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Uniqueness of L^p subsolutions to the heat equation on Finsler measure spaces

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Abstract. Let (M, F, m) be a forward complete Finsler measure space. In this paper, we prove that any nonnegative global subsolution in $L^p(M)(p > 1)$ to the heat equation on $\mathbb{R}^+ \times M$ is uniquely determined by the initial data. Moreover, we give an $L^p(0 < p \le 1)$ Liouville-type theorem for nonnegative subsolutions u to the heat equation on $\mathbb{R} \times M$ by establishing the local L^p mean value inequality for u on M with $\mathrm{Ric}_N \ge -K(K \ge 0)$.

1 Introduction

Finsler geometry is a Riemannian geometry without quadratic restriction. It is a natural generalization of Riemannian geometry. Studies show that there are similarities but essential differences between Riemannian and Finsler geometry [OS1, OS2, Sh, Xia1, Xia4, ZX]. One of the challenges is the nonlinearity of the Laplacian and the associated heat equation. Some questions related to nonlinear equations would be of interest in itself from the analytic viewpoint.

A *Finsler manifold* (M, F) means a smooth manifold M equipped with a Finsler metric (or Finsler structure) $F : TM \to [0, +\infty)$ such that $F_x = F|_{T_xM}$ is a Minkowski norm on T_xM at each point $x \in M$, that is to say, F is smooth on $TM\setminus\{0\}$ and $F_x(y) = 0$ if and only if $y = 0$, F satisfies $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$, and the matrix $\left(g_{ij}(x, y)\right) := \left(\frac{1}{2}(F^2)_{y^i y^j}\right)$ is positive definite for any nonzero $y \in T_xM$. Such a pair (M, F) is called a *Finsler manifold*. Given a smooth measure m, the triple (M, F, m) is called a *Finsler measure space*. In particular, if F is only a function of $x \in M$ independing of y, then it is Riemannian. A Finsler measure space is not a metric space in usual sense because F may be nonreversible, i.e., $F(x, y) \ne F(x, -y)$ may happen. This nonreversibility causes the asymmetry of the associated distance function. For any Finsler metric F, its dual $F^*(x, \xi) = \sup_{F_x(y)=1} \xi(y)$ is also a Finsler metric on M, where $\xi \in T_x^*(M)$.

In general, the space of functions $u \in L^p(M)$ with $\int_M [F^*(du)]^p dm < \infty$ with respect to the measure m is not a linear space over $\mathbb{R}$ because of the nonreversibility

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of F [KR]. We define

$$W^{1,p}(M) := \left\{ u \in L^p(M) \Big| \int_M [F^*(du)]^p dm + \int_M [F^*(-du)]^p dm < \infty \right\}$$

for $p \geq 1$. Thus, $W^{1,p}(M)$ is a Banach space with respect to the norm

$$\|u\|_{1,p} := \|u\|_{L^p} + \|F^*(du)\|_{L^p} + \|\overleftarrow{F}^*(du)\|_{L^p}. \tag{1.1}$$

It is a Sobolev space, which is the completion of the space $C^\infty(M)$ with respect to the Sobolev norm $\|\cdot\|_{1,p}$. Further, we may define $W_0^{1,p}(M)$ as the completion of the space $C_0^\infty(M)$ with respect to the norm $\|\cdot\|_{1,p}$, and hence $(W_0^{1,p}(M), \|u\|_{1,p})$ is also a Sobolev space. We denote $H^1(M) := W^{1,2}(M)$ and $H_0^1(M) := W_0^{1,2}(M)$ for simplicity. Likewise, $H^{-1}(M)$ means the dual Banach space of $H_0^1(M)$.

For any $T > 0$ (T maybe the infinity), a function $u \in [0,T] \times M$ is called a *global (super, sub)solution* to the heat equation $\partial_t u = \Delta u$ if it satisfies (i) $u \in L^2\left([0,T], H_0^1(M)\right) \cap H^1\left([0,T], H^{-1}(M)\right)$ and (ii) for almost every $t \in (0,T)$ and $0 \leq \phi \in C_0^\infty(M)$ (or, equivalently, for every $\phi \in H_0^1(M)$), it holds that

$$\int_M \phi \partial_t u dm \ (\geq, \leq) = -\int_M d\phi(\nabla u) dm. \tag{1.2}$$

