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ISOMETRIES OF $H^p(U^n)$

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1. Forelli in [**1**] has described the isometries of $H^p(U)$ into $H^p(U)$ for $p \neq 2, 0 < p < \infty$. We shall extend his methods to characterize the isometries of $H^p(U^n)$ onto $H^p(U^n)$.

The notation we shall use can be found in Rudin [**3**].

2. Let Π represent a permutation that induces a map on functions of n complex variables by

$$\Pi \cdot f(z_1, \ldots, z_n) = f(z_{i_1}, \ldots, z_{i_n}).$$

Clearly Π is an isometry of $H^p(U^n)$ onto $H^p(U^n)$.

THEOREM. *Suppose $p \neq 2$, $0 < p < \infty$ and T is a linear isometry of $H^p(U^n)$ onto $H^p(U^n)$. Then there is a permutation Π such that*

$$\text{(1)} \qquad \Pi \cdot T(f) = b\left(\frac{\partial \varphi_1}{\partial z}\right)^{1/p}(z_1) \ldots \left(\frac{\partial \varphi_n}{\partial z}\right)^{1/p}(z_n) f(\varphi_1(z_1), \varphi_2(z_2), \ldots, \varphi_n(z_n))$$

where the φ_i are conformal maps of the unit disc onto itself and b is a unimodular complex number. Conversely, (1) *defines a linear isometry of $H^p(U^n)$ onto $H^p(U^n)$.*

Proof. The converse is trivial. For the first part, let $F = T(1) \in H^p$. Let v be the measure $dv = |F|^p dm_n$ where m_n is Lebesque measure on the n-dimensional torus with

$$\int_{T^n} dm_n = 1.$$

Since $F \not\equiv 0$ and is in H^p, the linear transformation $S(f) = T(f)/F$ is well defined taking $H^p(U^n)$ into $L^p(v)$ isometrically with $S(1) = 1$, and v and m_n are mutually absolutely continuous.

Let $\psi_m(z) = z_m$ where $z = (z_1, \ldots, z_m)$. Then $\int |S(\psi_m{}^l)|^p dv = 1$ for all powers l as S is an isometry. From [**1**, Proposition 1] we see that since S is an isometry

$$\int |S(\psi_m{}^l)|^2 dv = 1$$

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and since $p \neq 2$, $|S(\psi_m{}^l)| = 1$. In particular S takes the algebra generated by all the ψ_m into $L^\infty(v)$.

By [**1**, Proposition 2] we see that S is multiplicative on this algebra.

We claim $S(\psi_m)$ is the boundary value of an analytic function in U^n. First note that

$$(T(\psi_m))^l = F^l(S(\psi_m))^l,$$
$$T(\psi_m{}^l) = FS(\psi_m{}^l) = F(S(\psi_m))^l$$

a.e. on the distinguished boundary T^n. Therefore

$$(T(\psi_m))^l/F^l = T(\psi_m{}^l)/F$$

a.e. on the distinguished boundary T^n or

$$F(T(\psi_m))^l = F^l T(\psi_m{}^l) \quad \text{a.e.}$$

but since both sides are in $N(U^n)$ they are equal as analytic functions in U^n. Now for $l \geqq 2$ this implies

$$(2) \qquad (T(\psi_m))^l = F^{l-1}T(\psi_m{}^l) \text{ in } U^n.$$

We wish to show that $T(\psi_m)/F$ is analytic in U^n. Since $S(\psi_m)$ is the boundary value function of $T(\psi_m)/F$ we will then have proved our assertion.

Suppose there is a point $p \in U^n$ where $T(\psi_m)/F$ is not analytic. We look at (2) in the local ring at the point p which is a unique factorization domain [**2**]. If Q is an irreducible factor of F then by (2) Q must be a factor of $(T(\psi_m))^l$ and by unique factorization a factor of $T(\psi_m)$. Therefore there must exist a positive t and s and some irreducible factor Q with Q^t and Q^s being the highest powers of Q in the factorization of $T(\psi_m)$ and F respectively with $t \leqq s - 1$. Pick l large enough so that $lt < (l-1)s$. Then from (2) $Q^{(l-1)s}$ must be a factor of $T(\psi_m))^l$ but in its unique factorization. Q^{lt} is the highest power of Q which gives a contradiction that shows $T(\psi_m)/F$ is analytic, and our original claim is proven. We shall show now that $S(\psi_m)$ is inner. Except for ω in a set of measure zero, for all l $F_\omega(S(\psi_m))_\omega^l$ is in $H^p(U)$, F_ω is in $H^p(U)$, and $S(\psi_m)_\omega$ is of modulus one a.e. on T. Now by the reasoning found in [**1**, p. 725], $(S(\psi_m))_\omega$ is inner for ω a.e.; but then for all $r < 1$

$$|(S(\psi_m))_r(\omega)| = |(S(\psi_m))_\omega(r)|.$$

Therefore $|S(\psi_m))_r(\omega)| \leqq 1$ for ω a.e. and by continuity for all ω. Hence $S(\psi_m)$ is in H^∞ and is inner.

