Canad. Math. Bull. Vol. **56** (1), 2013 pp. 127–135
http://dx.doi.org/10.4153/CMB-2011-162-6

Evolution of Eigenvalues along Rescaled Ricci Flow

Junfang Li

Abstract. In this paper, we discuss monotonicity formulae of various entropy functionals under various rescaled versions of Ricci flow. As an application, we prove that the lowest eigenvalue of a family of geometric operators $-4\Delta + kR$ is monotonic along the normalized Ricci flow for all $k \geq 1$ provided the initial manifold has nonpositive total scalar curvature.

1 Introduction

On a closed manifold M, let $g(t)$ be a smooth family of Riemannian metrics. The following Ricci flow equation was introduced by R. Hamilton:

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g),$$

where $\mathrm{Ric}(g)$ is the Ricci curvature tensor of the metric g. In this paper, we are interested in the following rescaled version of Ricci flow:

$$\frac{\partial g}{\partial t} = -2(\mathrm{Ric}(g) - \frac{s(t)}{n} g), \tag{1.1}$$

where $s(t)$ is a function only depending on time t. It is not hard to see the rescaled Ricci flow is equivalent to Ricci flow up to a homothetic rescale of the metric at each time. If one chooses

$$s = \frac{\int_M R d\mu}{\int_M d\mu},$$

then this is the normalized Ricci flow of R. Hamilton.

Special solutions of Ricci flow equation are called breathers if the metric $g(t)$ of the manifold M at two different times t_1, t_2 are the same up to a diffeomorphism and scaling. See the precise definition in Perleman's paper [10]. Inspired by Perelman's work, we will introduce various entropy functionals and discuss the monotonicity of these functionals along the Ricci flow and rescaled Ricci flow.

The purpose of studying entropy functionals is twofold. First, it leads to the monotonicity of lowest eigenvalues of geometric operators on the manifold along flows. Second, these monotonicity formulas can be used to classify Ricci breathers.

Received by the editors April 3, 2010.
Published electronically August 24, 2011.
The author's research was supported in part by NSF DMS-1007223.
AMS subject classification: **58C40**, 53C44.
Keywords: monotonicity formulas, Ricci flow.

In [8], to classify compact steady Ricci breathers, we introduced functionals

$$\mathcal{F}_k(g,f)=\int_M (kR+|\nabla f|^2)e^{-f}\,d\mu,$$

where $k\geq 1$ and derived the first variational formula under a coupled system (1.2) as follows:

$$\frac{d}{dt}\mathcal{F}_k(g_{ij},f)=2(k-1)\int_M |Rc|^2e^{-f}d\mu+2\int_M |R_{ij}+\nabla_i\nabla_j f|^2e^{-f}d\mu\geq 0. \tag{1.2}$$

See similar results in [9]. In particular, this yields the following theorem.

Theorem ([8]) *On a compact Riemannian manifold $(M,g(t))$, where $g(t)$ satisfies the Ricci flow equation for $t\in[0,T)$, the lowest eigenvalue λ_k of the operator $-4\Delta+kR$ is nondecreasing under the Ricci flow. The monotonicity is strict unless the metric is Ricci-flat.*

Moreover, to classify expanding Ricci breathers, in [8] we defined the following family of functionals $\mathcal{W}_{ek}$, which has monotonicity properties modelled on expanders:

$$\mathcal{W}_{ke}(g,\tau(t),f)=\tau^2\int_M\Big[k(R+\frac{n}{2\tau})+\Delta f\Big]\,e^{-f}\,d\mu.$$

Along the lines of [8], in this paper we will establish monotonicity formulae for various entropy functionals along rescaled Ricci flow. As applications, we obtain monotonicity formulae for the lowest eigenvalue of $-4\Delta+kR$ along rescaled Ricci flow for the case of $k\geq 1$ provided $\int_M R\,d\mu\leq 0$.