The existence and regularity of the global solution to the heat equation have been studied in [OS1] (see Section 2). Moreover, global solutions are constructed as gradient curves of the energy functional $\mathcal{E}(u) = \int_M F^2(\nabla u) dm$ on $u \in H^1_{loc}(M)$ in the Hilbert space $L^2(M)$ [OS1]. Comparing with the Riemannian case [SY], there are less results on the heat equation in Finsler geometry. An important progress is that Ohta and Sturm gave Li-Yau's gradient estimate for the positive solutions to the heat equation on a closed Finsler manifold with weighted Ricci curvature bounded below by establishing the Bochner–Weitzenböck formula in [OS2]. Recently, the author of the present paper gave the local and global Li-Yau's type gradient estimates on complete Finsler measure spaces with weighted Ricci curvature bounded below in [Xia2] and [Xia3] by Moser's iteration and a linearized heat semigroup approach. In this paper, we will focus on the uniqueness of nonnegative $L^p (p > 0)$ subsolutions to the heat equation on forward complete Finsler measure spaces.

For the case when $p > 1$, the uniqueness of L^p nonnegative subsolutions to the heat equation is determined by the initial data on a complete Finsler manifold. More precisely, we have the following theorem.

Theorem 1.1 *Let (M, F, m) be a forward or backward complete Finsler measure space. If $u(t,x)$ is a nonnegative global subsolution to the heat equation on $[0,\infty) \times M$ with $u(t,x) \in L^p(M)$ and $\lim_{t\to 0} \int_M u^p(t,x) dm = 0$ for all $t > 0$ and $p > 1$, then $u(t,x) \equiv 0$ on $[0,\infty) \times M$.*

For the case when $0 < p \leq 1$, we have the following theorem.

Theorem 1.2 *Let (M, F, m) be an n-dimensional forward or backward complete and noncompact Finsler measure space with finite reversibility Λ. If $Ric_N \geq 0$ for some $N \in$*

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$[n, \infty)$*, then every nonnegative* $L^p(\mathbb{R} \times M)(0 < p \leq 2)$ *subsolution to the heat equation on* $\mathbb{R} \times M$ *is identically zero.*

When $(M, F = \sqrt{g}, dV_g)$ is Riemannian and $N = n$, Theorem 1.1 is reduced to the result in [Li]. Since the Finsler Laplacian is nonlinear, we use some nonlinear approaches to prove Theorem 1.2. In fact, we shall establish the local L^p mean value inequality for the heat equation by means of Moser's iteration to prove this. In particular, if the (super, sub)solution u to the heat equation is independent of t, then u is a Finsler (super, sub)harmonic function. In this case, some Liouville-type theorems have been established in [Xc, Xia1, Xia4, ZX].

2 Preliminaries

In this section, we briefly review some basic concepts in Finsler geometry. For more details, we refer to [Sh].

2.1 Finsler metrics and curvatures

Let (M, F) be an n-dimensional Finsler manifold, and let TM be the tangent bundle of M. We define the *reverse metric* $\overleftarrow{F}$ of F by $\overleftarrow{F}(x, y) := F(x, -y)$ for all $(x, y) \in TM$. It is easy to see that $\overleftarrow{F}$ is also a Finsler metric on M. A Finsler metric F on M is said to be *reversible* if $\overleftarrow{F}(x, y) = F(x, y)$ for all $y \in TM$. Otherwise, we say F is nonreversible. In this case, we define the *reversibility* $\Lambda = \Lambda(M, F)$ of F by

$$\Lambda := \sup_{(x,y)\in TM\setminus\{0\}} \frac{F(x, -y)}{F(x, y)}.$$

Obviously, $\Lambda \in [1, \infty]$ and F is reversible if and only if $\Lambda = 1$.

Given a nonvanishing smooth vector field V, one introduces a weighted Riemannian metric g_V on M given by

$$g_V(y, w) = g_{ij}(x, V_x)y^i w^j, \quad \text{for } y, w \in T_xM \text{ and } x \in M. \tag{2.1}$$

Thus, we have $F^2(V) = g_V(V, V)$.

For $x_1, x_2 \in M$, the *distance* from x_1 to x_2 is defined by $d_F(x_1, x_2) := \inf_\gamma \int_0^1 F(\dot\gamma(t))dt$, where the infimum is taken over all C^1 curves $\gamma : [0, 1] \to M$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$. Note that $d_F(x_1, x_2) \neq d_F(x_2, x_1)$ unless F is reversible. $d_F(x, \cdot)$ is differentiable almost everywhere on M and $F(\nabla d_F) = 1$ at points where d_F is differentiable. If F has finite reversibility Λ, then

$$\Lambda^{-1} d_F(q, p) \leq d_F(p, q) \leq \Lambda d_F(q, p). \tag{2.2}$$

Now we define the *forward and backward geodesic balls* of radius R with center at $x \in M$ by

$$B_R^+(x) := \{z \in M \mid d_F(x, z) < R\}, \quad B_R^-(x) := \{z \in M \mid d_F(z, x) < R\}.$$

A C^1 curve $\eta : [0, \ell] \to M$ is called a *geodesic* if it has constant speed (i.e., $F(\eta, \dot\eta)$ is constant) and if it is locally minimizing. Such a geodesic is in fact a C^∞ curve.

