Can. J. Math., Vol. XXVI, No. 3, 1974, pp. 629-635

ON GAUSSIAN AND GEODESIC CURVATURE OF RIEMANNIAN MANIFOLDS

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Introduction. In [**1**], S. S. Chern gave a very elegant and simple proof of the Gauss-Bonnet formula for closed (i.e. compact without boundary) oriented Riemannian manifolds of even dimension:

$$\int_M \Omega = c\chi(M).$$

Here, c is a suitable constant depending on the dimension of M and Ω is an n-form ($n = \dim M$) which may be calculated from its curvature tensor. W. Greub gave a coordinate-free description of this integrand Ω (cf. [**4**]).

Chern generalized his result in [**2**] to smooth polyhedral regions G with boundary ∂G:

$$\int_G \Omega + \int_{\nu(\partial G)} \Pi = c\chi(G, \partial G).$$

Here, Π is a $(n - 1)$-form on the unit sphere bundle E over M and $\nu : \partial G \to E$ is the outer unit normal field on the boundary ∂G of G. Now, $\Omega = nKdV_n$, where dV_n is the oriented Riemannian volume on M and K is a smooth function on M, which may be considered as Gaussian curvature. In the same way, $\nu^*\Pi = \kappa dV_{n-1}$, where dV_{n-1} is the induced volume on ∂G. The function κ is then uniquely determined and corresponds to the geodesic curvature in the case $n = 2$, where ∂G is a curve. The aim of this article is to define the geodesic curvature for any oriented hypersurface in an even-dimensional oriented Riemannian manifold—without using the sphere bundle for this definition—and to state and prove the Gauss-Bonnet formula for compact regions with smooth boundary:

$$\int_{\partial G} \kappa dV_{n-1} + n \int_G KdV_n = c_{n-1}\chi(G, \partial G).$$

Here, $c_{n-1} = 2\,\pi^m/(m - 1)!$ is the volume of the unit $(n - 1)$-sphere and χ is the Euler-characteristic. The constants n and c_{n-1} appear in the formula to simplify notation in the definition of K and κ.

Part 1 of this article defines K and κ in terms of the Riemannian connexion and curvature tensor. Part 2 proves the Gauss-Bonnet formula. To do so, we

Received December 8, 1972 and in revised form, March 14, 1973. This research was supported by NRC Grant A-3018.

use Chern's idea to work with the unit sphere bundle. Furthermore we use details of Greub's proof in [**4**].

1. Gaussian and geodesic curvature and the Gauss-Bonnet formula. Throughout this paragraph, M denotes an oriented Riemannian manifold of dimension $n = 2m$, with 2-co- and 2-contravariant curvature tensor R, regarded as 2-form on M with values in $\Lambda^2 TM$ (TM the tangent bundle of M). To avoid unnecessary minus-signs, let us make the following sign-convention for R: If M is the n-sphere of radius r in Euclidean $(n + 1)$-space, its curvature tensor is given by

$$R(x; u_1, u_2) = +\frac{1}{r^2} u_1 \wedge u_2 \quad \text{for } x \in M, u_1, u_2 \in T_xM.$$

(This corresponds to the form $-\Lambda$ in [**4**]!) R induces an n-form R^m on M, with values in the line bundle $\Lambda^n TM$: define for $u_1, \ldots, u_n \in T_xM$

$$R^m(x; u_1, \ldots, u_n) := \frac{1}{2^m m!} \sum_{\sigma \in S_n} \epsilon_\sigma R(x; u_{\sigma 1}, u_{\sigma 2}) \wedge \ldots \wedge R(x; u_{\sigma n-1}, u_{\sigma n}).$$

Here, S_n is the symmetric group of permutations of n objects, and ϵ_σ is the sign of $\sigma \in S_n$.

The oriented Riemannian volume on M is a map $dV_n = e^*: \Lambda^n TM \to \mathbf{R}$, linear on each fibre. It is determined by the property $\langle e^*, e_1 \wedge \ldots \wedge e_n \rangle = 1$ for any positively oriented orthonormal basis $e_1, \ldots, e_n$ of T_xM.

Definition. Let $e_1, \ldots, e_n$ be an orthonormal basis of T_xM, and denote by $e_1^*, \ldots, e_n^*$ the dual basis. Then the Gaussian curvature of M at x is defined by

$$K(x) := \frac{1}{2^m m!} \langle e_1^* \wedge \ldots \wedge e_n^*, R^m(e_1, \ldots, e_n) \rangle.$$

(The choice of the constant factor is not the usual one; it is, however, useful in our context.)

