, $f \in \mathrm{GL}(M)$, then the induced homomorphism $f^*\colon \det(M) \to \det(M)$ coincides with the multiplication by $\mathrm{Det}_\tau(f) \in \mathbb{R}^{>0}$. Furthermore any exact sequence*

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

of finitely generated Hilbertian modules determines canonically an isomorphism

$$\det(M') \otimes \det(M'') \to \det(M),$$

which preserves the orientation of the determinant lines.

1.7 Extension to D-Admissible Scalar Products

Any D-admissible scalar product determines a non-zero element of the determinant line $\det(M)$ namely $\mathrm{Det}_\tau(A)^{-1/2}\langle\ ,\ \rangle_1$. A *$D$-admissible* isomorphism $f\colon M \to N$ is one for which the inner product $\langle v, w\rangle_M = \langle f(v), f(w)\rangle_N$ on M is D-admissible for some and hence any admissible inner product on N. Proposition 1.3 extends to D-admissible isomorphisms and to the obvious notion of D-admissible exact sequence.

2 Holomorphic Hilbertian $\mathcal{A}$-Bundles Bundles and $\mathcal{A}$-Linear Connections

In this section, we define Hilbertian $\mathcal{A}$-bundles and $\mathcal{A}$-linear connections on these. The definition of ($\mathcal{A}$-linear) connection is tricky in the infinite dimensional case, if one wants to be able to horizontally lift curves. We use some fundamental theorems in von Neumann algebras to make sense of our definition. We also define holomorphic Hilbertian $\mathcal{A}$-bundles bundles and holomorphic $\mathcal{A}$-linear connections on these.

2.1 Hilbertian $\mathcal{A}$-Bundles

A Hilbertian $\mathcal{A}$-bundle with fibre M over X is given by the following data.

(1) $p\colon \mathcal{E} \to X$ a smooth bundle of topological vector spaces, possibly infinite dimensional, such that each fibre $p^{-1}(x)$, $x \in X$ is a separable Hilbertian space (*cf.* [Lang]).
(2) There is a smooth fibrewise action $\mathcal{A} \times \mathcal{E} \to \mathcal{E}$ which endows each fibre $p^{-1}(x)$, $x \in X$ with a Hilbertian $\mathcal{A}$-module structure, such that for all $x \in X$, $p^{-1}(x)$ is isomorphic to M as Hilbertian $\mathcal{A}$-modules.

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(3) There is a local trivializing cover of $p\colon \mathcal{E} \to X$ which intertwines the $\mathcal{A}$-actions. More precisely, there is an open cover $\{U_\alpha\}$ of X such that for each α, there is a smooth isomorphism

$$\tau_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times M$$

which intertwines the $\mathcal{A}$-actions on $p^{-1}(U_\alpha) \subset \mathcal{E}$ and on $U_\alpha \times M$, and such that $\mathrm{pr}_1 \circ \tau_\alpha = p$, where $\mathrm{pr}_1\colon U_\alpha \times M \to U_\alpha$ denotes the projection onto the first factor. The restriction of τ_α

$$\tau_\alpha\colon p^{-1}(x) \to \{x\} \times M$$

is the isomorphism of Hilbertian $\mathcal{A}$-modules $\forall x \in U_\alpha$, as given in (2).

Remark 2.1 If $\{U_\alpha\}$ is a trivializing open cover of $p\colon \mathcal{E} \to X$, then the isomorphisms

$$\tau_\beta \circ \tau_\alpha^{-1}\colon (U_\alpha \cap U_\beta) \times M \to (U_\alpha \cap U_\beta) \times M$$

are of the form $\tau_\beta \circ \tau_\alpha^{-1} = (\mathrm{id}, g_{\alpha\beta})$ where $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(M)$ are smooth maps and are called the transition functions of $p\colon \mathcal{E} \to X$, and they satisfy the cocycle identity

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1 \quad \forall \alpha, \beta, \gamma.$$

Now suppose that $\{U_\alpha\}_\alpha$ is an open cover of X, and on each intersection $U_\alpha \cap U_\beta$, we are given smooth maps

$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(M)$$

satisfying $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$ and $g_{\alpha\alpha} = 1$ on U_α, then one can construct a Hilbertian $\mathcal{A}$-bundle $p\colon \mathcal{E} \to X$ via the clutching construction viz, consider the disjoint union $\tilde{\mathcal{E}} = \bigcup_\alpha (U_\alpha \times M)$ with the product topology, and define the equivalence relation $\sim$ on $\tilde{\mathcal{E}}$ by $(x, v) \sim (y, w)$ for $(x, v) \in U_\alpha \times M$ and $(y, w) \in U_\beta \times M$ if and only if $x = y$ and $w = g_{\alpha\beta}(x)v$. Then the quotient $\tilde{\mathcal{E}}/\!\sim\, = \mathcal{E} \to X$ is easily checked to be a Hilbertian $\mathcal{A}$-bundle over X.

Remark 2.2 This definition generalizes and is compatible with Breuer's definition of Hilbert $\mathcal{A}$-bundles (*cf.* [B], [BFKM]) and also with Lang's definition [Lang], where the action of the von Neumann algebra is not considered. Actually Breuer [B] considers von Neumann algebras $\mathcal{A}$ which are not necessarily finite.

Example 2.3 (a) It follows from Breuer's work [B] that there are many examples of Hilbertian $\mathcal{A}$-bundles, even in the case of simply connected manifolds. For example, on the 2-sphere S^2, the isomorphism classes of Hilbertian $\mathcal{A}$-bundles with fibre $\ell^2(A)$, are in 1–1 correspondence with homotopy classes of maps from S^1 to $\mathrm{GL}\big(\ell^2(A)\big)$. If $\mathcal{A}$ is a type II_1 factor, then by a result of Araki, Smith and Smith [ASS], it follows that the isomorphism classes of Hilbertian $\mathcal{A}$-bundle over S^2 is isomorphic to $\mathbb{R}$ (considered as a discrete group).

(b) Let $\mathcal{E} \to X$ be a Hilbertian $\mathcal{A}$-bundle over X. Then $\Lambda^j T^*_{\mathbb{C}} X \otimes \mathcal{E}$ is also a Hilbertian $\mathcal{A}$-bundle over X, where $\Lambda^j T^*_{\mathbb{C}} X$ denotes the j-th exterior power of the complexified cotangent bundle of X. This can be seen as follows. Let

$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(M)$$

denote the transition functions of the Hilbertian $\mathcal{A}$-bundle $\mathcal{E}$ with fibre M, and

$$g'_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{C})$$

denote the transition functions of the $\mathbb{C}$ bundle $\Lambda^j T^*_{\mathbb{C}} X \to X$. Then

$$g''_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{C}^r \otimes M)$$

denotes the transition functions of the Hilbertian $\mathcal{A}$-bundle $\Lambda^j T^*_{\mathbb{C}} X \otimes \mathcal{E}$ with fibre $\mathbb{C}^r \otimes M$.

2.2 Sections of Hilbertian $\mathcal{A}$-Bundles

A section of a Hilbertian $\mathcal{A}$-bundle $p\colon \mathcal{E} \to X$ is a smooth map $s\colon X \to \mathcal{E}$ such that $p \circ s$ is the identity map on X. Let $\{U_\alpha\}_\alpha$ be a local trivialization of $p\colon \mathcal{E} \to X$. Then a smooth section s is given on U_α by a smooth map $s_\alpha\colon U_\alpha \to M$. On $U_\alpha \cap U_\beta$ one has the relation $s_\alpha = g_{\alpha\beta} s_\beta$.

2.3 $\mathcal{A}$-Linear Connections on Hilbertian $\mathcal{A}$-Bundles

An $\mathcal{A}$-linear connection on a Hilbertian $\mathcal{A}$-bundle $p\colon \mathcal{E} \to X$ is an $\mathcal{A}$-morphism

$$\nabla\colon \Omega^j(X, \mathcal{E}) \to \Omega^{j+1}(X, \mathcal{E})$$

such that for any $A \in \Omega^0\big(X, \mathrm{End}_{\mathcal{A}}(\mathcal{E})\big)$ and $w \in \Omega^j(X, \mathcal{E})$, there is $\nabla A \in \Omega^1\big(X, \mathrm{End}_{\mathcal{A}}(\mathcal{E})\big)$ such that

$$\nabla(Aw) - A(\nabla w) = (\nabla A)w.$$

Here $\Omega^j(X, \mathcal{E})$ denotes the space of smooth sections of the Hilbertian $\mathcal{A}$-bundle $\Lambda^j T^*_{\mathbb{C}} X \otimes \mathcal{E}$, and $\Omega^1\big(X, \mathrm{End}_{\mathcal{A}}(\mathcal{E})\big)$ denotes the space of smooth sections of the Hilbertian $\mathcal{A}$-bundle $T^*_{\mathbb{C}} X \otimes \mathrm{End}_{\mathcal{A}}(\mathcal{E})$.

Remark 2.4 Let V be a vector field on X. Then

$$\nabla_V A \in \Omega^0\big(X, \mathrm{End}_{\mathcal{A}}(\mathcal{E})\big).$$

Proposition 2.5 *Let ∇, ∇' be two connections on the Hilbertian $\mathcal{A}$-bundle $p\colon \mathcal{E} \to X$ with fibre M. Then*

$$\nabla - \nabla' \in \Omega^1(X, \mathrm{End}_{\mathcal{A}}\, \mathcal{E}).$$

Proof Let V be a vector field on X. Then $\delta_V = \nabla_V - \nabla'_V$ in $C^\infty(X)$ linear, and hence by [Lang] is defined pointwise. $(\delta_V)_x$ is a derivation on the von Neumann algebra $\mathrm{End}_{\mathcal{A}}(\mathcal{E}_x)$. Since $(\delta_V)_x$ is everywhere defined, by Lemma 3, Part III, Chapter 9 of [Dix], $(\delta_V)_x$ is bounded. By Theorem 1, Part III, Chapter 9 of [Dix], there is an element $B_x(V) \in \mathrm{End}_{\mathcal{A}}(\mathcal{E}_x)$ such that $(\delta_V)_x = \mathrm{ad}\, B_x(V)$. That is, $x \to \mathrm{ad}\, B_x(V)$ is smooth. The remainder of the proof establishes that there is a smooth choice $x \to \tilde{\tilde{B}}_x(V)$ such that $\mathrm{ad}\, \tilde{\tilde{B}}_x(V) = (\delta_V)_x$. We first discuss the local problem.

Let U be an open subset of X and M be a Hilbertian $\mathcal{A}$-module. Consider the trivial bundle $U \times M \to U$ over U. By Dixmier's result cited above, there is a map

$$x \to \operatorname{ad} B_x(V) \quad x \in U$$

where $B_x(V) \in \operatorname{End}_{\mathcal{A}}(M)$ for all $x \in U$, such that

$$\operatorname{ad} B_x(V) = (\nabla_V - \nabla'_V)_x$$

since ∇, ∇' are connections and V is smooth, we deduce that $x \to \operatorname{ad} B_x(V)$ is smooth. However, it isn't *a priori* clear that one can choose $x \to B_x(V)$ to be smooth, as $B_x(V)$ is only defined modulo the centre of the von Neumann algebra $\operatorname{End}_{\mathcal{A}}(M) = \mathcal{B}(M)$. To complete the proof we need the next result.

Lemma 2.6 *Let $\mathcal{A}$ be a von Neumann algebra with centre Z. Then there is a smooth section $s\colon \mathcal{A}/Z \to \mathcal{A}$ to the natural projection $p\colon \mathcal{A} \to \mathcal{A}/Z$.*

Proof Let $Z \subset \mathcal{A} \subset B\big(\ell^2(\mathcal{A})\big)$, then since Z is a type I von Neumann algebra and hence injective, there exists a projection of norm 1, $P\colon B\big(\ell^2(\mathcal{A})\big) \to Z$ [HT]. Then $\mathcal{A} \cap \ker P$ is a complementary subspace to Z and one defines a section to the projection $p\colon \mathcal{A} \to \mathcal{A}/Z$.

$$s\colon \mathcal{A}/Z \to \mathcal{A} \quad \text{as } s([v]) = (1-P)v.$$

Then s is smooth since it is linear.

More explicitly, given a subgroup G of the unitaries in the commutant of Z, $U(Z')$, which is amenable and whose span is ultra weakly dense in Z', one can use the invariant mean on G to average over the closure of the orbit $\{uxu^* : u \in G\}$ and thus obtain a map P so that $P(x)$ is this average for each x and hence commutes with every $u \in Z'$. That is, $P(x)$ is in $Z'' = Z$. Such projections are called Schwartz projections, according to Kadison (*cf.* [Ph]). ∎

Returning now to the proof of Proposition 2.5, we define the smooth map $\tilde{B}(V)$ by

$$\tilde{B}(V) = s \circ \operatorname{ad} B(V),$$

where $s\colon \operatorname{End}_{\mathcal{A}}(M)/Z \to \operatorname{End}_{\mathcal{A}}(M)$ is the section as in Lemma 2.6 (with $\operatorname{End}_{\mathcal{A}}(M)$ replacing $\mathcal{A}$). Then clearly

$$\nabla_V = \nabla'_V = \operatorname{ad} \tilde{B}(V),$$

where $x \to \tilde{B}_x(V)$ is smooth. This solves the problem locally.

Let $\mathcal{E} \to X$ be a Hilbertian bundle with fibre M, and $\{U_\alpha\}$ be a trivialization of $\mathcal{E} \to X$. We have seen that on U_α, there is a smooth section

$$x \to \tilde{B}_{\alpha,x}(V) \quad \text{for } x \in U_\alpha$$

on $\mathcal{E}|_{U_\alpha \cap U_\beta}$, we can compare the 2 sections obtained, $x \to \tilde{B}_{\alpha,x}(V) - \tilde{B}_{\beta,x}(V) \in Z$, since $\operatorname{ad} \tilde{B}_{\alpha,x}(V) = \operatorname{ad} \tilde{B}_{\beta,x}(V)$. Therefore we can define $\lambda_{\alpha\beta}(x) = \tilde{B}_{\alpha,x}(V) - \tilde{B}_{\beta,x}(V)$

i.e. $\lambda_{\alpha\beta}\colon U_\alpha \cap U_\beta \to Z$ is a Cech 1-cocycle with values in the sheaf of smooth Z valued functions. As Z is contractable, lemme 22 of [DD] applies and so the first cohomology with values in the sheaf of smooth Z valued functions is trivial. Therefore $\lambda_{\alpha\beta}$ is a coboundary *i.e.* there are smooth maps

$$\varphi_\alpha\colon U_\alpha \to Z$$

such that $\lambda_{\alpha\beta} = \varphi_\beta - \varphi_\alpha$. Then $\{x \to \tilde{B}_{\alpha,x}(V) + \varphi_{\alpha,x}(V)\}_\alpha$ is a global section, since on $U_\alpha \cap U_\beta$, one has

$$\tilde{B}_{\alpha,x}(V) + \varphi_{\alpha,x}(V) = \tilde{B}_{\beta,x}(V) + \varphi_{\beta,x}(V),$$

i.e. one gets a smooth section

$$X \to \operatorname{End}_{\mathcal{A}}(\mathcal{E}), \quad x \to \tilde{\tilde{B}}_x(V)$$

where $\tilde{\tilde{B}}_x(V) = \tilde{B}_{\alpha,x}(V) + \varphi_{\alpha,x}(V)$ for $x \in U_\alpha$. It follows that $\tilde{\tilde{B}} \in \Omega^1\big(X, \operatorname{End}_{\mathcal{A}}(\mathcal{E})\big)$. ∎

Let ∇ be a connection on $p\colon \mathcal{E} \to X$ and let $\{U_\alpha\}_\alpha$ be a trivialization of $p\colon \mathcal{E} \to X$. Since $\mathcal{E}|_{U_\alpha} \cong U_\alpha \times M$, one sees that the differential d is a connection on $p\colon \mathcal{E}|_{U_\alpha} \to U_\alpha$. By Proposition 2.5, $\nabla - d \in \Omega^1(U_\alpha, \operatorname{End}_{\mathcal{A}} M)$ *i.e.* $\nabla = d + B_\alpha$ where $B_\alpha \in \Omega^1(U_\alpha, \operatorname{End}_{\mathcal{A}} M)$.