A Finsler manifold (M, F) is said to be *forward complete (or backward complete)* if each geodesic defined on $[0, \ell)$ (or $(-\ell, 0])$ can be extended to a geodesic defined on $[0, \infty)$ (or $(-\infty, 0])$. (M, F) is backward complete if and only if $(M, \overleftarrow{F})$ is forward complete. We say that (M, F) is complete if it is both forward complete and backward complete [Sh].

Given a smooth measure m, write $dm = \sigma(x)dx$. For any $v \in T_x M \backslash \{0\}$, let $\eta(t)$ be a geodesic with $\eta(0) = x$ and $\dot{\eta}(0) = v$. The *distortion* of F is defined by $\tau(x, v) := \log\left\{\frac{\sqrt{\det(g_{ij}(x,v))}}{\sigma(x)}\right\}$, and the *S-curvature* S is defined as the rate of change of the distortion τ along $\eta(t)$, i.e.,

$$S(x, v) = \frac{d}{dt}\tau(\eta(t), \dot{\eta}(t))|_{t=0}.$$

S-curvature disappears on a Riemannian manifold (M, g).

Next we recall the definition of the weighted Ricci curvature on a Finsler manifold, which was introduced by S. Ohta.

Definition 2.1 [Oh] Given a unit vector $v \in T_x M$, let $\eta : (-\varepsilon, \varepsilon) \to M$ be the geodesic with $\eta(0) = x$ and $\dot{\eta}(0) = v$. We set $dm = e^{-\Psi(\eta(t))}\mathrm{vol}_{\dot{\eta}}$ along η, where $vol_{\dot{\eta}}$ is the volume form of $g_{\dot{\eta}}$. Define the weighted Ricci curvature involving a parameter $N \in (n, \infty)$ by

$$\mathrm{Ric}_N(v) := \mathrm{Ric}(v) + (\Psi \circ \eta)''(0) - \frac{(\Psi \circ \eta)'(0)^2}{(N - n)},$$

where Ric is the Ricci curvature of F. Also, define $\mathrm{Ric}_\infty(v) := \mathrm{Ric}(v) + (\Psi \circ \eta)''(0)$ and

$$\mathrm{Ric}_n(v) := \lim_{N \to n} \mathrm{Ric}_N = \begin{cases} \mathrm{Ric}(v) + (\Psi \circ \eta)''(0), & \text{if } (\Psi \circ \eta)'(0) = 0, \\ -\infty, & \text{otherwise} \end{cases}$$

as limits. Finally, for any $\lambda \geq 0$ and $N \in [n, \infty]$, define $\mathrm{Ric}_N(\lambda v) := \lambda^2 \mathrm{Ric}_N(v)$.

We say that $\mathrm{Ric}_N \geq K$ for some $K \in \mathbb{R}$ if $\mathrm{Ric}_N(v) \geq KF^2(v)$ for all $v \in TM$. We remark that $(\Psi \circ \eta)'(0) = S(x, v)$ and $(\Psi \circ \eta)''(0)$ is exactly the change rate of the S-curvature along the geodesic $\eta(t)$.

2.2 Gradient and nonlinear Laplacian

Let $\mathcal{L} : T^*M \to TM$ be the *Legendre transform* associated with F and its dual norm F^*, that is, a transformation $\mathcal{L}$ from T^*M to TM, defined locally by $\mathcal{L}(x, \xi) = J^i(x, \xi)\frac{\partial}{\partial x^i}$, $J^i(x, \xi) := \frac{1}{2}[F^{*2}(x, \xi)]_{\xi_i}(x, \xi)$, is sending $\xi \in T^*M$ to a unique element $y \in T_x M$ such that $F(x, y) = F^*(x, \xi)$ and $\xi(y) = F^2(y)$ [Sh]. It is a diffeomorphism from $T^*M \backslash \{0\}$ to $TM \backslash \{0\}$ preserving the norm.

