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On Subcritically Stein Fillable 5-manifolds

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Abstract. We make some elementary observations concerning subcritically Stein fillable contact structures on 5-manifolds. Specifically, we determine the diffeomorphism type of such contact manifolds in the case where the fundamental group is finite cyclic, and we show that on the 5-sphere, the standard contact structure is the unique subcritically fillable one. More generally, it is shown that subcritically fillable contact structures on simply connected 5-manifolds are determined by their underlying almost contact structure. Along the way, we discuss the homotopy classification of almost contact structures.

1 Introduction

A *Stein domain* in the sense of [4, Definition 11.14] is a compact manifold W with boundary admitting a complex structure J and a J-convex Morse function for which the boundary $\partial W =: M$ is a regular level set. We shall write a Stein domain as a pair (W, J), although, strictly speaking, the J-convex Morse function is part of the data. The complex tangencies $TM \cap J(TM)$ define a contact structure.

A closed contact manifold (M, ξ) is said to be *Stein fillable* if it arises in this way as the boundary of a Stein domain. It is well known that a Stein domain of dimension $2n$ has a handle decomposition, adapted to the Stein structure, with handles of index at most equal to n. A Stein filling is called *subcritical* if there are no handles of index n.

In this note we are concerned with topological and contact geometric aspects of subcritically Stein fillable contact 5-manifolds. The first result we want to discuss gives a uniqueness statement for the diffeomorphism type of such contact manifolds when it has a finite cyclic fundamental group. This extends a corresponding result for simply connected contact manifolds due to Bowden–Crowley–Stipsicz [3]. The main issue is one of simple homotopy theory, which in our examples can be addressed with results of Hambleton–Kreck [7] on 2-complexes and, as in [3], the Mazur–Wall theory of thickenings.

In order to state the result, we need to introduce certain model manifolds. Let $m \geq 2$ be an integer. Write L_m for the 3-dimensional lens space $L(m,1)$ with an open 3-disc removed. This space L_m can be obtained from a solid torus $S^1 \times D^2$ by attaching a 2-handle along an $(m,-1)$-torus not in $\partial(S^1 \times D^2)$.

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Oriented D^3-bundles over L_m are classified by the second Stiefel–Whitney class w_2 (this standard fact will be elucidated in the proof of Theorem 1.1). Since $H^2(L_m;\mathbb{Z}_2)$ is trivial for m odd, and isomorphic to $\mathbb{Z}_2$ for m even, the only D^3-bundles are the product $L_m \times D^3$ and, for $m = 2n$ even, the non-trivial bundle $L_{2n}\widetilde{\times}D^3$. After rounding of corners, we can think of the total spaces of these bundles as manifolds with boundary.

Similarly, over S^2 we have the trivial S^3-bundle $S^2 \times S^3$, and the non-trivial one $S^2\widetilde{\times}S^3$. In [3, Proposition 7.4] it was shown that if (M,ξ) is a closed, *simply connected* 5-dimensional contact manifold admitting a subcritical Stein filling, then M is diffeomorphic to $\#_r S^2 \times S^3$ if M is spin, and $S^2\widetilde{\times}S^3\#_{r-1}S^2 \times S^3$ if M is not spin, where $r = \operatorname{rank} H_2(M;\mathbb{Z})$.

Our first result extends this to finite cyclic fundamental groups.

Theorem 1.1 *Suppose that (M,ξ) is a closed, connected contact 5-manifold admitting a subcritical Stein filling, with $\pi_1(M) \cong \mathbb{Z}_m$ for some integer $m \geq 2$. Set $r =$* $\operatorname{rank} H_2(M;\mathbb{Z})$.

(i) *If $m = 2n+1$ is odd, then M is diffeomorphic to*

$$\partial(L_{2n+1} \times D^3)\#_r(S^2 \times S^3) \quad \text{or} \quad \partial(L_{2n+1} \times D^3)\#(S^2\widetilde{\times}S^3)\#_{r-1}(S^2 \times S^3),$$

depending on whether M is spin or not.

(ii) *If $m = 2n$ is even, then M is diffeomorphic to*

$$\partial(L_{2n} \times D^3)\#_r(S^2 \times S^3)$$

if M is spin, and to

$$\partial(L_{2n}\widetilde{\times}D^3)\#_r(S^2 \times S^3) \quad \text{or} \quad \partial(L_{2n} \times D^3)\#(S^2\widetilde{\times}S^3)\#_{r-1}(S^2 \times S^3)$$

when M is not spin.

The diffeomorphism type of the subcritical Stein filling is determined by M.

On any of these manifolds, each homotopy class of almost contact structures contains a subcritically Stein fillable contact structure, unique up to isotopy, with a Stein filling, unique up to Stein homotopy.