On $U_\alpha \cap U_\beta$, one easily derives the relation

$$B_\beta = g_{\alpha\beta}^{-1} B_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta}. \tag{14}$$

So a connection can also be thought of as a collection $\{d + B_\alpha\}_\alpha$ where $B_\alpha \in \Omega^1(U_\alpha, \operatorname{End}_{\mathcal{A}} M)$ and satisfying the relation (14) on the intersection.

2.4 Parallel Sections and Horizontal Lifts of Curves

Let ∇ be a connection on $p\colon \mathcal{E} \to X$. Let $p\colon \mathcal{E} \to X$ be a Hilbertian $\mathcal{A}$-bundle and $I = [0, 1]$ be the unit interval. Let $\gamma\colon I \to X$ be a curve. Let $\xi\colon I \to \mathcal{E}$ be a curve such that $p_0\xi = \gamma$. Then ξ is called a *lift of* γ. ξ is said to be a *horizontal lift* of γ if it is parallel along γ, that is, if it satisfies the following equation,

$$\nabla_{\dot{\gamma}(t)}\xi(t) = 0 \quad \forall t \in I$$

where dot denotes the derivative with respect to t. In a local trivialization U_α, the equation looks as,

$$\dot{\xi}(t) + B_\alpha\big(\dot{\gamma}(t)\big)\xi(t) = 0 \quad \forall t \in I \tag{15}$$

where $\nabla = d + B_\alpha$ on U_α as before. Since $B_\alpha\big(\dot{\gamma}(t)\big)$ is *bounded*, we use a theorem of ordinary differential equations for Banach space valued functions (see Proposition 1.1, Chapter IV in [Lang]) to see that there is a unique solution to equation (15) with initial condition $\xi(0) = v \in M$. It follows that a connection enables one to lift curves horizontally. This enables one to define a "horizontal" subbundle $\mathcal{H}$ of $T\mathcal{E}$, which is a complement to the "vertical" subbundle $p^*\mathcal{E} \subset T\mathcal{E}$. This is how [Lang] discusses connections on infinite dimensional vector bundles. Conversely, given a choice of "horizontal" subbundle $\mathcal{H}$ of

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$T\mathcal{E}$, one can define a "covariant derivative" (that is, a connection) as follows. By hypothesis $T\mathcal{E} = \mathcal{H} \oplus p^*\mathcal{E}$. Let $\mathrm{pr}_2 \colon T\mathcal{E} \to p^*\mathcal{E}$ denote projection to the 2nd factor and $\kappa \colon T\mathcal{E} \to \mathcal{E}$ be the composition $p \circ \mathrm{pr}_2$ where $p \colon p^*\mathcal{E} \to \mathcal{E}$. Let V be a vector field on X. Define $\nabla_V s = \kappa\bigl(Ds(V)\bigr)$ where $s \colon X \to \mathcal{E}$ is a smooth section, and Ds is its differential. Then ∇ locally has the form $\{d + B_\alpha\}$ on a trivialization $\{U_\alpha\}$ of $p \colon \mathcal{E} \to X$, where $B_\alpha \in \Omega^1(U_\alpha, \mathrm{End}_{\mathcal{A}} M)$ (see [Lang, Chapter IV, Section 3]) and it satisfies relation (14). Therefore ∇ defines a connection on $p \colon \mathcal{E} \to X$ in the sense of Section 2.3.

2.5 Holomorphic Hilbertian $\mathcal{A}$-Bundles

A Hilbertian $\mathcal{A}$-bundle $p \colon \mathcal{E} \to X$ with fibre M, is said to be a *holomorphic Hilbertian $\mathcal{A}$-bundle* if the transition functions of $p \colon \mathcal{E} \to X$,

$$g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}(M)$$

are holomorphic maps. We call $\{U_\alpha\}_\alpha$ a holomorphic trivialization of $p \colon \mathcal{E} \to X$.

Remark 2.7 $\mathrm{GL}(M)$ is an open subset of a Banach space, and so it is a complex manifold (of infinite dimension).

2.6 Examples of Holomorphic Hilbertian $\mathcal{A}$-Bundles

(a) By using the clutching construction again, we see that holomorphic Hilbertian $\mathcal{A}$-bundles over S^2 correspond to holomorphic maps

$$g \colon A_\epsilon \to \mathrm{GL}(M)$$

where $A_\epsilon = \{z \in \mathbb{C} : 1 - \epsilon < |z| < 1 + \epsilon\}$ is an annulus, for some small $\epsilon > 0$. Therefore by Example 2.3, there are many examples of holomorphic Hilbertian $\mathcal{A}$-bundles over S^2.

(b) Let $p \colon \mathcal{E} \to X$ be a flat Hilbertian $\mathcal{A}$-bundle over X, *i.e.* M is a finitely generated $(\pi - \mathcal{A})$ bimodule, where $\varphi \colon \pi \to \mathrm{GL}(M)$ is the left action of π on M. Then

$$\mathcal{E} = (M \times \tilde{X})/\!\sim\; \to X$$

where $(v, x) \sim (\varphi(g)v, g.x)$ for $g \in \pi$, $v \in M$ and $x \in \tilde{X}$. Let

$$g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \pi$$

denote the transition functions of the universal cover $\tilde{X}$, which is a principal π bundle over X. Here $\{U_\alpha\}_\alpha$ forms an open cover of X. Since π is a discrete group and $g_{\alpha\beta}$ is smooth, it follows that $g_{\alpha\beta}$ is locally constant, and therefore holomorphic. The transition functions of $\mathcal{E}$ are $\varphi(g_{\alpha\beta})$, which again are locally constant, and therefore holomorphic.

(c) Let $E \to X$ be a holomorphic $\mathbb{C}$-vector bundle over X and $\mathcal{E} \to X$ a flat Hilbertian $\mathcal{A}$-bundle over X. Let

$$g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{C})$$

denote the holomorphic transition functions where $\{U_\alpha\}_\alpha$ form an open cover of X. Let

$$g'_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(M)$$

denote the transition functions of the flat Hilbertian $\mathcal{A}$-bundle $\mathcal{E} \to X$. Since $\mathcal{E} \to X$ is flat, $g'_{\alpha\beta}$ are locally constant and thus holomorphic (by the previous example). Consider the new bundle whose transition functions are given by

$$g''_{\alpha\beta} \equiv g_{\alpha\beta} \otimes g'_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{C}^r \otimes M).$$

Since the $g''_{\alpha\beta}$ are holomorphic, so is the new bundle which is the tensor product bundle, and which is denoted by

$$E \otimes_{\mathbb{C}} \mathcal{E} \to X.$$

We have shown that it is a holomorphic Hilbertian $\mathcal{A}$-bundle over X, with fibre $\mathbb{C}^r \otimes M$.

2.7 Holomorphic Sections of Holomorphic Hilbertian $\mathcal{A}$-Bundles

Let $p\colon \mathcal{E} \to X$ be a holomorphic Hilbertian $\mathcal{A}$-bundle. A section $a\colon X \to \mathcal{E}$ is said to be a *holomorphic section* if in a holomorphic local trivialization, $\{U_\alpha\}_\alpha$, the expression for s in U_α,

$$s_\alpha\colon U_\alpha \to M$$

is a holomorphic map. Note that M is a Banach space, and therefore a complex manifold. On $U_\alpha \cap U_\beta$, one has the relation

$$s_\alpha = g_{\alpha\beta} s_\beta$$

which is holomorphic, since $g_{\alpha\beta}$ is holomorphic.

2.8 $\mathcal{A}$-Linear Cauchy-Riemann Operators

Let $p\colon \mathcal{E} \to X$ be a holomorphic Hilbertian $\mathcal{A}$-bundle over X. With respect to the decomposition

$$T^*_{\mathbb{C}}X = T^*X \otimes_{\mathbb{R}} \mathbb{C} = (T^{1,0}X)^* \oplus (T^{0,1}X)^*, \tag{16}$$

the space of smooth differential j-forms on X with values in $\mathcal{E}$ decomposes as a direct sum of spaces of smooth differential (p,q)-forms on X with values in $\mathcal{E}$, where $p + q = j$. This space, which is an $\mathcal{A}$ module, will be denoted by $\Omega^{p,q}(X,\mathcal{E})$.

Then there is a unique operator

$$\bar\partial\colon \Omega^{p,q}(X,\mathcal{E}) \to \Omega^{p,q+1}(X,\mathcal{E})$$

which in any holomorphic trivialization of $p\colon \mathcal{E} \to X$, is equal to

$$\bar\partial = \sum_{i=1}^{n} e(d\bar z^i)\frac{\partial}{\partial \bar z^i}$$

where $e(d\bar z^i)$ denotes exterior multiplication by the 1-form $d\bar z^i$ and $n = \dim_{\mathbb{C}} X$. Note that $\bar\partial^2 = 0$.

2.9 Holomorphic $\mathcal{A}$-Linear Connections

Let $\nabla\colon \Omega^p(X,\mathcal{E}) \to \Omega^{p+1}(X,\mathcal{E})$ be an $\mathcal{A}$-linear connection on a holomorphic Hilbertian $\mathcal{A}$-bundle $p\colon \mathcal{E} \to X$. Then with respect to (16), there is a decomposition

$$\nabla = \nabla' + \nabla''.$$

Here

$$\nabla'\colon \Omega^{p,q}(X,\mathcal{E}) \to \Omega^{p+1,q}(X,\mathcal{E})$$

is an $\mathcal{A}$-morphism such that for $A \in \Omega^0(X, \operatorname{End}_{\mathcal{A}} \mathcal{E})$ and $w \in \Omega^j(X,\mathcal{E})$,

$$\nabla'(Aw) - A(\nabla' w) = (\nabla' A)w$$

where $\nabla' A \in \Omega^{1,0}(X, \operatorname{End}_{\mathcal{A}} \mathcal{E})$ is the $(1,0)$ component of ∇A, while

$$\nabla''\colon \Omega^{p,q}(X,\mathcal{E}) \to \Omega^{p,q+1}(X,\mathcal{E})$$

is an $\mathcal{A}$-morphism such that

$$\nabla''(Aw) - A(\nabla'' w) = (\nabla'' A)w$$

where $\nabla'' A \in \Omega^{0,1}(X, \operatorname{End}_{\mathcal{A}} \mathcal{E})$ is the $(0,1)$ component of ∇A.

An $\mathcal{A}$-linear connection ∇ on a holomorphic Hilbertian $\mathcal{A}$-bundle $p\colon \mathcal{E} \to X$ is said to be a *holomorphic $\mathcal{A}$-linear connection* if $(\nabla'')^2 = 0$.

Since every holomorphic Hilbertian $\mathcal{A}$-bundle has a $\mathcal{A}$-linear Cauchy-Riemann operator, it follows that it also has a holomorphic $\mathcal{A}$-linear connection.

2.10 Examples of Holomorphic $\mathcal{A}$-Linear Connections

(a) Let $\mathcal{E} \to X$ be a flat Hilbertian $\mathcal{A}$-bundle. Then $\mathcal{E}$ has a *canonical flat $\mathcal{A}$-linear connection* ∇ given by the de Rham exterior derivative, where we identify the space of smooth differential j-forms on X with values in $\mathcal{E}$, denoted $\Omega^j(X,\mathcal{E})$, as π-invariant differential forms in $M \otimes_{\mathbb{C}} \Omega^j(\tilde{X})$. Here $M \otimes_{\mathbb{C}} \Omega^j(\tilde{X})$ has the diagonal action. (See [CFM] for more details). Since the de Rham differential $d = \bar{\partial} + \partial$, it is a *canonical flat holomorphic $\mathcal{A}$-linear connection*.

(b) Let $E \to X$ be a holomorphic $\mathbb{C}$-vector bundle over X, and $\mathcal{E} \to X$ a flat Hilbertian $\mathcal{A}$-bundle over X. Then we have seen that $E \otimes_{\mathbb{C}} \mathcal{E} \to X$ is a holomorphic Hilbertian $\mathcal{A}$-bundle over X, with fibre $\mathbb{C}^r \otimes M$. Let $\tilde{\nabla}$ be a holomorphic connection on $E \to X$, and let $\tilde{\tilde{\nabla}}$ be the canonical flat $\mathcal{A}$-linear connection on $\mathcal{E} \to X$. Then $\nabla = \tilde{\nabla} \otimes 1 + 1 \otimes \tilde{\tilde{\nabla}}$ is easily checked to yield a holomorphic $\mathcal{A}$-linear connection on the holomorphic Hilbertian $\mathcal{A}$-bundle $E \otimes_{\mathbb{C}} \mathcal{E} \to X$.

3 Zeta Functions and D-Class Bundles

We now have most of the notation and preliminary results we need to generalize the classical construction of the holomorphic torsion of D. B. Ray and I. M. Singer [RS] to the infinite dimensional case. This section generalizes [BFKM] and [CFM] for the notion of a D-class holomorphic Hilbertian bundle and the definition of zeta-functions for complexes of such bundles.

3.1 Hermitian Metrics, Hilbert $\mathcal{A}$ Bundles, L^2 Scalar Products and the Canonical Holomorphic (Hermitian) $\mathcal{A}$-Linear Connection

A Hermitian metric h on a Hilbertian $\mathcal{A}$-bundle $p\colon \mathcal{E} \to X$ is a smooth family of admissible scalar products on the fibers. Any Hermitian metric on $p\colon \mathcal{E} \to X$ defines a wedge product

$$\wedge\colon \Omega^{p,q}(X,\mathcal{E}) \otimes \Omega^{r,s}(X,\mathcal{E}) \to \Omega^{p+r,q+s}(X)$$

similar to the finite dimensional case.

Let $p\colon \mathcal{E} \to X$ be a *holomorphic* Hilbertian $\mathcal{A}$-bundle and h be a Hermitian metric on $\mathcal{E}$. The Hermitian metric on $p\colon \mathcal{E} \to X$ determines a *canonical holomorphic $\mathcal{A}$-linear connection* on $\mathcal{E}$ as follows. Let ∇ be a holomorphic $\mathcal{A}$-linear connection on $\mathcal{E}$ which preserves the Hermitian metric $\mathcal{E}$, that is,

$$dh(\xi,\eta) = h(\nabla\xi,\eta) + h(\xi,\nabla\eta)$$

where ξ and η are smooth sections of $\mathcal{E}$. Equating forms of the same type, one has

$$\partial h(\xi,\eta) = h(\nabla'\xi,\eta) + h(\xi,\nabla''\eta)$$

and

$$\bar\partial h(\xi,\eta) = h(\nabla''\xi,\eta) + h(\xi,\nabla'\eta).$$

Since $\nabla'' = \bar\partial$, we see that a choice of Hermitian metric determines a holomorphic $\mathcal{A}$-linear connection, which is called the *canonical holomorphic $\mathcal{A}$-linear connection*.

The Hermitian metric on $p\colon \mathcal{E} \to X$ together with a Hermitian metric on X determines a scalar product on $\Omega^{p,q}(X,\mathcal{E})$ in the standard way; namely, using the Hodge star operator

$$*\colon \Omega^{p,q}(X,\mathcal{E}) \to \Omega^{n-q,n-p}(X,\mathcal{E})$$

one sets

$$(\omega,\omega') = \int_X \omega \wedge *\bar\omega'.$$

With this scalar product $\Omega^i(X,\mathcal{E})$ becomes a pre-Hilbert space. Define the space of L^2 differential p,q-forms on X with coefficients in $\mathcal{E}$, denoted $\Omega^{p,q}_{(2)}(X,\mathcal{E})$, to be the Hilbert space completion of $\Omega^{p,q}(X,\mathcal{E})$. We will tend to ignore the scalar product on $\Omega^{p,q}_{(2)}(X,\mathcal{E})$ and view it as an infinite Hilbertian $\mathcal{A}$ module.

