FREE DISTRIBUTIVE P-ALGEBRAS: A NEW APPROACH†

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It is well known (Lee [13]) that the class of all distributive p-algebras $\mathbf{B} = \mathbf{B}_\omega$ is a variety and that the class of all subvarieties of $\mathbf{B}$ forms a chain

$$\mathbf{B}_{-1} \subseteq \mathbf{B}_0 \subseteq \mathbf{B}_1 \subseteq \ldots \subseteq \mathbf{B}_n \subseteq \ldots \subseteq \mathbf{B}_\omega$$

where $\mathbf{B}_{-1}$ is the trivial class, $\mathbf{B}_0$ is the class of Boolean algebras, and $\mathbf{B}_1$ is the class of Stone algebras.

Urquhart [14] described the finitely generated free algebras in all classes $\mathbf{B}_n$ for $1 \leq n \leq \omega$ (see Berman and Dwinger [4] or Köhler [12]). The free Stone algebras were studied by Balbes and Horn [3], Chen [5] and Katriňák [9],[10]. Davey and Goldberg [7] gave a characterization of the free algebras $FDp_n(X)$ in the classes $\mathbf{B}_n$, $1 \leq n \leq \omega$, using the topological duality of Priestley.

In this paper we give an intrinsic algebraic characterization of $FDp_n(X)$ for all $1 \leq n \leq \omega$. We shall use the method of constructing the free extensions of posets in the class of distributive lattices and preserving some prescribed bounds (Dean [6]). This method has successfully been used to determine the free p-algebras $Fp_n(X)$ in [11].

1. Preliminaries. A (distributive) *p-algebra* is an algebra $L = (L; \vee, \wedge, ^*, 0, 1)$, where $(L; \vee, \wedge, 0, 1)$ is a bounded (distributive) lattice and the unary operation * is characterized by

$$a \leq b^* \text{if and only if} a \wedge b = 0$$

In any p-algebra L we can define the set of *closed* elements

$$B(L) = \{x \in L : x = x^{**}\}$$

It is known that $(B(L); +, \wedge, ^*, 0, 1)$ is a Boolean algebra, where

$$a + b = (a^* \wedge b^*)^*$$

As we mentioned above, the class $\mathbf{B} = \mathbf{B}_\omega$ of all distributive p-algebras is equational (see [2] or [8]). The subvariety $\mathbf{B}_n$, $1 \leq n < \omega$, is defined by the following identity

$$(L_n)(x_1 \wedge \ldots \wedge x_n)^* \vee \left(x_1^* \wedge \ldots \vee x_n\right)^* \vee \ldots \vee \left(x_1 \wedge \ldots \wedge x_n^*\right)^* = 1$$

(cf. [13]). Similarly as in [11], we shall work in the class of distributive lattices with the free extensions of posets preserving some bounds (see [6]): Consider a poset $(T; \leq)$ with families $\mathcal{L}$ and $\mathcal{U}$ of finite subsets of T such that

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(i) $p \leq q$ in T implies $\{p, q\} \in \mathcal{L}$ and $\{p, q\} \in \mathcal{U}$,
(ii) $S \in \mathcal{L}(S \in \mathcal{U})$ implies that there exists $\inf_T S(\sup_T S)$.

Following [6] a distributive lattice $\mathrm{FD}(T; \mathcal{L}, \mathcal{U})$ is called a *free distributive lattice* generated by T (a *free distributive extension of* T) and preserving bounds from $\mathcal{L}$ and $\mathcal{U}$, if it satisfies the following conditions:

(i) $T \subseteq \mathrm{FD}(T; \mathcal{L}, \mathcal{U})$ and for $a, b \in T$, $a \leq b$ in T if and only if $a \leq b$ in $\mathrm{FD}(T; \mathcal{L}, \mathcal{U})$;
(ii) For $S \in \mathcal{L}$, $\inf_T S = \bigwedge(s : S \in S)$ and for $S \in \mathcal{U}$, $\sup_T S = \bigvee(s : s \in S)$;
(iii) $[T] = \mathrm{FD}(T; \mathcal{L}, \mathcal{U})$, i.e. T generates $\mathrm{FD}(T; \mathcal{L}, \mathcal{U})$;
(iv) Let M be a distributive lattice and let $\varphi : T \to M$ be an isotone mapping with the properties $\varphi(\sup_T s) = \bigvee(\varphi(s) : s \in S)$ for every $S \in \mathcal{U}$ and $\varphi(\inf_T S) = \bigwedge(\varphi(s) : s \in S)$. Then there exists a (lattice) homomorphism $\eta : \mathrm{FD}(T; \mathcal{L}, \mathcal{U}) \to M$ extending φ, i.e. $\eta \restriction T = \varphi$.

