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A GENERALISATION OF FINITE *PT*-GROUPS

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Abstract

Let G be a group and $\sigma = \{\sigma_i \mid i \in I\}$ some partition of the set of all primes. A subgroup A of G is *σ-subnormal* in G if there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_m = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is a finite σ_j-group for some $j = j(i)$ for $i = 1, \ldots, m$, and it is *modular* in G if $\langle X, A \cap Z\rangle = \langle X, A\rangle \cap Z$ when $X \leq Z \leq G$ and $\langle A, Y \cap Z\rangle = \langle A, Y\rangle \cap Z$ when $Y \leq G$ and $A \leq Z \leq G$. The group G is called *σ-soluble* if every chief factor H/K of G is a finite σ_i-group for some $i = i(H/K)$. In this paper, we describe finite σ-soluble groups in which every σ-subnormal subgroup is modular.

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1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, σ is some partition of the set of all primes $\mathbb{P}$, that is, $\sigma = \{\sigma_i \mid i \in I\}$, where $\mathbb{P} = \bigcup_{i\in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for $i \neq j$. If G is a σ_i-group for some i, we say that G is *σ-primary* [13]. Following [10, page 54], we call G an *M-group* if the lattice $\mathcal{L}(G)$ of all subgroups of G is modular.

A subgroup H of G is said to be *quasinormal* (Ore) or *permutable* (Stonehewer) *in* G if H permutes with every subgroup L of G, that is, $HL = LH$. Quasinormal subgroups possess many interesting and useful properties. Every quasinormal subgroup is subnormal (Ore [7]) and so it is also σ-subnormal in the following sense.

Definition 1.1 [13]. A subgroup A of G is *σ-subnormal* in G if there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_n = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ-primary for $i = 1, \ldots, n$.

A subgroup M of G is called *modular in G* [9] if M is a modular element (in the sense of Kurosh [10, page 43]) of the lattice $\mathcal{L}(G)$, that is,

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(i) $\langle X, M \cap Z\rangle = \langle X, M\rangle \cap Z$ for all $X \leq G, Z \leq G$ such that $X \leq Z$; and
(ii) $\langle M, Y \cap Z\rangle = \langle M, Y\rangle \cap Z$ for all $Y \leq G, Z \leq G$ such that $M \leq Z$.

It is easy to show that every quasinormal subgroup of G is modular in G. Moreover, the following very interesting fact is true.

Theorem 1.2 [10, Theorem 5.1.1, page 43]. *The subgroup M of G is quasinormal in G if and only if M is modular and subnormal in G.*

The group G is called a *PT-group* [1, 2.0.2] if quasinormality is a transitive relation in G, that is, every subnormal subgroup of G is quasinormal in G. The description of PT-groups was first obtained by Zacher [15] for the soluble case (see Corollary 1.6 below), and by Robinson [8] for the general case on the basis of the classification of all nonabelian simple groups.

By Theorem 1.2, G is a PT-group if and only if every subnormal subgroup of G is modular in G. Bearing in mind this observation and the results in [8, 15], it seems to be natural to ask: what is the structure of G provided every σ-subnormal subgroup of G is modular in G? We will give a complete answer to this question in the case where G is σ-soluble in the following sense.

Definition 1.3. The group G is *σ-soluble* [13] if every chief factor of G is σ-primary, and *σ-decomposable* (Shemetkov [11]), or *σ-nilpotent* (Guo and Skiba [4]), if $G = G_1 \times \cdots \times G_t$ for some σ-primary groups $G_1, \ldots, G_t$.

Before continuing, we consider some examples.

Example 1.4. (i) In the classical case, when $\sigma = \sigma^0 = \{\{2\}, \{3\}, \{5\}, \ldots\}$, the group G is σ^0-soluble (respectively, σ^0-nilpotent) if and only if G is soluble (respectively, nilpotent). A subgroup A of G is σ^0-subnormal in G if and only if it is subnormal in G.