Theorem 1.1 *On a compact Riemannian manifold $(M^n,g(t))$, where $g(t)$ satisfies the normalized Ricci flow equation of Hamilton for $t\in[0,T)$, if $\lambda(t)$ denotes the lowest eigenvalue of the operator $-4\Delta+kR$ $(k\geq 1)$ at time t, then λ is nondecreasing under the normalized Ricci flow, provided the average total scalar curvature is nonpositive. The monotonicity is strict unless the metric is Einstein.*

2 Monotonicity Formulas under Rescaled Ricci Flow

In this section, we will use two different approaches to prove the monotonicity formulas along rescaled Ricci flow. The first approach is by direct computation: we first derive the evolution of various local and global quantities. Without loss of generality, all the global calculations can be understood to be carried out at one point with orthonormal coordinates which enable us to ignore the difference of upper and lower indices. We use R_{ij} to denote the Ricci curvature tensor under local coordinates, ∇f and f_i to denote the covariant derivative of f.

Lemma 2.1 *Under the coupled system*

$$\begin{aligned}\frac{\partial g_{ij}}{\partial t}&=-2\Big(R_{ij}-\frac{s}{n}g\Big),\\ \frac{\partial f}{\partial t}&=-\Delta f+|\nabla f|^2-R+s,\end{aligned}\tag{2.1}$$

https://doi.org/10.4153/CMB-2011-162-6 Published online by Cambridge University Press

the following evolution identities hold.

$$\partial_t g^{ij} = -2g^{ik}g^{jl}\Big(R_{kl} - \frac{s(t)}{n}g_{kl}\Big), \quad \partial_t(e^{-f}d\mu) = -\Delta e^{-f}d\mu,$$

$$\partial_t R = \Delta R + 2\Big(R_{ij} - \frac{s(t)}{n}g_{ij}\Big)R_{ij},$$

$$\partial_t \int_M Re^{-f}d\mu = 2\int_M \Big(|\operatorname{Ric}|^2 - \frac{s(t)}{n}R\Big)e^{-f}\,d\mu,$$

$$\partial_t|\nabla f|^2 = 2\Big(R_{ij}f_if_j - \frac{s(t)}{n}|\nabla f|^2\Big) + 2\nabla(-\Delta f + |\nabla f|^2 - R)\nabla f,$$

$$\partial_t \int_M |\nabla f|^2 e^{-f}\,d\mu = 2\int_M \Big(R_{ij}f_if_j - \frac{s(t)}{n}|\nabla f|^2 - \Delta^2 f + \frac{1}{2}|\nabla f|^2 - R\Big)e^{-f}\,d\mu,$$

$$\partial_t \int_M (R + |\nabla f|^2)e^{-f}\,d\mu = 2\int_M \Big(|\operatorname{Ric}|^2 - \frac{s(t)}{n}(R + |\nabla f|^2) + R_{ij}f_if_j$$
$$-\Delta^2 f + \frac{1}{2}|\nabla f|^2 - \Delta f\Big)e^{-f}\,d\mu,$$

Proof The first three identities can be calculated directly; see [4,8]. The fourth identity uses the second and third identities with integration by parts. The fifth identity is proved as follows.

$$\partial_t|\nabla f|^2 = \partial_t(g^{ij}f_if_j) = \partial_t g^{ij}f_if_j + 2\nabla f_t\nabla f$$
$$= 2\Big(R_{ij}f_if_j - \frac{s(t)}{n}|\nabla f|^2\Big) + 2\nabla(-\Delta f + |\nabla f|^2 - R)\nabla f.$$

The sixth identity is proved by the fifth identity and integration by parts. The last identity is based on the first six identities. ■

Recall the famous Ricci identities (see, for example, [4, p. 286]). We have

$$\nabla_i\nabla_j\nabla^i f = \nabla_j\nabla_i\nabla^i f - R^k_{iji}\nabla^k f = \nabla_j\nabla_i\nabla^i f + R_{kj}\nabla^k f,$$

where R^l_{ijk} represents the Riemann curvature $(3,1)$-tensor and R_{kj} denotes the Ricci curvature tensor. We use the following convention of Riemann curvature tensor throughout this paper:

$$R_m(X,Y)Z \equiv [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$

where $\{X,Y,Z\}$ are vector fields on the manifold. By using the contracted second Bianchi identity $\nabla^i R_{ij} = \frac{1}{2}\nabla_j R$, Ricci identities, and integration by parts, we can show the following lemma.