2. The poset associated with a set. In order to see how to introduce this concept, we begin by observation of four facts formulated in Lemmas 1–4.

LEMMA 1. *Let L be a distributive p-algebra. Then the following statements are equivalent:*

(i) *L satisfies the identity (L_n), $1 \leq n < \omega$;*
(ii) $(x_1 \wedge \ldots \wedge x_{n+1})^* = (x_2 \wedge \ldots \wedge x_{n+1})^* \vee \ldots \vee (x_1 \wedge \ldots \wedge x_n)^*$;
(iii) $(x_1 \vee \ldots \vee x_{n+1})^{**} = (x_2 \vee \ldots \vee x_{n+1})^{**} \vee \ldots \vee (x_1 \vee \ldots \vee x_n)^{**}$;
(iv) $a_1 \vee \ldots \vee a_{n+1} = t = a_1 + \ldots + a_{n+1}$, *whenever* $a_1, \ldots, a_{n+1} \in B(L)$ *and* $a_i + a_j = t$ *for any* $i \neq j$.

Proof. The equivalence of (i), (ii) and (iii) can be found in Wafker [15]. Assume now (iii). Take $t = a_1 + \ldots + a_{n+1}$ for $a_1, \ldots, a_{n+1} \in B(L)$ satisfying $a_i + a_j = t$ if $i \neq j$. We shall prove (iv.). For $n = 1$ it follows from (iii). Assume $n \geq 2$. Let $x_i = a_1 \wedge \ldots \wedge a_{i-1} \wedge a_{i+1} \wedge \ldots \wedge a_{n+1}$. Then

$$a_i = (x_1 \vee \ldots \vee x_{i-1} \vee x_{i+1} \vee \ldots \vee x_{n+1})^{**} = x_1 + \ldots + x_{i-1} + x_{i+1} + \ldots + x_{n+1}$$

by distributivity and the assumption. Clearly,

$$(x_1 \vee \ldots \vee x_{n+1})^{**} = (a_1 \vee \ldots \vee a_{n+1})^{**} = t$$

Therefore,

$$t = a_1 \vee \ldots \vee a_{n+1}$$

by (iii).

Conversely, suppose that (iv) is true. Take $x_1, \ldots, x_{n+1} \in L$ and put

$$a_i = (x_1 \vee \ldots \vee x_{i-1} \vee \ldots \vee x_{n+1})^{**}.$$

It is easy to verify that

$$\begin{aligned}\left(x_1^{**} + \ldots + x_{i-1}^{**} + x_{i+1}^{**} + \ldots + x_{n+1}^{**}\right) + \left(x_1^{**} + \ldots + x_{j-1}^{**} + x_{j+1}^{**} + \ldots + x_{n+1}^{**}\right) \\ = a_1 + a_j = x_1^{**} + \ldots + x_{n+1}^{**} \\ = (x_1 \vee \ldots \vee x_{n+1})^{**} = t\end{aligned}$$

for $i \neq j$, and this implies (iii).

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LEMMA 2. *Let the distributive p-algebra L be generated by a subset X, i.e.* $[X] = L$. *Then the set* $X^{**} = \{z \in L : z = x^{**}, x \in X\}$ *generates* $B(L)$ *in the class of Boolean algebras, i.e.* $[X^{**}]_{Bool} = B(L)$.

LEMMA 3. *Let* $L = \mathrm{FD}p_n(X)$ *be a free p-algebra freely generated by X in the variety* $\mathbf{B}_n$, $1 \leq n \leq \omega$. *Then* $B(L) = \mathrm{FB}(X^{**})$ (= *the free Boolean algebra freely generated by* X^{**}).

LEMMA 4. *Let L be a distributive p-algebra generated by as subset X. Then* $X \cup B(L)$ *generates L in the class of (distributive) lattices, i.e.* $L = [X \cup B)L)]_{Lat}$.