Call $S(\psi_m) = \varphi_m$. S is multiplicative on the algebra generated by ψ_m. Since polynomials are dense in H^p, $p < \infty$, and T is bounded, T is given by

$$T(f) = F \cdot f(\varphi_1(z_1, \ldots, z_n), \ldots, \varphi_n(z_1, \ldots, z_n))$$

for all $f \in H^p(U^n)$. Since T^{-1} is an isometry there are $\theta_1, \ldots, \theta_n$ inner functions so that $T^{-1}(f) = G \cdot f(\theta_1, \ldots, \theta_n)$ all $f \in H^p(U^n)$.

Now $TT^{-1}(f) = T^{-1}T(f) = f$. Let $f = 1$ and we see that

$$F \cdot G(\varphi_1, \ldots, \varphi_n) = G \cdot F(\theta_1, \ldots, \theta_n) = 1.$$

Therefore

$$(3) \quad f(\varphi_1(\theta_1, \ldots, \theta_n), \ldots, \varphi_n(\theta_1, \ldots, \theta_n)) = f(\theta_1(\varphi_1, \ldots, \varphi_n), \ldots, \theta_n(\varphi_1, \ldots, \varphi_n)) = f$$

for all $f \in H^p$. Let

$$\Phi = (\varphi_1, \ldots, \varphi_n)\colon U^n \to U^n$$
$$\Theta = (\theta_1, \ldots, \theta_n)\colon U^n \to U^n.$$

Therefore (3) implies $\Phi \cdot \Theta = \Theta \cdot \Phi =$ the identity and since Φ is then an automorphism of U^n the Corollary of [**3**, p. 167] gives

$$\Phi(z) = (\varphi_1(z_{i_1}), \ldots, \varphi_n(z_{i_n}),$$

where the φ_i are conformal maps of U onto U.

There is then a permutation Π such that

$$\Pi \cdot T(f) = H \cdot f(\varphi_1(z_1), \ldots, \varphi_n(z_n))$$

where $H \in H^p$ and the φ_i are conformal maps of U onto U that are permutations of the original φ. We shall abuse notation and denote these permutation as φ_i also. For all $f \in H^p$,

$$(4) \quad \int_{T^n} |H|^p |f \cdot \Phi|^p dm_n = \int |f|^p dm_n = \int \left|\prod_{i=1}^n \frac{\partial \varphi_i}{\partial z}(z_i)\right| |f \cdot \Phi|^p dm_n$$

Let $\mathcal{O}$ be any open set on T^n. Let g_m be the function equal to 1 on $\mathcal{O}$ and $1/m$ off $\mathcal{O}$. By [**3**, Theorem 3.53], $g_m = |h_m{}^*|$ for some $h_m \in H^\infty(U^n)$. But $h_m = f \cdot \Phi$ for some $f \in H^\infty(U^n)$. Using (4) we see that

$$\int_{T^n} |H|^p |h_m|^p dm_n = \int_{T^n} \left|\prod_{i=1}^n \frac{\partial \varphi_i}{\partial z}(z_i)\right| |h_m|^p dm_n,$$

and letting m go to infinity we obtain

$$\int_{\mathcal{O}} |H|^p = \int_{\mathcal{O}} \left|\prod_{i=1}^n \frac{\partial \varphi_i}{\partial z}(z_i)\right|$$

for all open sets $\mathcal{O}$. By standard measure theoretic arguments this shows

$$|H|^p = \left|\prod_{i=1}^n \frac{\partial \varphi_i}{\partial z}(z_i)\right| \text{ a.e.}$$

Now $H \cdot f \cdot \Phi = 1$ for some $f \in H^p(U^n)$. Since $f \cdot \Phi$ is in $H^p(U^n)$ we see that $1/H$ is in $H^p(U^n)$. This shows that H is outer. $(\partial\varphi_i/\partial z)^{1/p}$ is also outer. By [**3**, Lemma 4.4.4], almost every slice function H_ω and

$$\prod_{i=1}^n \left(\frac{\partial \varphi_i}{\partial z}\right)_\omega^{1/p}$$

is outer, and almost everywhere for almost all ω

$$|H_\omega| = \left| \prod_{i=1}^{n} \left(\frac{\partial \varphi_i}{\partial z}\right)_\omega^{1/p} \right|.$$

Thus for almost all ω,

$$H_\omega = b_\omega \prod \left(\frac{\partial \varphi_i}{\partial z}\right)_\omega^{1/p},$$

where the b_ω are unimodular complex numbers. But $H(0) = b_\omega \prod (\partial \varphi_i/\partial z)^{1/p}(0)$ for almost all ω implies that $b_\omega = b$ and $H = b\prod (\partial\varphi_i/\partial z)^{1/p}(z_i)$.

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