3.2 Reduced L^2 Dolbeault Cohomology

Given a holomorphic Hilbertian $\mathcal{A}$ bundle $p\colon \mathcal{E} \to X$ together with a Hermitian metric on $\mathcal{E}$, one defines the *reduced L^2 Dolbeault cohomology with coefficients in $\mathcal{E}$* as the quotient

$$H^{p,q}(X,\mathcal{E}) = \frac{\ker \nabla''/\Omega^{p,q}_{(2)}(X,\mathcal{E})}{\operatorname{cl}\big(\operatorname{im} \nabla''/\Omega^{p,q-1}(X,\mathcal{E})\big)},$$

where the Cauchy-Riemann operator ∇'' is associated to the canonical $\mathcal{A}$-linear connection ∇ on $\mathcal{E}$. ∇'' on $\mathcal{E}$ extends to an unbounded, densely defined operator $\Omega^{p,q}_{(2)}(X,\mathcal{E}) \to \Omega^{p,q+1}_{(2)}(X,\mathcal{E})$. Then $H^{p,q}(X,\mathcal{E})$ is naturally defined as a Hilbertian module over $\mathcal{A}$. It can also be considered as the cohomology of X with coefficients in a locally constant sheaf, determined by $\mathcal{E}$.

3.3 Hodge Decomposition

The Laplacian $\square_{p,q}$ acting on L^2 $\mathcal{E}$-valued (p,q)-forms on X is defined to be

$$\square_{p,q} = \nabla''\nabla''^* + \nabla''^*\nabla'' \colon \Omega^{p,q}_{(2)}(X,\mathcal{E}) \to \Omega^{p,q}_{(2)}(X,\mathcal{E})$$

where ∇''^* denotes the formal adjoint of ∇'' with respect to the L^2 scalar product on $\Omega^{p,q}_{(2)}(X,\mathcal{E})$. Note that by definition, the Laplacian is a formally self-adjoint operator which is densely defined. We also denote by $\square_{p,q}$ the self adjoint extension of the Laplacian.

Let $\mathcal{H}^{p,q}(X,\mathcal{E})$ denote the closed subspace of L^2 harmonic p,q-forms with coefficients in $\mathcal{E}$, that is, the kernel of $\square_{p,q}$. Note that $\mathcal{H}^{p,q}(X,\mathcal{E})$ is a Hilbertian $\mathcal{A}$-module. By elliptic regularity (*cf.* [BFKM, Section 2]), one sees that $\mathcal{H}^{p,q}(X,\mathcal{E}) \subset \Omega^{p,q}(X,\mathcal{E})$, that is, every L^2 harmonic (p,q)-form with coefficients in $\mathcal{E}$ is smooth. Standard arguments then show that one has the following Hodge decomposition (*cf.* [D]; [BFKM, Section 4] and also [GS, Section 3])

$$\Omega^{p,q}_{(2)}(X,\mathcal{E}) = \mathcal{H}^{p,q}(X,\mathcal{E}) \oplus \mathrm{cl}\big(\mathrm{im}\,\nabla''/\Omega^{p,q-1}(X,\mathcal{E})\big) \oplus \mathrm{cl}\big(\mathrm{im}\,\nabla''^*/\Omega^{p,q+1}(X,\mathcal{E})\big).$$

Therefore it follows that the natural map $\mathcal{H}^{p,q}(X,\mathcal{E}) \to H^{p,q}(X,\mathcal{E})$ is an isomorphism Hilbertian $\mathcal{A}$-modules. The corresponding L^2 Betti numbers are denoted by

$$b^{p,q}(X,\mathcal{E}) = \dim_\tau\big(H^{p,q}(X,\mathcal{E})\big).$$

Definition 3.1 Let $\square_{p,q} = \int_0^\infty \lambda\, dE_{p,q}(\lambda)$ denote the spectral decomposition of the Laplacian. The *spectral density function* is defined to be $N_{p,q}(\lambda) = \mathrm{Tr}_\tau\big(E_{p,q}(\lambda)\big)$ and the *theta function* is defined to be $\theta_{p,q}(t) = \int_0^\infty e^{-t\lambda}\, dN_{p,q}(\lambda) = \mathrm{Tr}_\tau(e^{-t\square_{p,q}}) - b^{p,q}(X,\mathcal{E})$. Here we use the well known fact that the projection $E_{p,q}(\lambda)$ and the heat operator $e^{-t\square_{p,q}}$ have smooth Schwartz kernels which are smooth sections of a bundle over $X \times X$ with fiber the commutant of M, *cf.* [BFKM], [GS], [Lu]. The symbol Tr_τ denotes application of the canonical trace on the commutant to the restriction of the kernels to the diagonal followed by integration over the manifold X. This is a trace; it vanishes on commutators of smoothing operators. See also [M], [L] and [GS] for the case of the flat holomorphic Hilbertian bundle defined by the regular representation of the fundamental group.

Definition 3.2 A holomorphic Hilbertian $\mathcal{A}$-bundle $\mathcal{E} \to X$ together with a choice of Hermitian metric h on $\mathcal{E}$, is said to be *D-class* if

$$\int_0^1 \log(\lambda)\, dN_{p,q}(\lambda) > -\infty$$

or equivalently

$$\int_1^\infty t^{-1}\theta_{p,q}(t)\, dt < \infty$$

for all $p,q = 0,\dots,n$.

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Note that the D-class property of a holomorphic Hilbertian $\mathcal{A}$ bundle does not depend on the choice of metrics g on X and h on $\mathcal{E}$.

For most of the paper, we make the assumption that the holomorphic Hilbertian $\mathcal{A}$-bundle $\mathcal{E} \to X$ is D-class. It is easy to write down examples of flat bundles on the circle that are not of *D-class*, as these are determined by the value of the associated representation at the identity. By taking Cartesian products, we obtain such examples on tori of any dimension. Closed Kähler hyperbolic manifolds (discussed further in Section 7) and closed Kähler manifolds with residually amenable fundamental group [Lu], [DM], [Cl], [Sc] are of *D-class* for the flat holomorphic Hilbertian bundle defined by the regular representation of the fundamental group. In fact, in the case of Kähler hyperbolic manifolds, Gromov [G] proves that there are spectral gaps for the Laplacian in all degrees. It follows that there are non-flat holomorphic Hilbertian bundles that are of *D-class* on such manifolds.

Under the *D-class* assumption, we will next define and study the zeta function of the Laplacian $\square_{p,q}$ acting on $\mathcal{E}$ valued L^2 differential forms on X.

Definition 3.3 For $\lambda > 0$ the *zeta function of the Laplacian* $\square_{p,q}$ is defined on the half-plane $\Re(s) > n$ as

$$\zeta_{p,q}(s,\lambda,\mathcal{E}) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-\lambda t} \theta_{p,q}(t)\, dt. \tag{17}$$

Lemma 3.4 *$\zeta_{p,q}(s,\lambda,\mathcal{E})$ is a holomorphic function in the half-plane $\Re(s) > n$ (where $n = \dim_{\mathbb{C}} X$) and has a meromorphic continuation to $\mathbb{C}$ with no pole at $s = 0$. If we assume that the holomorphic Hilbertian $\mathcal{A}$-bundle $\mathcal{E} \to X$ is D-class then $\lim_{\lambda\to 0} \zeta'_{p,q}(0,\lambda,\mathcal{E})$ exists (where the prime denotes differentiation with respect to s)*

Proof There is an asymptotic expansion as $t \to 0^+$ of the trace of the heat kernel $\mathrm{Tr}_\tau(e^{-t\square_{p,q}})$ (*cf.* [BFKM] and [R, Chapter 13]),

$$\mathrm{Tr}_\tau(e^{-t\square_{p,q}}) \sim t^{-n} \sum_{i=0}^\infty t^i c_{i,p,q}. \tag{18}$$

In particular, $\mathrm{Tr}_\tau(e^{-t\square_{p,q}}) \leq Ct^{-n}$ for $0 < t \leq 1$. From this we deduce that $\zeta_{p,q}(s,\lambda,\mathcal{E})$ is well defined on the half-plane $\Re(s) > n$ and it is holomorphic there. The meromorphic continuation of $\zeta_{p,q}(s,\lambda,\mathcal{E})$ to the half-plane $\Re(s) > n - N$ is obtained by considering the first N terms of the small time asymptotic expansion (18) of $\mathrm{Tr}_\tau(e^{-t\square_{p,q}})$,

(19)

$$\zeta_{p,q}(s,\lambda,\mathcal{E}) = -\frac{1}{\Gamma(s)} \sum_{j=0}^\infty \frac{b^{p,q}(X,\mathcal{E})(-\lambda)^j}{(s+j)\,j!} + \frac{1}{\Gamma(s)} \Big[\sum_{0\leq i+j\leq N} \frac{(-\lambda)^j c_{i,p,q}}{(s+i+j-n)\,j!} + R_N(s,\lambda) \Big] + \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} \theta_{p,q}(t) e^{-t\lambda}\, dt$$

where $R_N(s,\lambda)$ is holomorphic in the half plane $\Re(s) > n - N$ with a meromorphic extension to a neighbourhood of $s = 0$. Since the Gamma function has a simple pole at $s = 0$,

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we observe that the meromorphic continuation of $\zeta_{p,q}(s,\lambda,\mathcal{E})$ has no pole at $s=0$. The last part of the lemma now follows *cf.* [BFKM]. ■

Let $\zeta'_{p,q}(0,0,\mathcal{E}) = \lim_{\lambda\to 0}\zeta'_{p,q}(0,\lambda,\mathcal{E})$. The following corollary is clear from (19).

Corollary 3.5 *One has*

$$\zeta_{p,q}(0,0,\mathcal{E}) = -b^{p,q}(X,\mathcal{E}) + c_{n,p,q}$$

where $c_{n,p,q}$ is the n-th coefficient in the small time asymptotic expansion of the theta function, cf. (18).

4 Holomorphic L^2-Torsion

In this section, we define and study the generalization of Ray-Singer holomorphic torsion to the case of holomorphic Hilbertian $\mathcal{A}$-bundles. *For the rest of the section, we make the assumption that the holomorphic Hilbertian $\mathcal{A}$-bundle $\mathcal{E}\to X$ is D-class.* Given a Hermitian manifold X and a metric on a holomorphic Hilbertian $\mathcal{A}$-bundle $\mathcal{E}$ over X with fibre a Hilbertian $\mathcal{A}$ module M, the holomorphic L^2 torsion $\rho^p_{\mathcal{E}}$ defined in this section is a *positive element of the determinant line*

$$\det\bigl(H^{p,*}(X,\mathcal{E})\bigr).$$

We also prove a variational formula for the holomorphic L^2 torsion.

4.1 The Construction of Holomorphic L^2 Torsion

Let (X,g) be a compact, connected Hermitian manifold of complex dimension n with $\pi = \pi_1(X)$. Let $\mathcal{E}\to X$ be a holomorphic Hilbertian $\mathcal{A}$-bundle over X with fibre M and let h be a Hermitian metric on $\mathcal{E}$. We assume that $\mathcal{E}$ is of D-class.

As before, let $H^{p,q}(X,\mathcal{E})$ denote the L^2 cohomology groups of X with coefficients in $\mathcal{E}$. Then we know that $H^{p,q}(X,\mathcal{E})$ is a Hilbertian $\mathcal{A}$-module. If $\mathcal{H}^{p,q}(X,\mathcal{E})$ denotes the space of L^2 harmonic p,q-forms with coefficients in $\mathcal{E}$, then it is a Hilbert $\mathcal{A}$-module with the admissible scalar product induced from $\Omega^{p,q}_{(2)}(X,\mathcal{E})$. By the Hodge theorem, the natural map

$$\mathcal{H}^{p,q}(X,\mathcal{E})\to H^{p,q}(X,\mathcal{E})$$

is an isomorphism of Hilbertian $\mathcal{A}$-modules. Thus, we may identify these modules via this isomorphism, or equivalently, we may say that this isomorphism defines an admissible scalar product on the reduced L^2 cohomology $H^{p,q}(X,\mathcal{E})$. These admissible scalar products on $H^{p,q}(X,\mathcal{E})$ for all p,q, determine elements of the determinant lines $\det\bigl(H^{p,q}(X,\mathcal{E})\bigr)$ and thus, their product in

$$\det\bigl(H^{p,*}(X,\mathcal{E})\bigr) = \prod_{q=0}^{n}\det\bigl(H^{p,q}(X,\mathcal{E})\bigr)^{(-1)^q}$$

is defined. This last element we will denote $\rho'^p(g,h)$; the notation emphasizing the dependence on the metrics g and h.

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Using the results of the previous section, we introduce the graded zeta function

$$\zeta^p(s,\lambda,\mathcal{E}) = \sum_{q=0}^{n} (-1)^q q \zeta_{p,q}(s,\lambda,\mathcal{E}).$$

It is a meromorphic function with no pole at $s = 0$. Note also that this zeta-function depends on the choice of the trace τ and on the metrics g and h.

Definition 4.1 Define the *holomorphic L^2 torsion* to be the element of the determinant line

$$\rho^p_{\mathcal{E}}(g,h) \in \det\bigl(H^{p,*}(X,\mathcal{E})\bigr), \quad \rho^p_{\mathcal{E}}(g,h) = e^{\frac{1}{2}\zeta^{p\,\prime}(0,0,\mathcal{E})} \cdot \rho'^p(g,h),$$

where $\zeta^{p\,\prime}$ denotes the derivative with respect to s. Thus, the holomorphic L^2 torsion is a volume form on the reduced L^2 Dolbeault cohomology.

Remark 4.2 1. In the case when $\mathcal{A} = \mathbb{C}$, we arrive at the classical definition of the Ray-Singer-Quillen metric on the determinant of the Dolbeault cohomology.

2. We will prove later in this section a metric variation formula for the holomorphic L^2 torsion as defined in Definition 4.1. Using this, we prove that a relative version of the holomorphic L^2 torsion is independent of the choice of Hermitian metric.

3. Assuming that the reduced L^2 Dolbeault cohomology $H^{p,*}(X,\mathcal{E})$ vanishes, we can identify canonically the determinant line $\det\bigl(H^{p,*}(X,\mathcal{E})\bigr)$ with $\mathbb{R}$, and so the torsion $\rho^p_{\mathcal{E}}$ in this case is just a number.

4.2 Metric Variation Formulae

Suppose that a holomorphic Hilbertian $\mathcal{A}$-bundle $\mathcal{E} \to X$ of D-class is given. This property does not depend on the choice of the metrics. Consider a smooth 1-parameter family of metrics g_u on X and h_u on $\mathcal{E}$, where u varies in an interval $(-\epsilon,\epsilon)$. Let $(\ ,\ )_u$ denote the L^2 scalar product on $\Omega^{p,*}_{(2)}(X,\mathcal{E})$ determined by g_u and h_u. This family determines an invertible, positive, self-adjoint bundle map $A_u \colon \mathcal{E} \to \mathcal{E}$ which is uniquely determined by the relation

$$(\omega,\omega')_u = (A_u\omega,\omega')_0$$

for $\omega,\omega' \in \Omega^{p,*}_{(2)}(X,\mathcal{E})$; it depends smoothly on u.

Let ∇ be the canonical $\mathcal{A}$-linear connection on $\mathcal{E}$. Define the operator

$$D_u = \nabla'' + \nabla''^*_u \colon \Omega^{p,*}_{(2)}(X,\mathcal{E}) \to \Omega^{p,*}_{(2)}(X,\mathcal{E})$$

where ∇''^*_u denotes the formal adjoint of ∇'' with respect to the L^2 scalar product $(\ ,\ )_u$ on $\Omega^{p,*}_{(2)}(X,\mathcal{E})$. Then $\nabla''^*_u = A_u^{-1}\nabla''^*_0 A_u$ acting on $\Omega^{p,*}_{(2)}(X,\mathcal{E})$. Denote $Z_u = A_u^{-1}\dot{A}_u$, where the dot means the derivative with respect to u.

As in Section 4.1, let $\zeta^p_u(s,\lambda,\mathcal{E})$ denote the graded zeta function with respect to the metrics g_u, h_u. The scalar product $(\ ,\ )_u$ induces a scalar product on the space of harmonic forms $\mathcal{H}^{p,*}_u(X,\mathcal{E})$, and via the canonical isomorphism $\mathcal{H}^{p,*}_u(X,\mathcal{E}) \to H^{p,*}(X,\mathcal{E})$, it induces an admissible scalar product on the reduced L^2 cohomology $H^{p,*}(X,\mathcal{E})$. Let $\rho'(u)$ denote

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the class in $\det\big(H^{p,*}(X,\mathcal{E})\big)$ of this scalar product. Then the holomorphic L^2 torsion with respect to the metrics g_u, h_u is given, as in Definition 4.1, by

$$\rho^p_{\mathcal{E}}(u) = e^{\frac{1}{2}\zeta^{p}{}'_u(0,0,\mathcal{E})}\rho'^p(u) \in \det\big(H^{p,*}(X,\mathcal{E})\big),$$

where $\zeta^{p}{}'$ means the derivative with respect to s.