Lemmas 2–4 are straightforward consequences of [**11**; Lemmas 2–4]. We recall that a poset associated with a set X was defined in [**11**] as follows:

DEFINITION 1. Let X be a set. Take a disjoint copy $\bar{X} = \{\bar{x} : x \in X\}$ and construct a free Boolean algebra $\mathrm{FB}(\bar{X})$. We can assume $X \cap \mathrm{FB}(\bar{X}) = \emptyset$. Now we define the poset $P(X) = (P(X); \leq)$ *associated with* X as follows:

(i) $P(X)$ is bounded, i.e. $0 \leq u \leq 1$ for every $u \in P(X)$ and $0, 1 \in \mathrm{FB}(\bar{X})$;
(ii) $a \leq u$ and $0 \neq a \in \mathrm{FB}(\bar{X})$ if and only if $u \in \mathrm{FB}(\bar{X})$ and $a \leq u$ in $\mathrm{FB}(\bar{X})$;
(iii) $x \leq \bar{x}$ for every $x \in X$;
(iv) $x \leq u$ for $x \in X$ if and only if $\bar{x} \leq u$ or $x = u$.

Denote by $\mathrm{FB}(\bar{X})$ the free Boolean algebra $(\mathrm{FB}(\bar{X}); +, ., ', 0, 1)$. It remains to be said which existing glb's and lub's in $P(X)$ should be preserved.

DEFINITION 2. Let $P(X)$ be the poset associated with the given set X. Set

(i) $\mathcal{L} = \mathcal{L}_\omega = \mathcal{L}_1 = \ldots = \mathcal{L}_n \ldots$ and $A \in \mathcal{L}$ if and only if A is a finite subset of $\mathrm{FB}(\bar{X})$ or $A = \{a, b\} \subseteq P(X)$ and $a \leq b$ in $P(X)$;
(ii) $\mathcal{U} = \mathcal{U}_\omega$, where $A \in \mathcal{U}$ if and only if $A = \{a, b\} \subseteq P(X)$ and $a \leq b$ in $P(X)$;
(iii) $A \in \mathcal{U}_n$ for $1 \leq n < \omega$ if and only if $A \in \mathcal{U}$ or $A = \{a_1, \ldots, a_{n+1}\}$ such that $a_1, \ldots, a_{n+1} \in \mathrm{FB}(\bar{X})$ and $a_1 + \ldots + a_{n+1} = a_i + a_j$ for any $i \neq j$.

Now we shall show that the lattice $\mathrm{FD}(P(X); \mathcal{L}_n, \mathcal{U}_n)$ for $1 \leq n \leq \omega$ do exist. Note that $a \wedge b = a \cdot b$ for any $a, b \in \mathrm{FB}(\bar{X})$ in $\mathrm{FD}(P(X); \mathcal{L}_n, \mathcal{U}_n)$.

First we need a new concept from Balbes [**1**]. Suppose that we have a set I. Let (E) denote a set of lattice inequalities of the form.

$$x_{i_1} \wedge \ldots \wedge x_{i_n} \leq y_{j_1} \vee \ldots \vee y_{j_m}$$

where $\{i_1, \ldots, i_n\}$, $\{j_1, \ldots, j_m\} \subseteq I$. A distributive lattice L is called (E)-*free* if there exists a subset $A = \{a_i\}_{i \in J}$ of L with $I \subseteq J$ such that.

(i) $[A] = L$;
(ii) the set $\{a_i\}_{i \in J}$ satisfies (E) i.e.

$$a_{i_1} \wedge \ldots \wedge a_{i_n} \leq a_{j_1} \vee \ldots \vee a_{j_m}$$

for every inequality from (E);

(iii) whenever $\{b_i\}_{i \in J}$ is a subset of a distributive lattice M such that $\{b_i\}_{i \in J}$ satisfies (E), then there exists a homomorphism $f : L \to M$ such that $f(a_i) = b_i$ for all $i \in J$.

THEOREM 1. *Let $P(X)$ be the poset associated with a set X. Then* $\mathrm{FD}(P(X); \mathcal{L}_n, \mathcal{U}_n)$ *exists for every* $1 \leq n \leq \omega$.