(ii) In the other standard case, when $\sigma = \sigma^\pi = \{\pi, \pi'\}$, the group G is σ^π-soluble (respectively, σ^π-nilpotent) if and only if G is π-separable (respectively, π-decomposable, that is, $G = O_\pi(G) \times O_{\pi'}(G)$). A subgroup A of a π-separable group G is σ^π-subnormal in G if and only if there is a subgroup chain

$$A = A_0 \leq A_1 \leq \cdots \leq A_n = G$$

such that $A_i/(A_{i-1})_{A_i}$ is either a π-group or a π'-group for $i = 1, \ldots, n$.

(iii) In the theory of π-soluble groups ($\pi = \{p_1, \ldots, p_n\}$), we deal with the partition $\sigma = \sigma^{0\pi} = \{\{p_1\}, \ldots, \{p_n\}, \pi'\}$ of $\mathbb{P}$. Note that G is $\sigma^{0\pi}$-soluble (respectively, $\sigma^{0\pi}$-nilpotent) if and only if G is π-soluble (respectively, π-nilpotent, that is, $G = O_{p_1}(G) \times \cdots \times O_{p_n}(G) \times O_{\pi'}(G)$). A subgroup A of G is $\sigma^{0\pi}$-subnormal in G if and only if there is a subgroup chain

$$A = A_0 \leq A_1 \leq \cdots \leq A_n = G$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is a π'-group for $i = 1, \ldots, n$. Therefore, A is $\sigma^{0\pi}$-subnormal in G if and only if it is $\mathfrak{F}$-subnormal in G in the sense of Kegel [6], where $\mathfrak{F}$ is the class of all π'-groups.

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(iv) Let p, q, r, t be distinct primes, where q divides $p-1$ and t divides $r-1$. Let Q be a simple $\mathbb{F}_q C_p$-module which is faithful for C_p, let $C_r \rtimes C_t$ be a nonabelian group of order rt, and let $A = C_t$. Finally, let $G = (Q \rtimes C_p) \times (C_r \rtimes C_t)$ and let B be a subgroup of order q in Q. Then $B < Q$ since $p > q$. It is not difficult to show that A is modular in G (see [10, Lemma 5.1.8]). On the other hand, A is σ-subnormal in G, where $\sigma = \{\{q, r, t\}, \{q, r, t\}'\}$, and so A is σ-quasinormal in G. It is clear also that A is not subnormal in G, so A is not quasinormal in G. Finally, note that B is subnormal but it is not modular in G by Lemma 2.2(i) below.

Now we can give an answer to the question posed above.

Theorem 1.5. *Let D be the σ-nilpotent residual of G, that is, the intersection of all normal subgroups N of G with σ-nilpotent quotient G/N. If G is σ-soluble and every σ-subnormal subgroup is modular in G, then:*

(i) *$G = D \rtimes L$, where D is an abelian Hall subgroup of G of odd order and L is a σ-nilpotent M-group;*
(ii) *every element of G induces a power automorphism in D; and*
(iii) *$O_{\sigma_i}(D)$ has a normal complement in a Hall σ_i-subgroup of G for all i.*

Conversely, if (i), (ii) and (iii) hold for some subgroups D and L of G, then every σ-subnormal subgroup is modular in G.

In view of [10, 2.3.2, 2.4.4], if G is a nilpotent M-group, then G is an *Iwasawa group* [1, 1.4.2], that is, every subgroup of G is quasinormal in G. Therefore in the case $\sigma = \sigma^0$ (see Example 1.4(i)), Theorem 1.5 gives the following well-known result.

Corollary 1.6 (Zacher [15]). *A group G is a soluble PT-group if and only if the following conditions hold:*

(i) *the nilpotent residual $D = G^{\mathfrak{N}}$ of G is an abelian Hall subgroup of odd order;*
(ii) *every element of G induces a power automorphism in D; and*
(iii) *G/D is an Iwasawa group.*

In the case $\sigma = \sigma^{\pi}$ (Example 1.4(ii)), Theorem 1.5 gives the following corollary.