Lemma 2.2 *On a closed Riemannian manifold* (M^n, g)*, the following identities hold.*

$$\int_M |f_{ij}|^2 e^{-f}\,d\mu = \int_M \Big(\frac{1}{2}\Delta|\nabla f|^2 - \Delta^2 f - R_{ij}f_i f_j\Big)\, e^{-f}\,d\mu,$$

$$2\int_M R_{ij}f_{ij}e^{-f}d\mu = \int_M (2R_{ij}f_i f_j - \Delta R)e^{-f}\,d\mu,$$

$$\int_M |R_{ij} + f_{ij}|^2 e^{-f}d\mu = \int_M \Big(|\operatorname{Ric}|^2 + R_{ij}f_i f_j - \Delta R + \frac{1}{2}\Delta|\nabla f|^2 - \Delta^2 f\Big)\, e^{-f}\,d\mu,$$

where f_{ij} *is the Hessian of* f.

Proof We proved the first identity as follows.

$$\begin{aligned}\int_M |f_{ij}|^2 e^{-f}d\mu &= \int_M [\nabla_i(f_{ij}f_j) - \nabla_i(f_{ij})\nabla_j f)e^{-f}\,d\mu \\ &= \int_M \Big[\frac{1}{2}\Delta|\nabla f|^2 - (\nabla_j\Delta f + R_{ij}\nabla_i f)\nabla_j f\Big]\, e^{-f}\,d\mu \\ &= \int_M \Big[\frac{1}{2}\Delta|\nabla f|^2 - \Delta^2 f - R_{ij}f_i f_j\Big]\, e^{-f}\,d\mu.\end{aligned}$$

The second identity is proved by integration by parts and contracted Bianchi identity.

$$\begin{aligned}2\int_M R_{ij}f_{ij}e^{-f}d\mu &= -2\int_M \nabla_j R_{ij}f_i e^{-f}d\mu + 2\int_M R_{ij}f_i f_i e^{-f}\,d\mu \\ &= \int_M (2R_{ij}f_i f_j - \nabla_i R f_i)e^{-f}\,d\mu \\ &= \int_M (2R_{ij}f_i f_j - \Delta R)e^{-f}\,d\mu.\end{aligned}$$

Adding the first two identities with $\int_M |\operatorname{Ric}|^2 e^{-f}\,d\mu$, one can prove the third identity. ∎

Now we are ready to prove the following first variation formula for the entropy functionals $\mathcal{F}_k$.

Proposition 2.3 *Under the coupled system* (2.1)*, the first variation formula of*

$$\mathcal{F}_k(f, g) = \int_M (kR + \Delta f)e^{-f}d\mu$$

is the following

$$\frac{d}{dt}\mathcal{F}_k = -\frac{2s(t)}{n}\mathcal{F}_k + 2(k-1)\int_M |Rc|^2 e^{-f}\,d\mu + 2\int_M |R_{ij} + \nabla_i\nabla_j f|^2 e^{-f}\,d\mu. \tag{2.2}$$

Proof From the fourth identity of Lemma 2.1, we have

$$\partial_t (k-1)\int_M Re^{-f}\,d\mu = 2(k-1)\int_M (|\operatorname{Ric}|^2 - \frac{s(t)}{n}R)e^{-f}\,d\mu. \tag{2.3}$$

Combining the last identity of Lemma 2.1 and the last identity of Lemma 2.2, we have

$$\partial_t \int_M (R+|\nabla f|^2)e^{-f}\,d\mu = -\frac{2s(t)}{n}\int_M (R+|\nabla f|^2)e^{-f}\,d\mu + \int_M |R_{ij}+f_{ij}|^2 e^{-f}\,d\mu. \tag{2.4}$$

Adding (2.3) and (2.4), we prove the variation formula in the proposition. ∎

In the rest of this section, we will reveal the closed relation between the usual Ricci flow and rescaled Ricci flow. This will yield a new approach to deriving Proposition 2.3.

