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WEAKLY PRIME SUBMODULES AND PRIME SUBMODULES

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Abstract. A proper submodule N of an R-module M is called a *weakly prime* submodule, if for each submodule K of M and elements a, b of R, $abK \subseteq N$, implies that $aK \subseteq N$ or $bK \subseteq N$. In this paper we will study weakly prime submodules and we shall compare weakly prime submodules with prime submodules.

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1. Introduction. Throughout this paper all rings are commutative with identity and all modules are unitary. Also we consider R to be a ring and M a unitary R-module.

Let N be a proper submodule of M. It is said that N is a *prime* submodule of M, if the condition $ra \in N$, $r \in R$ and $a \in M$ implies that $a \in N$ or $rM \subseteq N$. In this case, if $P = (N : M) = \{t \in R | \ tM \subseteq N\}$, we say that N is a P-prime submodule of M, and it is easy to see that P is a prime ideal of R. Prime submodules have been studied in several papers such as [**1**–**4**], [**6**–**8**], [**10**].

A proper submodule N of M is called a *weakly prime* submodule, if for each submodule K of M and elements a, b of R, $abK \subseteq N$, implies that $aK \subseteq N$, or $bK \subseteq N$.

Weakly prime submodules have been introduced and studied in [**5**]. If we consider R as an R-module, then prime submodules and weakly prime submodules are exactly prime ideals of R. More generally for every multiplication module any submodule is a prime submodule if and only if it is a weakly prime submodule. For every R-module, it is easy to see that any prime submodule is a weakly prime submodule, but the converse is not always correct. For example let R be a ring with $dim\ R \neq 0$, and $P \subset Q$ a chain of prime ideals of R. Then it is easy to see that for the free R-module $R \oplus R$, the submodule $P \oplus Q$ is a weakly prime submodule which is not a prime submodule.

Recall that a proper submodule N of a module M is said to be a *primary* submodule if the condition $ra \in N$, $r \in R$ and $a \in M$, implies that $a \in N$ or $r^n M \subseteq N$, for some positive number n.

In this note, we will find some relations between prime submodules and weakly prime submodules. It is proved that any weakly prime submodule is a prime submodule if and only it is a primary submodule. Also any irreducible and weakly prime submodule is a prime submodule.

It is proved that:

(1) If F is a flat R-module and N a weakly prime submodule of M such that $F \otimes N \neq F \otimes M$, then $F \otimes N$ is a weakly prime submodule of $F \otimes M$.

(2) If F is a faithfully flat R-module and N a submodule of M, then N is a weakly prime submodule of M, if and only if $F \otimes N$ is a weakly prime submodule of $F \otimes M$.

2. Some comparisons. In the following, we compare some properties of weakly prime submodules with properties of prime submodules.

LEMMA 2.1. *Let M be an R-module and N a proper submodule of M.*

(i) N is a weakly prime submodule if and only if for every submodule K of M not contained in N, $(N : K)$ is a prime ideal of R. In particular $(N : M)$ is a prime ideal of R.

(ii) Let N be a weakly prime submodule of M. Then for all submodules K and L of M not contained in N, $(N : K) \subseteq (N : L)$ or $(N : L) \subseteq (N : K)$.

Proof. The proof is obvious. □

COROLLARY 2.2. *Let M be an R-module and N a proper submodule of M. Then N is a prime submodule if and only if N is primary and weakly prime.*

Proof. Let N be primary and weakly prime, and $rx \in N$, where $x \notin N$. Then there exists a positive number n such that for each $y \in M \setminus N$, $r^n y \in N$, i.e., $r^n \in (N : y)$. By Lemma 2.1, (i), $(N : y)$ is a prime ideal, then $r \in (N : y)$. Hence for each $y \in M$, we have, $ry \in N$, that is, $rM \subseteq N$. The converse is clear. □

THEOREM 2.3. *Let M be an R-module and N a proper submodule of M. The following are equivalent.*

(i) N is a weakly prime submodule.

(ii) For any $x, y \in M$, if $(N : x) \neq (N : y)$, then $N = (N + Rx) \cap (N + Ry)$.

Proof. $(i) \Longrightarrow (ii)$ Let $r \in (N : x) \setminus (N : y)$, where $r \in R$, i.e., $rx \in N$ and $ry \notin N$. Since by Lemma 2.1, (i), $(N : y)$ is a prime ideal, it is easy to see that $(N : y) = (N : ry)$. If $t \in (N + Rx) \cap (N + Ry)$, then $t = n_1 + r_1 x = n_2 + r_2 y$, where $n_1, n_2 \in N$ and $r_1, r_2 \in R$. Note that $rt = rn_1 + r_1 rx = rn_2 + r_2 ry$ and $r_1 rx, rn_1, rn_2 \in N$, so $r_2 ry \in N$, that is $r_2 \in (N : ry) = (N : y)$. Since $r_2 y \in N$, we have $t = n_2 + r_2 y \in N$.