Corollary 1.7. *Suppose that G is π-separable and let D be the π-decomposable residual of G, that is, the intersection of all normal subgroups N of G with π-decomposable quotient G/N. Then every σ^{π}-subnormal subgroup of G is modular in G if and only if the following conditions hold:*

(i) *$G = D \rtimes M$, where D is an abelian Hall subgroup of G of odd order and $M = O_{\pi}(M) \times O_{\pi'}(M)$ and every element of G induces a power automorphism in D;*
(ii) *$O_{\pi}(D)$ has a normal complement in a Hall π-subgroup of G;*
(iii) *$O_{\pi'}(D)$ has a normal complement in a Hall π'-subgroup of G.*

In the case $\sigma = \sigma^{0\pi}$ (Example 1.4(iii)), Theorem 1.5 gives the following corollary.

COROLLARY 1.8. *Suppose that G is π-soluble and let D be the π-nilpotent residual of G, that is, the intersection of all normal subgroups N of G with π-nilpotent quotient G/N. Then every $\sigma^{0\pi}$-subnormal subgroup of G is modular in G if and only if the following conditions hold:*

(i) *$G = D \rtimes M$, where D is an abelian Hall subgroup of G of odd order and $M = O_{p_1}(M) \times \cdots \times O_{p_n}(M) \times O_{\pi'}(M)$ and every element of G induces a power automorphism in D;*
(ii) *$O_{\pi'}(D)$ has a normal complement in a Hall π'-subgroup of G.*

2. Preliminaries

If $G = A \rtimes \langle t \rangle$ is nonabelian, where A is an elementary abelian p-group and t is an element of prime order $q \neq p$ which induces a nontrivial power automorphism on A, then we say that G is a *P-group of type* (p, q) (see [10, page 49]).

LEMMA 2.1 [10, Lemma 2.2.2(d)]. *If $G = A \rtimes \langle t \rangle$ is a P-group of type (p, q), then $\langle t \rangle^G = G$.*

The next two lemmas collect the properties of modular subgroups which we use in our proofs.

LEMMA 2.2 [10, Theorems 5.1.14 and 5.2.5]. *Let M be a modular subgroup of G. Then:*

(i) *M/M_G is nilpotent and every chief factor of G between M^G and M_G is cyclic.*
(ii) *If $M_G = 1$, then $G = S_1 \times \cdots \times S_r \times K$, where $0 \leq r \in \mathbb{Z}$ and for all $i, j \in \{1, \ldots, r\}$,*
    (a) *S_i is a nonabelian P-group,*
    (b) *$(|S_i|, |S_j|) = 1 = (|S_i|, |K|)$ for $i \neq j$,*
    (c) *$M = Q_1 \times \cdots \times Q_r \times (M \cap K)$ and Q_i is a nonnormal Sylow subgroup of S_i,*
    (d) *$M \cap K$ is quasinormal in G.*

LEMMA 2.3 [10, page 201]. *Let A, B and N be subgroups of G, where A is modular in G and N is normal in G.*

(i) *If B is modular in G, then $\langle A, B \rangle$ is modular in G.*
(ii) *AN/N is modular in G/N.*
(iii) *If $N \leq B$ and B/N is modular in G/N, then B is modular in G.*
(iv) *If $A \leq B$, then A is modular in B.*

LEMMA 2.4 [13, Lemma 2.6]. *Let A, K and N be subgroups of G. Suppose that A is σ-subnormal in G and N is normal in G.*

(i) *If $N \leq K$ and K/N is σ-subnormal in G/N, then K is σ-subnormal in G.*
(ii) *$A \cap K$ is σ-subnormal in K.*
(iii) *If A is a σ-*Hall subgroup *of G, then A is normal in G.*
(iv) *If $H \neq 1$ is a Hall σ_i-subgroup of G and A is not a σ_i'-group, then $A \cap H \neq 1$ and $A \cap H$ is a Hall σ_i-subgroup of A.*

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(v) AN/N *is* σ*-subnormal in* G/N.
(vi) *If* K *is a* σ*-subnormal subgroup of* A*, then* K *is* σ*-subnormal in* G.
(vii) *If* A *is a* σ_i*-group, then* $A \leq O_{\sigma_i}(G)$.