For a smooth function $u : M \to \mathbb{R}$, the *gradient vector* $\nabla u(x)$ of u is defined by $\nabla u := \mathcal{L}(du) \in T_x M$. Obviously, $\nabla u = 0$ if $du = 0$. In a local coordinate system, we reexpress ∇u as

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$$\nabla u := \begin{cases} g^{ij}(\nabla u)\frac{\partial u}{\partial x^i}\frac{\partial}{\partial x^j}, & x \in M_u, \\ 0, & x \in M\setminus M_u, \end{cases}$$

where $M_u = \{x \in M | du(x) \neq 0\}$. In general, ∇u is only continuous on M, but smooth on M_u.

Recall that $dm = \sigma(x)dx$. The *Finsler Laplacian* Δ of a function u on M is formally defined by $\Delta u = \mathrm{div}(\nabla u)$. It should be understood in the sense that

$$\int_M \varphi \Delta u dm = -\int_M d\varphi(\nabla u) dm \tag{2.3}$$

for any $u \in H^1_{\mathrm{loc}}(M)$ and $\varphi \in C_0^\infty(M)$ [Sh]. Obviously, Δ is a nonlinear divergence-type operator. Locally,

$$\Delta u = \frac{1}{\sigma}\frac{\partial}{\partial x^i}\left(\sigma g^{ij}(\nabla u)\frac{\partial u}{\partial x^j}\right) \quad \text{on } M_u. \tag{2.4}$$

We say that a function u on $[0, T] \times M$ is a *global (super, sub)solution* to the heat equation $\partial_t u = \Delta u$ if it satisfies (1.2). When $\Lambda < \infty$, the global solutions enjoy the H^2_{loc}-regularity in x as well as the $C^{1,\alpha}$-regularity in both t and x on $(0, \infty) \times M$. Moreover, $\partial_t u$ lies in $H^1_{loc}(M)$ [OS1]. We also may define the local solutions to the heat equation as follows.

Given an open subset $\Omega \subset M$ and an open interval $I \subset \mathbb{R}$, we say that a function u on $I \times \Omega$ is a *local (super, sub)solution* to the heat equation $\partial_t u = \Delta u$ on $I \times \Omega$ if $u \in L^2_{loc}(I \times \Omega)$ with $F^*(du) \in L^2_{loc}(I \times \Omega)$, and for any $\phi \in C_0^\infty(I \times \Omega)$ (or, equivalently, for any $\phi \in H^1_0(I \times \Omega)$),

$$\int_I \int_\Omega u \partial_t \phi dm dt \ (\leq, \geq) = \int_I \int_\Omega d\phi(\nabla u) dm dt. \tag{2.5}$$

Every continuous local solution to the heat equation $\partial_t u = \Delta u$ on $I \times \Omega$ is $C^{1,\beta}$ in t and x [OS1].

3 Mean value inequality

In this section, we establish the local L^p $(p > 0)$ mean value inequalities for nonnegative subsolutions to the heat equation on Finsler measure spaces (M, F, m). Let us first recall the local Sobolev inequality, which was due to Xia [Xc]. It is worth mentioning that it suffices to assume the finite reversibility instead of the uniform convexity and uniform smoothness for the Finsler metrics from the proof of Theorem 3.3 in [Xc]. For the sake of simplicity, we shall denote by $B_R := B_R^+(x_0)$ the forward geodesic ball of radius R centered at some point $x_0 \in M$.

Lemma 3.1 *[Xc] Let (M, F, m) be a forward complete Finsler measure space with finite reversibility Λ satisfying $Ric_N \geq -K$ for some $K > 0$. Then there exist constants $\nu = \nu(N) > 2$ and $c_0 = c_0(N, \Lambda)$ depending on N and Λ, such that*

$$\left(\int_{B_R} |u|^{\frac{2\nu}{\nu-2}} dm\right)^{\frac{\nu-2}{\nu}} \leq e^{c_0(1+\sqrt{K}R)} R^2 m(B_R)^{-\frac{2}{\nu}} \int_{B_R} \{F^{*2}(x, du) + R^{-2}u^2\} dm \tag{3.1}$$

for any forward geodesic ball B_R and $u \in W^{1,2}_{loc}(M)$.

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With this, we can prove the local L^2 mean value inequality for nonnegative subsolutions to the heat equation.