Theorem 4.3 *Let $\mathcal{E} \to X$ be a holomorphic Hilbert bundle of D-class. Then in the notation above, $u \mapsto \rho^p_{\mathcal{E}}(u)$ is a smooth map and one has*

$$\frac{\partial}{\partial u}\rho^p_{\mathcal{E}}(u) = c^p_{\mathcal{E}}(u)\rho^p_{\mathcal{E}}(u),$$

where $c^p_{\mathcal{E}}(u) \in \mathbb{R}$ (cf. (24)) is a local term.

The proof of this theorem will follow from two propositions which we will prove in this section.

Let $P_p(u)$ denote the orthogonal projection from $\Omega^{p,*}_{(2)}(X,\mathcal{E})$ onto $\ker D^2_u$ and $\mathrm{Tr}^s_\tau(\cdot)$ denote the graded trace, that is the alternating sum of the von Neumann traces Tr_τ on operators on $\Omega^{p,*}_{(2)}(X,\mathcal{E})$ having smooth Schwartz kernels.

Proposition 4.4 *Let $\mathcal{E} \to X$ be a holomorphic Hilbert bundle of D-class. Then in the notation above, one has*

$$\frac{\partial}{\partial u}\zeta^{p}{}'_u(0,0,\mathcal{E}) = \mathrm{Tr}^s_\tau\big(Z_u P_p(u)\big) - 2c^p_{\mathcal{E}}(u)$$

where $c^p_{\mathcal{E}}(u) \in \mathbb{R}$ (cf. (24)) is a local term.

Proof We consider the function

$$F(u,\lambda,s) = \sum_{q=0}^{n}(-1)^q q\int_0^\infty t^{s-1}e^{-t\lambda}\,\mathrm{Tr}_\tau\big(e^{-t\square_{p,q}(u)} - P_{p,q}(u)\big)\,dt$$

which is defined on the half-plane $\Re(s) > n$ and is holomorphic there. As in (18), one has for each u, the small time asymptotic expansion of the heat kernel,

$$\mathrm{Tr}_\tau(e^{-t\square_{p,q}(u)}) \sim \sum_{k=0}^{\infty} c_{k,p,q}(u)t^{-n+k}. \tag{20}$$

Using (20), we see that $F(u,\lambda,s)$ has a meromorphic continuation to $\mathbb{C}$ with no pole at $s = 0$. This assertion is analogous to that in Lemma 2.8, and is proved by an easy modification of that proof.

If we know that $u \to F(u,\lambda,s)$ is a smooth function then

$$\frac{\partial}{\partial u}\zeta^{p}{}'_u(0,0,\mathcal{E}) = \lim_{\lambda\to 0}\frac{\partial}{\partial s}\left(\frac{1}{\Gamma(s)}\frac{\partial}{\partial u}F(u,\lambda,s)\right)\bigg|_{s=0}$$

by the D-class assumption. Hence:

$$\frac{\partial}{\partial u}\zeta^{p}{}'_{u}(0,0,\mathcal{E}) = \lim_{\lambda\to 0}\frac{\partial}{\partial u}F(u,\lambda,s)|_{s=0}.$$

Observing that $\operatorname{Tr}_\tau\big(P_{p,q}(u)\big) = b^{p,q}(X,\mathcal{E})$ is independent of u we see that $u \to F(u,s)$ is smooth provided we can show that $u \to \operatorname{Tr}_\tau(e^{-t\Box_{p,q}(u)})$ is a smooth function. By an application of Duhamel's principle, one has

(21)
$$\begin{aligned}\frac{1}{u'-u}&\Big(\operatorname{Tr}_\tau\big((e^{-\frac{t}{2}\Box_{p,q}(u')} - e^{-\frac{t}{2}\Box_{p,q}(u)})e^{-\frac{t}{2}\Box_{p,q}(u)}\big)\Big)\\ &= -\int_0^{\frac{t}{2}}\operatorname{Tr}_\tau\Big(e^{-s\Box_{p,q}(u')}\frac{1}{u'-u}\big(\Box_{p,q}(u')-\Box_{p,q}(u)\big)e^{-\frac{t}{2}\Box_{p,q}(u)}e^{-(\frac{t}{2}-s)\Box_{p,q}(u)}\Big)\,ds.\end{aligned}$$

Since $\Big\|\Big(\frac{1}{u'-u}\big(\Box_{p,q}(u')-\Box_{p,q}(u)\big)-\dot{\Box}_{p,q}(u)\Big)e^{-\frac{t}{2}\Box_{p,q}(u)}\Big\|$ is $O(u'-u)$ as $u'\to u$, one sees that the limit as $u'\to u$ of (21) exists and

$$\begin{aligned}\operatorname{Tr}_\tau\Big(\Big(\frac{\partial}{\partial u}e^{-\frac{t}{2}\Box_{p,q}(u)}\Big)e^{-\frac{t}{2}\Box_{p,q}(u)}\Big) &= -\int_0^{\frac{t}{2}}\operatorname{Tr}_\tau\big(e^{-s\Box_{p,q}(u)}\dot{\Box}_{p,q}(u)e^{-\frac{t}{2}\Box_{p,q}(u)}e^{-(\frac{t}{2}-s)\Box_{p,q}(u)}\big)\,ds\\ &= -\frac{t}{2}\operatorname{Tr}_\tau\big(\dot{\Box}_{p,q}(u)e^{-t\Box_{p,q}(u)}\big).\end{aligned}$$

Therefore $u\to\operatorname{Tr}_\tau(e^{-t\Box_{p,q}(u)})$ is a smooth function (and hence so is $u\to F(u,s)$) and by the semigroup property of the heat kernel, one has

$$\begin{aligned}\frac{\partial}{\partial u}\operatorname{Tr}_\tau\big(e^{-t\Box_{p,q}(u)}-P_{p,q}(u)\big) &= \frac{\partial}{\partial u}\operatorname{Tr}_\tau(e^{-t\Box_{p,q}(u)})\\ &= 2\operatorname{Tr}_\tau\Big(\Big(\frac{\partial}{\partial u}e^{-\frac{t}{2}\Box_{p,q}(u)}\Big)e^{-\frac{t}{2}\Box_{p,q}(u)}\Big)\\ &= -t\operatorname{Tr}_\tau(\dot{\Box}_{p,q}(u)e^{-t\Box_{p,q}(u)}).\end{aligned}$$

A calculation similar to [RS, p. 152] yields

$$\dot{\Box}_{p,q}(u) = -Z_u\nabla''^*_u\nabla'' + \nabla''^*_uZ_u\nabla'' - \nabla''Z_u\nabla''^*_u + \nabla''\nabla''^*_uZ_u.$$

Since $\nabla''\Box_{p,q}(u) = \Box_{p,q+1}(u)\nabla''$ and $\nabla''^*_u\Box_{p,q}(u) = \Box_{p,q-1}(u)\nabla''^*_u$ and using the fact that Tr_τ is a trace, one has

$$\begin{aligned}\operatorname{Tr}_\tau(\dot{\Box}_{p,q}(u)e^{-t\Box_{p,q}(u)}) &= \operatorname{Tr}_\tau(Z_u\nabla''\nabla''^*_ue^{-t\Box_{p,q}(u)}) - \operatorname{Tr}_\tau(Z_u\nabla''^*_u\nabla''e^{-t\Box_{p,q-1}(u)})\\ &\quad + \operatorname{Tr}_\tau(Z_u\nabla''\nabla''^*_ue^{-t\Box_{p,q+1}(u)}) - \operatorname{Tr}_\tau(Z_u\nabla''^*_u\nabla''_ue^{-t\Box_{p,q}(u)}).\end{aligned}$$

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So one sees that

$$\begin{aligned}\frac{\partial}{\partial u}\sum_{q=0}^{n}(-1)^q q\operatorname{Tr}_\tau\big(e^{-t\Box_{p,q}(u)}-P_{p,q}(u)\big) &= -t\sum_{q=0}^{n}(-1)^q q\operatorname{Tr}_\tau\big(\dot{\Box}_{p,q}(u)e^{-t\Box_{p,q}(u)}\big)\\ &= -t\sum_{q=0}^{n}(-1)^q q\operatorname{Tr}_\tau\big(Z_u\Box_{p,q}(u)e^{-t\Box_{p,q}(u)}\big)\\ &= t\frac{\partial}{\partial t}\sum_{q=0}^{n}(-1)^q\operatorname{Tr}_\tau(Z_u e^{-t\Box_{p,q}(u)}).\end{aligned}$$

Using this, one sees that for $\Re(s) > n$,

$$\frac{\partial}{\partial u}F(u,\lambda,s) = \sum_{q=0}^{n}(-1)^q\int_0^\infty t^s e^{-t\lambda}\frac{\partial}{\partial t}\operatorname{Tr}_\tau(Z_u\big(e^{-t\Box_{p,q}(u)}-P_{p,q}(u)\big)\,dt. \tag{22}$$

Since Z_u is a bounded endomorphism, by a straightforward generalization of Lemma 1.7.7 in [Gi], there is a small time asymptotic expansion

$$\operatorname{Tr}_\tau(Z_u e^{-t\Box_{p,q}(u)}) \sim \sum_{k=0}^{\infty} m_{k,p,q}(u)t^{-n+k}. \tag{23}$$

In particular, one has

$$|\operatorname{Tr}_\tau(Z_u e^{-t\Box_{p,q}(u)})| \le ct^{-n}$$

for all $0 < t \le 1$. If $\Re(s) > n$, we can integrate the right-hand side of (22) by parts to get

$$\sum_{q=0}^{n}(-1)^{q+1}\int_0^\infty (st^{s-1}-\lambda t^s)e^{-t\lambda}\operatorname{Tr}_\tau\Big(Z_u\big(e^{-t\Box_{p,q}(u)}-P_{p,q}(u)\big)\Big)\,dt.$$

By splitting the integral into two parts, one from 0 to 1 and the other from 1 to ∞ and using (23) on the first integral together with the observations above, one gets the following explicit expression for the meromorphic continuation of $\frac{\partial}{\partial u}F(u,s)$ to the half-plane $\Re(s) > n-N$

$$\begin{aligned}\frac{\partial}{\partial u}F(u,s) &= \sum_{q=0}^{n}(-1)^q\operatorname{Tr}_\tau\big(Z_uP_{p,q}(u)\big)\frac{1}{\lambda^s}\int_0^\lambda (st^{s-1}-t^s)e^{-t}\,dt\\ &\quad+\sum_{q=0}^{n}(-1)^{q+1}\sum_{0\le k+r\le N}\frac{(-\lambda)^r m_{k,p,q}(u)}{r!}\Big(\frac{s}{s-n+k+r}-\frac{\lambda}{s-n+k+r+1}\Big)\\ &\quad+R_N(u,\lambda,s)\end{aligned}$$

where $R_N(u,\lambda,s)$ is holomorphic in a neighbourhood of zero. At $s=0$ we have

$$R_N(u,\lambda,0) = \int_1^\infty \operatorname{Tr}_\tau\Big(Z_u\big(e^{-t\Box_{p,q}(u)}-P_{p,q}(u)\big)\Big)e^{-t\lambda}\,dt.$$

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Thus we have

$$\begin{aligned}\frac{\partial}{\partial u}\zeta'^{p}_{u}(0,\lambda,\mathcal{E}) &= \frac{\partial}{\partial u}F(u,0)\\ &= \sum_{q=0}^{n}(-1)^{q+1}\Big(\sum_{k+r=n}\frac{(-\lambda)^r}{r!}(1-\lambda)m_{k,p,q}(u) - \operatorname{Tr}_\tau\big(Z_u P_{p,q}(u)\big)\Big)\\ &\qquad + R_N(u,\lambda,0).\end{aligned}$$

Hence

$$\zeta^{p}{}'_{u}(0,0,\mathcal{E}) = \sum_{q=0}^{n}(-1)^q \operatorname{Tr}_\tau\big(Z_u P_{p,q}(u)\big) - 2c^p_{\mathcal{E}}(u)$$

where

$$c^p_{\mathcal{E}}(u) = \frac{1}{2}\sum_{q=0}^{n}(-1)^q m_{n,p,q}(u). \tag{24}$$

This completes the proof of the proposition. ■

The 1-parameter family of scalar products on $\Omega^{p,*}_{(2)}(X,\mathcal{E})$ which are induced by the 1-parameter family of metrics on X and $\mathcal{E}\to X$, defines an inclusion isomorphism of Hilbertian modules

$$I_u \colon \mathcal{H}^{p,*}_u(X,\mathcal{E}) \to H^{p,*}(X,\mathcal{E}).$$

Here $\mathcal{H}^{p,q}_u(X,\mathcal{E})$ denotes the kernel of $\Box_{p,q}(u)$. There is an induced isomorphism of determinant lines *cf.* (13) and the discussion in the paragraph above it.

$$I_u^* \colon \det\big(H^{p,*}(X,\mathcal{E})\big) \to \det\big(\mathcal{H}^{p,*}_u(X,\mathcal{E})\big).$$

We first identify $H^{p,*}(X,\mathcal{E})$ with $\mathcal{H}^{p,*}_0(X,\mathcal{E})$. Then I_u defines a 1-parameter family of admissible scalar products on $H^{p,*}(X,\mathcal{E})$, which we can write explicitly as follows:

$$\langle\eta,\eta'\rangle_u = \big(P(u)\eta, P(u)\eta'\big)_u = \big(A_u P(u)\eta, P(u)\eta'\big)_0$$

where η,η' are harmonic forms in $\mathcal{H}^{p,*}_0(X,\mathcal{E})$. The relation between these scalar products in the determinant line $\det\big(H^{p,*}(X,\mathcal{E})\big)$ is given as in Proposition 1.3 and (11), by

$$\langle\ ,\ \rangle_u = \prod_{q=0}^{n}\operatorname{Det}_{\tau'}\big(P_{p,q}(u)^\dagger A_u P_{p,q}(u)\big)^{\frac{(-1)^{q+1}}{2}}\langle\ ,\ \rangle_0. \tag{25}$$

where $P_{p,q}(u)^\dagger$ denotes the adjoint of $P_{p,q}(u)$ with respect to the fixed admissible scalar product $\langle\ ,\ \rangle_0$ and $\operatorname{Tr}_{\tau'}(\cdot)$ is the trace on $H^{p,q}(X,\mathcal{E})$. Using the fact that $H^{p,q}(X,\mathcal{E})$ is isomorphic to a submodule of a free Hilbertian module as is $\mathcal{H}^{p,q}_u(X,\mathcal{E})$, it follows that $\operatorname{Tr}_{\tau'}(\cdot)$ is equal to $\operatorname{Tr}_\tau\big(P_{p,q}(u)\cdot P_{p,q}(u)\big)$. We begin with the following:

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Proposition 4.5 *Let $\mathcal{E} \to X$ be a holomorphic Hilbert bundle of D-class. Then the function $u \to P_{p,q}(u)$ is smooth and in the notation of Section 4.2 and Theorem 4.3, one has*

$$\frac{\partial}{\partial u} \rho'^{p}(u) = -\frac{1}{2} \operatorname{Tr}^{s}_{\tau}\big(Z_u P_p(u)\big) \rho'^{p}(u).$$

Proof We will first prove that $u \to P_{p,q}(u)$ is a smooth function. First consider the Hodge decomposition in the u-metric in the context,

$$\Omega^{p,q}_{(2)}(X, \mathcal{E}) = \mathcal{H}^{p,q}_{u}(X, \mathcal{E}) \oplus \operatorname{cl}(\operatorname{im} \nabla'') \oplus \operatorname{cl}(\operatorname{im} {\nabla''}^{*}_{u})$$

and let π denote the projection onto $\operatorname{cl}(\operatorname{im} \nabla'')$, which does not depend on the u-metric. Let $h \in \mathcal{H}^{p,q}_{0}(X, \mathcal{E})$ be harmonic in the $u = 0$ metric. We will arrive at a formula for $h_u \equiv P_{p,q}(u)h$, from which which the differentiability of $u \to P_{p,q}(u)$ will be clear. Now define r_u by the equation

$$h_u = h + r_u.$$

Since h_u is harmonic in the u-metric, one has ${\nabla''}^{*}_{u}(h_u) = 0$. By the formula for ${\nabla''}^{*}_{u}$ in 4.4, one sees that ${\nabla''}^{*}_{0} A_u(h + r_u) = 0$. Since ${\nabla''}^{*}_{0}$ is injective on $\operatorname{cl}(\operatorname{im} \nabla'')$, one has that $\pi\big(A_u(h + r_u)\big) = 0$. Since $B_u \equiv \pi A_u \pi \colon \operatorname{cl}(\operatorname{im} \nabla'') \to \operatorname{cl}(\operatorname{im} \nabla'')$ is an isomorphism, one sees that $r_u = -B_u^{-1} \pi A_u(h)$ and therefore

$$h_u = h - B_u^{-1} \pi A_u(h).$$

Since $u \to A_u$ is smooth, it follows that $u \to B_u$ is smooth and by the formula above, one concludes that $u \to P_{p,q}(u)$ is also smooth.