LEMMA 2.5 [5, Proposition 3.4]. *Every subgroup of a* σ*-nilpotent group is* σ*-subnormal.*

LEMMA 2.6 [13, Corollary 2.4 and Lemma 2.5]. *The class of all* σ*-nilpotent groups* $\mathfrak{N}_\sigma$ *is closed under taking products of normal subgroups, homomorphic images and subgroups. Moreover, if* E *is a normal subgroup of* G *and* $E/E \cap \Phi(G)$ *is* σ*-nilpotent, then* E *is* σ*-nilpotent.*

We will use $G^{\mathfrak{N}_\sigma}$ to denote the σ-nilpotent residual of G. In view of Lemma 2.6, the following lemma is a consequence of [2, Proposition 2.2.8].

LEMMA 2.7. *If* N *is a normal subgroup of* G*, then* $(G/N)^{\mathfrak{N}_\sigma} = G^{\mathfrak{N}_\sigma}N/N$.

LEMMA 2.8.

(i) *Every* M*-group is soluble.*
(ii) *If* $G = A \times B$*, where* A *is a Hall subgroup of* G *and* A *and* B *are* M*-groups, then* G *is an* M*-group.*
(iii) *Every subgroup and every quotient of an* M*-group is an* M*-group.*

PROOF. Statements (i) and (ii) are corollaries of Iwasawa's theorem on the structure of M-groups [10, 2.4.4].

As in the Introduction, we use $\mathcal{L}(G)$ to denote the lattice of all subgroups of G. Suppose that R is a subgroup of an M-group G. Then $\mathcal{L}(R) \subseteq \mathcal{L}(G)$, so R is an M-group. Finally, suppose that R is normal in G. Then $\mathcal{L}(G/R)$ is isomorphic to the interval $[G/R]$ in the modular lattice $\mathcal{L}(G)$. Hence G/N is an M-group. □

LEMMA 2.9 [12, Theorem A]. *If* G *is* σ*-soluble, then* G *possesses a Hall* σ_i*-subgroup for all* i.

A subgroup H of a σ-soluble group G is said to be *σ-permutable in G* [13] if H permutes with every Hall σ_i-subgroup of G for all i.

LEMMA 2.10 [14, Theorem A]. *Suppose that* G *is* σ*-soluble and let* $D = G^{\mathfrak{N}_\sigma}$*. If* D *is nilpotent and every* σ*-subnormal subgroup of* G *is* σ*-permutable in* G*, then:*

(i) $G = D \rtimes L$*, where* D *is an abelian Hall subgroup of* G *of odd order and* L *is a* σ*-nilpotent group;*
(ii) *every element of* G *induces a power automorphism in* D*; and*
(iii) $O_{\sigma_i}(D)$ *has a normal complement in a Hall* σ_i*-subgroup of* G *for all* i.

PROPOSITION 2.11. *Suppose that the subgroup* H *of* G *is modular and* σ*-subnormal in* G*. If* G *possesses a Hall* σ_i*-subgroup, then* H *permutes with every Hall* σ_i*-subgroup of* G.

PROOF. Suppose the statement is false and let G be a counterexample of minimal order. Then $HV \neq VH$ for some Hall σ_i-subgroup V of G.

It is clear that V is a Hall σ_i-subgroup of $\langle H, V\rangle$. On the other hand, H is modular and σ-subnormal in $\langle H, V\rangle$ by Lemmas 2.3(iv) and 2.4(ii). In the case where $\langle H, V\rangle < G$, the choice of G implies $HV = VH$. Therefore $\langle H, V\rangle = G$.

Since H is σ-subnormal in G, there is a subgroup chain $H = H_0 \leq H_1 \leq \cdots \leq H_n = G$ such that either $H_{i-1} \trianglelefteq H_i$ or $H_i/(H_{i-1})_{H_i}$ is σ-primary for $i = 1, \ldots, n$.