Proposition 3.1 *Let (M, F, m) be an n-dimensional forward complete Finsler measure space with finite reversibility Λ. Fix $R > 0$, and assume that (3.1) holds. If u is a nonnegative local subsolution of the heat equation in the cylinder $Q := (s - r^2, s + \varepsilon) \times B_r^+(x_0)$ for any $s \in \mathbb{R}$ and $0 < r < R$, where ε is a positive constant, then there are constants $\nu > 2$, and $c = c(N, \nu, \Lambda) > 0$ such that for any $0 < \delta < \delta' \le 1$, we have*

$$\sup_{Q_\delta} u^2 \le \frac{e^{c(1+r\sqrt{K})}}{(\delta' - \delta)^{2+\nu} r^2 m(B_r^+(x_0))} \int_{Q_{\delta'}} u^2 dm dt, \tag{3.2}$$

where $Q_\delta := (s - \delta r^2, s] \times B_{\delta r}^+(x_0)$ and $Q_{\delta'} := (s - \delta' r^2, s] \times B_{\delta' r}^+(x_0)$. In particular, when $Ric_N \ge -K$ for some $N \in [n, \infty)$ and $K > 0$, (3.2) holds.

Proof For the sake of simplicity, we denote $B_\bullet := B_\bullet^+(x_0)$ in the following. Since u is a nonnegative local subsolution to the heat equation in Q, we have

$$\int_{s-r^2}^{t} \int_{B_r} (\phi \partial_t u + d\phi(\nabla u)) dm dt \le 0 \tag{3.3}$$

for any $t \in (s - \delta r^2, s]$ and nonnegative function $\phi \in W_0^{1,2}(Q)$. Let $\lambda = \lambda(t) \in C_0^\infty(\mathbb{R})$ and $\varphi = \varphi(x) \in C_0^\infty(B_r)$ be cutoff functions. Replacing ϕ with $u\varphi^2\lambda^2$ in (3.3) and using the Cauchy–Schwarz inequality yield

$$\begin{aligned}
&\int_{s-r^2}^{t} \int_{B_r} \lambda^2 \varphi^2 F^2(\nabla u) dm dt + \int_{s-r^2}^{t} \int_{B_r} \lambda^2 \varphi^2 u \partial_t u dm dt \\
&\quad \le -2 \int_{s-r^2}^{t} \int_{B_r} \lambda^2 \varphi u d\varphi(\nabla u) dm dt \le 2 \int_{s-r^2}^{t} \int_{B_r} \lambda^2 \varphi u F^*(-d\varphi) F(\nabla u) dm dt \\
&\quad \le \frac{1}{3} \int_{s-r^2}^{t} \int_{B_r} \lambda^2 \varphi^2 F^{*2}(du) dm dt + 3 \int_{s-r^2}^{t} \int_{B_r} \lambda^2 u^2 F^{*2}(-d\varphi) dm dt,
\end{aligned}$$

where we used $F(\nabla u) = F^*(du)$. Thus,

$$\begin{aligned}
&\int_{s-r^2}^{t} \int_{B_r} \lambda^2 \varphi^2 \partial_t(u^2) dm dt + \frac{4}{3} \int_{s-r^2}^{t} \int_{B_r} \lambda^2 \varphi^2 F^{*2}(du) dm dt \\
&\quad \le 6 \int_{s-r^2}^{t} \int_{B_r} \lambda^2 u^2 F^{*2}(-d\varphi) dm dt.
\end{aligned}$$

Note that $F^*(d(\varphi u)) \le u F^*(d\varphi) + \varphi F^*(du)$. Hence,

$$\begin{aligned}
&\int_{s-r^2}^{t} \int_{B_r} \frac{\partial}{\partial t} (\lambda \varphi u)^2 dm dt + \int_{s-r^2}^{t} \int_{B_r} \lambda^2 F^{*2}(d(\varphi u)) dm dt \\
&\quad \le 2 \int_{s-r^2}^{t} \int_{B_r} \lambda \lambda' \varphi^2 u^2 dm dt + \int_{s-r^2}^{t} \int_{B_r} \lambda^2 \varphi^2 \partial_t(u^2) dm dt \\
&\qquad + \frac{4}{3} \int_{s-r^2}^{t} \int_{B_r} \lambda^2 \varphi^2 F^{*2}(du) u^2 dm dt + 4 \int_{s-r^2}^{t} \int_{B_r} \lambda^2 F^{*2}(d\varphi) dm dt \\
&\quad \le 2 \int_{s-r^2}^{t} \int_{B_r} \lambda u^2 \left[\lambda' \varphi^2 + 5 \Lambda^2 \lambda F^{*2}(d\varphi) \right] dm dt. \qquad (3.4)
\end{aligned}$$