Observe that

$$P_{p,q}(u)^2 = P_{p,q}(u).$$

Differentiating with respect to u, one has

$$\dot{P}_{p,q}(u) = P_{p,q}(u)\dot{P}_{p,q}(u) + \dot{P}_{p,q}(u)P_{p,q}(u).$$

Therefore

$$P_{p,q}(u)\dot{P}_{p,q}(u)P_{p,q}(u) = 0.$$

Therefore

$$\begin{aligned} \operatorname{Tr}_\tau\big(\dot{P}_{p,q}(u)\big) &= 2\operatorname{Tr}_\tau\big(P_{p,q}(u)\dot{P}_{p,q}(u)\big) \\ &= 2\operatorname{Tr}_\tau\big(P_{p,q}(u)\dot{P}_{p,q}(u)P_{p,q}(u)\big) \\ &= 0. \end{aligned}$$

A similar argument shows that the projection $P^{\dagger}_{p,q}(u)$ also satisfies

$$\operatorname{Tr}_\tau\big(\dot{P}^{\dagger}_{p,q}(u)\big) = 0.$$

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By definition, $\rho'^p(u) = \langle \ , \ \rangle_u \in \det\big(H^{p,*}(X,\mathcal{E})\big)$, and therefore by differentiating the relation (25), one has

$$\frac{\partial}{\partial u}\rho'^p(u) = -\frac{1}{2}\operatorname{Tr}^s_{\tau'}\left(C_u^{-1}\frac{\partial}{\partial u}C_u\right)\rho'^p(u)$$

where $C_u \equiv P_p(u)^\dagger A_u P_p(u)$ and $\operatorname{Tr}^s_{\tau'}(\cdot)$ denotes the graded von Neumann trace on $H^{p,\bullet}(X,\mathcal{E})$. Therefore one sees that

$$\begin{aligned}\frac{\partial}{\partial u}\rho'^p(u) &= -\frac{1}{2}\operatorname{Tr}^s_{\tau'}\big(Z_u P_p(u) + P_p(u)\dot{P}_p(u) + P_p^\dagger(u)\dot{P}_p^\dagger(u)\big)\rho'^p(u)\\ &= -\frac{1}{2}\operatorname{Tr}^s_{\tau'}\big(Z_u P_p(u)\big)\rho'^p(u) = -\frac{1}{2}\operatorname{Tr}^s_{\tau}\big(Z_u P_p(u)\big)\rho'^p(u).\end{aligned}$$ ■

Proof of Theorem 4.3 By Proposition 4.5, one calculates

$$\begin{aligned}\frac{\partial}{\partial u}\rho^p_{\mathcal{E}}(u) &= \frac{1}{2}e^{\frac{1}{2}\zeta^{p\,\prime}_u(0,0,\mathcal{E})}\frac{\partial}{\partial u}\zeta^{p\,\prime}_u(0,0,\mathcal{E})\rho'^p(u) + e^{\frac{1}{2}\zeta^{p\,\prime}_u(0,0,\mathcal{E})}\frac{\partial}{\partial u}\rho'^p(u)\\ &= \frac{1}{2}\Big[\frac{\partial}{\partial u}\zeta^{p\,\prime}_u(0,0,\mathcal{E}) - \operatorname{Tr}^s_\tau\big(Z_u P_p(u)\big)\Big]e^{\frac{1}{2}\zeta^{p\,\prime}_u(0,0,\mathcal{E})}\rho'^p(u)\\ &= \frac{1}{2}\Big[\frac{\partial}{\partial u}\zeta^{p\,\prime}_u(0,0,\mathcal{E}) - \operatorname{Tr}^s_\tau\big(Z_u P_p(u)\big)\Big]\rho^p_{\mathcal{E}}(u).\end{aligned}$$

Therefore by Proposition 4.4, one has

$$\frac{\partial}{\partial u}\rho^p_{\mathcal{E}}(u) = c^p_{\mathcal{E}}(u)\rho^p_{\mathcal{E}}(u)$$

where $c^p_{\mathcal{E}}(u) \in \mathbb{R}$ is as in (24). This completes the proof of the theorem. ■

5 Flat Hilbert $\mathcal{A}$-Bundles and Relative Holomorphic L^2 Torsion

In this section, we define the relative holomorphic L^2 torsion with respect to a pair of *flat Hilbert* (unitary) $\mathcal{A}$-bundles $\mathcal{E}$ and $\mathcal{F}$, and we prove that it is independent of the choice of Hermitian metric on the complex manifold. Thus it can be viewed as an invariant volume form on the reduced L^2 cohomology $H^{p,*}(X,\mathcal{E}) \oplus H^{p,*}(X,\mathcal{F})'$. In Section 6, we will prove the relative holomorphic L^2 torsion with respect to a pair of *flat Hibertian* $\mathcal{A}$-bundles $\mathcal{E}$ and $\mathcal{F}$, is independent of the choice of almost Kähler metric on an almost Kähler manifold and on the choice of Hermitian metrics on $\mathcal{E}$ and $\mathcal{F}$.

5.1 Relative Holomorphic L^2 Torsion

It follows from Theorem 4.3 that the holomorphic L^2 torsion is *not* independent of the choice of metrics on the complex manifold and on the flat Hilbertian bundle. Therefore in order to obtain an invariant, we now consider the *relative* holomorphic L^2 torsion for

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a pair of *unitary* flat Hilbertian bundles over a complex manifold. In the next section, we will study the *relative* holomorphic L^2 torsion for an arbitrary pair of flat Hilbertian bundles over a complex manifold.

A *distance function* r on a manifold X is a map $r\colon X \times X \to \mathbb{R}$ such that

(1) Its square $r^2(x, y)$ is smooth on $X \times X$.
(2) $r(x, x) = 0$ and $r(x, y) > 0$ if $x \neq y$.
(3) $\frac{\partial^2}{\partial x_i \partial x_j} r^2(x, y)|_{x=y} = g_{ij}(x)$

Condition (3) says essentially that $r(x, y)$ coincides with the geodesic distance from x to y, whenever x and y are close. One can easily construct such a function using local coordinates and a partition of unity. Let

$$k(t, x, y) = c_1 t^{-n} e^{-c_2 \frac{r^2(x,y)}{t}}, \quad t > 0$$

and c_1, c_2 are some positive constants. Then one has the following basic theorem about the fundamental solution of the heat equation,

Proposition 5.1 *The heat kernel $e^{-t\Box^{\mathcal{E}}_{p,q}}(x, y)$ is a smooth, symmetric double form on X and has the property*

$$\nabla''_x e^{-t\Box^{\mathcal{E}}_{p,q}}(x, y) = \nabla''^*_y e^{-t\Box^{\mathcal{E}}_{p,q}}(x, y). \tag{26}$$

It satisfies the bounds

$$|De^{-t\Box^{\mathcal{E}}_{p,q}}(x, y)| \le c_3 t^{-\frac{1}{2}} k(t, x, y) \tag{27}$$

for $D = \nabla''$ or ∇''^*, x, y close to each other and $0 < t \le 1$. Finally, there is a small time asymptotic expansion

$$e^{-t\Box^{\mathcal{E}}_{p,q}}(x, x) \sim \sum_{j=0}^{\infty} t^{-n+j} C_{j,p,q}(x) \tag{28}$$

as $t \to 0$, where $C_{j,p,q}$ is a smooth double form on X, for all j.

Proof The result is local, and in a local normal coordinate neighborhood of a point $x \in X$, where the bundle $\mathcal{E}$ is also trivialized, one can proceed exactly as in [RS1, Proposition 5.3] (*cf.* [R], [BFKM]). ■

5.2

By Theorem 4.3, we see that the holomorphic L^2 torsion is not necessarily independent of the choice of Hermitian metrics on X and $\mathcal{E} \to X$. We will now study the case when the flat Hilbertian $\mathcal{A}$-bundle $\mathcal{E} \to X$ with fiber M is defined by a *unitary representation* $\pi \to \mathcal{B}_{\mathcal{A}}(M)$, that is, M is a *unitary* Hilbertian $(\mathcal{A} - \pi)$ bimodule. That is,

$$\mathcal{E} \equiv (M \times \tilde{X})/\sim \to X$$

where $(v, x) \sim (vg^{-1}, gx)$ for all $g \in \pi$, $x \in \tilde{X}$ and $v \in M$. The unitary representation defines a flat Hermitian metric h on $\mathcal{E} \to X$. We call such a bundle a *flat Hilbert bundle*, or sometimes a *unitary* flat Hilbertian bundle. Then by definition (*cf.* Definition 4.1), one has

$$\rho^p_{\mathcal{E}}(g, h) \in \det\big(H^{p,*}(X, \mathcal{E})\big).$$

Let $\mathcal{F} \to X$ be another flat Hilbert $\mathcal{A}$ bundle with fibre N, such that $\dim_\tau(M) = \dim_\tau(N)$. Let $\Box^{\mathcal{E}}_{p,q}(u)$ and $\Box^{\mathcal{F}}_{p,q}(u)$ denote the Laplacians in the metric g_u, acting on $\Omega^{p,q}_{(2)}(X, \mathcal{E})$ and $\Omega^{p,q}_{(2)}(X, \mathcal{F})$ respectively. We first prove the following Proposition.

Proposition 5.2 *Let $\mathcal{E}$ and $\mathcal{F}$ be a pair of flat Hilbert bundles over X, as above. Then there are positive constants C_1, C such that*

$$\left|\operatorname{Tr}_\tau\Big(Z_u \exp\big(-t\Box^{\mathcal{E}}_{p,q}(u)\big)\Big) - \operatorname{Tr}_\tau\Big(Z_u \exp\big(-t\Box^{\mathcal{F}}_{p,q}(u)\big)\Big)\right| \le C_1 e^{-\frac{C}{t}}$$

for all $0 < t \le 1$.

Proof Let $x \in X$ and assume that the ball $U_\delta = \{y \in X : r^2(x, y) < \delta\}$ is simply connected, where r is a distance function on X which coincides with the geodesic distance on U_δ. Since the Laplacian is a local operator, it follows that $\Box^{\mathcal{E}}_{p,q}$ acting on $\Omega^{p,q}_{(2)}(X, \mathcal{E})$ over U_δ coincides with $\Box^{\mathcal{F}}_{p,q}$ acting on $\Omega^{p,q}_{(2)}(X, \mathcal{F})$ over U_δ. By Duhamel's Principle and by applying Green's theorem, one has for $x, y \in U_\delta$, one has

$$\begin{aligned} e^{-t\Box^{\mathcal{E}}_{p,q}(u)}(x, y) - e^{-t\Box^{\mathcal{F}}_{p,q}(u)}(x, y) = \int_0^t \int_{r^2(x,z)=\delta} &[e^{-(t-s)\Box^{\mathcal{F}}_{p,q}(u)}(z, y) \wedge *\nabla''^* e^{-s\Box^{\mathcal{E}}_{p,q}(u)}(x, z) \\ &- \nabla''^* e^{-s\Box^{\mathcal{E}}_{p,q}(u)}(x, z) \wedge *e^{-(t-s)\Box^{\mathcal{F}}_{p,q}(u)}(z, y) \\ &- e^{-s\Box^{\mathcal{E}}_{p,q}(u)}(x, z) \wedge *\nabla''^* e^{-(t-s)\Box^{\mathcal{F}}_{p,q}(u)}(z, y) \\ &+ \nabla''^* e^{-(t-s)\Box^{\mathcal{F}}_{p,q}(u)}(z, y) \wedge *e^{-s\Box^{\mathcal{E}}_{p,q}(u)}(x, z)]. \end{aligned}$$

Using the basic estimate (27) for heat kernels, one has

$$|\operatorname{Tr}_\tau(Z_u e^{-t\Box^{\mathcal{E}}_{p,q}(u)}) - \operatorname{Tr}_\tau(Z_u e^{-t\Box^{\mathcal{F}}_{p,q}(u)})| \le c_1 t^{-\frac{1}{2}} e^{-\frac{c_2\delta}{t}} \le C_1 e^{-\frac{C}{t}}$$

for all $0 < t \le 1$. ■

Theorem 5.3 *In the notation of Remark 4.2, if $\mathcal{E}$ and $\mathcal{F}$ are a pair of flat Hilbert bundles over X which are of D-class, then the relative holomorphic L^2 torsion*

$$\rho^p_{\mathcal{E},\mathcal{F}} = \rho^p_{\mathcal{E}} \otimes (\rho^p_{\mathcal{F}})^{-1} \in \det\big(H^{p,*}(X, \mathcal{E})\big) \otimes \det\big(H^{p,*}(X, \mathcal{F})\big)^{-1}$$

is independent of the choice of Hermitian metric on X which is needed to define it.

Proof Let $u \to g_u$ be a smooth family of Hermitian metrics on X and $\Box^{\mathcal{E}}_{p,q}(u)$ and $\Box^{\mathcal{F}}_{p,q}(u)$ denote the Laplacians on $\mathcal{E}$ and $\mathcal{F}$ respectively, as before.

By Proposition 5.2, one has

$$\Big|\operatorname{Tr}_\tau\Big(Z_u \exp\big(-t\Box^{\mathcal{E}}_{p,q}(u)\big)\Big) - \operatorname{Tr}_\tau\Big(Z_u \exp\big(-t\Box^{\mathcal{F}}_{p,q}(u)\big)\Big)\Big| \le C_1 e^{-\frac{C}{t}}$$

as $t \to 0$. That is, $\operatorname{Tr}^s_\tau\Big(Z_u \exp\big(-t\Box^{\mathcal{E}}_{p,q}(u)\big)\Big)$ and $\operatorname{Tr}^s_\tau\Big(Z_u \exp\big(-t\Box^{\mathcal{F}}_{p,q}(u)\big)\Big)$ have the same asymptotic expansion as $t \to 0$. In particular, one has in the notation of Theorem 4.3,

$$c_{\mathcal{E}}(u) = c_{\mathcal{F}}(u).$$

Then the relative holomorphic L^2 torsion

$$\rho^p_{\mathcal{E},\mathcal{F}} \in \det H^{p,*}(X,\mathcal{E}) \otimes \big(\det H^{p,*}(X,\mathcal{F})\big)^{-1}$$
$$\rho^p_{\mathcal{E},\mathcal{F}}(u) = \rho^p_{\mathcal{E}}(u) \otimes \big(\rho^p_{\mathcal{F}}(u)\big)^{-1}$$

satisfies

$$\begin{aligned}\frac{\partial}{\partial u}\rho^p_{\mathcal{E},\mathcal{F}}(u) &= \Big(\frac{\partial}{\partial u}\rho^p_{\mathcal{E}}(u)\Big) \otimes \big(\rho^p_{\mathcal{F}}(u)\big)^{-1} - \rho^p_{\mathcal{E}}(u) \otimes \frac{\partial}{\partial u}\rho^p_{\mathcal{F}}(u) \cdot \rho^p_{\mathcal{F}}(u)^{-2}\\ &= \big(c_{\mathcal{E}}(u) - c_{\mathcal{F}}(u)\big)\rho^p_{\mathcal{E},\mathcal{F}}(u)\\ &= 0\end{aligned}$$

using Theorem 4.3 and the discussion above. This proves the theorem. ∎

6 Determinant Line Bundles, Correspondences and Relative Holomorphic L^2 Torsion

In this section, we introduce the notion of determinant line bundles of Hilbertian $\mathcal{A}$-bundles over compact manifolds. A main result in this section is Theorem 6.6, which says that the holomorphic L^2 torsion associated to a flat Hilbertian bundle over a compact almost Kähler manifold, depends only on the class of the Hermitian metric in the determinant line bundle of the flat Hilbertian bundle. This enables us to show that a correspondence of determinant line bundles is well defined on almost Kähler manifolds. Finally, using such a correspondence of determinant line bundles, we prove in Theorem 6.9 that the relative holomorphic L^2 torsion is independent of the choices of almost Kähler metrics on the complex manifold and Hermitian metrics on the pair of flat Hilbertian bundles over the complex manifold.