Proof. First we set $I = P(X)$. Now we define the set (E_n) of inequalities in $\{x_i\}_{i\in I}$ as follows:

(a) $x_i \leq x_j$ if and only if $i \leq j$ in $P(X)$;
(b) $x_{i_j} \wedge \ldots \wedge x_{i_k} \leq x_j$ if and only if $\{i_1, \ldots, i_k\} \subseteq \mathrm{FB}(\bar{X})$ and $j = i_1 \ldots i_k$ in $\mathrm{FB}(\bar{X})$;
(c) $x_t \leq x_{j_1} \vee \ldots \vee x_{j_{n+1}}$ if and only if $t = j_1 + \ldots + j_{n+1}$ for $j_1, \ldots, j_{n+1} \in FB(\bar{X})$ and $t = j_i + j_k$ whenever $i \neq k$.

By [**1**; Theorem 1.9], there exists an (E_n)-free distributive lattice H_n. The properties of H_n can be summarized in other words as follows [**1**; Theorem 1.8]:

(i) there exists an order preserving embedding $\varepsilon : P(X) \to H_n$;
(ii) $\{\varepsilon(i) : i \in P(X)\}$ satisfies (E_n);
(iii) whenever $\{b_i\}_{i\in I} \subseteq M$ and M is a distributive lattice such that $\{b_i\}_{i\in I}$ satisfies (E_n), then there exists a homomorphism $f : H_n \to M$ such that $f(\varepsilon(i)) = b_i$.

Now it is easy to verify that $H_n = \mathrm{FD}(P(X2); \mathcal{L}_n, \mathcal{U}_n)$, and the proof is complete.

REMARK 1. Following [**1**; Theorem 1.9] the lattice H_n from the proof of Theorem 1 can be constructed as follows: Consider mappings

$$s : P(X) \to \mathbf{2} = (\{0, 1\})$$

i.e. $s \in \mathbf{2}^{P(X)}$. We say that $s \in \mathbf{2}^{P(X)}$ satisfies the inequality

$$x_{i_j} \wedge \ldots \wedge x_{i_n} \leq x_{j_1} \vee \ldots \vee x_{j_m}$$

for $i_1, \ldots, i_n, j_1, \ldots, j_m \in P(X)$, if

$$s(i_1) = \ldots = s(i_n) = 1 \text{ and } s(j_1) = \ldots = s(j_m) = 0.$$

Let $Iq(E_n)$ denote the set of $s \in \mathbf{2}^{P(X)}$ which satisfy the inequalities from (E_n).
Now, define $A_i^{(n)} = \{s \in \mathbf{2}^{P(X)} : s(i) = 1 \text{ and } s \in \mathbf{2}^{P(X)} - Iq(E_n)\}, i \in P(X)$. The sublattice of $\mathbf{2}^{P(X)}$ generated by the set $\{A_i^{(n)} : i \in P(X)\}$ is a bounded distributive lattice. This lattice, $[\{A^{(n)} : i \in P(X)\}]$, is H_n.
Our next task is to establish that $\mathrm{FD}(P(X); \mathcal{L}_n, \mathcal{U}_n)$ is isomorphic to $\mathrm{FD}p_n(X)$. For the sake of clarity we shall adapt [**11**; Lemmas 5–7] for the distributive case.

LEMMA 5. *Let $P(X)$ denote the poset associated with a set X. Then there exists a (unique) lattice-epimorphism*

$$\pi : H_n = \mathrm{FD}(P(X); \mathcal{L}_n, \mathcal{U}_n) \to \mathrm{FB}(\bar{X})$$

for every $1 \leq n \leq \omega$ *such that*

(i) $\pi(x) = \bar{x}$ *for every* $x \in X$,
(ii) $\pi(a) = a$ *for every* $a \in \mathrm{FB}(\bar{X})$,
(iii) $u \leq \pi(u)$ *for every* $u \in H_n$,
(iv) $u \leq a$ *and* $a \in \mathrm{FB}(\bar{X})$ *implies* $\pi(u) \leq a$.

Now, we are in position to formulate the next result.

THEOREM 2. *Let $P(X)$ denote the poset associated with a set X. Then the free distributive p-algebra* $\mathrm{FD}p_n(X)$ *in the class* $\mathbf{B}_n$, $1 \le n \le \omega$, *and the distributive lattice* $H_n = \mathrm{FD}(P(X); \mathcal{L}_n, \mathcal{U}_n)$ *are isomorphic as lattices. More precisely,* H_n *can be considered as a p-algebra such that*

(i) $u^* = \pi(u)'$ *for every* $u \in H_n$,
(ii) $H_n = [X]$,
(iii) $B(H_n) \simeq (\mathrm{FB}(\bar{X}); +, ., ', 0, 1)$,
(iv) $H_n \in \mathbf{B}_n$, *i.e.* H_n *satisfies the identity* (L_n).