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Now we choose $\lambda(t)$ on $\mathbb{R}$ with $0 \le \lambda \le 1$, $\lambda = 0$ in $(-\infty, s - \delta' r^2)$, $\lambda = 1$ in $(s - \delta r^2, +\infty)$ and $|\lambda'(t)| \le \frac{1}{[(\delta'-\delta)r]^2}$, and $\varphi = \varphi(x)$ on B_r defined by

$$\varphi(x) := \begin{cases} 1, & \text{on } B_{\delta r}, \\ \frac{\delta' r - d_F(x_0,x)}{(\delta'-\delta)r}, & \text{on } B_{\delta' r} \setminus B_{\delta r}, \\ 0, & \text{on } B_{\delta' r}. \end{cases}$$

Then $F^*(-d\varphi) \le \frac{1}{(\delta'-\delta)r}$ and $F^*(d\varphi) \le \Lambda F^*(-d\varphi) \le \frac{\Lambda}{(\delta'-\delta)r}$ a.e. on $B_{\delta' r}$. Denote $I_\delta := (s - \delta r^2, s]$ and $I_{\delta'} := (s - \delta' r^2, s]$. Taking the supremum with respect to t on both sides of (3.5) yields

$$\sup_{t \in I_\delta} \left\{ \int_{B_r} (\varphi u)^2 dm \right\} + \int_{I_\delta \times B_r} F^{*2}(d(\varphi u)) dm dt \le \frac{12\Lambda^2}{(\delta'-\delta)^2 r^2} \int_{Q_{\delta'}} u^2 dm dt. \tag{3.5}$$

On the other hand, by Hölder's inequality and Soblev's inequality (3.1), we have

$$\begin{aligned} \int_{B_r} (u\varphi)^{2(1+\frac{2}{\nu})} dm &\le \left(\int_{B_r} (u\varphi)^{\frac{2\nu}{\nu-2}} dm \right)^{\frac{\nu-2}{\nu}} \cdot \left(\int_{B_r} (u\varphi)^2 dm \right)^{\frac{2}{\nu}} \\ &\le \left(A \int_{B_r} [F^{*2}(d(u\varphi)) + r^{-2}(u\varphi)^2] dm \right) \cdot \left(\int_{B_r} (u\varphi)^2 dm \right)^{\frac{2}{\nu}} \end{aligned} \tag{3.6}$$

for some $\nu > 2$, where $A := e^{c_0(1+r\sqrt{K})} r^2 m(B_r)^{-2/\nu}$, where $c_0 = c_0(N, \Lambda)$ is a positive constant. From (3.5) and (3.6), one obtains

$$\begin{aligned} \int_{Q_\delta} u^{2(1+\frac{2}{\nu})} dm dt &= \int_{Q_\delta} (u\varphi)^{2(1+\frac{2}{\nu})} dm dt \\ &\le \left(A \int_{I_\delta \times B_r} [F^{*2}(d(u\varphi)) + r^{-2}(u\varphi)^2] dm dt \right) \left(\sup_{t \in I_\delta} \int_{B_r} (u\varphi)^2 dm \right)^{2/\nu} \\ &\le A \left(\frac{12\Lambda^2}{((\delta'-\delta)r)^2} \int_{Q_{\delta'}} u^2 dm dt \right)^{1+\frac{2}{\nu}}. \end{aligned}$$

For any $\tau \ge 1$, it is easy to see that u^τ is also a nonnegative subsolution of the heat equation. Let $\chi := 1 + \frac{2}{\nu}$. The above inequality implies that

$$\int_{Q_\delta} u^{2\chi\tau} dm dt \le A \left(\frac{12\Lambda^2}{((\delta'-\delta)r)^2} \int_{Q_{\delta'}} u^{2\tau} dm dt \right)^{\chi}. \tag{3.7}$$

For any $0 < \delta < \delta' \le 1$, let $\delta_0 = \delta'$, $\delta_{i+1} = \delta_i - \frac{\delta'-\delta}{2^{i+1}}$, and $\tau = \chi^i > 1$ on $B_{\delta_i r}$ for $i = 0, 1, \ldots$. Applying (3.7) for $\delta' = \delta_i$, $\delta = \delta_{i+1}$, and $\tau = \chi^i$, we get

$$\int_{Q_{\delta_{i+1}}} u^{2\chi^{i+1}} dm dt \le A \left(\frac{(12\Lambda^2) 4^{i+1}}{((\delta'-\delta)r)^2} \int_{Q_{\delta_i}} u^{2\chi^i} dm dt \right)^{\chi}.$$

By iteration, one obtains that

$$\left(\int_{Q_{\delta_{i+1}}} u^{2\chi^{i+1}} dm dt \right)^{\frac{1}{\chi^{i+1}}} \le C^{\sum j\chi^{1-j}} (A)^{\sum \chi^{-j}} [(\delta'-\delta)r]^{-2\sum \chi^{1-j}} \cdot \int_{Q_{\delta'}} u^2 dm dt, \tag{3.8}$$

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in which $\sum$ denotes the summation on j from 1 to $i+1$ and C is a positive constant depending on Λ. Since $\sum_{j=1}^{\infty}\chi^{1-j} = 1 + \frac{\nu}{2}$, $\sum_{j=1}^{\infty}\chi^{-j} = \frac{\nu}{2}$ and $\sum_{j=1}^{\infty} j\chi^{1-j}$ converges, (3.8) implies (3.2) by taking $i \to \infty$. This finishes the proof. ∎

Based on Proposition 3.1, we have the following local $L^p(0 < p \le 2)$ mean value inequality.

