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On Convolutions of Convex Sets and Related Problems

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Abstract. We prove some results concerning convolutions, additive energies, and sumsets of convex sets and their generalizations. In particular, we show that if a set $A = \{a_1, \ldots, a_n\}_< \subseteq \mathbb{R}$ has the property that for every fixed $1 \le d < n$, all differences $a_i - a_{i-d}$, $d < i < n$, are distinct, then $|A + A| \gg |A|^{3/2+c}$ for a constant $c > 0$.

1 Introduction

We say that a set $A = \{a_1, \ldots, a_n\}$ of real numbers is *convex* if

$$a_i - a_{i-1} < a_{i+1} - a_i$$

for every $1 < i < n$. It is known that sumsets of convex sets are large, see [2–8]. The current best bounds

$$|A - A| \ge |A|^{8/5-o(1)} \quad \text{and} \quad |A + A| \ge |A|^{14/9-o(1)}$$

were proved in [11]. Furthermore, it was proved in [3, 8], that the additive energy of every convex set A satisfies $\mathsf{E}(A) \ll |A|^{5/2}$. Very recently it was improved by Shkredov [12], who showed that

$$\mathsf{E}(A) \ll |A|^{32/13+o(1)}.$$

Solymosi [13] proposed to consider the following wide generalization of a convex set. We call a monotone increasing set $A = \{a_1, \ldots, a_n\} \subseteq \mathbb{R}$ a *dcd*-set (distinct consecutive differences) if all consecutive differences of A are distinct *i.e.*, $a_i - a_{i-1} = a_j - a_{j-1}$ implies $i = j$. Solymosi [13] proved that if A is *dcd*-set, then for every set B we have

$$|A + B| \gg |A||B|^{1/2}.$$

As showed by Ruzsa [13], the above bound is best possible. However, Solymosi conjectured that $|A + A| \gg |A|^{3/2+c}$. One cannot extend the method used in [11] for *dcd*-sets for many reasons. The simplest one is that there exist *dcd*-sets with large additive energy. Let us consider the following example of a *dcd*-set: $A = P_1 \cup P_2$, where

$$P_1 = \{n, 2n, \ldots, (n/2)n\},$$
$$P_2 = \{n - 1, 2(n - 1), \ldots, (n/2)(n - 1)\},$$

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and n is an even integer. Since P_1 and P_2 are arithmetic progressions, we have $\mathsf{E}(A) \gg |A|^3$.

Here we consider another generalization of convex sets. We impose a stronger condition than Solymosis's, which has a combinatorial nature rather than a geometric one. We call a monotone increasing set $A = \{a_1, \dots, a_n\} \subseteq \mathbb{R}$, a *tdcd*-set (totaly distinct consecutive differences) if for every fixed $1 \le d < n$, all differences $a_i - a_{i-d}$, $d < i < n$, are distinct. However, such sets have also a geometric motivation. In the well-known Szemerédi-Trotter theorem [14] one considers a system of pseudo-lines *i.e.*, a family of continuous plane curves with the property that each two curves share at most one point in common. Every convex set of reals generates a convex curve in a natural way; it is enough to take the graph of any convex function with $f(i) = a_i$. Then, clearly any family of shifts of a convex curve is a pseudo-line system. If we consider a discrete version of the above construction, then a family of shifts of discrete graph $(i, f(i)) + (\alpha, \beta)$, $(\alpha, \beta) \in X$, is a discrete pseudo-line system if for all $(\alpha, \beta), (\alpha', \beta') \in X$ there is at most one solution to the equation

$$\big(i, f(i)\big) + (\alpha, \beta) = \big(j, f(j)\big) + (\alpha', \beta'),$$

which is equivalent to being $f(i)$ a *tdcd*-set.

We shows that there is a deeper difference in additive behavior of *dcd*-sets and *tdcd*-sets. We prove that for a *tdcd*-set A we have even $\mathsf{E}(A) \ll |A|^{5/2-c}$ for a constant $c > 0$, which clearly implies that $|A \pm A| \gg |A|^{3/2+c}$.