We can assume without loss of generality that $M = H_{n-1} < G$. Then H permutes with every Hall σ_i-subgroup U of M for every i. Moreover, the modularity of H in G implies that

$$M = M \cap \langle H, V\rangle = \langle H, M \cap V\rangle.$$

On the other hand, by Lemma 2.4(iv), $M \cap V$ is a Hall σ_i-subgroup of M. Hence $M = H(M \cap V) = (M \cap V)H$. If $V \leq M_G$, then $H(M \cap V) = HV = VH$ and so $V \nleq M_G$.

Now note that $VM = MV$. Indeed, if M is normal in G, it is clear. Otherwise, G/M_G is σ-primary and so $G = MV = VM$ since $V \nleq M_G$ and V is a Hall σ_i-subgroup of G. Therefore

$$VH = V(M \cap V)H = VM = MV = H(M \cap V)V = HV.$$

This contradiction completes the proof of the lemma. □

3. Proof of Theorem 1.5

PROOF OF NECESSITY. First suppose that G is a σ-soluble group such that every σ-subnormal subgroup of G is modular in G. We show that conditions (i), (ii) and (iii) hold for G. Assume that this is false and let G be a counterexample of minimal order. Then $D = G^{\mathfrak{N}_\sigma} \neq 1$, that is, G is not σ-nilpotent.

Claim (a). The hypothesis holds for every quotient G/N of G.

Let H/N be a σ-subnormal subgroup of G/N. Then H is a σ-subnormal subgroup of G by Lemma 2.4(i), so H is modular in G by hypothesis. Hence H/N is modular in G/N by Lemma 2.3(ii) and this proves (a).

Claim (b). G/D is an M-group and therefore $D \neq 1$.

In view of Lemmas 2.5 and 2.6, every subgroup of G/D is σ-subnormal in G/D. Therefore G/D is an M-group by claim (a), so $D \neq 1$ by the choice of G.

Claim (c). D is nilpotent.

Assume this is false and let R be a minimal normal subgroup of G. First note that $RD/R = (G/R)^{\mathfrak{N}_\sigma}$ is abelian by Lemma 2.7 and claim (a). Therefore $R \leq D$ and R is the unique minimal normal subgroup of G. For otherwise, if N is any other minimal normal subgroup of G, then $D \simeq D/1 = D/R \cap N$, so that D is abelian. Finally, $R \nleq \Phi(G)$ by Lemma 2.6. Therefore $C_G(R) \leq R$ by [3, A, 15.2]. Now let V be a maximal subgroup of R. Suppose that $V \neq 1$. Then $V_G = 1$ and $R \leq V^G$. Since G is σ-soluble, R is σ-primary and so V is σ-subnormal in G by Lemma 2.4(vi). Therefore

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V is modular in G by hypothesis, so $|R| = p$ for some prime p by Lemma 2.2(i). Hence $C_G(R) = R$ and so $G/R = C_G(R)$ is cyclic, which implies that G is supersoluble. But then $D = G^{\mathfrak{N}_\sigma} \leq G' \leq F(G)$ and so D is nilpotent, a contradiction. This proves (c).

Final contradiction for the necessity.

Since G is σ-soluble by hypothesis, from Lemma 2.9 and Proposition 2.11 it follows that every σ-subnormal subgroup of G is σ-permutable in G. Therefore, in view of Lemma 2.10 and claim (c), $G = D \rtimes L$, where $L \simeq G/D$ is an M-group and conditions (i), (ii) and (iii) hold for G. □

Proof of sufficiency. Now we show that if conditions (i), (ii) and (iii) hold for some subgroups D and L, then every σ-subnormal subgroup H of G is modular in G. Suppose that this is false, that is, some σ-subnormal subgroup H of G is not modular in G. Let G be a counterexample with $|G| + |H|$ minimal. Then $D \neq 1$. Moreover, G is soluble by Lemma 2.8, and the following statement holds.

Claim (1). Either for some subgroups $X \leq G, Z \leq G$, where $X \leq Z$,

$$\langle X, H \cap Z\rangle \neq \langle X, H\rangle \cap Z, \tag{$*$}$$

or for some subgroups $X \leq G, Z \leq G$, where $H \leq Z$,

$$\langle H, X \cap Z\rangle \neq \langle H, X\rangle \cap Z. \tag{$**$}$$

Claim (2). The hypothesis holds on every quotient G/N of G.