The proofs of Lemma 5 and Theorem 2 are essentially the same as of [**11**; Lemmas 5–7] and of [**11**; Theorem 1] (see also Lemma 1).

3. Construction of $\mathbf{FD}p_n(X)$. Theorem 2 lacks a certain simplicity which it ought to have. Our work in this section will remedy this defect.

The following definition is crucial.

DEFINITION 3. Let $P(X)$ denote the poset associated with a set X. A subset $\emptyset \neq S \subseteq P(X)$ is said to be an *n-order-filter* $(1 \le n \le \omega)$ if

(i) S is increasing, i.e. $S = [S)$,
(ii) $S \cap \mathrm{FB}(\bar{X})$ is a filter (= dual ideal) of the Boolean algebra $\mathrm{FB}(\bar{X})$,
(iii) $t = a_1 + \ldots + a_{n+1} \in S$ and $t = a_i + a_j$ for any $i \neq j$ imply $a_i \in S$ for some $1 \le i \le n+1$, whenever $1 \le n < \omega$ and $a_1, \ldots, a_{n+1} \in \mathrm{FB}(\bar{X})$.

REMARK 2. It is easy to see that a k-order-filter is an n-order-filter, whenever $k \le n$. For the sake of simplicity we shall often say "order-filter" instead of "ω-order-filter". Next we shall consider mappings $s : P(X) \to \mathbf{2}$. Let $Ker(s)$, for $s \in \mathbf{2}^{P(X)}$, denote the set $\{i \in P(X) : s(i) = 1\}$.

LEMMA 6. *Let $P(X)$ be the poset associated with a set X. Then $s \in \mathbf{2}^{P(X)} - Iq(E_n)$*[1] *if and only if $Ker(s)$ is an n-order-filter* $(1 \le n \le \omega)$.

Proof. Suppose that $s \in \mathbf{2}^{P(X)} - Iq(E_n)$. Consider $Ker(s) = S \subseteq P(X)$. Let $i \le j$ in $P(X)$ and $i \in Ker(s)$. Since s does not satisfy

$$x_i \le x_j$$

and $s(i) = 1$, we get $s(j) = 1$. Thus, $j \in Ker(s)$ and $Ker(s)$ is increasing. Similarly, $i_1, i_2 \in S \cap \mathrm{FB}(\bar{X})$ and $j = i_1 . i_j$ in $\mathrm{FB}(\bar{X})$ imply $j \in S$, because s does not satisfy

$$x_{i_1} \wedge x_{i_2} \le x_j.$$

Assume now that $1 \le n < \omega$ and $t = a_1 + \ldots + a_{n+1} \in S$ such that $a_i + a_j = t$ for any $i \neq j$ and $a_1, \ldots, a_{n+1} \in \mathrm{FB}(\bar{X})$. Since s does not satisfy

$$x_t \le x_{a_1} \vee \ldots \vee x_{a_{n+1}}$$

[1]The symbol $-$ refers to the set-theoretic difference.

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for $t = a_1 + \ldots + a_{n+1}$ and $s(t) = 1$, we get $(a_i) = 1$ for some $1 \leq i \leq n+1$. Thus, $Ker(s)$ is an n-order-filter of $P(X)$.

The converse statement follows easily from properties of n-order-filters, and the proof is complete.

We now denote the set of all n-order-filters S containing the given order-filter $M \subseteq P(X)$ by $u^{(n)}(M)$. $u^{(n)}(i)$ will simply mean $u^{(n)}(\{i\})$. Since $u^{(n)}(i)$ is a family of subsets of $P(X)$, we can write $u^{(n)}(i) \subseteq \mathbf{2}^{P(X)}$. Let $K_n(X)$ denote the (distributive) sublattice of $\mathbf{2}^{P(X)}$ generated by $\{u^{(n)}(i) : i \in P(X)\}$, i.e. $K_n(X) = [\{u^{(n)}(i) : i \in P(X)\}]$.