The strategy for proving this theorem is as follows. First, we use homotopy-theoretic methods to arrive at a topological classification of the potential subcritical fillings. We then appeal to the fundamental work of Cieliebak–Eliashberg [4] that reduces the existence and classification question for Stein structures in the subcritical case, yet again, to a problem of homotopy theory. The relevant results from [4] will be recalled below. That second homotopy-theoretic problem, the homotopy classification of almost contact and almost complex structures on 5-dimensional and 6-dimensional manifolds, respectively, is a matter of classical obstruction theory; see Section 2.

The same strategy, combined with the results of [3], allows us to complete our discussion of subcritical Stein fillings of simply connected 5-manifolds in [5], which was written previous to [4] being available.

Theorem 1.2 *Let M be a closed, simply connected 5-manifold admitting a subcritical Stein filling, that is, one of the manifolds $\#_r S^2 \times S^3$ or $S^2 \widetilde{\times} S^3 \#_{r-1} S^2 \times S^3$. Then each homotopy class of almost contact structures contains a subcritically Stein fillable contact structure, unique up to isotopy, with a Stein filling, unique up to Stein homotopy.*

Two particular consequences (or special cases) of this result are worth noting. Here, for (M, ξ) a contact manifold, $c_1(\xi) \in H^2(M;\mathbb{Z})$ denotes the first Chern class of ξ; recall that a contact structure carries a complex bundle structure, unique up to homotopy [6, Proposition 2.4.8].

Corollary 1.3 (i) *Any subcritically Stein fillable contact structure on S^5 is isotopic to the standard contact structure.*

(ii) *Let (M_i, ξ_i), $i = 1, 2$, be two simply connected subcritically Stein fillable contact 5-manifolds. If there is an isomorphism $\phi: H^2(M_1;\mathbb{Z}) \to H^2(M_2;\mathbb{Z})$ such that $\phi(c_1(\xi_1)) = c_1(\xi_2)$, then (M_1, ξ_1) and (M_2, ξ_2) are contactomorphic.*

Part (i) confirms an expectation from [5, Section 6]. Part (ii) was proved in [5, Theorem 4.8] under the additional assumption that the fillings contain no 1-handles. As pointed out by the referee, an isomorphism ϕ as required by part (ii) of the corollary exists if and only if the Chern classes $c_1(\xi_1)$ and $c_1(\xi_2)$ have the same divisibility in the free abelian group $H^2(M_i;\mathbb{Z})$; see [8, Theorem 8.20]. Thus, the divisibility of the first Chern class is the only contactomorphism invariant of a subcritically Stein fillable contact structure on a simply connected 5-manifold.

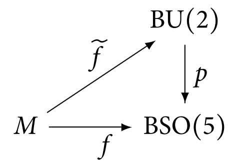
2 Homotopy Classification of Almost Contact Structures

In this section we discuss the homotopy classification of almost contact structures on 5-manifolds, correcting a negligence in [6]. Likewise, we describe the classification of almost complex structures on 6-manifolds, correcting a similar oversight in [17]. These classification results are key ingredients in the proof of Theorems 1.1 and 1.2.

A careful discussion of this homotopy classification can be found in M. Hamilton's thesis [8, VIII.4], and our reasoning goes along the same lines. We show that by a closer look at this obstruction-theoretic argument, one can in fact exhibit a free and transitive action of the second cohomology group on the space of almost contact (resp. complex) structures.

Let M be a compact (not necessarily closed), oriented 5-manifold. A choice of Riemannian metric on M, or equivalently, a reduction of the structure group of the tangent bundle to $SO(5)$, allows us to describe the tangent bundle TM in terms of a classifying map $f: M \to BSO(5)$. Define the inclusion $U(2) \subset SO(5)$ by the embedding $\mathbb{C}^2 \equiv \mathbb{R}^4 \times \{0\} \subset \mathbb{R}^5$. Any subgroup $G \subset O(5)$ acts on the space $V(5)$ of orthonormal 5-frames in $\mathbb{R}^\infty$, and this defines the universal bundles $V(5) \to BG := V(5)/G$. The quotient BG is the classifying space for G-bundles; see [18, Section A.2]. The inclusion $\mathrm{U}(2) \subset \mathrm{SO}(5)$ defines a fibration $p\colon \mathrm{BU}(2) \to \mathrm{BSO}(5)$ with fibre $F_5 := \mathrm{SO}(5)/\mathrm{U}(2)$.

An almost contact structure on M is a reduction of the structure group of TM from $\mathrm{SO}(5)$ to $\mathrm{U}(2)$, which amounts to a lift $\widetilde{f}$ of the classifying map f:



The lifting condition $p \circ \widetilde{f} = f$ is equivalent to saying that the map

$$M \ni x \longmapsto \sigma(x) := \big(x, \widetilde{f}(x)\big) \in M \times V(5)/\mathrm{U}(2)$$

is a section of the induced bundle $E := f^* V(5)/U(2) = f^*\mathrm{BU}(2)$ over M with fibre F_5. This is the obstruction-theoretic setting of [15, Part III]. Notice that $f^* V(5)$ is the frame bundle of M.