Furthermore, we will also study the additive energy of sets introduced by Bochkarev in [1]. For a given $\alpha > 0$, we call a set $A = \{a_1, \dots, a_n\}$, α-set if for every $i > j$ the equation $a_i - a_j = a_r - a_s$ with $r > s \ge j$ has $O((i-j)^\alpha)$ solutions. Hence if $i = j+1$, there are $O(1)$ solutions, so every α-set is almost a *dcd*-set for any α. Bochkarev, among other things, proved that if A is any α-set then $\mathsf{E}(A) \ll |A|^{3-\frac{1}{1+\alpha}}$. We improve this estimate for $\alpha \ge 1$ by showing that $\mathsf{E}(A) \ll |A|^{3-\frac{1}{1+\alpha}-c}$ for a constant $c > 0$ depending on α only.

Notation By $A(x)$ we denote the indicator function of a set $A \subseteq \mathbb{R}$. Let

$$(A * A)(x) = \sum_t A(t)A(x-t),$$
$$(A \circ A)(x) = \sum_t A(t)A(x+t).$$

The additive energy of a set A is defined by

$$\mathsf{E}(A) = \sum_x (A \circ A)(x)^2 = \sum_x (A * A)(x)^2.$$

We will also use higher additive energy introduced in [9, 11]

$$\mathsf{E}_3(A) = \sum_x (A \circ A)(x)^3.$$

2 Auxiliary Results

Let $A = \{a_1, \dots, a_n\}$ be a set of real numbers, $a_i > a_{i-1}$. Let $A - A = \{x_1, \dots, x_s\}$ and

$$(A \circ A)(x_1) \ge (A \circ A)(x_2) \ge \cdots \ge (A \circ A)(x_s).$$

The first result we will use is a version of Garaev's result (see [4, Theorem 2]), who used it to bound the additive energy of convex sets. Let J_H denote the number of solutions to

$$a_i - a_j = a_{i+h_1} - a_{j+h_2}, \quad 1 \le h_1, h_2 \le H. \tag{2.1}$$

Lemma 2.1 *Let $A \subseteq \mathbb{R}$ be a finite set. Then for every H*

$$(A \circ A)(x_r) \ll \frac{n}{H} + \frac{J_H}{r}.$$

Lemma 2.2 *Let $A \subseteq \mathbb{R}$ be a finite tdcd-set. Then $(A \circ A)(x_r) \ll |A|/r^{1/3}$. In particular, $\mathsf{E}(A) \ll |A|^{5/2}$ and $\mathsf{E}_3(A) \ll |A|^3 \log |A|$.*

Proof By the definition it follows that for fixed i, h_1, and h_2 there is at most one j such that

$$a_i - a_j = a_{i+h_1} - a_{j+h_2}, \quad 1 \le h_1, h_2 \le H.$$

Thus, we have at most $\le H^2 n$ solutions to (2.1), hence by Lemma 2.1,

$$(A \circ A)(x_r) \ll \frac{n}{H} + \frac{H^2 n}{r}.$$

Putting $H = \lceil r^{1/3} \rceil$, we obtain the required bound. ∎

Lemma 2.3 *Let $A \subseteq \mathbb{R}$ be a finite α-set. Then*

$$(A \circ A)(x_r) \ll |A|/r^{1/(1+\alpha)}.$$

In particular $\mathsf{E}(A) \ll |A|^{3-\frac{1}{1+\alpha}}$ and $\mathsf{E}_{2+\alpha}(A) \ll |A|^{2+\alpha} \log |A|$.

Proof The number of solutions to (2.1) equals the number of solutions to

$$a_{i+h_1} - a_i = a_{j+h_2} - a_j, \ 1 \le h_1, h_2 \le H.$$

Again, by the definition, assuming $i > j$, for fixed j and h_2 there are $O(h_2^\alpha) = O(H^\alpha)$ such solutions, so that by Lemma 2.1

$$(A \circ A)(x_r) \ll \frac{n}{H} + \frac{H^{1+\alpha} n}{r}.$$

Putting $H = \lceil r^{1/1+\alpha} \rceil$, we obtain the required bound. ∎

It is easy to observe that Lemma 2.2 holds for $(A * A)$ as well.