Suppose $g(\,\cdot\,,t)$ is a solution of Ricci flow equation

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij}.$$

For any given function $s(t)$, if

$$\varphi(t) = \frac{1}{1-\frac{2}{n}\int_0^t s(t)\,dt},$$

and $\bar t = \int_0^t \varphi(t)dt$, then $\bar g(\,\cdot\,,\bar t) = \varphi(t)g(\,\cdot\,,t)$ solves the rescaled Ricci flow equation

$$\frac{\partial}{\partial \bar t}\bar g_{ij} = -2\Big(\bar R_{ij} - \frac{s}{n}\bar g_{ij}\Big). \tag{2.5}$$

Clearly, there is a one-to-one relation between the Ricci flow equation and the rescaled Ricci flow. By direct computations, we can show the correspondence between the coupled system under Ricci flow and under the rescaled Ricci flow.

$$\left.\begin{aligned} &\mathcal{F}_k(g,f) = \textstyle\int_M \big(kR+\Delta f\big)\,e^{-f}d\mu\\ &\frac{\partial g_{ij}}{\partial t} = -2R_{ij}\\ &\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R \end{aligned}\right\} \overset{\varphi g=\bar g}{\Longleftrightarrow} \left\{\begin{aligned} &\mathcal{F}_k(g,f) = \varphi\mathcal{F}_k(\bar g,\bar f)\\ &\frac{\partial \bar g_{ij}}{\partial \bar t} = -2\Big(\bar R_{ij} - \frac{s}{n}\bar g_{ij}\Big)\\ &\frac{\partial \bar f}{\partial \bar t} = -\bar\Delta \bar f + |\bar\nabla \bar f|^2 - \bar R + s \end{aligned}\right. \tag{2.6}$$

Recall that the evolution equation of f and $\bar f$ guarantees that $\int_M e^f d\mu$ and $\int_M e^{\bar f}\,d\mu_{\bar g}$ are constants along Ricci flow and rescaled Ricci flow, respectively. Now we give a second proof of Proposition 2.3.

Second proof of Proposition 2.3 We use the relation between Ricci flow and rescaled Ricci flow and the first variation formula (1.2) (proved in [8]) to prove the formula (2.2). Introduce φ, $\bar{t}$ as above and we first prove (2.2) for metric $\bar{g}$. From the definition of φ, $\frac{d}{dt}(\frac{1}{\varphi}) = -\frac{2s}{n}$. Using the chart (2.6), under

$$\begin{cases} \dfrac{\partial \bar{g}_{ij}}{\partial t} = & -2\Big(\bar{R}_{ij} - \dfrac{s}{n}\bar{g}\Big) \\ \dfrac{\partial \bar{f}}{\partial t} = & -\bar{\Delta} f + |\bar{\nabla} f|^2 - \bar{R} + s, \end{cases}$$

by formula (2.2) we have

$$\begin{aligned} \frac{d}{d\bar{t}}\mathcal{F}_k(\bar{g}, \bar{f}) &= \frac{d}{d\bar{t}}\Big(\frac{1}{\varphi}\mathcal{F}_k(g, f)\Big) \\ &= \frac{1}{\varphi}\frac{d}{dt}\Big(\frac{1}{\varphi}\mathcal{F}_k(g, f)\Big) \\ &= -\frac{2s}{n}\frac{1}{\varphi}\mathcal{F}_k(g, f) + \frac{1}{\varphi^2}\frac{d}{dt}\mathcal{F}_k(g, f) \\ &= -\frac{2s}{n}\mathcal{F}_k(\bar{g}, \bar{f}) \\ &\quad + \frac{1}{\varphi^2}\Big[2(k-1)\int_M |Rc|^2 e^{-f}\, d\mu + 2\int_M |R_{ij} + \nabla_i\nabla_j f|^2 e^{-f}\, d\mu\Big] \\ &= -\frac{2s}{n}\mathcal{F}_k(\bar{g}, \bar{f}) + 2(k-1)\int_M |\bar{Rc}|^2 e^{-\bar{f}}\, d\bar{\mu} + 2\int_M |\bar{R}_{ij} + \bar{\nabla}_i\bar{\nabla}_j \bar{f}|^2 e^{-\bar{f}}\, d\bar{\mu}, \end{aligned}$$

where the last step follows from the rescaling properties of related geometric quantities under the change of the metric. Without ambiguity, we change the notation of $\bar{g}$ back to g. The formula still holds which finishes the proof. ■

By completing the square in the first variation of $\mathcal{F}_k$, we can show the following.