From now on we will interpret almost contact structures on M as sections σ of this bundle $E \to M$. Homotopy of almost contact structures means homotopy of such sections.

Lemma 2.1 *The* $\mathrm{U}(2)$*-bundle* $\widetilde{f}^* V(5) \to M$ *corresponding to the almost contact structure defined by* $\widetilde{f}$ *equals the pull-back of the* $\mathrm{U}(2)$*-bundle* $f^* V(5) \to E$ *under the map* $\sigma\colon M \to E$.

Proof Write the two relevant universal bundles as

$$\pi_{\mathrm{SO}}\colon V(5) \longrightarrow \mathrm{BSO}(5) \quad \text{and} \quad \pi_{\mathrm{U}}\colon V(5) \longrightarrow \mathrm{BU}(2).$$

Then

$$f^* V(5) = \big\{(x, v) \in M \times V(5) \colon f(x) = \pi_{\mathrm{SO}}(v)\big\},$$

and the bundle projection $\pi_E\colon f^* V(5) \to E$ is given by

$$\pi_E(x, v) = \big(x, \pi_{\mathrm{U}}(v)\big) \in f^*\mathrm{BU}(2) = E.$$

Under σ this pulls back to

$$\big\{(x, w) \in M \times f^* V(5) \colon \sigma(x) = \pi_E(w)\big\},$$

with the obvious projection map to M. This space can be rewritten as

$$\big\{(x, v) \in M \times V(5) \colon \widetilde{f}(x) = \pi_{\mathrm{U}}(v)\big\},$$

which is the total space of the bundle $\widetilde{f}^* V(5) \to M$. ■

The fibre F_5 of the bundle $E \to M$ is diffeomorphic to $\mathbb{C}\mathrm{P}^3$; see [6, Lemma 8.1.2 and Proposition 8.1.3]. From the homotopy exact sequence of the generalised Hopf fibration $S^1 \hookrightarrow S^7 \to \mathbb{C}\mathrm{P}^3$, one then sees that the homotopy groups $\pi_i(F_5)$ are trivial for $i = 0, 1, 3, 4, 5$, and $\pi_2(F_5) \cong \mathbb{Z}$.

Since the fibre F_5 is simply connected, it is in particular 2-simple in the sense of [15, §16.5]; *i.e.,* the fundamental group operates trivially on $\pi_2(F_5)$. Moreover, the structure group $\mathrm{SO}(5)$ of the bundle $E \to M$ is connected. From [15, §30.4] it then follows that the bundle of coefficients over M whose fibre over x is the homotopy group $\pi_2(E_x) \cong \mathbb{Z}$ of the fibre E_x of E is actually a trivial bundle. This implies that the obstruction to extending a section of E over the 2-skeleton of M to the 3-skeleton is a cohomology class in $H^3(M;\mathbb{Z})$. Given two sections of $E \to M$ that are homotopic over the 1-skeleton, the obstruction to homotopy over the 2-skeleton lives in $H^2(M;\mathbb{Z})$. Similarly, the obstruction cocycles are simply integral chains.

The obstruction class for the existence of a section over the 3-skeleton can be identified with the third integral Stiefel–Whitney class $W_3(M)$; see [6, p. 370]. By the vanishing of the other relevant homotopy groups of F_5, this class is the only obstruction to the existence of an almost contact structure. Likewise, the only obstruction to homotopy of two almost contact structures is the primary difference class in $H^2(M;\mathbb{Z})$.

We can now formulate the homotopy classification of almost contact structures. Regarding an almost contact structure as a $\mathrm{U}(2)$-bundle, we can sensibly speak of its first Chern class c_1, which is a homotopy invariant. In the following statement and its proof we allow ourselves to identify an almost contact structure with the homotopy class it represents. The k-skeleton of M will be denoted by $M^{(k)}$.

Proposition 2.2 *Let M be a compact, oriented 5-manifold with $W_3(M) = 0$. There is a free and transitive action of $H^2(M;\mathbb{Z})$ on the set $\mathcal{A}(M)$ of almost contact structures on M. Write $u * \sigma \in \mathcal{A}(M)$ for the image of $\sigma \in \mathcal{A}(M)$ under the action of $u \in H^2(M;\mathbb{Z})$. Then $c_1(u * \sigma) = c_1(\sigma) + 2u$.*

Proof Fix a reference element $\sigma_0 \in \mathcal{A}(M)$. For any $u \in H^2(M;\mathbb{Z})$, by the first extension theorem of obstruction theory [15, §37.2], we can find a section σ'_u of E over $M^{(3)}$ such that the primary difference class $d(\sigma_0, \sigma'_u) \in H^2(M;\mathbb{Z})$ equals u. Since the higher relevant homotopy groups of F_5 vanish, σ'_u can be extended to a section σ_u over all of M.