Lemma 6.1 *The subgroup* $\mathrm{SL}(M) = \operatorname{Det}_\tau^{-1}(1)$ *of* $\mathrm{GL}(M)$ *is connected.*

Proof Let $U(M)$ denote the subgroup of all unitary elements in $\mathrm{GL}(M)$. Recall the standard retraction of $\mathrm{GL}(M)$ onto $U(M)$, which is given by

$$T_s\colon\ \mathrm{GL}(M) \to \mathrm{GL}(M)$$
$$A \to |A|^s \frac{A}{|A|}$$

where $T_0\colon \mathrm{GL}(M) \to U(M)$ is onto and $T_1 = \text{identity}$. Clearly $U(M) \subset \mathrm{SL}(M)$ and the retraction T_s above restricts to be a retraction of $\mathrm{SL}(M)$ onto $U(M)$. By the results of [ASS], it follows that $\mathrm{SL}(M)$ is connected. ■

Let $\mathcal{E} \to X$ be a Hilbertian $\mathcal{A}$-bundle over X and $\mathrm{GL}(\mathcal{E})$ denote the space of complex A-linear automorphisms of $\mathcal{E}$ which induce the identity map on X, that is, $\mathrm{GL}(\mathcal{E})$ is the gauge group of $\mathcal{E}$. The Fuglede-Kadison determinant, *cf.* Theorem 1.2.

$$\mathrm{Det}_\tau\colon \mathrm{GL}(M) \to \mathbb{R}^+$$

extends to a homomorphism

$$\mathrm{Det}_\tau\colon \mathrm{GL}(\mathcal{E}) \to C^\infty(X, \mathbb{R}^+)$$

where $C^\infty(X, \mathbb{R}^+)$ denotes the space of smooth positive functions on X. This extension has all the properties listed in Theorem 1.2. Using the long exact sequence in homotopy and the lemma above, one has

Corollary 6.2 *Let $\mathcal{E} \to X$ be a Hilbertian $\mathcal{A}$-bundle over X (recall that X is assumed to be connected). Then the subgroup $\mathrm{SL}(\mathcal{E}) = \mathrm{Det}_\tau^{-1}(1)$ of $\mathrm{GL}(\mathcal{E})$ is connected.*

6.1 Determinant Line Bundles

Let $\mathcal{E} \to X$ be a Hilbertian $\mathcal{A}$-bundle over X. Then we can define a natural *determinant line bundle* of $\mathcal{E}$ as follows:

Let $\mathrm{Herm}(\mathcal{E})$ denote the space of all Hermitian metrics on $\mathcal{E}$. Clearly $\mathrm{Herm}(\mathcal{E})$ is a convex set and $\mathrm{GL}(\mathcal{E})$ acts on $\mathrm{Herm}(\mathcal{E})$ by

$$\mathrm{GL}(\mathcal{E}) \times \mathrm{Herm}(\mathcal{E}) \to \mathrm{Herm}(\mathcal{E})$$
$$(a, h) \to \bar{a}^t ha.$$

That is, $(a.h)_x(v, w) = h_x(av, aw)$ for all $v, w \in \mathcal{E}_x$.

The action of $\mathrm{GL}(\mathcal{E})$ on $\mathrm{Herm}(\mathcal{E})$ is transitive, that is, one can identify $\mathrm{Herm}(\mathcal{E})$ with the quotient

$$\mathrm{GL}(\mathcal{E})\big/U(\mathcal{E}, h_0) \quad \text{where } U(\mathcal{E}, h_0)$$

is the subgroup of $\mathrm{GL}(\mathcal{E})$ which leaves $h_0 \in \mathrm{Herm}(\mathcal{E})$ invariant, that is, $U(\mathcal{E}, h_0)$ is the unitary transformations with respect to h_0.

For a Hilbertian bundle $\mathcal{E}$ over X, we define $\det(\mathcal{E})$ to be the real vector space generated by the symbols h, one for each Hermitian metric on $\mathcal{E}$, subject to the following relations: for any pair h_1, h_2 of Hermitian metrics on $\mathcal{E}$, we write the following relation

$$h_2 = \sqrt{\mathrm{Det}_\tau(A)}^{-1} h_1$$

where $A \in \mathrm{GL}(\mathcal{E})$ is positive, self-adjoint and satisfies

$$h_2(v, w) = h_1(Av, w)$$

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for all $v, w \in \mathcal{E}_x$.

Assume that we have three different Hermitian metrics h_1, h_2 and h_3 on $\mathcal{E}$.

Suppose that

$$h_2(v,w) = h_1(Av,w) \quad \text{and} \quad h_3(v,w) = h_2(Bv,w)$$

for all $v, w \in \mathcal{E}_x$ and $A, B \in \mathrm{GL}(\mathcal{E})$. Then $h_3(v,w) = h_1(ABv,w)$a and we have the following relations in $\det(\mathcal{E})$,

$$h_2 = \sqrt{\mathrm{Det}_\tau(A)}^{-1} h_1$$
$$h_3 = \sqrt{\mathrm{Det}_\tau(B)}^{-1} h_2$$
$$h_3 = \sqrt{\mathrm{Det}_\tau(AB)}^{-1} h_1.$$

The third relation follows from the first two, from which it follows that $\det(\mathcal{E})$ is a line bundle over X.

To summarize, $\det(\mathcal{E})$ is a real line bundle over X, which has nowhere zero sections h, where h is any Hermitian metric on $\mathcal{E}$. It has a canonical orientation, since the transition functions $\sqrt{\mathrm{Det}_\tau(A)}^{-1}$ are always positive.

Non zero elements of $\det(\mathcal{E})$ should be viewed as volume forms on $\mathcal{E}$.

For *flat* Hilbertian $\mathcal{A}$ bundles, the determinant line bundle can be described in the following alternate way.

Then $\mathcal{E} = M \times_\rho \tilde{X}$, where $\rho\colon \pi \to \mathrm{GL}(M)$ is a representation. The associated *determinant line bundle* is defined as

$$\det \mathcal{E} = \det(M) \times_{\mathrm{Det}_\tau(\rho)} \tilde{X}.$$

Here $\mathrm{Det}_\tau(\rho)\colon \pi \to \mathbb{R}^+$ is a representation which is defined as

$$\mathrm{Det}_\tau(\rho)(\gamma) = \mathrm{Det}_\tau\big(\rho(\gamma)\big)$$

for $\gamma \in \pi$. Then $\det(\mathcal{E})$ has the property that

$$\det(\mathcal{E})_x = \det(\mathcal{E}_x) \quad \forall x \in X.$$

Clearly $\det(\mathcal{E})$ coincides with the construction given in the beginning of Section 6.1, and $\det(\mathcal{E})$ is a *flat* real line bundle over X.

6.2 Almost Kähler Manifolds

A Hermitian manifold (X, g) is said to be *almost Kähler* if the Kähler 2-form ω is not necessarily closed, but instead satisfies the weaker condition $\bar\partial\partial\omega = 0$. Gauduchon (*cf.* [Gau]) proved that every complex manifold of real dimension less than or equal to 4, is almost Kähler.

Let ∇^B denote the holomorphic Hermitian connection on TX with the torsion tensor T^B and curvature tensor R^B. Define the smooth 3-form B by

$$B(U,V,W) = \big(T^B(U,V),W\big)$$

for all $U,V,W \in TX$. Let ω denote the Kähler 2-form on X. Then one has

$$B = i(\partial - \bar{\partial})\omega.$$

Since X is almost Kähler, it follows that B is closed and therefore the following curvature identity holds

$$\big(R^B(U,V)W,Z\big) = \big(R^{-B}(Z,W)V,U\big)$$

for all $U,V,W,Z \in TX$, where R^{-B} denotes the curvature of the holomorphic Hermitian connection ∇^{-B} on TX with the torsion tensor $T^{-B} = -T^B$ *cf.* [Bi]. The Dolbeault operator $\sqrt{2}(\nabla'' + \nabla''^*)$ is a Dirac type operator. More precisely, let $\Lambda = (\det T''^0 X)^{\frac{1}{2}}$ and $\mathcal{S}$ denote the bundle of spinors on X, then as $\mathbb{Z}_2$ graded bundles on X, one has

$$\Lambda^{p,*}T^*X \otimes \mathcal{E} = \mathcal{S} \otimes \Lambda \otimes \Lambda^{p,0}T^*X \otimes \mathcal{E}.$$

Let ∇^L denote the Levi-Civita connection on X and $\mathcal{D}^L$ the Dirac operator with respect to this connection. Then using the connection ∇^B on Λ and $\Lambda^{p,0}T^*X$, the Dirac operator $\mathcal{D}^L$ extends as an operator

$$\mathcal{D}^L\colon \Gamma(X,\mathcal{S}^+ \otimes \Lambda \otimes \Lambda^{p,0}T^*X \otimes \mathcal{E}) \to \Gamma(X,\mathcal{S}^- \otimes \Lambda \otimes \Lambda^{p,0}T^*X \otimes \mathcal{E})$$

and one has the formula

$$\sqrt{2}(\nabla'' + \nabla''^*) = \mathcal{D}^L - \frac{1}{4}c(B) = \mathcal{D}^L + \frac{1}{2}\sum_{i=1}^{n} c\big(S(e_i)e_i\big)$$

where $c(B)$ denotes Clifford multiplication by the 3-form B and $S = \nabla^B - \nabla^L$ is a 1-form on X with values in skew-Hermitian endomorphisms of TX. We now work in a local normal coordinate ball, where we trivialize the bundles using parallel transport along geodesics. Scale the metric on X by r^{-1} and let I_r denote the operator $2\Box_{p,*} = \big(\sqrt{2}(\nabla'' + \nabla''^*)\big)^2$ in this scaled metric. In local normal coordinates, one has the following expression for I_r (*cf.* [Bi])

$$\begin{aligned}
I_r = -rg^{ij}&\left(\partial_i + \frac{1}{4}\Gamma_{iab}c(e_a \wedge e_b) + A_i + \frac{1}{2\sqrt{r}}c\big(S_{il\alpha}(e_l)e(f_\alpha)\big) + \frac{1}{4r}S_{i\beta\gamma}e(f_\beta \wedge f_\gamma)\right)\\
&\times\left(\partial_j + \frac{1}{4}\Gamma_{jab}c(e_a \wedge e_b) + A_j + \frac{1}{2\sqrt{r}}S_{jl\alpha}c(e_l)e(f_\alpha) + \frac{1}{4r}S_{j\beta\gamma}e(f_\beta \wedge f_\gamma)\right)\\
&+\frac{1}{4}rk - \frac{1}{2}rc(e_i \wedge e_j)L_{ij} - \frac{1}{2}e(f_\alpha \wedge f_\beta)L_{\alpha\beta} - \sqrt{r}c(e_i)e(f_\alpha \wedge L_{i\alpha})\\
&+rg^{ij}\Gamma^k_{ij}\left(\partial_k + \frac{1}{4}\Gamma_{kab}c(e_a \wedge e_b) + A_k + \frac{1}{2\sqrt{r}}S_{kl\alpha}c(e_l)e(f_\alpha) + \frac{1}{4r}S_{k\beta\gamma}e(f_\beta \wedge f_\gamma)\right)
\end{aligned}$$

where k denotes the scalar curvature of X.

Consider the heat equation on sections of $\mathcal{S} \otimes \Lambda \otimes \Lambda^{p,0} T^* X \otimes \mathcal{E}$,

$$(\partial_t + I_r) g(x,t) = 0$$
$$g(x,0) = g(x).$$

By parabolic theory, there is a fundamental solution $e^{-tI_r}(x,y)$ which is smooth for $t > 0$. We will consider the case when $t = 1$, $e^{-I_r}(x,y)$ and prove the existence of an asymptotic expansion on the diagonal, as $r \to 0$. A difficulty arises because of the singularities arising in the coefficients of I_r, as $r \to 0$.

Proposition 6.3 *For some positive integer $p \geq n$, one has the following asymptotic expansion as $r \to 0$,*

$$e^{-I_r}(x,x) \sim r^{-p} \sum_{i=0}^{\infty} r^i E_i(x,x)$$

where E_i are endomorphisms of $\mathcal{S} \otimes \Lambda \otimes \Lambda^{p,0} T^* X \otimes \mathcal{E}$.

Proof Consider the operator

$$\begin{aligned} J_r = -r g^{ij} \Big(\delta_i + \frac{1}{4} \Gamma_{iab} c(e_a \wedge e_b) + A_i \Big) \times \Big(\partial_j + \frac{1}{4} \Gamma_{jab} c(e_a \wedge e_b) + A_j \Big) \\ + r g^{ij} \Gamma_{ij}^k \Big(\partial_k + \frac{1}{4} \Gamma_{kab} c(e_a \wedge e_b) + A_k \Big) + \frac{1}{4} r k - \frac{1}{2} c(e_i \wedge e_j) L_{ij}. \end{aligned}$$

Since J_r has no singular terms as $r \to 0$, it has a well known asymptotic expansion, as $r \to 0$ with $p = n$.

We can construct $\exp(-I_r)$ as a perturbation of $\exp(-J_r)$, using Duhamel's principle. More precisely,

$$\exp(-I_r) = \exp(-J_r) + \sum_{k=1}^{\infty} \underbrace{e^{-J_r}(J_r - I_r) e^{-J_r} \cdots e^{-J_r}}_{k \text{ terms}}.$$

Each coefficient in the difference $J_r - I_r$ contains at least one term which is exterior multiplication by f_α. Therefore the infinite series on the right hand side collapses to a finite number of terms. The proposition then follows from the asymptotic expansion for $\exp(-J_r)(x,x)$. ∎

Let R^B denote the curvature of the holomorphic Hermitian connection and R^L denote the curvature of the Levi-Civita connection. Let $\hat{A}$ denote $\hat{A}$-invariant polynomial and ch the Chern character invariant polynomial. Then

$$\hat{A}(R^{-B}) \operatorname{ch}\big(\operatorname{Tr}(R^L)\big) \operatorname{ch}(\Lambda^{p,0} R^L) \in \Lambda^* T^* X.$$

The goal is to prove the following decoupling result in the adiabatic limit. It resembles the local index theorem for almost Kähler manifolds by Bismut [Bi] (he calls them non-Kähler manifolds). However, we use instead the techniques of the proofs in [BGV], [Ge] and [D] of the local index theorem for families. In particular, we borrow a local conjugation trick due to Donnelly [D], which is adjusted to our situation.

Theorem 6.4 **(Adiabatic decoupling)** *Let (X, g) be an almost Kähler manifold. In the notation above, one has the following decoupling result in the adiabatic limit*

$$\lim_{r\to 0} \operatorname{Tr}^s_\tau\big(Z_u e^{-I_r}(x,x)\big) = \operatorname{Tr}_\tau(Z_u)(x)\big[\hat{A}(R^{-B})\operatorname{ch}\big(\operatorname{Tr}(R^L)\big)\operatorname{ch}(\Lambda^{p,0}R^L)\big]^{\max}_x \in \Lambda^{2n}T^*_x X$$

for all $x \in X$.

Proof We first consider the corresponding problem on $\mathbb{R}^{2n}$, using the exponential map. Let $\bar{I}_r$ denote the operator on $\mathbb{R}^{2n}$, whose expressions agrees with the local coordinate expression for I_r near p, where p is identified with the origin in $\mathbb{R}^{2n}$.

Consider the heat equation on $\mathbb{R}^{2n}$,

$$(\partial_t + \bar{I}_r)g(x,t) = 0$$
$$g(x,0) = g(x).$$

Then one has ■

Proposition 6.5 *There is a unique fundamental solution $e^{-t\bar{I}_r}(x,y)$ which satisfies the decay estimate*

$$|e^{-t\bar{I}_r}(x,y)| \le c_1 t^{-n} e^{-\frac{c_2|x-y|^2}{t}}$$

as $t \to 0$, with similar estimates for the derivatives in x, y, t.