First note that $G/N = (DN/N) \rtimes (LN/N)$, where $DN/N \simeq D/D \cap N$ is an abelian Hall subgroup of G/N of odd order and $LN/N \simeq L/L \cap N$ is a σ-nilpotent M-group by Lemma 2.8(iii) and so condition (i) holds for G/N. Moreover, if V/N is any subgroup of DN/N, then $V = N(D \cap V)$ and so, in fact, V/N is normal in G/N since $D \cap V$ is normal in G by condition (ii). Hence condition (ii) holds for G/N.

Condition (iii) implies that $O_{\sigma_i}(D)$ has a normal complement S in a Hall σ_i-subgroup E of G for every i. Then EN/N is a Hall σ_i-subgroup of G/N and SN/N is normal in EN/N. Since D is nilpotent, $O_{\sigma_i}(D)N/N = O_{\sigma_i}(DN/N)$. Hence

$$(SN/N)(O_{\sigma_i}(DN/N)) = (SN/N)(O_{\sigma_i}(D)N/N) = EN/N$$

and

$$\begin{aligned}(SN/N) \cap O_{\sigma_i}(DN/N) &= (SN/N) \cap (O_{\sigma_i}(D)N/N) = N(S \cap O_{\sigma_i}(D)N)/N \\ &= N(S \cap O_{\sigma_i}(D))(S \cap N)/N = N/N.\end{aligned}$$

Hence condition (iii) also holds on G/N.

Claim (3). $H_G = 1$.

Assume $H_G \neq 1$. The hypothesis holds for G/H_G by claim (2). On the other hand, H/H_G is σ-subnormal in G/H_G by Lemma 2.4(v) and so H/H_G is modular in G/H_G by the choice of G. But then H is modular in G by Lemma 2.3(iii), a contradiction, and this proves claim (3).

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Claim (4). H is a σ_i-group for some i and $H \leq L^x$ for all $x \in G$.

Claim (3) implies that $H \cap D = 1$, so $H \simeq HD/D \leq G/D$ is σ-nilpotent by Lemma 2.6 and hence $H = A_1 \times \cdots \times A_n$ for some σ-primary groups $A_1, \ldots, A_n$. Then $H = A_1$ is a σ_i-group for some i since otherwise H is modular in G by Lemma 2.3(i) and the choice of (G, H).

Let M_i be the Hall σ_i-subgroup of L and E be a Hall σ_i-subgroup of G containing M_i. Lemma 2.4(iv) implies that $H \leq E^x$ for all $x \in G$. Therefore, if $E \cap D = 1$, then M_i is a Hall σ_i-subgroup of G and so $H \leq L^x$ for all $x \in G$.

Now suppose that $E \cap D \neq 1$. Then $H \leq E^x = O_{\sigma_i}(D) \times M_i^x$ by condition (iii) since D is a nilpotent Hall subgroup of G, so $H \leq M_i^x \leq L^x$.

Claim (5). The Hall σ_j-subgroups of G are M-groups for all j.

Let A be a Hall σ_j-subgroup of G. If $A \cap D = 1$, then $A \simeq AD/D \leq G/D$, where G/D is an M-group. Hence A is an M-group by Lemma 2.8(iii). Now let $A \cap D \neq 1$. Then $A = (A \cap D) \times S$ by condition (iii), where S is a Hall subgroup of A. Then A is an M-group by Lemma 2.8(ii) because $A \cap D$ and $S \simeq DS/D \leq G/D$ are M-groups.

Claim (6). The subgroup H is modular in every proper subgroup E of G containing H.