Given any other $\tau_u \in \mathcal{A}(M)$ with primary difference $d(\sigma_0, \tau_u) = u$, the addition formula [15, §36.6] implies $d(\sigma_u, \tau_u) = 0$, and hence that σ_u and τ_u are homotopic. Thus, σ_u denotes a well-defined homotopy class. This allows us to define a free and transitive action of $H^2(M;\mathbb{Z})$ on $\mathcal{A}(M)$ by $u * \sigma_v := \sigma_{u+v}$.

It remains to prove the formula for the first Chern class. From the homotopy exact sequence of the universal bundles, we have $\pi_2(\mathrm{BU}(2)) \cong \pi_1(\mathrm{U}(2)) \cong \mathbb{Z}$ and $\pi_2(\mathrm{BSO}(5)) \cong \pi_1(\mathrm{SO}(5)) \cong \mathbb{Z}_2$. The homotopy exact sequence of the bundle

$$\mathbb{C}\mathrm{P}^3 = F_5 \hookrightarrow \mathrm{BU}(2) \longrightarrow \mathrm{BSO}(5)$$

then gives us

$$\begin{array}{ccccccc} \pi_2(F_5) & \longrightarrow & \pi_2(\mathrm{BU}(2)) & \longrightarrow & \pi_2(\mathrm{BSO}(5)) & \longrightarrow & \pi_1(F_5) \\ \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 0. \end{array}$$

It follows that the first homomorphism in this sequence is multiplication by 2.

The inclusion map $\iota\colon F_5 \to \mathrm{BU}(2)$ is covered by a bundle map of $\mathrm{U}(2)$-bundles:

$$\begin{array}{ccc} \mathrm{SO}(5) & \longrightarrow & V(5) \\ \downarrow & & \downarrow \\ F_5 & \xrightarrow{\ \iota\ } & \mathrm{BU}(2). \end{array}$$

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This means that the bundle $\mathrm{SO}(5) \to F_5$ can be regarded as the induced bundle $\iota^* V(5) \to F_5$. By the observation on the homomorphism $\pi_2(F_5) \to \pi_2(\mathrm{BU}(2))$, the homomorphism

$$\iota^*\colon \mathbb{Z} \cong H^2(\mathrm{BU}(2);\mathbb{Z}) \longrightarrow H^2(F_5;\mathbb{Z}) \cong \mathbb{Z}$$

is likewise multiplication by 2. Since $H^2(\mathrm{BU}(2);\mathbb{Z})$ is generated by the first Chern class, it follows that $c_1(\iota^* V(5))$ is twice a generator of $H^2(F_5;\mathbb{Z}) \cong \mathbb{Z}$. Choose a generator of $\pi_2(F_5) = H_2(F_5;\mathbb{Z}) \cong \mathbb{Z}$ and the corresponding dual generator of $H^2(F_5;\mathbb{Z}) \cong \mathbb{Z}$—in other words, fix an identification of these groups with $\mathbb{Z}$—in such a way that $c_1(\iota^* V(5)) = -2$.

By construction, we have the formula $d(\sigma, u * \sigma) = u$ for the difference class. We therefore need to show that

$$(*) \qquad c_1(\tau) - c_1(\sigma) = 2d(\sigma,\tau) \quad \text{for any } \sigma, \tau \in \mathcal{A}(M).$$

It suffices to prove this formula over the 2-skeleton $M^{(2)}$. Indeed, the inclusion $M^{(2)} \to M$ induces an injective homomorphism $H^2(M;\mathbb{Z}) \to H^2(M^{(2)};\mathbb{Z})$.

The bundle $E|_{M^{(1)}}$ is trivial; moreover, the fibre F_5 is simply connected. Thus, we can assume that the sections σ and τ are constant (and identical) over the 1-skeleton $M^{(1)}$.

Recall from [15, §36] the definition of the primary difference class $d(\sigma,\tau)$, represented by a cochain with values in $\pi_2(F_5)$. Any oriented 2-cell $\Delta \subset M^{(2)} \subset M$ is described by a characteristic map $\varphi_\Delta\colon D^2 \to M$ sending $\mathrm{Int}(D^2)$ homeomorphically onto Δ, and ∂D^2 into $M^{(1)}$. The section σ of the bundle $E \to M$ defines a section σ_Δ of the pull-back bundle $\varphi_\Delta^* E \to D^2$ via

$$\sigma_\Delta(x) := \big(x, \sigma \circ \varphi_\Delta(x)\big).$$

Likewise for τ,

$$\begin{array}{ccc} D^2 \times F_5 \cong \varphi_\Delta^* E & \xrightarrow{\ \overline{\varphi}_\Delta\ } & E \\ \sigma_\Delta, \tau_\Delta \uparrow\downarrow & & \downarrow\uparrow \sigma,\tau \\ D^2 & \xrightarrow[\ \varphi_\Delta\ ]{} & M. \end{array}$$

Notice that the pull-back bundle over D^2 is trivial, and in the trivialisation $\varphi_\Delta^* E \cong D^2 \times F_5$ the sections $\sigma_\Delta, \tau_\Delta$ may be regarded as maps $D^2 \to F_5$. However, there is no *a priori* relation between this trivialisation and that of $E|_{M^{(1)}}$, so $\sigma_\Delta, \tau_\Delta$ coincide over ∂D^2, but they will not, in general, be constant along ∂D^2.