Lemma 7. *Let $P(X)$ denote the poset associated with a set X. Then there exists a lattice isomorphism $\varphi : \mathrm{FD}p_n(X) \to K_n(X)$, $1 \leq n \leq \omega$, such that*

(i) *the restriction $\varphi \restriction P(X)$ is an order-isomorphism between $P(X)$ and $\{u^{(n)}(i) : i \in P(X)\}$;*
(ii) *the restriction $\varphi \restriction \mathrm{FB}(X)$ is an order-isomorphism between $\mathrm{FB}(X)$ and $\{u^{(n)}(i) : i \in \mathrm{FB}(X)\}$. Moreover, there exists a lattice epimorphism*

$$\tau : K_n(X) \to \{u^{(n)}(i) : i \in FB(\bar{X})\}$$

such that

(iii) $\tau({}^{(n)}(x)) = u^{(n)}(\bar{x})$ *for every* $x \in X$;
(iv) $\tau(u^{(n)}(a)) = u^{(n)}(a)$ *for every* $a \in \mathrm{FB}(\bar{X})$;
(v) $v \leq \tau(v)$ *for every* $v \in K_n(X)$;
(vi) $u^{(n)}(i) \subseteq u^{(n)}(a)$ *and* $a \in \mathrm{FB}(\bar{X})$ *imply* $\tau(u^{(n)}(i)) \subseteq u^{(n)}(a)$ *for any* $i \in P(X)$.

Proof. According to Lemma 6, the mapping

$$s \to Ker(s)$$

is an order-isomorphism between $\mathbf{2}^{P(X)} - Iq(E_n)$ and the set of all n-order-filters of $P(X)$. Therefore, by Remark 1, H_n and $K_n(X)$ are isomorphic as lattices. Eventually, by Theorem 2, there exists a lattice isomorphism

$$\varphi : \mathrm{FD}p_n(X) \to K_n(X)$$

It is easy to verify that this isomorphism satisfies (i) and (ii). The last conditions and Lemma 5 imply (iii)–(iv).

The condition (ii) of Lemma 7 shows that $\{u^{(n)}(i) : i \in \mathrm{FB}(\bar{X})\}$ is a Boolean algebra isomorphic to $\mathrm{FB}(\bar{X})$. In addition, $u^{(n)}(0)$ and $u^{(n)}(1)$ are the smallest and the greatest elements of it, respectively, and $u^{(n)}(a')$ is the complement of $u^{(n)}(a)$, $a \in \mathrm{FB}(\bar{X})$.

We are now in position to state the main facts about n-order-filters.

Theorem 3. *Let $P(X)$ denote the poset associated with a set X,* let $1 \leq n \leq \omega$. *Then the free distributive p-algebra $\mathrm{FD}p_n(X)$ in the class $\mathbf{B}_n$ and the distributive lattice $K_n(X)$ are isomorphic as lattices. More precisely, $K_n(X)$ can be considered as a p-algebra such that*

(i) $v^* = \tau(v)'$ *for every* $v \in K_n(X)$,
(ii) $\{u^{(n)}(i) : i \in X\}$ *freely generates* $K_n(X)$ *as a p-algebra in the class* $\mathbf{B}_n$,
(iii) $B(K_n(X)) = \{u^{(n)}(i) : i \in \mathrm{FB}(\bar{X})\} \simeq \mathrm{FB}(\bar{X})$.

The proof follows directly from Lemma 7.

4. Finite Algebras. It is known (see [4] that a finitely generated distributive p-algebra is finite. Obviously, every finite distributive p-algebra is determined by its poset of non-zero join-irreducible elements. Theorem 3 suggests a natural way of describing this set.

LEMMA 8. *Let X be a finite set. Assume that $i_1, \ldots, i_k \in P(X)$ and $1 \leq n \leq \omega$. Then there exists an order-filter $M \subseteq P(X)$ such that*

$$u^{(n)}(i_1) \cap \ldots \cap u^{(n)}(i_k) = u^{(n)}(M). \qquad (*)$$

Conversely, for every order-filter $M \subseteq P(X)$ there exists $i_1, \ldots, i_k \in P(X)$ such that (*) is true.

Proof. Let $i_1, \ldots, i_k \in P(X)$ be given. Evidently,

$$[i_j) \subseteq \{i_1, \ldots, i_k\} \cup [\tau(i_1) \ldots \tau(i_k)) = M \subseteq P(X)$$

for every $j = 1, \ldots, k$. M is an order-filter and (*) can be easily verified.