Corollary 2.4 *Let $g(t)$ be a smooth solution of the rescaled Ricci flow. Then under the coupled system* (2.1), *we have formula*

$$\text{(2.7)}\quad \frac{d}{dt}\mathcal{F}_k = \frac{2s}{n}(\mathcal{F}_k - ks) + 2(k-1)\int_M \Big|Rc - \frac{s}{n}g\Big|^2 e^{-f}\, d\mu + 2\int_M \Big|R_{ij} + \nabla_i\nabla_j f - \frac{s}{n}g\Big|^2 e^{-f}\, d\mu.$$

Remark 2.5 The first variation of (2.7) is interesting, because the second integrand vanishes on Einstein manifolds, the third integrand vanishes on gradient Ricci solitons. If one has control on the sign of the first quantity in the first variation, then we can derive a monotonicity formula for all $k \geq 1$. This will be achieved in the next section.

3 Monotonicity of Lowest Eigenvalues under Rescaled Ricci Flow

A direct consequence of (2.7) yields the following monotonicity property for lowest eigenvalues.

Theorem 3.1 *On a compact Riemannian manifold $(M^n, g(t))$, where $g(t)$ satisfies the rescaled Ricci flow equation* (2.5) *for $t \in [0, T)$, we denote $\lambda(t)$ to be the lowest eigenvalue of the operator $-4\Delta + kR$ ($k \geq 1$) at time t. Assume that there exists a function $\varphi(x,t) \in C^\infty(M^n \times [0,T))$, such that*

$$s(t) = \frac{\int_M (kR + |\nabla \varphi|^2) e^{-\varphi} d\mu}{k \int e^{-\varphi}\, d\mu}.$$

Then λ is nondecreasing under the rescaled Ricci flow (1.1)*, provided $s \leq 0$. The monotonicity is strict unless the metric is Einstein.*

Proof The lowest eigenvalue of the operator $-4\Delta + kR$ on a closed Riemannian manifold is $\lambda(g(t)) = \inf \mathcal{F}_k(g, f)$, where the infimum is taken among functions satisfying the normalization $\int_M e^{-f}\, d\mu$.

By eigenvalue perturbation theory, the lowest eigenvalue of the entropy functional can be attained by smooth functions. Assuming eigenvalues, $\lambda(t)$ is attained by $f(\cdot, t)$. One can derive the evolution of the eigenvalue along time by Corollary 2.4. We have

$$\begin{aligned}\frac{d\lambda}{dt} = \frac{2s}{n}(\lambda - ks) + 2(k-1)\int_M \Big| Rc - \frac{s}{n} g\Big|^2 e^{-f}\, d\mu \\ + 2\int_M \Big| R_{ij} + \nabla_i \nabla_j f - \frac{s}{n} g\Big|^2 e^{-f}\, d\mu,\end{aligned}$$

where $e^{-f/2}$ is the eigenfunction of λ. By assumption $s \leq 0$ and by definitions of λ, s, we have $\lambda \leq ks$ and $\frac{2s}{n}(\lambda - ks) \geq 0$. Hence,

$$\frac{d\lambda}{dt} \geq 2(k-1)\int_M \Big| Rc - \frac{s}{n} g\Big|^2 e^{-f}\, d\mu + 2\int_M \Big| R_{ij} + \nabla_i \nabla_j f - \frac{s}{n} g\Big|^2 e^{-f}\, d\mu \geq 0.$$

It is clear that the equality is attained if and only if $Rc - \frac{s(t)}{n} g \equiv 0$, which is an Einstein metric. ∎

In case of Hamilton's normalized Ricci flow, where

$$s = \frac{\int_M R d\mu}{\int_M d\mu}$$

is the average total scalar curvature and one can choose $\varphi(t) = \ln \mathrm{Vol}(M^n)$, we prove Theorem 1.1.