Write $\pi_\pm\colon S^2_\pm \to D^2$ for the projection of the upper and lower hemisphere of the 2-sphere, respectively, onto the equatorial disc. Then the class $d(\sigma,\tau)$ is represented by the cocycle whose value on Δ is the element of $\pi_2(F_5)$ given by the map

$$d(\sigma,\tau)(\Delta) = \begin{cases} \sigma_\Delta \circ \pi_+ & \text{on } S^2_+, \\ \tau_\Delta \circ \pi_- & \text{on } S^2_-. \end{cases}$$

Here, by a slight abuse of notation, we do not distinguish between cocycles and the cohomology classes they represent. The sign convention for the difference class is the standard one as in [15, §33.4].

Over E we have the $U(2)$-bundle $f^*V(5) \to E$, which we shall now denote by η. Our aim is to compute the difference $c_1(\tau) - c_1(\sigma)$, which by definition equals $c_1(\tau^*\eta) - c_1(\sigma^*\eta)$. These Chern classes live in the cohomology of M with coefficients in the coefficient bundle $\eta(\pi_1)$ in the notation of [15, §30.2]. Since the structure group $U(2)$ has abelian fundamental group $\pi_1(U(2)) \cong \mathbb{Z}$, hence is 1-simple, and is connected, again by [15, §30.4] this coefficient bundle is trivial and we are simply dealing with integral cohomology classes. (This is, of course, well known.)

With $\overline{\varphi}_\Delta$ defined by the diagram above, the pull-back bundle $\overline{\varphi}_\Delta^*\eta = D^2 \times \iota^*V(5)$ restricts to a trivial bundle over either $\sigma_\Delta(D^2)$ and $\tau_\Delta(D^2)$, and we have sections of these bundles over $\sigma_\Delta(\partial D^2) = \tau_\Delta(\partial D^2)$, since $\sigma \circ \varphi_\Delta|_{\partial D^2} = \tau \circ \varphi_\Delta|_{\partial D^2}$ is a constant section of $E|_{M^{(1)}} \cong M^{(1)} \times F_5$. These sections define elements of $\pi_1(U(2)) \cong \mathbb{Z}$, and the classes $c_1(\sigma), c_1(\tau)$ are represented by the cochains whose value on Δ is precisely that respective element.

It follows that $c_1(\sigma) - c_1(\tau)$ is represented by a cochain whose value on Δ is given by the first Chern class of the $U(2)$-bundle $\iota^*V(5)$ over the 2-sphere $d(\sigma,\tau)(\Delta) \in \pi_2(F_5)$. Since $c_1(\iota^*V(5)) = -2$, this implies $(*)$. ■

The following corollary is then immediate; see [8, Theorem 8.18].

Corollary 2.3 *In the absence of* 2*-torsion in* $H^2(M;\mathbb{Z})$*, almost contact structures are determined up to homotopy by the first Chern class.*

By completely analogous arguments, one can also prove the following homotopy classification of almost complex structures on 6-manifolds. Again, the third integral Stiefel–Whitney class is the only obstruction to the existence of an almost complex structure.

Proposition 2.4 *Let W be a compact, oriented* 6*-manifold with* $W_3(W) = 0$*. There is a free and transitive action of* $H^2(W;\mathbb{Z})$ *on the set* $\mathcal{A}(W)$ *of almost complex structures on W. Write* $u * \sigma \in \mathcal{A}(W)$ *for the image of* $\sigma \in \mathcal{A}(W)$ *under the action of* $u \in H^2(W;\mathbb{Z})$*. Then* $c_1(u*\sigma) = c_1(\sigma) + 2u$*.* ■

Remark 2.5 Let M be a closed, connected 5-manifold with a subcritical filling W, *i.e.,* a topological filling made up of handles of index at most two. Dually, W can be obtained from M by attaching handles of index at least four. The particular consequences relevant to the discussion below are that the inclusion $M \to W$ induces isomorphisms both on fundamental groups and on the second cohomology groups (with any coefficients).