Now fix an orthonormal basis $e_1, \ldots, e_n$ of T_xM, and select $2p$ pairwise different indices $j_1, \ldots, j_{2p}$ $(1 \leqq p \leqq m)$. $e_{j_1}, \ldots, e_{j_{2p}}$ span a $2p$-dimensional subspace of T_xM, and for a sufficiently small neighbourhood U of 0 in that subspace, $\exp_x(U)$ is a $2p$-dimensional submanifold of an open neighbourhood of x in M. We denote it by $M_{j_1 \ldots j_{2p}}$ or M_J, if J denotes the $2p$-tuple $J = (j_1, \ldots, j_{2p})$ and call it the submanifold spanned by $e_{j_1}, \ldots, e_{j_{2p}}$. M_J shall be endowed with the induced Riemannian metric. In particular, it has a well-defined Gaussian curvature at x.

LEMMA. *The Gaussian curvature of M_J at x is*

$$K_J(x) = \frac{1}{2^p p!} \langle e_{j_1}^* \wedge \ldots \wedge e_{j_{2p}}^*, R^p(e_{j_1}, \ldots, e_{j_{2p}}) \rangle.$$

https://doi.org/10.4153/CJM-1974-060-x Published online by Cambridge University Press

Proof. Denote the curvature tensor of M_J by $\tilde{R}$. $\Lambda^{2p}T_xM_J$ may be regarded as a one-dimensional subspace of $\Lambda^{2p}T_xM$, and if p_*: $\Lambda^{2p}T_xM \to \Lambda^{2p}T_xM_J$ denotes the map induced by the orthogonal projection $T_xM \to T_xM_J$, one checks that

$$\tilde{R}^p(e_{j_1}, \ldots, e_{j_{2p}}) = p_*R^p(e_{j_1}, \ldots, e_{j_{2p}}).$$

(To do so, one needs the fact that M_J is geodesic at x, i.e. it contains the geodesics passing through x in directions $e_{j_1}, \ldots, e_{j_{2p}}$.) Now the lemma follows, because $(T_xM_J)^*$ can be regarded as a subspace of $(T_xM)^*$, $e_{j_1}^*, \ldots, e_{j_{2p}}^*$ being the dual basis to $e_{j_1}, \ldots, e_{j_{2p}}$.

Before defining the geodesic curvature, we introduce some notational conventions: For $p, r \in \mathbf{N}$, $p \leqq r$, denote by $A\binom{r}{p}$ the set of ordered p-tuples $(i_1, \ldots, i_p)$ with $1 \leqq i_1 < \ldots < i_p \leqq r$, and for $I \in A\binom{r}{p}$ let $J(I)$ be the complementary $(r - p)$-tuple in

$$A\binom{r}{r-p}: J = (j_1, \ldots, j_{r-p}), 1 \leqq j_1 < \ldots < j_{r-p} \leqq r,$$
$$\{i_1, \ldots, i_p, j_1, \ldots, j_{r-p}\} = \{1, \ldots, r\}.$$

If $p = r$, $J(I)$ is not defined since $A\binom{r}{r} = \emptyset$! For $I \in A\binom{r}{p}$ and real numbers $\lambda_{i_1}, \ldots, \lambda_{i_p}$ set $\lambda_I := \lambda_{i_1} \ldots \lambda_{i_p}$.

Next consider an oriented hypersurface N of M. N has an upper unit normal field ν. For $x \in N$, define $L_x: T_xN \to T_xN$ by $L_x(u) := D_u\nu$, where D is the Levi-Civita connexion on M. L_x is the so-called Weingarten map, which is self-adjoint with respect to the induced metric on N (see [**6**]). Therefore there exists an orthonormal basis $e_1, \ldots, e_{n-1}$ of T_xN, consisting of eigenvectors of L_x. Denote the respective eigenvalues by $\lambda_1, \ldots, \lambda_{n-1}$.

Definition. With the foregoing notations, the geodesic curvature of N at x is defined by

$$\kappa_N(x) := \sum_{k=0}^{m-1} \binom{m-1}{k}^{-1} \sum_{I \in A\binom{n-1}{n-1-2k}} \lambda_I K_{J(I)}(x).$$

(Note that for $k = 0$, $I = (1, \ldots, n - 1)$ and that $J(I)$ is not defined. So the term for $k = 0$ is simply $\lambda_1 \ldots \lambda_{n-1}$!)