On the other hand, let $M \subseteq P(X)$ be an order-filter. Clearly, $M \cap \mathrm{FB}(\bar{X}) = [a)$ for some $a \in \mathrm{FB}(\bar{X})$. If $M \cap X = \emptyset$, then $u^{(n)}(a) = u^{(n)}(M)$. Let $M \cap X = \{i_1, \ldots, i_k\}$. Evidently, $\tau(i_j) \geq a$ for every $j = 1, \ldots, k$. A simple verification shows that (*) holds true.

Recall that for $t \in \mathrm{FB}(\bar{X}) \subseteq P(X)$ and finite X, we can define the *height* function: Let $h_B(t)$ denote the length of a longest maximal chain in $[0, t] \cap \mathrm{FB}(\bar{X})$. It is easy to see that $h_B(t)$ is the number of all atoms $a \in \mathrm{FB}(\bar{X})$ such that $a \leq t$.

LEMMA 9. *Let X be a finite set and let M be an order-filter of $P(X)$. Assume that $M \cap \mathrm{FB}(\bar{X}) = [t)$. Then M is an n-order-filter for some $1 \leq n < \omega$ if and only if*

$$h_B(t) \leq n$$

Proof. Suppose that M is an n-order-filter and let $1 \leq n < \omega$ be the smallest integer with this property. We have to show that $h_B(t) = n$. For $t = 0$ this is true. Assume that $t \neq 0$. Let $a_1, \ldots, a_k$ be all atoms of $\mathrm{FB}(\bar{X})$ with property: $a_j \leq t$. Consider elements $b_1 = t \cdot a_1'$, $\ldots, b_k = t \cdot a_k' \in FB(\bar{X})$. Clearly,

$$t = b_1 + \ldots + b_k \text{ and } b_i + b_j = t \text{ for any } i \neq j.$$

Since $b_i \leq t$ for all $i = 1, \ldots, k$, we have

$$k = h_B(t) \leq n$$

We now go in the other direction. According to the choice of n there exist distinct $b_1, \ldots, b_n \in \mathrm{FB}(\bar{X})$ such that

$$t = b_1 + \ldots + b_n, b_i + b_j = t$$

for any $i \neq j$ and no $b_i = t$. Set

$$a_1 = t \cdot b_1', \ldots, a_n = t \cdot b_n' \in \mathrm{FB}(\bar{X})$$

$a_1, \ldots, a_n$ are distinct and $a_i \cdot a_j = 0$ in $\mathrm{FB}(\bar{X})$ whenever $i \neq 0$. Since $a_i \neq 0$ for all $i = 1, \ldots, n$, we see that $n \leq k = h_B(t)$. Thus $h_B(t) = n$.

Conversely, suppose that $h_B(t) \leq n$. Take $b_1, \ldots, b_{n+1} \in \mathrm{FB}(\bar{X})$ such that

$$t = b_1 + \ldots + b_{n+1} \text{ and } t = b_i + b_j \text{ whenever } i \neq j.$$

We can also assume that $b_1, \ldots, b_{n+1}$ are distinct. Therefore, the elements

$$a_1 = t \cdot b_1', \ldots, a_{n+1} = t \cdot b_{n+1}' \in \mathrm{FB}(\bar{X})$$

are distinct and $a_i \cdot a_j = 0$ in $\mathrm{FB}(\bar{X})$ whenever $i \neq j$. Since $h_B(t) \leq n$ we see that $a_i = 0$ for some $1 \leq i \leq n+1$. Hence $b_i = t \in M$ and M is an n-order-filter.

As our final result, we have the following theorem.

THEOREM 4. *Let $P(X)$ denote the poset associated with a finite set X. Then $A \in K_n(X)$, $1 \leq n \leq \omega$, is a join-irreducible element in the lattice $K_n(X)$ if and only if $A = u^{(n)}(M)$ for some n-order-filter M.*

Proof. In view of Lemma 8, every element $A \in K_n(X)$ can be written in the form

$$A = u^{(n)}(M_1) \cup \ldots \cup u^{(n)}(M_r)$$

for some order-filters $M_1, \ldots, M_r \subseteq P(X)$. Suppose now that $A \in K_n(X)$ is join-irreducible. Therefore, $A = u^{(n)}(M)$ for some order-filter ($=\omega$-order-filter) M of $P(X)$, and the assertion of the theorem is clear for $n = \omega$. We therefore assume that $1 \leq n < \omega$. Our aim is to show that there exists an n-order filter $T \subseteq P(X)$ such that

$$A = u^{(n)}(M) = u^{(n)}(T).$$

Let $[t) = M \cap FB(\bar{X})$. We shall proceed by induction on $h_B(t)$.