It is enough to show that the hypothesis holds for E. First note that $D \cap E$ is a normal abelian Hall π-subgroup of E of odd order, where $\pi = \pi(D)$, and if V is a Hall π'-subgroup of E, then $V \leq L^x$ for some $x \in G$ since G is soluble and L is a Hall π'-subgroup of G. Therefore $E = (D \cap E) \rtimes V$, where V is an M-group by Lemma 2.8(iii). Hence condition (i) holds for $(E, D \cap E, V)$. It is clear also that condition (ii) holds for $D \cap E$. Finally, let $E_i \leq H_i$, where E_i is a Hall σ_i-subgroup of E and H_i is a Hall σ_i-subgroup of G. Then, by condition (iii), $H_i = O_{\sigma_i}(D) \times S$ and so $E_i = E_i \cap (O_{\sigma_i}(D) \times S) = (E_i \cap O_{\sigma_i}(D)) \times (E_i \cap S)$, where $E_i \cap O_{\sigma_i}(D) = O_{\sigma_i}(D \cap E)$. Hence condition (iii) also holds for $(E, D \cap E, V)$. This proves (6).

Claim (7). $\langle X, H \rangle = G$.

Suppose that $E = \langle X, H \rangle < G$ and let $Z_0 = Z \cap E$. Then H is modular in E by claim (6). In the case where $X \leq Z$,

$$\langle X, H \rangle \cap Z = Z_0 = Z_0 \cap \langle X, H \rangle = \langle X, Z_0 \cap H \rangle = \langle X, (Z \cap \langle H, X \rangle) \cap H \rangle = \langle X, H \cap Z \rangle,$$

contrary to ($*$). On the other hand, in the case where $H \leq Z$, similarly

$$\langle X, H \rangle \cap Z = Z_0 = Z_0 \cap \langle H, X \rangle = \langle H, Z_0 \cap X \rangle = \langle H, X \cap Z \rangle,$$

which is impossible by ($**$). Hence $\langle H, X \rangle = G$.

Claim (8). $D \leq X$.

It is clear that $X = (D \cap X) \rtimes X_1$, where $X_1 \leq L^x$ for some $x \in G$. Claim (4) implies that $H \leq L^x$. Hence $\langle X_1, H \rangle \leq L^x$ and, from claim (7),

$$G = \langle X, H \rangle = \langle (D \cap X) \rtimes X_1, H \rangle = (D \cap X)\langle X_1, H \rangle = D \rtimes L^x.$$

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Thus,

$$D = D \cap (D \cap X)\langle X_1, H\rangle = (D \cap X)(D \cap \langle X_1, H\rangle) = D \cap X$$

and so $D \leq X$.

Claim (9). $Z \cap D = 1$ *and therefore* $Z \leq L^x$ *for some* $x \in G$.

Suppose that $Z_0 = Z \cap D \neq 1$. Claim (8) implies that $Z_0 \leq X$. In view of Lemma 2.4(v), HZ_0/Z_0 is σ-subnormal in G/Z_0. Therefore from claim (2) and the choice of G it follows that HZ_0/Z_0 is modular in G/Z_0. Hence in the case $X \leq Z$,

$$\langle X/Z_0, (HZ_0/Z_0) \cap (Z/Z_0)\rangle = \langle X/Z_0, HZ_0/Z_0\rangle \cap (Z/Z_0),$$

which implies that

$$\langle X, (H \cap Z)\rangle = \langle XZ_0, H \cap Z\rangle = \langle X, Z_0(H \cap Z)\rangle = \langle X, HZ_0 \cap Z\rangle = \langle X, HZ_0\rangle \cap Z,$$

and so

$$\langle X, (H \cap Z)\rangle = \langle X, H\rangle \cap Z$$

since evidently

$$\langle X, H \cap Z\rangle \leq \langle X, H\rangle \cap Z.$$

In the case $H \leq Z$, similarly,

$$\langle H, X \cap Z\rangle = \langle H, Z_0(X \cap Z)\rangle = \langle HZ_0, X \cap Z\rangle = \langle HZ_0, X\rangle \cap Z = \langle H, X\rangle \cap Z.$$

But this situation is impossible by claim (1). This contradiction shows that $Z \cap D = 1$. Hence $Z \leq L^x$ for some $x \in G$ since G is soluble and L is a Hall π'-subgroup of G where $\pi = \pi(D)$.