This definition reduces to the usual one in the case $n = 2$, where N is an oriented curve on a surface. More generally, the eigenvalues λ_i can be interpreted as geodesic curvatures of certain curves on surfaces: Take a smooth curve γ in N, passing through x in direction e_i, and attach to its points the geodesics passing through it in direction ν. This yields a surface $M_{\nu,\gamma}$ whose tangent space at x is spanned by $\nu(x)$ and e_i. Endow $M_{\nu,\gamma}$ with the induced Riemannian metric and orient it by requiring $(\nu(x), e_i)$ to represent the orientation at x. Then, if γ is oriented by its tangent vector e_i at x, λ_i is the geodesic curvature of γ at x, regarded as curve on the surface $M_{\nu,\gamma}$. We leave the verification to the reader.

Now, let G be a compact domain in M, with smooth boundary ∂G, and denote by ν the outwards pointing unit normal field on ∂G. (Recall that M is oriented!) It can be extended to a vector field $\hat{\nu}$ on G with a single singularity. The index of this singularity does not depend on the particular extension, but only on ν, and hence on G. We define the Euler-characteristic of $(G, \partial G)$ by

$$\chi(G, \partial G) := \text{index } (\hat{\nu}).$$

In this definition, ν can be replaced by any tangent field on ∂G without zeroes. Such tangent fields exist, because ∂G has odd dimension.

THEOREM (Gauss-Bonnet formula). *Let M be an oriented Riemannian manifold of even dimension $n = 2m$, and G a compact domain in M with smooth boundary ∂G, oriented by the outwards pointing normal field. Then*

$$\int_{\partial G} \kappa_{\partial G} dV_{n-1} + n \int_G K dV_n = c_{n-1}\chi(G, \partial G).$$

Here, dV_n and dV_{n-1} denote the oriented Riemannian volume on M and ∂G, respectively, and $c_{n-1} = 2\,\pi^m/(m-1)!$ is the volume of the unit $(n-1)$-sphere.

2. The proof of the Gauss-Bonnet formula. Denote by (E, p, M) the unit sphere bundle over M, whose fibre at x is the unit sphere S_x in the tangent space T_xM.

For $v \in E$, the Levi-Civita connexion D on M defines a decomposition of the tangent space

$$T_vE = H_vE \oplus V_vE$$

into horizontal and vertical part. The horizontal part, H_vE, is isomorphic to $T_{p(v)}M$, the isomorphism being given by the derivative of p at v, $p_*: T_vE \to T_{p(v)}M$. We therefore regard p_* as the projection of T_vE onto its horizontal part and write $H := p_*$. The vertical part can be identified with the subspace $v^\perp$ of $T_{p(v)}M$, and we denote by $V: T_vE \to T_{p(v)}M$ the corresponding projection map.

If $v: U \to E$ is a differentiable section (U an open subset of M), for its derivative $v_*: TU \to TE$ and its covariant derivative $Dv: TU \to TM$ the following relations hold:

$$V \circ v_* = Dv, \quad H \circ v_* = \text{id}. \tag{1}$$

On the fibre product $TE \times_E TE$ we define the alternating bilinear bundle map ("Alternating bilinear" means alternating and **R**-bilinear on each fibre.)

$$W := V \wedge V: TE \times_E TE \to \Lambda^2 TM$$

over $p: E \to M$, i.e., $W(w_1, w_2) = 2V(w_1) \wedge V(w_2) \in \Lambda^2 T_{p(v)}M$ for $v \in E$,

$w_1, w_2 \in T_vE$. In the same way,

$$R \circ H := R \circ (H \times_p H) : TE \times_E TE \to \Lambda^2 TM$$

is an alternating bilinear bundle map over $p : E \to M$.

Following the main ideas of [**4**], we construct for $\lambda \in \mathbf{R}$ the alternating $(n-1)$-linear bundle map

$$\Phi_\lambda := V \wedge (\lambda W + R \circ H)^{m-1} : TE \times_E \ldots \times_E TE \to \Lambda^{n-1} TM.$$

Here, the "exterior power" is defined as

$$(\lambda W + R \circ H)^{m-1} := \frac{1}{(m-1)!} (\lambda W + R \circ H) \wedge \ldots \wedge (\lambda W + R \circ H),$$

and the binomial formula holds:

$$(\lambda W + R \circ H)^{m-1} = \sum_{k=0}^{m-1} \lambda^k W^k \wedge (R \circ H)^{m-1-k}.$$

(See [**4**].)