(I) Suppose that $h_B(t) \leq n$. It follows by Lemma 9 that M is an n-order-filter. Hence $T = M$.

(II) Assume that $h_B(t) > n$. Moreover, if there exists an order-filter $T \subseteq P(X)$ such that $[t_1) = T \cap FB(\bar{X})$, $t > t_1$ and

$$A = u^{(n)}(M) = u^{(n)}(T),$$

then T is an n-order-filter of $P(X)$. Without loss of generality we can assume that M is no n-order-filter of $P(X)$. Then there exist distinct elements $b_1, \ldots, b_{n+1} \in \mathrm{FB}(\bar{X})$ satisfying the following conditions: $b_i < t$ for every $i = 1, \ldots, n+1$ and

$$t = b_1 + \ldots + b_{n+1} = b_i + b_j, \text{ whenever } i \neq j$$

Form the following order-filters:

$$M_1 = Y \cup [b_1), \ldots, M_{n+1} = Y \cup [b_{n+1}) \text{ for } Y = M \cap X.$$

We claim that

$$u^{(n)}(M) = u^{(n)}(M_1) \cup \ldots \cup u^{(n)}(M_{n+1}).$$

Obviously, $u^{(n)}(M) \supseteq u^{(n)}(M_1) \cup \ldots \cup u^{(n)}(M_{n+1})$. On the other hand, let $S \in u^{(n)}(M)$. Since S is an n-order-filter, there is $b_i \in S$ for some $1 \le i \le n+1$. It follows from this that $S \supseteq M_i$. Therefore, $S \in u^{(n)}(M_i)$ and

$$u^{(n)}(M) = u^{(n)}(M_1) \cup \ldots \cup u^{(n)}(M_{n+1}),$$

as claimed. The hypothesis that $u^{(n)}(M)$ is join-irreducible implies that $A = u^{(n)}(M_j)$ for some $1 \le j \le n+1$. Clearly, $[b_j) = M_j \cap (\mathrm{FB}(\bar{X})$ and $t > b_j$. By induction hypothesis is M_j an n-order-filter of $P(X)$ and we can put $T = M_j$. This shows that for a join-irreducible $A \in K_n(X)$ there exists an n-order-filter T of $P(X)$ with $A = u^{(n)}(T)$.

Conversely, let $A = u^{(n)}(M)$ for some n-order-filter M of $P(X)$. Suppose that $A = C \cup D$ for some $C, D \in K_n(X)$. In view of Lemma 9 we can write

$$A = u^{(n)}(M_1) \cup \ldots \cup u^{(n)}(M_r)$$

for some order-filters $M_1, \ldots, M_r$ of $P(X)$. Since $M \in A$, there is $1 \le j \le r$ such that $M \in u^{(n)}(M_j)$. It follows that $A \subseteq u^{(n)}(M_j)$, and consequently, $A = u^{(n)}(M_j)$. This shows that A is join-irreducible in $K_n(X)$.

REMARK 3. Another characterization of join-irreducibles from $\mathrm{FD}p_n(X)$ for finite X is given in Urquhart **[14]**. A transformation which converts n-order-filters into elements of $\mathrm{FD}p_n(X)$ can be easily established: Let $M \subseteq P(X)$ be an n-order-filter and let $[t) = M \cap FB(\bar{X})$. Moreover, let

$$\varphi : (\mathrm{FB}(\bar{X}); +, ., ', 0, 1) \to (\mathrm{FB}(X^{**}); +, \wedge, ^*, 0, 1)$$

be a Boolean isomorphism given by

$$\varphi : \bar{x} \to x^{**} \text{ for } \bar{x} \in \bar{X}.$$

Define

$$p(M) = \bigwedge (x : x \in M \cap X) \wedge \varphi(t).$$

Then $p(M)$ is a join-irreducible element in $\mathrm{FD}p_n(X)$ (in the characterization of **[14]**) which corresponds to $u^{(n)}(M)$.

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