With this observation, we can formulate a relation between the sets $\mathcal{A}(M)$ and $\mathcal{A}(W)$. We have a restriction map $\mathcal{A}(W) \to \mathcal{A}(M)$ defined by $J \mapsto J|_{TM}$. Here, by slight abuse of notation, $J \mapsto J|_{TM}$ denotes the almost contact structure on M given by the coorientable hyperplane field $TM \cap J(TM)$ with the complex bundle structure

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given by J. The isomorphism $H^2(W;\mathbb{Z}) \cong H^2(M;\mathbb{Z})$ in the following proposition is understood to be the one induced by the inclusion map $M \to W$.

Proposition 2.6 *Let M be a closed, connected 5-manifold having a subcritical filling W with $W_3(W) = 0$. Then the restriction map $\mathcal{A}(W) \to \mathcal{A}(M)$, $J \mapsto J|_{TM}$ is an equivariant bijection with respect to the respective actions of $H^2(W;\mathbb{Z}) \cong H^2(M;\mathbb{Z})$.*

Proof This is immediate from the construction of the action of $H^2(M;\mathbb{Z})$ on $\mathcal{A}(M)$ in the proof of Proposition 2.2 and the analogous construction for W to prove Proposition 2.4, given that W is obtained from M by attaching handles of index at least four. ■

3 Topology of Subcritically Stein Fillable 5-manifolds

We first recall the two pertinent results from [4] in the form in which we need them.

Theorem 3.1 ([4, Theorem 1.5]) *Let W be a compact manifold with boundary, of dimension $2n \geq 6$, equipped with an almost complex structure J. If W admits a handle decomposition with handles of index $\leq n$ only, then J is homotopic to a complex structure J' making (W, J') a Stein domain. The Stein structure can be chosen compatible with the given handle decomposition.*

The second theorem deals with subcritical Stein domains, where we have a decomposition into Stein handles of index at most $n-1$. Notice that the preceding theorem says that if we start with a subcritical handle decomposition and an almost complex structure, we can find a subcritical Stein structure.

Theorem 3.2 ([4, Theorem 15.14]) *Let W be a compact manifold with boundary, of dimension $2n \geq 6$, equipped with almost complex structures J_1, J_2 making (W, J_1) and (W, J_2) subcritical Stein domains. If J_1 and J_2 are homotopic as almost complex structures, they are homotopic as Stein structures.*

In fact, Cieliebak–Eliashberg prove this theorem for so-called *flexible* Stein domains [4, Definition 11.29], which by [4, Remark 11.30] includes all subcritical ones.

Proof of Theorem 1.1 Let (M, ξ) be a closed, connected, 5-dimensional contact manifold with finite cyclic fundamental group, admitting a subcritical Stein filling (W, J). This Stein filling is made up of handles of index at most two, so W is simple homotopy equivalent to a finite 2-complex; see [11, p. 7]. Dually, as observed in Remark 2.5, W can be obtained from M by attaching handles of index at least four.

For $m \geq 2$ an integer, we write X_m for the m-fold dunce cap. This complex is obtained by attaching a 2-disc D^2 to the circle S^1 with an attaching map $\partial D^2 \to S^1$ of degree m. Equivalently, X_m can be obtained as a quotient space of D^2 by identifying each point $x \in S^1 = \partial D^2$ with its rotate through an angle $2\pi/m$.

According to [7, Theorem 2.1], given two finite 2-complexes K, K' of the same Euler characteristic $\chi(K) = \chi(K')$ and any isomorphism $\pi_1(K) \to \pi_1(K')$, where the

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fundamental group is a finite subgroup of $SO(3)$, there is a simple homotopy equivalence $K \to K'$ inducing the given isomorphism on fundamental groups. This implies that if $\pi_1(M)$ is the cyclic group of order m, the Stein filling W is simple homotopy equivalent to the 2-complex $X_m \vee_r S^2$, where $r := \chi(W) - 1$. Thus, W is a thickening of this 2-complex in the sense of [11] or [16].

Because of $6 \geq 2 \cdot 2 + 1$, 6-dimensional thickenings of a 2-complex are in the stable range, and by [11, Lemma 11.29] or [16, Proposition 5.1], oriented thickenings of $X_m \vee_r S^2$ are classified up to diffeomorphism by $[X_m \vee_r S^2, \mathrm{BSO}]$. This set of homotopy classes is isomorphic to

$$\big[X_m \vee_r S^2, K(\mathbb{Z}_2, 2)\big] \cong H^2(X_m \vee_r S^2; \mathbb{Z}_2),$$

since the homotopy groups $\pi_k(\mathrm{BSO})$ coincide with those of the Eilenberg–MacLane space $K(\mathbb{Z}_2, 2)$ for $k \leq 2$. The isomorphism

$$[X_m \vee_r S^2, \mathrm{BSO}] \cong H^2(X_m \vee_r S^2; \mathbb{Z}_2)$$

is given by the second Stiefel–Whitney class, because this obstruction class detects the non-trivial oriented $\mathbb{R}^\infty$-bundle over S^2; see [6, Lemma 8.2.5], for instance.