Proof The proof is standard, as in [D], [RS]. ■

By Duhamel's principle applied in a small enough normal coordinate neighborhood, there is a positive constant c such that

$$e^{-\bar{I}_r}(0,0) = e^{-I_r}(x,x) + O(e^{-c/r}) \quad \text{as } r \to 0.$$

Therefore

$$\lim_{r\to 0} \operatorname{Tr}^s_\tau\big(Z_u e^{-I_r}(x,x)\big) = \lim_{r\to 0} \operatorname{Tr}^s_\tau\big(Z_u e^{-\bar{I}_r}(0,0)\big) \tag{29}$$

and it suffices to compute the right hand side of (29). This is done using Getzler's scaling idea [Ge], $x \to \epsilon x$, $t \to \epsilon^2 t$ $e_i \to \epsilon^{-1}e_i$. Then Clifford multiplication scales as $c_\epsilon(\cdot) = e(\cdot) + \epsilon^2 i(\cdot)$, where $e(\cdot)$ denotes exterior multiplication by the covector $\cdot$ and $i(\cdot)$ denotes contraction by the dual vector.

$$\begin{aligned}\bar{I}_\epsilon = -rg^{ij}(\epsilon x)\bigg(&\partial_i + \frac{\epsilon^{-1}}{4}\Gamma_{iab}(\epsilon x)c_\epsilon(e_a\wedge e_b) + \epsilon A_i(\epsilon x)\\ &+ \frac{\epsilon^{-1}}{2\sqrt{r}}c_\epsilon\big(S_{il\alpha}(\epsilon x)e_i\big)e(f_\alpha) + \frac{\epsilon^{-1}}{4r}S_{i\beta\gamma}(\epsilon x)c(f_\beta\wedge f_\beta)\bigg)\\ \times\bigg(&\partial_j + \frac{\epsilon^{-1}}{4}\Gamma_{jab}(\epsilon x)c_\epsilon(e_a\wedge e_b) + \epsilon A_j(\epsilon x)\end{aligned}$$

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$$+ \frac{\epsilon^{-1}}{2\sqrt{r}} c_\epsilon \big(S_{jl\alpha}(\epsilon x) e_l \big) e(f_\alpha) + \frac{\epsilon^{-1}}{4r} S_{j\beta\gamma}(\epsilon x) e(f_\beta \wedge f_\gamma) \Big)$$
$$+ r g^{ij}(\epsilon x) \Gamma^k_{ij}(\epsilon x) \Big(\epsilon \partial_k + \frac{1}{4} \Gamma_{kab}(\epsilon x) c_\epsilon(e_a \wedge e_b) + \epsilon^2 A_k(\epsilon x)$$
$$+ \frac{1}{2\sqrt{r}} S_{kl\alpha}(\epsilon x) c_\epsilon(e_i) e(f_\alpha) + \frac{1}{4r} S_{k\beta\gamma}(\epsilon x) e(f_\beta \wedge f_\gamma) \Big)$$
$$+ \frac{\epsilon^2}{4} r k(\epsilon x) - \frac{r}{2} c_\epsilon(e_i \wedge e_j) L_{ij}(\epsilon x) - \frac{1}{2} f_\alpha \wedge f_\beta \wedge L_{\alpha\beta}(\epsilon x) - \sqrt{r} c_\epsilon(e_i) f_\alpha L_{i\alpha}(\epsilon x).$$

The asymptotic expansion in r as in Propositions 6.3 and 6.5, for $e^{-\bar{I}_r}(0,0)$ yields an asymptotic expansion in ϵ for $e^{-\bar{I}_\epsilon}(0,0)$ and one has

$$\lim_{r\to 0} \operatorname{Tr}^s_\tau \big(Z_u e^{-\bar{I}_r}(0,0) \big) = \lim_{\epsilon \to 0} \operatorname{Tr}^s_\tau \big(Z_u e^{-\bar{I}_\epsilon}(0,0) \big). \tag{30}$$

That is, if either limit exists, then both exist and are equal.

However, in the limit as $\epsilon \to 0$, there are singularities in the coefficients of S tensor in the expression for $\bar{I}_\epsilon$ and one cannot immediately apply Getzler's theorem. Therefore one first makes the following local conjugation trick, as in Donnelly [D].

Define the expression

$$h(x, \epsilon, r) = \exp\Big(\frac{\epsilon^{-1}}{2\sqrt{r}} S_{il\alpha}(0) x_i e_l \wedge f_\alpha + \frac{\epsilon^{-1}}{4r} S_{i\beta\gamma}(0) x_i f_\beta \wedge f_\gamma \Big).$$

Note that $h(x,\epsilon,r)$ has polynomial growth in x, since its expression contains exterior multiplication. We claim that if the operator $\bar{I}_\epsilon$ is conjugated by h, then the resulting operator is *not* singular as $\epsilon \to 0$. More precisely,

$$\begin{aligned} J_\epsilon &= h \bar{I}_\epsilon h^{-1} \\ &= r g^{ij}(\epsilon x) \Big(\partial_i + \frac{\epsilon^{-1}}{4} \Gamma_{iab}(\epsilon x) e_a \wedge e_b + \frac{\epsilon^{-1}}{2\sqrt{r}} \big(S_{il\alpha}(\epsilon x) - S_{il\alpha}(0) \big) e_l \wedge f_\alpha \\ &\qquad + \frac{\epsilon^{-1}}{4r} \big(S_{i\beta\gamma}(\epsilon x) - S_{i\beta\gamma}(0) \big) f_\beta \wedge f_\gamma - \frac{1}{4r} S_{il\alpha}(0) S_{kl\beta}(0) x_k f_\alpha \wedge f_\beta \Big) \\ &\quad \times \Big(\partial_j + \frac{\epsilon^{-1}}{4} \Gamma_{jab}(\epsilon x) e_a \wedge e_b \\ &\qquad + \frac{\epsilon^{-1}}{2\sqrt{r}} \big(S_{jl\alpha}(\epsilon x) - S_{i\beta\gamma}(0) \big) e_l \wedge f_\alpha + \frac{\epsilon^{-1}}{4r} \big(S_{j\beta\gamma}(\epsilon x) - S_{j\beta\gamma}(0) \big) f_\beta \wedge f_\gamma \\ &\qquad - \frac{1}{4r} S_{jl\alpha}(0) S_{kl\beta}(0) x_k f_\alpha \wedge f_\beta \Big) - \frac{1}{2} r e_i \wedge e_j L_{ij}(\epsilon x) \\ &\quad - \frac{1}{2} f_\alpha \wedge f_\beta L_{\alpha\beta}(\epsilon x) - \sqrt{r} e_i \wedge f_\alpha L_{i\alpha}(\epsilon x) + R(x, \epsilon). \end{aligned} \tag{31}$$

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Here $R(x,\epsilon)$ denotes the terms which vanish as $\epsilon \to 0$, and which therefore do not contribute to the limit. Clearly there are no singular terms in J_ϵ as $\epsilon \to 0$.

A fundamental solution for the heat equation for J_ϵ can be obtained by conjugating the one for $\bar{I}_\epsilon$, that is

$$e^{-tJ_\epsilon}(x,y) = h(x,\epsilon,r)e^{-t\bar{I}_\epsilon}(x,y)h^{-1}(y,\epsilon,r).$$

The right hand side satisfies the heat equation $(\partial_t + J_\epsilon)g(x,t) = 0$, $g(x,0) = \delta_x$. Since $h(0) = 1$, one has $\forall \epsilon > 0$,

$$\operatorname{Tr}^s_\tau\big(Z_u e^{-\bar{I}_\epsilon}(0,0)\big) = \operatorname{Tr}^s_\tau\big(Z_u e^{-J_\epsilon}(0,0)\big). \tag{32}$$

Therefore it suffices to compute the limit as $\epsilon \to 0$ of the right hand side of (32).

Using the following Taylor expansions in a normal coordinate neighborhood,

$$\Gamma_{iab}(\epsilon x) = -\frac{1}{2}R_{ijab}(0)\epsilon x_j + R(x,\epsilon^2)$$
$$S_{il\alpha}(\epsilon x) = S_{il\alpha}(0) + S_{il\alpha,j}(0)\epsilon x_j + R(x,\epsilon^2)$$
$$S_{i\beta\gamma}(\epsilon x) = S_{i\beta\gamma}(0) + S_{i\beta\gamma,j}(0)\epsilon x_j + R(x,\epsilon^2)$$

one sees that

$$J_0 = \lim_{\epsilon\to 0} J_\epsilon = -r\sum_i\Big(\partial_i - \frac{1}{4}B_{ij}x_j\Big)^2 + r\mathcal{L}$$

where

$$B_{ij} = \frac{1}{2}R_{ijab}(0)e_a \wedge e_b - \frac{2}{\sqrt{r}}S_{il\alpha,j}(0)e_l \wedge f_\alpha$$
$$- \frac{1}{r}\big(S_{i\beta\gamma,j}(0) - S_{il\beta}(0)S_{jl\gamma}(0)\big) f_\beta \wedge f_\gamma$$

and

$$\mathcal{L} = \frac{1}{2}L_{ij}(0)e_i \wedge e_j + \frac{1}{\sqrt{r}}L_{i\alpha}(0)e_i \wedge f_\alpha + \frac{1}{2r}L_{\alpha\beta}(0)f_\alpha \wedge f_\beta.$$

Using Mehler's formula (*cf.* [Ge]), one can obtain an explicit fundamental solution $e^{-sJ_0}(x,y)$. First decompose B into its symmetric and skew symmetric parts, that is $B = C + D$ where $C = \frac{1}{2}(B + B^t)$ and $D = \frac{1}{2}(B - B^t)$, where B, C, D are matrices of 2-forms. Then

$$e^{-J_0}(x,0) = (4\pi r)^{-n/2}\hat{A}(rD)e^{\frac{x^tCx}{8}} \times \exp\bigg(r\mathcal{L} - \frac{1}{4r}x^t\Big(\frac{rD/2}{\tanh(rD/2)}\Big)x\bigg).$$

Now $\lim_{\epsilon\to 0} e^{-J_\epsilon}(0,0) = e^{-J_0}(0,0)$. Therefore

$$\lim_{\epsilon\to 0}\operatorname{Tr}^s_\tau\big(Z_u e^{-J_\epsilon}(0,0)\big) = \Big(\frac{2}{i}\Big)^{n/2}(4\pi r)^{-n/2}\operatorname{Tr}_\tau(Z_u)(0)\big[\hat{A}(rD)\operatorname{ch}(r\mathcal{L})\big]^{\max}. \tag{33}$$

Here $D = R^{-B}(0)$ and $\mathcal{L} = \operatorname{Tr}\big(R^L(0)\big) + \Lambda^{p,0}R^L(0)$. Using (29), (30), (32) and (33), one completes the proof of Theorem 6.4. ■

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Theorem 6.6 *Let $\mathcal{E}$ be a flat Hilbertian bundle of D-class, over an almost Kähler manifold (X, g) and let h, h' be Hermitian metrics on $\mathcal{E}$ such that $h = h'$ in* $\det(\mathcal{E})$. *Then*

$$\rho^p_{\mathcal{E}}(g, h) = \rho^p_{\mathcal{E}}(g, h') \in \det\big(H^{p,*}(X, \mathcal{E})\big).$$

Proof Since $h = h'$ in $\det(\mathcal{E})$, there is a positive, self-adjoint bundle map $A\colon \mathcal{E} \to \mathcal{E}$ satisfying

$$h(Av, w) = h'(v, w) \quad \forall v, w \in \mathcal{E} \quad \text{and} \quad \mathrm{Det}_\tau(A) = 1.$$

By Corollary 6.2, there is a smooth 1-parameter family of positive, self-adjoint bundle maps $u \to A_u\colon \mathcal{E} \to \mathcal{E}$ joining A to the identity and satisfying

$$\mathrm{Det}_\tau(A_u) = 1. \tag{34}$$

for all $u \in (-\epsilon, 1 + \epsilon)$. Here $A_0 = I$ and $A_1 = A$. Let $u \to h_u$ be a smooth family of Hermitian metrics on $\mathcal{E}$ defined by

$$h(A_u v, w) = h_u(v, w) \quad \forall v, w \in \mathcal{E}.$$

Then $h_0 = h$, $h_1 = h'$ in $\mathcal{E}$ and $h = h_u$ in $\det(\mathcal{E})$ for all $u \in (-\epsilon, 1 + \epsilon)$ by (72). Note that by differentiating (34), one has

$$0 = \frac{\partial}{\partial u}\mathrm{Det}_\tau(A_u) = \mathrm{Tr}_\tau(Z_u) \tag{35}$$

where $Z_u = A_u^{-1}\dot{A_u}$.

We wish to compute $\frac{\partial}{\partial u}\rho^p_{\mathcal{E}}(g, h_u)$. By Theorem 4.3, one has

$$\frac{\partial}{\partial u}\rho^p_{\mathcal{E}}(g, h_u) = c^p_{\mathcal{E}}(g, h_u)\rho^p_{\mathcal{E}}(g, h_u).$$

By Theorem 6.4 and (35), one sees that

$$\lim_{t \to 0} \mathrm{Tr}^s_\tau(Z_u e^{-t\Box(u)}) = 0. \tag{36}$$

By the small time asymptotic expansion of the heat kernel, one has

$$\begin{aligned}\lim_{t \to 0} \mathrm{Tr}^s_\tau(Z_u e^{-t\Box(u)}) &= \sum_{q=0}^{n} (-1)^q m_{n,p,q}(u) \\ &= c^p_{\mathcal{E}}(g, h_u).\end{aligned} \tag{37}$$

Therefore by (36) and (37), one has $c^p_{\mathcal{E}}(g, h_u) = 0$, that is,

$$\frac{\partial}{\partial u}\rho^p_{\mathcal{E}}(g, h_u) = 0.$$

■

Remark 6.7 Theorem 6.6 says that on an almost Kähler manifold (X, g), the holomorphic L^2 torsion $\rho^p_{\mathcal{E}}(g, h)$ depends only on the equivalence class of the Hermitian metric h in $\det(\mathcal{E})$. We do not believe that the almost Kähler hypothesis in Theorem 6.6 is necessary. However, we use the techniques of the proof of the local index theorem, and the situation to date is that the local index theorem for the operator $\bar{\partial} + \bar{\partial}^*$ has not yet been established for a general Hermitian manifold.

6.3

Let $\mathcal{E}$ and $\mathcal{F}$ be two flat Hilbertian bundles of D-class over an almost Kähler manifold (X, g) and $\varphi\colon \det(\mathcal{E}) \to \det(\mathcal{F})$ be an isomorphism of the determinant line bundles. Then using the theorem above, we will construct a canonical isomorphism between determinant lines

$$\hat{\varphi}^p\colon \det H^{p,*}(X, \mathcal{E}) \to \det H^{p,*}(X, \mathcal{F})$$
$$\hat{\varphi}^p\big(\lambda\rho^p_{\mathcal{E}}(g, h)\big) = \lambda\rho^p_{\mathcal{F}}(g, h'), \quad \lambda \in \mathbb{R}$$

where h and h' are Hermitian metrics on $\mathcal{E}$ and $\mathcal{F}$ respectively, such that $\varphi(h) = h'$ in $\det(\mathcal{F})$. Then $\hat{\varphi}$ is called a *correspondence* between determinant line bundles. It is well defined by Theorem 6.6 and Remark 6.7. We next state some obvious properties of correspondences.

Proposition 6.8 *Let $\mathcal{E}$ be a flat Hilbertian bundle of D-class over an almost Kähler manifold (X, g) and $\varphi\colon \det(\mathcal{E}) \to \det(\mathcal{E})$ be the identity map. Then*

$$\hat{\varphi}^p = \text{identity}\,.$$

Let $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ be flat Hilbertian bundles of D-class over an almost Kähler manifold (X, g) and $\varphi\colon \det(\mathcal{E}) \to \det(\mathcal{F})$, $\psi\colon \det(\mathcal{F}) \to \det(\mathcal{G})$ be isomorphisms of the determinant line bundles. Then the composition satisfies

$$\widehat{\varphi \circ \psi}^p = \hat{\varphi}^p \circ \hat{\psi}^p.$$

We next prove one of the main results in the paper.