By a *consecutive difference* in a set $A = \{a_1, \dots, a_n\}_<$ we mean any difference of the form $a_i - a_{i-1}$. The next result can be easily extracted from the main theorem of [13].

Lemma 2.4 *Suppose that $A \subseteq \mathbb{R}$ has $\delta|A|$ distinct consecutive differences. Then for every finite set $B \subseteq \mathbb{R}$, $|A + B| \gg \delta|A||B|^{1/2}$.*

As mentioned in the introduction, for any α, each α-set A has $\Omega(|A|)$ distinct consecutive differences and therefore by Lemma 2.4, for every finite set $B \subseteq \mathbb{R}$

$$|A + B| \gg |A||B|^{1/2}.$$

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Lemma 2.5 *Let $A \subseteq \mathbb{R}$ be a finite set and suppose that $A' \subseteq A$, $|A'| = \delta|A|$. If A is a tdcd-set, then A' has at least $\frac{1}{2}\delta|A'| - 1$ distinct consecutive differences. If A is an α-set then A' has at least $\Omega((\delta/2)^\alpha(\frac{1}{2}\delta|A'| - 1))$ distinct consecutive differences.*

Proof Write $A' = \{a_{i_1}, \dots, a_{i_t}\}$, then $\{i_1, \dots, i_t\} \subseteq [n]$, $t \geq \delta n$. Since

$$\sum_{k=2}^{t}(i_k - i_{k-1}) \leq n,$$

it follows that at least $\frac{1}{2}t - 1$ differences $i_k - i_{k-1}$ are less than $2/\delta$. Therefore, there exist $1 \leq d \leq 2/\delta$ and a set $S \subseteq [t]$ such that $|S| \geq \frac{1}{2}\delta t - 1$, and for every consecutive elements s and s' in S we have $i_s - i_{s'} = d$. If A is a *tdcd*-set, then clearly, the consecutive differences $a_{i_s} - a_{i_{s'}}$ are distinct.

Next, if A is an α-set, then each consecutive difference has $O((i_s - i_{s'})^\alpha) = O((2/\delta)^\alpha)$ representations in the form $a_{i_s} - a_{i_{s'}}$ and therefore A' has

$$\Omega((\delta/2)^\alpha(\frac{1}{2}\delta|A'| - 1))$$

distinct consecutive differences. ∎

The next two lemmas that we will use in the proof of our main theorems were proved in [12, Theorem 34] and [10, Theorem 54], respectively.

Lemma 2.6 *Let A be a subset of an abelian group. Suppose that $\mathsf{E}(A) = |A|^3/K$ and $\mathsf{E}_3(A) = M|A|^4/K^2$. Then there exists $A' \subseteq A$ such that*

$$|A'| \gg |A|/M^{11} \quad \text{and} \quad |kA' - lA'| \ll M^{60(k+l)}K|A'|,$$

for every $k, l \in \mathbb{N}$.

Lemma 2.7 *Let A be a subset of an abelian group and $\alpha > 1$. Suppose that $\mathsf{E}(A) = |A|^3/K$ and $\mathsf{E}_{2+\alpha}(A) = M|A|^{3+\alpha}/K^{1+\alpha}$. Then there exists $A' \subseteq A$ such that*

$$|A'| \gg M^{-\frac{6\alpha-3}{\alpha(\alpha-1)}}|A| \quad \text{and} \quad |kA' - lA'| \ll M^{6(k+l)\frac{4\alpha-1}{\alpha(\alpha-1)}}K|A'|$$

for every $k, l \in \mathbb{N}$.