Remark 3.2 The above monotonicity property of lowest eigenvalues under normalized Ricci flow is dimensionless and works for all $k \geq 1$ and the case of $k > 1$ can be used to classify compact steady or expanding Ricci breathers directly, which, in fact, yields another proof for [8, Corollary 8.1]. See references for related results in [2, 5–8, 10].

Remark 3.3 Recently, similar results for Theorem 1.1 with $k = 1$ were obtained independently in [3].

As a special case, on a Riemann surface with negative Euler characteristic $\chi < 0$, we have the following corollary.

Corollary 3.4 *On a compact Riemann surface with negative Euler characteristic $\chi <$ 0, the lowest eigenvalue $\lambda(t)$ of $-4\Delta + kR$ $(k \geq 1)$ at time t is nondecreasing under the normalized Ricci flow. The monotonicity is strict unless the metric has constant curvature.*

In the case of $k = 1$, recall Perelman's $\bar{\lambda} = \lambda V^{2}/n$ invariant [10]. Since normalized Ricci flow preserves the volume and $\bar{\lambda}$ is a scale invariant, the monotonicity of lowest eigenvalues under normalized Ricci flow is equivalent to the monotonicity of $\bar{\lambda}$ under Ricci flow. Naturally, one could extend this definition and introduce a new scale invariant constant $\bar{\lambda}_k \equiv \lambda_k V^{2}/n$ which also has monotonicity properties along (normalized) Ricci flow and vanishes if and only if on an expanding *Einstein* manifold (instead of expanding gradient solitons in the $k = 1$ case).

Remark 3.5 There is some fundamental relation between $\bar{\lambda}$ and the Yamabe constant $\mathcal{Y}$ discussed in [1] and the references therein. Similar results can also be obtained for $\bar{\lambda}_k$.

Acknowledgements Part of this work was done when the author was a postdoctoralfellow at CRM and McGill University. We would like to thank both institutes for the support. We also thank Pengfei Guan for his encouragement and Xiaodong Cao for discussions.

References

[1] K. Akutagawa, M. Ishida, and C. LeBrun, *Perelman's invariant, Ricci flow, and the Yamabe invariants of smooth manifolds.* Arch. Math. **88**(2007), no. 1, 71–76.

[2] X. Cao, *Eigenvalues of* $(-\Delta + \frac{R}{2})$ *on manifolds with nonnegative curvature operator.* Math. Ann. **337**(2007), no. 2, 435–441.

[3] ———, *First eigenvalues of geometric operators under Ricci flow.* arxiv:math.DG/0710.3947v1.

[4] B. Chow and D. Knopf, *The Ricci Flow: An Introduction.* Mathematical Surveys and Monographs 110. American Mathematical Society, Providence, RI, 2004.

[5] B. Chow, S.-C. Chu, D. Glickenstein, et al., *The Ricci Flow: Techniques and Applications. Part I. Geometric Aspects.* Mathematical Surveys and Monographs 135. American Mathematical Society, Providence, RI, 2007.

[6] R. S. Hamilton, *The formation of singularities in the Ricci flow.* In: Surveys in Differential Geometry, Vol. II. Internat. Press, Cambridge, MA, 1995, pp. 7–136.

[7] T. Ivey, *Ricci solitons on compact three-manifolds.* Differential Geom. Appl. **3**(1993), no. 4, 301–307. http://dx.doi.org/10.1016/0926-2245(93)90008-O

[8] J.-F. Li, *Eigenvalues and energy functionals with monotonicity formulae under Ricci flow*. Math. Ann. **338**(2007), 927–946. http://dx.doi.org/10.1007/s00208-007-0098-y
[9] T. Oliynyka, V. Suneetab, and E. Woolgara, *Irreversibility of world-sheet renormalization group flow*. Phys. Lett. B,**610**(2005), no. 1-2, 115–121. http://dx.doi.org/10.1016/j.physletb.2005.01.077
[10] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*. arxiv:math.DG/0211159.

Department of Mathematics, University of Alabama at Birmingham, Birmingham, AL 35294
e-mail: jli@math.uab.edu