For fixed $v \in E$, $\varphi_\lambda(v) := \langle e^*, v \wedge \Phi_\lambda \rangle$ is a well-defined alternating $(n-1)$-form on T_vE, i.e. φ_λ is an $(n-1)$-form on the manifold E, depending on the parameter λ. With the inclusion map $J : E \to TM$, we can write

$$\varphi_\lambda = \langle e^*, J \wedge \Phi_\lambda \rangle.$$

From (1) we obtain for any local differentiable section v in E the relation

(2) $$v^*(\varphi_\lambda) = \langle e^*, v \wedge Dv \wedge (\lambda Dv \wedge Dv + R)^{m-1} \rangle.$$

Greub proved in [**4**] the formula

(3) $$d\langle e^*, v \wedge Dv \wedge (\lambda Dv \wedge Dv + R)^{m-1} \rangle = \sum_{k=0}^{m-1} \frac{(2k+1)!}{k!} \lambda^k \langle e^*, 2(k+1)(Dv)^{2(k+1)} \wedge R^{m-1-k} - (Dv)^{2k} \wedge R^{m-k} \rangle.$$

(Note our sign-convention for R!) Since (2) and (3) hold for any local differentiable section v in E, (3) determines $d\varphi_\lambda$ uniquely:

(4) $$d\varphi_\lambda = \sum_{k=0}^{m-1} \frac{(2k+1)!}{k!} \lambda^k \langle e^*, 2(k+1) V^{2(k+1)} \wedge (R \circ H)^{m-1-k} - V^{2k} \wedge (R \circ H)^{m-k} \rangle.$$

Comparing the coefficients of λ^k in φ_λ and $d\varphi_\lambda$ leads to the definition

(5) $$\varphi_k := \langle e^*, J \wedge V^{2k+1} \wedge (R \circ H)^{m-1-k} \rangle, \quad 0 \leqq k \leqq m-1,$$

and the relation

(6) $$d\varphi_k = \langle e^*, 2(k+1) V^{2(k+1)} \wedge (R \circ H)^{m-1-k} - V^{2k} \wedge (R \circ H)^{m-k} \rangle.$$

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Now set

$$(7)\qquad \varphi := \sum_{k=0}^{m-1} \frac{1}{2^k k! \binom{m-1}{k}} \varphi_{m-1-k}.$$

Let us integrate φ over a fibre in the bundle (E, p, M): The integrals of the terms containing $R \circ H$ vanish, because $H(T_v(S_x)) = 0$ for $x \in M$, $v \in S_x$. So we obtain

$$(8)\qquad \int_{S_x} \varphi = \int_{S_x} \varphi_{m-1} = \int_{S_x} \langle e^*, J \wedge V^{n-1} \rangle = \int_{S_x} dV_{n-1} = c_{n-1}.$$

From (5) and (6) we find that

$$d\varphi = \frac{-1}{2^{m-1}(m-1)!} \langle e^*, (R \circ H)^m \rangle = \frac{-1}{2^{m-1}(m-1)!} p^* \langle e^*, R^m \rangle,$$

and our definition of the Gaussian curvature turns this into

$$(9)\qquad d\varphi = -np^*(KdV_n).$$

If N is any oriented hypersurface in M with upper normal field ν, the $(n-1)$-form $\nu^*\varphi$ on N can be written as

$$(10)\qquad \nu^*\varphi = \tilde{\kappa}_N dV_{n-1},$$

where $\tilde{\kappa}_N$ is a well-defined smooth function on N.