3 Proofs of the Main Results

Theorem 3.1 *Let $A \subseteq \mathbb{R}$ be a finite tdcd-set. Then there exists a positive constant c such that $\mathsf{E}(A) \ll |A|^{5/2-c}$.*

Proof Write $\mathsf{E}(A) = |A|^3/K$ and $M = K^2|A|^{-1}\log|A|$. Then by Lemma 2.6 there exists $A' \subseteq A$ such that

$$|A'| \gg |A|/M^{11} \quad \text{and} \quad |kA'| \ll M^{70k}K|A'|,$$

for every $k \in \mathbb{N}$. By Lemma 2.5 the set A' has at least $\Omega(|A'|/M^{11})$ distinct consecutive differences. By a straightforward induction and Lemma 2.4 we infer that

$$|kA'| \gg M^{-22}|A'|^{2-2^{-k+1}},$$

for every $k \in \mathbb{N}$. Comparing the upper and the lower bound on $|3A'|$ we obtain that $K \geq |A|^{1/2+c}$ for some positive constant c. ■

As an immediate consequence we obtain that there exists a constant $c > 0$ such that for every finite *tdcd*-set $A \subseteq \mathbb{R}$, we have $|A \pm A| \gg |A|^{3/2+c}$.

Theorem 3.2 *Let $\alpha \geq 1$. Then there exists a positive constant $c = c(\alpha)$ such that for every α-set, $A \subseteq \mathbb{R}$ $\mathsf{E}(A) \ll |A|^{3-\frac{1}{1+\alpha}-c}$.*

Proof Write $\mathsf{E}(A) = |A|^3/K$ and $M = K^{1+\alpha}|A|^{-1}\log|A|$. For $\alpha = 1$ we apply Lemma 2.6 as in Theorem 3.1, so we can assume that $\alpha > 0$. Then by Lemma 2.7 there exists $A' \subseteq A$ such that

$$|A'| \gg M^{-\frac{6\alpha-3}{\alpha(\alpha-1)}}|A| \quad \text{and} \quad |kA'| \ll M^{7k\frac{4\alpha-1}{\alpha(\alpha-1)}}K|A'|,$$

for every $k \in \mathbb{N}$. By Lemma 2.5 the set A' has at least $\Omega((2M)^{-1-\frac{6\alpha-3}{\alpha-1}}|A'|)$ distinct consecutive differences. By a straightforward induction and Lemma 2.4 we infer that

$$|kA'| \gg (2M)^{-2-\frac{12\alpha-6}{\alpha-1}}|A'|^{2-2^{-k+1}},$$

for every $k \in \mathbb{N}$. Again, comparing the upper and the lower bound on $|3A'|$ we obtain that $K \geq |A|^{\frac{1}{1+\alpha}+c}$ for some positive constant c, and the proof is completed. ■

Using a standard argument we get an estimate on L_1-norm of exponential sums over *tdcd*-sets and α-sets.

Corollary 3.3 *Let $A \subseteq \mathbb{R}$ be a finite tdcd-set. Then there exists a constant $c > 0$ such that for arbitrary coefficients $\gamma(a)$, $|\gamma(a)| = 1$,*

$$\int_0^1 \Big| \sum_{a \in A} \gamma(a) e^{2\pi i a x} \Big| \, dx \gg |A|^{1/4+c}.$$

If $A \subseteq \mathbb{R}$ be a finite α-set, then there exists a constant $c = c(\alpha) > 0$ such that for arbitrary coefficients $\gamma(a)$, $|\gamma(a)| = 1$

$$\int_0^1 \Big| \sum_{a \in A} \gamma(a) e^{2\pi i a x} \Big| \, dx \gg |A|^{\frac{1}{2(1+\alpha)}+c}.$$

Proof By the Parseval formula and Hölder's inequality we have

$$\begin{aligned} |A| &= \int_0^1 \Big| \sum_{a \in A} \gamma(a) e^{2\pi i a x} \Big|^2 dx \\ &\leq \Big(\int_0^1 \Big| \sum_{a \in A} \gamma(a) e^{2\pi i a x} \Big|^4 \Big)^{1/3} \Big(\int_0^1 \Big| \sum_{a \in A} \gamma(a) e^{2\pi i a x} \Big| \, dx \Big)^{2/3} \\ &\leq \mathsf{E}(A)^{1/3} \Big(\int_0^1 \Big| \sum_{a \in A} \gamma(a) e^{2\pi i a x} \Big| \, dx \Big)^{2/3}. \end{aligned}$$

Now the required inequality follows from Theorem 3.1. The second assertion can be proved in the same way. ■

4 Maximal Value of Convolution of Convex Sets

Here we are interested in the largest number of representation of a number as a sum of two elements from a convex set A. In particular, our bound improves Lemma 2.2 for $r \ll |A|$; however, it does not provide any better estimate for the additive energies.