Theorem 3.1 *Let (M, F, m) be an n-dimensional forward complete Finsler measure space with finite reversibility Λ. Fix $R > 0$, and assume that (3.1) holds. If u is a nonnegative L^p subsolution of the heat equation in the cylinder $Q = (s - r^2, s + \varepsilon) \times B_r^+(x_0)$ for any $s \in \mathbb{R}$ and $0 < r < R$, where ε is a positive constant, then there are constants $\nu > 2$ and $c = c(N, p, \nu, \Lambda) > 0$ such that*

$$\sup_{Q_\delta} u^p \le \frac{e^{c(1+r\sqrt{K})}}{(\delta' - \delta)^{2+\nu} r^2 m(B_r)} \int_{Q_{\delta'}} u^p \, dm dt \tag{3.9}$$

for any $0 < p \le 2$ and $0 < \delta < \delta' \le 1$, where Q_δ and $Q_{\delta'}$ are as in Proposition 3.1. In particular, when $Ric_N \ge -K(K > 0)$ for some $N \in [n, \infty)$, (3.9) holds.

Proof The case when $p = 2$ has been proved in Proposition 3.1. It suffices to prove (3.9) for any $0 < p < 2$.

We denote $B_\bullet := B_\bullet^+(x_0)$ as in the proof of Proposition 3.1. For any $0 < \delta < \delta'' \le \delta' \le 1$, it follows from Proposition 3.1 that there are positive constants $\nu = \nu(N) > 2$ and $c = c(N, \nu, \Lambda)$ such that

$$\sup_{Q_\delta} u^2 \le \frac{e^{c(1+r\sqrt{K})}}{(\delta'' - \delta)^{2+\nu} r^2 m(B_r)} \int_{Q_{\delta''}} u^2 \, dm dt.$$

On the other hand,

$$\int_{Q_{\delta''}} u^2 \, dm dt \le \sup_{Q_{\delta''}} u^{2-p} \int_{Q_{\delta''}} u^p \, dm dt \le \left(\sup_{Q_{\delta''}} u^2\right)^{1-p/2} \int_{Q_{\delta'}} u^p \, dm dt$$

Thus,

$$\sup_{Q_\delta} u^2 \le \frac{e^{c(1+r\sqrt{K})}}{(\delta'' - \delta)^{2+\nu} r^2 m(B_r)} \left(\sup_{Q_{\delta''}} u^2\right)^{1-p/2} \int_{Q_{\delta'}} u^p \, dm dt. \tag{3.10}$$

Let $\mu = 1 - p/2 > 0$ and

$$\mathcal{M}(\delta) := \sup_{Q_\delta} u^2, \qquad \mathcal{R} = \frac{e^{c(1+r\sqrt{K})}}{r^2 m(B_r)} \int_{Q_{\delta'}} u^p \, dm dt.$$

Let $\delta_0 = \delta$ and $\delta_i = \delta_{i-1} + \frac{\delta' - \delta}{2^i}$ for $i = 1, 2, \ldots$. Applying (3.10) for $\delta = \delta_{i-1}$ and $\delta'' = \delta_i$ yields

$$\mathcal{M}(\delta_{i-1}) \le \mathcal{R} 2^{i(2+\nu)} (\delta' - \delta)^{-2-\nu} \mathcal{M}(\delta_i)^\mu.$$

By iterating, we get

(3.11) $$\mathcal{M}(\delta_0) \le \mathcal{R}^{\sum \mu^{i-1}} 2^{(2+\nu)\sum i\mu^{i-1}} (\delta' - \delta)^{(-2-\nu)\sum \mu^{i-1}} \mathcal{M}(\delta_j)^{\mu^j},$$

in which $\sum$ denotes the summation on i from 1 to j. Obviously, $\lim_{j\to\infty} \delta_j = \delta'$ and $\lim_{j\to\infty} \mu^j = 0$. Moreover, $\sum_{i=1}^{\infty} \mu^{i-1} = 2/p$ and $\sum_{i=1}^{\infty} i\mu^{i-1}$ converges. Letting $j \to \infty$ on both sides of (3.11), there is a positive constant $c = c(N, \nu, p)$ such that

$$\sup_{Q_\delta} u^p \le \mathcal{M}(\delta_0)^{p/2} \le e^{c(1+r\sqrt{K})} (\delta' - \delta)^{-2-\nu} r^{-2} m(B_r)^{-1} \int_{Q_{\delta'}} u^p dmdt.$$