Theorem 6.9 *Let $\mathcal{E}$ and $\mathcal{F}$ be two flat Hilbertian bundles of D-class over over an almost Kähler manifold (X, g) and $\varphi\colon \det(\mathcal{E}) \to \det(\mathcal{F})$ be an isomorphism of the corresponding determinant line bundles. Consider smooth 1-parameter families of almost Kähler metrics g_u on X and Hermitian metrics $h_{1,u}$ on $\mathcal{E}$, where u varies in an internal $(-\epsilon, \epsilon)$. Choose a smooth family of Hermitian metrics $h_{2,u}$ on $\mathcal{F}$ in such a way that $\varphi(h_{1,u}) = h_{2,u}$ in $\det(\mathcal{F})$. Then the relative holomorphic torsion*

$$\rho^p_\varphi(u) = \rho^p_{\mathcal{E}}(g_u, h_{1,u}) \otimes \rho^p_{\mathcal{F}}(g_u, h_{2,u})^{-1} \in \det H^{p,*}(X, \mathcal{E}) \otimes \det H^{p,*}(X, \mathcal{F})^{-1}$$

is a smooth function of u and satisfies $\frac{\partial}{\partial u}\rho_\varphi(u) = 0$. That is, the relative holomorphic L^2 torsion ρ^p_φ is independent of the choices of metrics on X, $\mathcal{E}$ and $\mathcal{F}$ which are needed to define it.

Proof From the data in the theorem, one can define a correspondence as in Section 6.3,

$$\hat{\varphi}^p\colon \det\big(H^{p,*}(X, \mathcal{E})\big) \to \det\big(H^{p,*}(X, \mathcal{F})\big)$$

which is an isomorphism of determinant lines. It is defined as

$$\hat{\varphi}^p\big(\lambda\rho^p_{\mathcal{E}}(g_u, h_{1,u})\big) = \lambda\rho^p_{\mathcal{F}}(g_u, h_{2,u}) \tag{38}$$

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for $\lambda \in \mathbb{R}$ and $u \in (-\epsilon, \epsilon)$. Therefore using Theorem 4.3 and (38) above, one has

$$
\begin{aligned}
\frac{\partial}{\partial u}\hat{\varphi}^p\big(\rho^p_{\mathcal{E}}(g_u, h_{1,u})\big) &= \hat{\varphi}^p\Big(\frac{\partial}{\partial u}\rho^p_{\mathcal{E}}(g_u, h_{1,u})\Big) \\
&= c_{\mathcal{E}}(g_u, h_{1,u})\hat{\varphi}^p\big(\rho^p_{\mathcal{E}}(g_u, h_{1,u})\big) \\
&= c_{\mathcal{E}}(g_u, h_{1,u})\rho^p_{\mathcal{F}}(g_u, h_{2,u}).
\end{aligned} \tag{39}
$$

But by differentiating equation (38) above, one has

$$
\begin{aligned}
\frac{\partial}{\partial u}\hat{\varphi}\big(\rho^p_{\mathcal{E}}(g_u, h_{1,u})\big) &= \frac{\partial}{\partial u}\rho^p_{\mathcal{F}}(g_u, h_{2,u}) \\
&= c_{\mathcal{F}}(g_u, h_{2,u})\rho^p_{\mathcal{F}}(g_u, h_{2,u}).
\end{aligned} \tag{40}
$$

Equating (39) and (40), one has

$$
c_{\mathcal{E}}(g_u, h_{1,u}) = c_{\mathcal{F}}(g_u, h_{2,u}). \tag{41}
$$

Then the relative holomorphic L^2 torsion

$$
\begin{aligned}
\rho^p_{\varphi} &\in \det H^{p,*}(X, \mathcal{E}) \otimes \big(\det H^{p,*}(X, \mathcal{F})\big)^{-1} \\
\rho^p_{\varphi}(u) &= \rho^p_{\mathcal{E}}(g_u, h_{1,u}) \otimes \big(\rho^p_{\mathcal{F}}(g_u, h_{2,u})\big)^{-1}
\end{aligned}
$$

satisfies

$$
\begin{aligned}
\frac{\partial}{\partial u}\rho^p_{\varphi}(u) &= \Big(\frac{\partial}{\partial u}\rho^p_{\mathcal{E}}(g_u, h_{1,u})\Big) \otimes \big(\rho^p_{\mathcal{F}}(g_u, h_{2,u})\big)^{-1} \\
&\qquad - \rho^p_{\mathcal{E}}(g_u, h_{1,u}) \otimes \frac{\partial}{\partial u}\rho^p_{\mathcal{F}}(g_u, h_{2,u}) \cdot \rho^p_{\mathcal{F}}(g_u, h_{2,u})^{-2} \\
&= c_{\mathcal{E}}(g_u, h_{1,u})\rho^p_{\varphi}(u) - c_{\mathcal{F}}(g_u, h_{2,u})\rho^p_{\varphi}(u) \\
&= 0
\end{aligned}
$$

using Theorem 4.3 and (41) above. This proves the theorem. ■

7 Calculations

In this section, we calculate the holomorphic L^2 torsion for Kähler locally symmetric spaces. We do this within a wider framework which enables us, at the same time, to indicate an extension to the situation where one deals with a family of operators.

We will restrict ourselves to the special case of the Hilbert $\big(\mathcal{U}(\Gamma) - \Gamma\big)$-bimodule $\ell^2(\Gamma)$, where Γ is a countable discrete group. Let $\mathcal{E} \to X$ denote the associated flat Hilbert $\mathcal{U}(\Gamma)$-bundle over the compact complex manifold X. Then it is well known that the Hilbert $\mathcal{U}(\Gamma)$-complexes $\big(\Omega^{\bullet,\bullet}_{(2)}(X, \mathcal{E}), \nabla''\big)$ and $\big(\Omega^{\bullet,\bullet}_{(2)}(\tilde{X}), \bar{\partial}\big)$ are canonically isomorphic, where $\Gamma \to \tilde{X} \to X$ denotes the universal covering space of X with structure group Γ. We will denote the $\bar{\partial}$-Laplacian acting on $\Omega^{p,q}_{(2)}(\tilde{X})$ by $\square_{p,q}$.

Firstly, we will discuss the D-class condition in this case. Let X be a Kähler hyperbolic manifold. Recall that this means that X is a Kähler manifold with Kähler form ω, which has the property that $p^*(\omega) = d\eta$, where $\Gamma \to \tilde{X} \to X$ denotes the universal cover of X and η is a bounded 1-form on $\tilde{X}$. Any Riemannian manifold of negative sectional curvature, which also supports a Kähler metric, is a Kähler hyperbolic manifold. Note that the Kähler metric is not assumed to be compatible with the Riemannian metric of negative sectional curvature. Then Gromov [G] proved that on the universal cover of a Kähler hyperbolic manifold, the Laplacian $\square_{p,q}$ has a spectral gap at zero on all L^2 differential forms. Therefore it follows that the associated flat bundle $\mathcal{E} \to X$ is of D-class. By a vanishing theorem of Gromov [G] for the L^2 Dolbeault cohomology of the universal cover, one has

$$H^{p,q}_{(2)}(\tilde{X}) = 0$$

unless $p + q = n$, where n denotes the complex dimension of X.

Again following Gromov we can 'twist' the canonical connection d on $\big(\Omega^{\bullet}_{(2)}(\tilde{X}), \bar{\partial}\big)$ by the one form $i\alpha\eta$ where $\alpha \in \mathbb{R}$. This means considering the new connection $d^{\alpha} = d + i\alpha\eta$ which will be a holomorphic connection if we can choose η so that $(d^{\alpha\,\prime\prime})^2 = 0$. Of course the corresponding Laplacian $\square^{\alpha}_{p,q}$ is no longer Γ invariant in general. However there is a projective action of Γ on $\big(\Omega^{\bullet;\bullet}_{(2)}(\tilde{X}), \bar{\partial}\big)$ under which $\square^{\alpha}_{p,q}$ is invariant.

As an example, let G be a connected semisimple Lie group, and K be a maximal compact subgroup such that G/K carries an invariant complex structure, and let Γ be a torsion-free uniform lattice in G. Then it is known that $\Gamma \setminus G/K$ is a Kähler hyperbolic manifold (*cf.* [BW]) and therefore the canonical flat Hilbert bundle $\mathcal{E} \to X$ is of D-class. In this Kähler metric, the Laplacian $\square_{p,q}$ is G-invariant. If now we consider the twisted connection d^{α} we find that the projective action of Γ on $\big(\Omega^{\bullet;\bullet}_{(2)}(\tilde{X}), \bar{\partial}\big)$ extends to a projective action of G. Assume the twisted connection is holomorphic. Then $\square^{\alpha}_{p,q}$ is invariant under this projective action of G so it follows that the theta function

$$\theta^{\alpha}_{p,q}(t) = C^{\alpha}_{p,q}(t)\,\mathrm{vol}\,(\Gamma \setminus G/K)$$

is proportional to the volume of $\Gamma \setminus G/K$. Here $C^{\alpha}_{p,q}(t)$ depends only on α, t and on G and K, but *not* on Γ. For small α the *D-class* condition is preserved and it follows that the zeta function $\zeta^{\alpha}_{p,q}(s, \lambda, \mathcal{E})$ is also proportional to the volume of $\Gamma \setminus G/K$. Therefore the holomorphic twisted L^2 torsion is given by

$$\rho^p_{\mathcal{E}} = e^{C^{\alpha}_p \Gamma \setminus G/K)} \rho'^p \in \det\big(H^{p,n-p}_{(2)}(G/K)\big)^{(-1)^{n-p}}$$

where we have used the twisted version of the vanishing theorem of Gromov [G]. Here C^{α}_p is a constant that depends only on α, G and K, but *not* on Γ. Using representation theory, as for instance in [M], [L] and [Fr], it is possible to determine C_p explicitly at least for $\alpha = 0$. This will be done elsewhere. Using the proportionality principle again, one sees that the Euler characteristic of $\Gamma \setminus G/K$ is proportional to its volume. By a theorem of Gromov [G], the Euler characteristic of $\Gamma \setminus G/K$ is non-zero. Therefore we can also express the holomorphic L^2 torsion as

$$\rho^p_{\mathcal{E}} = e^{\tilde{C}^{\alpha}_p \chi(\Gamma \setminus G/K)} \rho'^p \in \det\big(H^{p,n-p}_{(2)}(G/K)\big)^{(-1)^{n-p}}$$

where $\chi(\Gamma \setminus G/K)$ denotes the Euler characteristic of $\Gamma \setminus G/K$, and $\tilde{C}_p^\alpha$ is a constant that depends only on α, G and K, but *not* on Γ. This discussion is summarized in the following proposition.

Proposition 7.1 *In the notation above, the holomorphic twisted L^2 torsion of the semisimple locally symmetric space $\Gamma \setminus G/K$, which is assumed to carry an invariant complex structure, is given, for α sufficiently small, by*

$$\rho_{\mathcal{E}}^p = e^{C_p^\alpha \Gamma\setminus G/K)}\rho'^p \in \det\big(H_{(2)}^{p,n-p}(G/K)\big)^{(-1)^{n-p}}.$$

Here C_p^α is a constant that depends only on α, G and K, but not on Γ. Equivalently, the holomorphic twisted L^2 torsion of $\Gamma \setminus G/K$ is given as

$$\rho_{\mathcal{E}}^p = e^{\tilde{C}\alpha_p \chi(\Gamma\setminus G/K)}\rho'^p \in \det\big(H_{(2)}^{p,n-p}(G/K)\big)^{(-1)^{n-p}}$$

where $\chi(\Gamma \setminus G/K)$ denotes the Euler characteristic of $\Gamma \setminus G/K$, and $\tilde{C}_p^\alpha$ is a constant that depends only on α, G and K, but not on Γ.

We will now compute the holomorphic twisted L^2 torsion for a Riemann surface, which is a special case of the proposition above, and we will show that the constants C_p^α and $\tilde{C}_p^\alpha$ are not zero at least for α small.

Let X be a closed Riemann surface of genus g, which is greater than 1, which can be realised as a compact quotient complex hyperbolic space $\mathbb{H}$ of complex dimension 1, by the torsion-free discrete group Γ. The volume from ω on $\mathbb{H}$ is the Kähler form required in the analysis above and $\omega = \partial\eta$ where $\eta \in \Omega_{(2)}^{0,1}(\mathbb{H})$ as may be verified by direct calculation. Then the twisted connection d^α is holomorphic for $\alpha \in \mathbb{R}$ and $(d^\alpha)'' = \bar{\partial} + i\alpha\eta$. It is not difficult to see that, by Hodge theory for the complex $\big(\Omega_{(2)}^{0,\bullet}(\tilde{X}), \bar{\partial}\big)$ the operator $\Box_{0,1}^\alpha$ on the orthogonal complement of the L^2 cohomology is isospectral to $\Box_{0,0}^\alpha$. Now we see that in order to calculate the von Neumann determinant of the operator $\Box_{0,1}^\alpha$ we need only calculate the von Neumann determinant of the twisted Laplacian Δ_0^α acting on L^2 functions on the hyperbolic disk (these differ by a factor of $\frac{1}{2}$). Recall that the von Neumann determinant of the operator A is by definition $e^{-\zeta_A'(0)}$, where $\zeta_A'(s)$ denotes the zeta function of the operator A.

Using the work of Comtet and Houston [CH], one can see that there is a gap in the spectrum of Δ_0^α near zero so the D-class condition holds for all α. (In fact we believe that this stability of the D-class condition is true in the holomorphic setting under much more general conditions.) They also obtain the following expression for the meromorphic continuation of the zeta function of Δ_0^α to the half-plane $\Re(s) < 1$. With V denoting the hyperbolic volume let

$$\kappa(r) = \frac{V}{4\pi^2 i}\left[r\frac{d}{dr}\left(\log\Gamma\left(\frac{1}{2} + ir - \alpha\right) + \Gamma\left(\frac{1}{2} + ir + \alpha\right)\right) - V\log r\right]$$

where the principal branch of the logarithm is used. Then

$$(42) \quad -\zeta_0'(0, 0, \mathcal{E}, \alpha) = \frac{V}{4\pi}\left(\frac{1}{4} + \alpha^2\right)\left(1 - \ln\left(\frac{1}{4} + \alpha^2\right)\right) + \int_C \ln\left(\frac{1}{4} + r^2 + \alpha^2\right)\kappa(r)\,dr$$

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where C is a contour in the complex plane passing from $-\infty$ to ∞ below the real axis and such that it passes below the poles of the integrand (all of these lie on the imaginary axis). Note that one can do better if $\alpha = 0$ when, using [Ran],

$$\zeta_0'(0,0,\mathcal{E}) = \lim_{s\to 0}\big(\zeta_0(s,0,\mathcal{E}) - \zeta_0(0,0,\mathcal{E})\big)\Gamma(s)$$
$$= (g-1)\pi \int_0^\infty \Big(\frac{1}{4}+r^2\Big) sech^2(\pi r)\Big(-1+\log\Big(\frac{1}{4}+r^2\Big)\Big)\, dr.$$

A numerical approximation for the last integral shows that $\zeta'(0,0,\mathcal{E}) \sim -0.677(g-1)$. Hence for α small we know that twisted torsion is non-zero. We can summarize the discussion in the following proposition.

Proposition 7.2 *In the notation above, the holomorphic twisted L^2 torsion of a compact Riemann surface $X = \Gamma \setminus \mathbb{H}$ of genus g, is given by*

$$\rho_{\mathcal{E}}^{0,\alpha} = e^{C^\alpha}\rho'^{0} \in \det\big(H_{(2)}^{0,1}(\mathbb{H})\big)^{(-1)}. \tag{43}$$

Here $C^\alpha = \frac{1}{2}\zeta_0'(0,0,\mathcal{E},\alpha)$ is a constant that depends only on $\mathbb{H}$ and α, but not on Γ. When $\alpha = 0$, C^α is approximately -0.338, and in particular, it is not equal to zero for small α. Also,

$$\rho_{\mathcal{E}}^{1,\alpha} = e^{-C^\alpha}\rho'^{1} \in \det\big(H_{(2)}^{1,0}(\mathbb{H})\big)^{(-1)}.$$

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