$(ii) \Longleftarrow (i)$ It is enough to show that if $r_1 r_2 a \in N$, where $r_1, r_2 \in R$, $a \in M$ and $r_1 a \notin N$, then $r_2 a \in N$. We have, $r_1 \in (N : r_2 a) \setminus (N : a)$, so $(N : r_2 a) \neq (N : a)$. Put $x = r_2 a$, $y = a$, then by our assumption we have, $N = (N + Rr_2 a) \cap (N + Ra)$. Evidently, $r_2 a \in (N + Ra) \cap (N + Rr_2 a) = N$. □

From the definition of prime submodule, it is easy to see that if N is a prime submodule of an R-module M and $x, y \in M$ such that $rx \in N$, where $r \in R$, then $N = N + Rx$, or $N = N + Rry$. Compare this note with the following corollary, part (i).

COROLLARY 2.4. *Let M be an R-module, N a weakly prime submodule of M and $x, y \in M$.*

(i) If $rx \in N$ where $r \in R$, then $N = (N + Rx) \cap (N + Rry)$.

(ii) If N is an irreducible submodule, then N is a prime submodule.

Proof. (i) If $ry \in N$, then obviously, $N = (N + Rx) \cap (N + Rry)$. Now let $ry \notin N$. So $(N : x) \neq (N : y)$, and by Theorem 2.3, we have $N \subseteq (N + Rx) \cap (N + Rry) \subseteq (N + Rx) \cap (N + Ry) = N$.

(ii) Let $rx \in N$ where $r \in R$. By part (i), for each $y \in M$, we have, $N = (N + Rx) \cap (N + Rry)$, and since N is irreducible, $x \in N$ or $ry \in N$. □

PROPOSITION 2.5. *Let A_i, $1 \leq i \leq n$ be a finite collection of ideals of a ring R and let M be the free R-module $\oplus_{i=1}^{n} R$. Then $\oplus_{i=1}^{n} A_i$ is a weakly prime submodule of M if and only if $\{A_i \mid A_i \neq R\}$ is a non-empty chain of prime ideals of R.*

Proof. The proof is straightforward. □

3. Weakly prime submodules and flat modules. Let M be an R-module and N a submodule of M. In this section for every $a \in R$, we consider $(N : a)$ to be:

$$(N : a) = \{m \in M | \ am \in N\}.$$

It is easy to see that $(N : a)$ is a submodule of M containing N. The following lemma will give us a characterization of weakly prime submodules.

LEMMA 3.1. *Let M be an R-module and N a proper submodule of M. Then N is a weakly prime submodule of M if and only if for every $a, b \in R$, $(N : ab) = (N : a)$ or $(N : ab) = (N : b)$.*

Proof. Let N be a weakly prime submodule of M. It is easy to see that $(N : ab) = (N : a) \cup (N : b)$. Now since $(N : a) \cup (N : b) = (N : ab)$ is a submodule of M, we have $(N : a) \subseteq (N : b)$ or $(N : b) \subseteq (N : a)$. Hence $(N : ab) = (N : a)$, or $(N : ab) = (N : b)$.

For the converse let $abm \in N$, where $a, b \in R$ and $m \in M$. By our assumption we may suppose that $(N : ab) = (N : a)$. Thus $m \in (N : ab) = (N : a)$, that is, $am \in N$. So N is a weakly prime submodule of M. □

LEMMA 3.2. *Let M be an R-module, N a submodule of M and $a \in R$. Then for every flat R-module F, we have $F \otimes (N : a) = (F \otimes N : a)$.*

Proof. Clearly $F \otimes (N : a) \subseteq (F \otimes N : a)$. Consider the exact sequence $0 \longrightarrow (N : a) \xrightarrow{\subseteq} M \xrightarrow{g_a} \frac{M}{N}$, where $g_a(m) = am + N$, $\forall m \in M$. Since F is a flat module and $\theta : F \otimes \frac{M}{N} \longrightarrow \frac{F\otimes M}{F\otimes N}$ induced by $\theta(f \otimes (m + N)) = (f \otimes m) + F \otimes N$, $\forall m \in M$, $\forall f \in F$ is an isomorphism, we have the following exact sequence

$$0 \longrightarrow F \otimes (N : a) \xrightarrow{\subseteq} F \otimes M \xrightarrow{1\otimes g'_a} \frac{F \otimes M}{F \otimes N},$$

where $(1 \otimes g'_a)(f \otimes m) = a(f \otimes m) + F \otimes N$, $\forall m \in M, \forall f \in F$. Consequently $F \otimes (N : a) = Ker(1 \otimes g'_a) = (F \otimes N : a)$. □