Write $\natural$ for the boundary connected sum of manifolds with boundary, and $S^2 \widetilde{\times} D^4$ for the non-trivial D^4-bundle over S^2. In the case where $m = 2n + 1$ is odd, we have $H^2(X_{2n+1}; \mathbb{Z}_2) = 0$, so a thickening of $X_{2n+1} \vee_r S^2$ is determined by the tangent bundle over each of the r 2-spheres being trivial or not. There is a well-known diffeomorphism

$$S^2 \widetilde{\times} D^4 \natural S^2 \widetilde{\times} D^4 \cong S^2 \widetilde{\times} D^4 \natural S^2 \times D^4$$

(see [5, Proposition 4.7]). This diffeomorphism can also be derived from the argument we will use presently in the case where m is even. It follows that W diffeomorphic to

$$(L_{2n+1} \times D^3) \natural_r (S^2 \times D^4) \quad \text{or} \quad (L_{2n+1} \times D^3) \natural (S^2 \widetilde{\times} D^4) \natural_{r-1} (S^2 \times D^4),$$

depending on whether W is spin or not. Since the inclusion $M \to W$ induces an isomorphism on $H^2(\,\cdot\,; \mathbb{Z}_2)$, this proves part (i) of the proposition.

If $m = 2n$ is even, we have $H^2(X_{2n}; \mathbb{Z}_2) = \mathbb{Z}_2$, so there is now also a choice of two thickenings over X_{2n}. The same argument as before shows that W is diffeomorphic to

$$(L_{2n} \times D^3) \natural_r (S^2 \times D^4)$$

if W is spin, or, in the non-spin case, to one of the three manifolds

$$\begin{aligned} W_{1,0} &:= (L_{2n} \widetilde{\times} D^3) \#_r (S^2 \times D^4), \\ W_{0,1} &:= (L_{2n} \times D^3) \# (S^2 \widetilde{\times} D^4) \#_{r-1} (S^2 \times D^4), \\ W_{1,1} &:= (L_{2n} \widetilde{\times} D^3) \# (S^2 \widetilde{\times} D^4) \#_{r-1} (S^2 \times D^4). \end{aligned}$$

The manifolds $W_{1,0}$ and $W_{0,1}$ are not diffeomorphic, since there is no isomorphism $H^2(W_{1,0}; \mathbb{Z}) \to H^2(W_{0,1}; \mathbb{Z})$ of $\mathbb{Z}_{2n} \oplus \mathbb{Z}^r$ whose mod 2 reduction sends $w_2(W_{1,0})$ to $w_2(W_{0,1})$; the same argument applies to the boundaries of these manifolds.

We claim, however, that $W_{1,1}$ is diffeomorphic to $W_{0,1}$; it suffices to prove this for $r = 1$. Both manifolds are obtained from a 6-ball by first attaching a 1-handle to produce $S^1 \times D^5$, and then a couple of 2-handles $h_1, h_2 \cong D^2 \times D^4$. In order to obtain $W_{0,1}$, we attach h_1 by an attaching map $\varphi_1\colon \partial D^2 \times D^4 \to \partial(S^1 \times D^5)$ that sends $\partial D^2 \times \{0\}$ to a $(2n, -1)$-torus knot on

$$S^1 \times \partial D^2 \times \{0\} \subset S^1 \times \partial D^2 \times D^3 \subset \partial(S^1 \times D^5),$$

extended to an embedding of $\partial D^2 \times D^4$ with trivial framing. The handle h_2 is attached by a map φ_2 sending $\partial D^2 \times \{0\}$ to a homotopically trivial circle in

$$S^1 \times \partial D^2 \times \{0\} \setminus \varphi_1\big(\partial D^2 \times D^4\big),$$

extended to an embedding of $\partial D^2 \times D^4$ with twisted framing, corresponding to the non-trivial element of $\pi_1(\mathrm{SO}(4)) = \mathbb{Z}_2$.

By sliding h_1 over h_2 we get a diffeomorphic manifold where the framing of the first handle is now also twisted, in other words, the manifold $W_{1,1}$. This proves part (ii) of the proposition.

Finally, we come to the statement about the existence of subcritically fillable contact structures. Let M be one of the 5-manifolds in (i) or (ii), and W the corresponding 6-manifold discussed in the course of proving this classification, with boundary M. This manifold W admits an almost complex structure, since $H^3(W; \mathbb{Z}) = 0$. By Theorem 3.1, any almost complex structures on W is homotopic to a subcritical Stein structure. From Proposition 2.6 it follows that every homotopy class of almost contact structures on M contains a subcritically Stein fillable contact structure. According to Theorem 3.2, the subcritically Stein fillable contact structure within a given homotopy class is unique up to isotopy, and its Stein filling is unique up to Stein homotopy. ∎

Remark 3.3 (i) Let M be one of the 5-manifolds in Theorem 1.1 and ξ a subcritically Stein fillable contact structure as just described. Then by [2, Theorem 5.3], any symplectically aspherical filling is homotopy equivalent to the corresponding W, and even diffeomorphic to W if the Whitehead group of $\mathbb{Z}_m$ vanishes, which by [13, Corollary 6.5] happens exactly for $m \in \{2, 3, 4, 6\}$.