Theorem 4.1 *Let $A \subseteq \mathbb{R}$ be a finite convex-set. Then for every x, we have*

$$(A * A)(x) \ll |A|^{2/3}.$$

Proof Suppose that $x \in \mathbb{R}$ has t distinct representations in $A + A$,

$$x = a_{i_1} + a_{j_1} = \cdots = a_{i_t} + a_{j_t},$$

where $i_1 < \cdots < i_t$ and $i_1 \le j_1, \ldots, i_t \le j_t$. Observe also that $i_u - i_v \ge j_u - j_v$ for all $u > v$. Arguing as in Lemma 2.5, there exist $1 \le d \le 2n/t$ and a set $S \subseteq [t]$ such that $|S| \ge \frac{t^2}{2n} - 1$ and for all $s \in S$ we have $i_s - i_{s-1} = d$. Thus, there are $m \gg t^2/n$ numbers $k_{i-1} < k_i \le l_i$ and $l'_i < l_i < l_{i-1}$, $i = 2, \ldots, m$ such that

$$x = a_{k_i} + a_{l_i} = a_{k_i+d} + a_{l'_i}.$$

Observe that by convexity

$$a_{l'_{i-1}} - a_{l_{i-1}} = a_{k_{i-1}+d} - a_{k_{i-1}} < a_{k_i+d} - a_{k_i} = a_{l'_i} - a_{l_i},$$

so that $l'_{i-1} - l_{i-1} < l'_i - l_i$. Therefore, we have

$$d = (k_m + d) - k_m \ge l_m - l'_m > \cdots > l_1 - l'_1 > 0,$$

hence $t^2/n \ll m \le d \le 2n/t$, and the assertion follows. ∎

Unlike Lemma 2.2, the above theorem does not hold for the convolution $(A \circ A)(x)$. To see this consider the following simple example. Let $k, l, m \in \mathbb{N}$ be such that

$$kl + 2\binom{k-1}{2} - \binom{k-2}{2} < m < kl + \binom{k-1}{2} + l + 1$$

and let $a_i = il + \binom{i-1}{2}$. Put

$$A = \{a_1, \ldots a_k\} \cup \{m + a_1, m + a_3, \ldots, m + a_t\},$$

where $t = 2\lceil k/2 \rceil - 1$. Then clearly A is a convex set, and $(A \circ A)(m) = \lceil k/2 \rceil \gg |A|$.

Furthermore, Theorem 4.1 cannot be extended for *tdcd*-sets. Indeed, let k and m be positive integers such that $2^{2k} < m$ and let

$$A = \{1, 2, 2^2, \ldots, 2^{2(k-1)}\} \cup \{m - 2^{2(k-1)}, m - 2^{2(k-2)}, \ldots, m - 2^2, m - 1\}.$$

We denote by X and Y the first and the second parts of the set A, respectively. Inside X and Y all differences are distinct, so it is enough to check the *tdcd* condition between X and Y. If $a_i, a_{i+d} \in X$ and $a_j, a_{j+d} \in Y$, then

$$a_{i+d} - a_i = 2^{i+d} - 2^i \neq 2^{2(2k-j)} - 2^{2(2k-j-d)} = a_{j+d} - a_j$$

for $d > 0$. Next, if $a_i, a_j \in X$ and $a_{i+d}, a_{j+d} \in Y$ then it is easy to see that

$$a_{i+d} - a_i = m - 2^{2(2k-i-d)} - 2^i \neq m - 2^{2(2k-j-d)} - 2^j = a_{j+d} - a_j.$$

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The condition is also satisfied for $a_i, a_j, a_{i+d} \in X$ and $a_{j+d} \in Y$ or $a_i \in X$ and $a_j, a_{j+d}, a_{i+d} \in Y$, because $m > 2^{2k}$. Thus, A is a *tdcd*-set, and clearly $(A * A)(m) \gg |A|$.

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