THEOREM 3.3. *Let M be an R-module.*

(i) If F is a flat R-module and N a weakly prime submodule of M such that $F \otimes N \neq F \otimes M$, then $F \otimes N$ is a weakly prime submodule of $F \otimes M$.

(ii) Let F be a faithfully flat R-module. Then N is a weakly prime submodule of M if and only if $F \otimes N$ is a weakly prime submodule of $F \otimes M$.

Proof. (i) Let $a, b \in R$. By Lemma 3.1, we may suppose that $(N : ab) = (N : a)$. Now, by Lemma 3.2, we have $(F \otimes N : ab) = F \otimes (N : ab) = F \otimes (N : a) = (F \otimes N : a)$, that is, $(F \otimes N : ab) = (F \otimes N : a)$. Hence by Lemma 3.1, $F \otimes N$ is a weakly prime submodule of $F \otimes M$.

(ii) Let N be a weakly prime submodule of M and $F \otimes N = F \otimes M$. Therefore, $0 \longrightarrow F \otimes N \xrightarrow{\subseteq} F \otimes M \longrightarrow 0$ is an exact sequence, and since F is a faithfully flat module, then $0 \longrightarrow N \xrightarrow{\subseteq} M \longrightarrow 0$ is an exact sequence. Hence $N = M$, which is a contradiction. So $F \otimes N \neq F \otimes M$. Now by part i), $F \otimes N$ is a weakly prime submodule of $F \otimes M$.

Conversely suppose that $F \otimes N$ is a weakly prime submodule of $F \otimes M$. We have, $F \otimes N \neq F \otimes M$ and obviously $N \neq M$. Let $a, b \in R$. We may assume that $(F \otimes N :$

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$ab) = (F \otimes N : a)$, by Lemma 3.1. Then by Lemma 3.2, we have $F \otimes (N : a) = (F \otimes N : a) = (F \otimes N : ab) = F \otimes (N : ab)$. So $0 \longrightarrow F \otimes (N : a) \xrightarrow{\subseteq} F \otimes (N : ab) \longrightarrow 0$ is an exact sequence, and since F is faithfully flat, $0 \longrightarrow (N : a) \xrightarrow{\subseteq} (N : ab) \longrightarrow 0$ is an exact sequence, which implies that $(N : a) = (N : ab)$. Hence by Lemma 3.1, N is a weakly prime submodule of M. □

A theorem similar to Theorem 3.3 for prime submodules has been proved in [**2**]. It is easy to see that a proper submodule N of an R-module M is a prime submodule if and only if for every $a \in R$, $(N : a) = N$ or $(N : a) = M$. Now by a proof similar to that of Theorem 3.3, we can show this theorem for prime submodules, which is different from the mentioned proof in [**2**].

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REFERENCES

1. A. Azizi, Intersectin of prime submodules and dimension of modules, *Acta Math. Sci.* **25B** (3) (2005), 385–394.

2. A. Azizi, Prime submodules and flat modules, *Acta Math. Sinica, English Series*, to appear.

3. A. Azizi, Weak multiplication modules, *Czech Math. J.* **53** (**128**) (2003), 529–534.

4. A. Azizi and H. Sharif, On prime submodules, *Honam Math. J.* **21** (1) (1999), 1–12.

5. M. Behboodi and H. Koohi, Weakly prime submodules, *Vietnam J. Math.* **32** (2) (2004), 185–195.

6. C. P. Lu, Prime submodules of modules, *Comment. Math. Univ. St. Paul* **33** (1) (1984), 61–69.

7. C. P. Lu, Spectra of modules, *Comm. Algebra* **23** (10) (1995), 3741–3752.

8. R. L. McCasland and M. E. Moore, Prime submodules, *Comm. Algebra* **20** (6) (1992), 1803–1817.

9. H. Matsumura, *Commutative ring theory* (Cambridge University Press, 1992).

10. P. F. Smith, Primary modules over commutative rings, *Glasgow Math. J.* **43** (2001), 103–111.