Claim (10). $X \nleq Z$.

Otherwise, we have $D \leq Z$, which is impossible by claim (9) since $D \neq 1$.

Claim (11). $L^x = \langle H, L^x \cap X\rangle$.

Let $1 < Z_0 \leq D$. Claim (2) and the choice of G imply that HZ_0/Z_0 is modular in G/Z_0. Moreover, claims (1) and (10) imply that $H \leq Z$. Also, in view of claim (4), $H \leq L^x$. Therefore from claim (7),

$$\begin{aligned} L^xZ_0/Z_0 &= (L^xZ_0/Z_0) \cap \langle HZ_0/Z_0, X/Z_0\rangle \\ &= \langle HZ_0/Z_0, (L^xZ_0/Z_0) \cap (X/Z_0)\rangle = \langle HZ_0/Z_0, Z_0(L^x \cap X)/Z_0\rangle, \end{aligned}$$

and so

$$Z_0 \rtimes L^x = Z_0 \rtimes \langle H, (L^x \cap X)\rangle,$$

where L^x and $\langle H, (L^x \cap X)\rangle$ are Hall π'-subgroups of $Z_0 \rtimes L^x$ and $\pi = \pi(D)$. This proves (11).

Final contradiction for the sufficiency.

Claim (9) implies that $Z \leq L^x$ for some $x \in G$. Then

$$Z = \langle H, Z \cap (L^x \cap X)\rangle = \langle H, Z \cap X\rangle$$

by claim (11) since $L^x \simeq G/D$ is an M-group. But this is impossible by claims (1) and (10). □

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References

[1] A. Ballester-Bolinches, R. Esteban-Romero and M. Asaad, *Products of Finite Groups* (Walter de Gruyter, Berlin, 2010).
[2] A. Ballester-Bolinches and L. M. Ezquerro, *Classes of Finite Groups* (Springer, Dordrecht, 2006).
[3] K. Doerk and T. Hawkes, *Finite Soluble Groups* (Walter de Gruyter, Berlin, 1992).
[4] W. Guo and A. N. Skiba, 'Finite groups with permutable complete Wielandt sets of subgroups', *J. Group Theory* **18** (2015), 191–200.
[5] W. Guo and A. N. Skiba, 'Finite groups whose n-maximal subgroups are σ-subnormal', Preprint, 2016, arXiv:1608.03353 [math.GR].
[6] O. H. Kegel, 'Untergruppenverbände endlicher Gruppen, die den Subnormalteilerverband echt enthalten', *Arch. Math.* **30**(3) (1978), 225–228.
[7] O. Ore, 'Contributions in the theory of groups of finite order', *Duke Math. J.* **5** (1939), 431–460.
[8] D. J. S. Robinson, 'The structure of finite groups in which permutability is a transitive relation', *J. Aust. Math. Soc.* **70** (2001), 143–159.
[9] R. Schmidt, 'Modulare Untergruppen endlicher Gruppen', *Illinois J. Math.* **13** (1969), 358–377.
[10] R. Schmidt, *Subgroup Lattices of Groups* (Walter de Gruyter, Berlin, 1994).
[11] L. A. Shemetkov, *Formations of finite groups* (Nauka, Main Editorial Board for Physical and Mathematical Literature, Moscow, 1978).
[12] A. N. Skiba, 'A generalization of a Hall theorem', *J. Algebra Appl.* **15**(4) (2015), 21–36.
[13] A. N. Skiba, 'On σ-subnormal and σ-permutable subgroups of finite groups', *J. Algebra* **436** (2015), 1–16.
[14] A. N. Skiba, 'Characterizations of some classes of finite σ-soluble $P\sigma T$-groups', *J. Algebra*, to appear.
[15] G. Zacher, 'I gruppi risolubili finiti in cui i sottogruppi di composizione coincidono con i sottogruppi quasi-normali', *Atti Accad, Naz. Lincei Rend. cl. Sci. Fis. Mat. Natur.* **37**(8) (1964), 150–154.

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