(ii) From [7, Theorem B] one can derive the following stabilisation result. Let (M, ξ) be a closed, connected contact 5-manifold with finite fundamental group, admitting a subcritical Stein filling (W_0, J_0). Then any subcritical Stein filling of any contact structure on $M \# S^2 \times S^3$ is simple homotopy equivalent to $W_0 \natural S^2 \times D^4$. Homotopically, the additional summand amounts to a one-point union with S^2.

4 Uniqueness of Subcritically Stein Fillable 5-manifolds

Proof of Theorem 1.2 By the proof of [3, Proposition 7.4], the filling W is diffeomorphic to $\natural_r(S^2 \times D^4)$ or $(S^2 \widetilde{\times} D^4) \natural_r (S^2 \times D^4)$, where r is the same non-negative integer as in the description of M; this also follows from [2, Theorem 1.5].

The theorem then follows by the same argument as the one we used at the end of the proof of Theorem 1.1. ∎

Proof of Corollary 1.3 (i) This is an immediate consequence of Theorem 1.2. Alternatively, here is a more direct proof. Suppose that ξ is a contact structure on S^5 that admits a subcritical Stein filling (W, J). This means that there is a contactomorphism $f\colon (S^5, \xi) \to (\partial W, \xi_J)$, where ξ_J denotes the contact structure induced by J.

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By [2, Theorem 1.2] or an earlier result of Oancea–Viterbo [14] (see the discussion in [2, Section 3.3]), the manifold W is a simply connected homology ball, and hence diffeomorphic to the standard ball D^6 by [12, Proposition A, p. 108]. Choose an orientation-preserving diffeomorphism $G\colon D^6 \to W$.

Recall that any diffeomorphism of S^5 can be extended to a diffeomorphism of D^6; this corresponds with the fact that there are no exotic 6-spheres, see [9] and [10, Corollary VIII.(5.6)]. Let $\Phi\colon D^6 \to D^6$ be an orientation-preserving diffeomorphism extending the diffeomorphism $(G|_{S^5})^{-1} \circ f\colon S^5 \to S^5$. Notice that the diffeomorphism $F := G \circ \Phi\colon D^6 \to W$ restricts to f on S^5.

It follows that the subcritical Stein structure F^*J on D^6 induces the contact structure ξ on S^5. Write J_{st} for the standard complex structure on D^6 inducing the standard contact structure ξ_{st} on S^5. The two complex structures F^*J and J_{st} on D^6 are homotopic as almost complex structures. Thus, by Theorem 3.2, the respective induced contact structures ξ and ξ_{st} on S^5 are isotopic.

(ii) One way to prove this is by appealing to the results of Barden [1] on the classification and the diffeomorphisms of simply connected 5-manifolds, as nicely expounded in [8, Chapter VII]; see in particular [8, Theorem 7.16]. Under the assumptions of the corollary (and given the fact from [3] that simply connected 5-manifolds admitting a subcritical Stein filling necessarily have torsion-free homology), there is a diffeomorphism $M_2 \to M_1$ that induces the given isomorphism on H^2. Then argue as in the proof of Theorem 1.2. ∎

Here is an alternative argument for part (ii) of the corollary that avoids having to cite the result of Barden on the diffeomorphisms of simply connected 5-manifolds. This argument is, in some sense, more constructive, since it reduces the problem to the diagrammatic language of [5].

We want to show that any closed, simply connected contact 5-manifold (M,ξ) that admits a subcritical Stein filling (W,J) also admits a subcritical Stein filling without 1-handles. Then the theorem follows from the corresponding result [5, Theorem 4.8], which made precisely this additional assumption on the absence of 1-handles.

Again we use the fact (as in the proof of Theorem 1.2) that for a given M the topology of the filling W is known. As shown in the proof of [5, Proposition 4.5], for any class $c \in H^2(W;\mathbb{Z})$ that reduces modulo 2 to the Stiefel–Whitney class $w_2(W)$, there is a subcritical Stein structure on W *without* 1*-handles* with first Chern class c. (Moreover, the cited proposition shows directly that the contact structure induced on the boundary is determined by c.)

In particular, we find such a subcritical Stein structure J' with $c_1(J') = c_1(J)$. Since W is simply connected, the analogue of Corollary 2.3 shows that J and J' are homotopic as almost complex structures. By Theorem 3.2, this implies that J and J' are actually Stein homotopic. Thus, as claimed, the stipulation that there be no 1-handles poses no restriction.

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