The proof is finished. ■

4 Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1 Let $B_R := B_R^+(x_0)$, and let $\psi(x)$ be a cutoff function defined by

$$\psi(x) = \begin{cases} 1, & \text{on } B_R, \\ \frac{2R - d_F(x_0, x)}{R}, & \text{on } B_{2R} \setminus B_R, \\ 0, & \text{on } M \setminus B_{2R}. \end{cases}$$

Obviously, $0 \le \psi \le 1$ and $F^*(-d\psi) \le 1/R$ a.e., on B_{2R}. Since $\lim_{t\to 0} \int_M u^p(t,x)dm = 0$, we have

$$0 \le \lim_{t\to 0} \int_M \psi^2 u^p dm \le \lim_{t\to 0} \int_M u^p(t,x) dm = 0,$$

which means that $\lim_{t\to 0} \int_M \psi^2 u^p dm = 0$. Choosing $\phi := \psi^2 u^{p-1}$ as a test function in (1.2) yields

$$\begin{aligned} -\int_0^T \int_M d(\psi^2 u^{p-1})(\nabla u) dmdt &\ge \int_0^T \int_M \psi^2 u^{p-1} \frac{\partial u}{\partial t} dmdt \\ &\ge \frac{1}{p} \int_0^T \frac{\partial}{\partial t} \left(\int_M \psi^2 u^p dm \right) dt \\ &= \frac{1}{p} \int_M \psi^2 u^p(T,x) dm \end{aligned}$$

by the assumption. On the other hand, the LHS of the above inequality is equal to

$$\begin{aligned} &-2\int_0^T \int_M \psi u^{p-1} d\psi(\nabla u) dmdt - (p-1) \int_0^T \int_M \psi^2 u^{p-2} F^2(\nabla u) dmdt \\ &\le \frac{2}{p-1} \int_0^T \int_M u^p F^{*2}(-d\psi) dmdt - \frac{2(p-1)}{p^2} \int_0^T \int_M \psi^2 F^2(\nabla u^{p/2}) dmdt, \end{aligned}$$

where we used the inequality $-d\psi(\nabla u) \le F^*(-d\psi)F(\nabla u)$. Therefore,

$$\int_M \psi^2 u^p(T,x) dm + \frac{2(p-1)}{p} \int_0^T \int_M \psi^2 F^2(\nabla u^{p/2}) dmdt \le \frac{2p}{(p-1)R^2} \int_0^T \int_M u^p dmdt.$$

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Letting $R \to \infty$, one obtains

$$\int_M u^p(T,x)dm = 0, \quad \text{and} \quad \int_0^T \int_M F^2(\nabla u^{p/2})dmdt = 0,$$

which implies that $u(T,x) = 0$ and $\nabla u^{p/2} = 0$. Hence, $u(t,x) \equiv 0$. ∎

Proof of Theorem 1.2 By Theorem 3.1 and the assumption, we have

$$\begin{aligned} u(t,x_0) \leq \sup_{Q_\delta} u^p &\leq \frac{c}{(\delta' - \delta)^{2+\nu} r^2 m(B_r)} \int_{Q_{\delta'}} u^p dmdt \\ &= \frac{c}{(\delta' - \delta)^{2+\nu} r^2 m(B_r)} \int_{\mathbb{R}\times M} u^p dmdt \end{aligned} \tag{4.1}$$

for any $t \in (s - \delta r^2, s]$ and $0 < \delta < \delta' \leq 1$. Since $\mathrm{Ric}_N \geq 0$ and $\Lambda < \infty$, $m(B_r) \geq Cr$ by Theorem 5.4 in [Wu], where C is a positive constant depending on N and Λ. Note that $0 \leq u(t,x) \in L^p(\mathbb{R} \times M)$ by the assumption. Letting $\delta \to 0$ and then $r \to \infty$ on both sides of (4.1) yields $u(s,x_0) \equiv 0$. Since $s \in \mathbb{R}$ and $x_0 \in M$ are arbitrary, we have $u(t,x) \equiv 0$ on $\mathbb{R} \times M$. ∎

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