Now it is easy to prove the theorem with $\tilde{\kappa}_{\partial G}$ instead of $\kappa_{\partial G}$: Extend the outwards pointing normal field ν on ∂G to a unit vector field $\hat{\nu}: G - x_0 \rightarrow E$ with a singularity of index $\chi := \chi(G, \partial G)$ at $x_0 \in G$. $\hat{\nu}(G)$ is an n-dimensional submanifold of E with boundary $\partial\hat{\nu}(G) = \nu(\partial G) - \chi S_{x_0}$. Hence, by Stokes' Theorem, and (8), (9), (10),

$$-n \int_G KdV_n = \int_{\hat{\nu}(G)} d\varphi = \int_{\nu(\partial G)} \varphi - \chi \int_{S_{x_0}} \varphi = \int_{\partial G} \tilde{\kappa}_{\partial G} dV_{n-1} - c_{n-1}\chi.$$

Our proof will be completed, if we can show $\tilde{\kappa}_N = \kappa_N$, for any oriented hypersurface N in M with upper normal field ν. Fix a point $x \in N$ and set $e_0 := \nu(x)$, and let $e_1, \ldots, e_{n-1}$ be any positively oriented orthonormal basis of T_xN. Then, by definition,

$$\begin{aligned} \tilde{\kappa}_N(x) &= \nu^*\varphi(e_1, \ldots, e_{n-1}) \\ (11)\qquad &= \sum_{k=0}^{m-1} \frac{1}{2^k k! \binom{m-1}{k}} \nu^*\varphi_{m-1-k}(e_1, \ldots, e_{n-1}). \end{aligned}$$

To express $\nu^*\varphi_{m-1-k}(e_1, \ldots, e_{n-1})$, let us write

$$e_{0,I}^* = e_{0,i_1\ldots i_r}^* := e_0^* \wedge e_{i_1}^* \wedge \ldots \wedge e_{i_r}^* \quad \text{for} \quad I = (i_1, \ldots, i_r),$$

and similarly

$$R^k_J = R^k_{j_1\ldots j_{2k}} = R^k(e_{j_1}, \ldots, e_{j_{2k}}),$$
$$L_{x,I} = L_{x,i_1\ldots i_r} = L_x(e_{i_1}) \wedge \ldots \wedge L_x(e_{i_r}),$$

where $L_x : T_xN \to T_xN$ is the Weingarten map as defined above. Recall also the definition of the index set $A\binom{r}{p}$ for $0 \leqq p \leqq r$ and the map

$$J : A\binom{r}{p} \to A\binom{r}{r-p}.$$

Thus we have

$$\nu^*\varphi_{m-1-k}(e_1, \ldots, e_{n-1})$$
$$= \frac{1}{(2k)!(n-1-2k)!} \sum_{i_1,\ldots,i_{n-1}} \langle \overset{*}{e}_{0,i_1\ldots i_{n-1}}, e_0 \wedge L_{x,i_1\ldots i_{n-1-2k}} \wedge R^k_{i_{n-2k}\ldots i_{n-1}} \rangle$$
$$= \sum_{I \in A\binom{n-1}{n-1-2k}} \langle \overset{*}{e}_{0,I,J(I)}, \overset{*}{e}_0 \wedge L_{x,I} \wedge R^k_{J(I)} \rangle.$$

Now take as $e_1, \ldots, e_{n-1}$ the eigenvectors of L_x, with respective eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$. Then we obtain

$$\nu^*\varphi_{m-1-k}(e_1, \ldots, e_{n-1}) = \sum_{I \in A\binom{n-1}{n-1-2k}} \lambda_I \langle \overset{*}{e}_{J(I)}, R^k_{J(I)} \rangle$$
$$= 2^k k! \sum_{I \in A\binom{n-1}{n-1-2k}} \lambda_I K_{J(I)}(x),$$

according to the lemma. Inserting this in (11) shows $\tilde{\kappa}_N = \kappa_N$. In particular, κ_N is a smooth function, which is not immediately clear from its definition.

References

1. S. S. Chern, *A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds*, Ann. of Math. *45* (1944), 747–752.

2. ——— *On the curvatura integra in a Riemann manifold*, Ann. of Math. *46* (1945), 674–684.

3. ——— *On curvature and characteristic classes of a Riemann manifold*, Abh. Math. Sem. Univ. Hamburg *20* (1955), 117–126.

4. W. H. Greub, *Zur Theorie der linearen Uebertragungen*, Ann. Acad. Sci. Fenn. Ser. A. I. *346* (1964), 3–32.

5. ——— *Multilinear algebra* (Springer, Berlin, Heidelberg, New York, 1967),

6. N. J. Hicks, *Notes on differential geometry* (Van Nostrand, New York, 1965).

7. H. Holmann and H. Rummler, *Differentialformen*, Bibliographisches Institut, Mannheim, 1972.

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