NATURALLY ORDERED BANDS

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In the terminology of Clifford and Preston [**2**], a *band B* is a semigroup in which every element is idempotent. On such a semigroup there is a natural (partial) order relation defined by the rule

$$e \leqq f \quad \text{if and only if} \quad ef = fe = e.$$

If the order relation $\leqq$ is compatible with the multiplication in B, in the sense that $e \leqq f$ and $g \leqq h$ together imply that $eg \leqq fh$, we shall say that B is a *naturally ordered* band. The object of this note is to describe the structure of naturally ordered bands.

It is clear that a semilattice is a naturally ordered band. It is also the case that a *rectangular* band, which it is convenient to define here as a band in which the relation $xyz = xz$ holds identically, is naturally ordered, since in such a band $e \leqq f$ if and only if $e = f$. The structure of a rectangular band can be described completely in terms of sets: it is the cartesian product $I \times J$ of two sets I and J, with multiplication defined by

$$(i_1, j_1)(i_2, j_2) = (i_1, j_2).$$

It is known (Clifford [**1**], McLean [**2**]) that an arbitrary band is a semilattice of rectangular bands. Investigations into what Clifford and Preston [**2**, p. 28] have called the " fine structure " of unions of groups (of which bands are a special case) have been made by Clifford, particularly in the final section of his paper [**1**], and, more recently, by Fantham [**3**] and Petrich [**7**]. Both Fantham and Petrich give descriptions of the structure of certain types of unions of groups *in terms of bands*, so that their theorems become trivial when applied to bands. Clifford considers the structure of a semigroup which is the disjoint union of an arbitrary semigroup S_α and a completely simple semigroup S_β, in which $S_\alpha S_\beta$ and $S_\beta S_\alpha$ are both contained in S_β. Some of the steps in the proofs below can be deduced from results of Clifford, but it seemed easier to derive them independently.

The first theorem characterises naturally ordered bands in such a way as to show that they form a subvariety of the variety of bands.

THEOREM 1. *A band B is naturally ordered if and only if the identical relation*

$$xzxyxztzxz = xyxztz \tag{1}$$

holds in B.

Proof. For any x, y, z, t in B, we have

$$xyx \leqq x, \quad ztz \leqq z.$$

If $\leqq$ is compatible, it follows that $xyxztz \leqq xz$, and hence $xzxyxztzxz = xyxztz$ as required.

Conversely, if (1) holds identically in B, and if $y \leqq x$, $t \leqq z$, then $y = xyx$, $t = ztz$.

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Hence

$$yt = xyxztz = xzxyxztzxz, \quad \text{by (1)},$$
$$= xzytxz,$$

from which it follows that $yt \leqq xz$. Thus $\leqq$ is compatible.

It follows incidentally that not all bands are naturally ordered: the free band on four generators (see Green and Rees [**4**]) clearly does not satisfy the identical relation (1).

THEOREM 2. *Let* $Y = \{\alpha, \beta, \gamma, \dots\}$ *be a semilattice and let* $\{B_\alpha : \alpha \in Y\}$ *be a family of disjoint rectangular bands, indexed by* Y. *If* $\alpha > \beta$ *in* Y, *let* $\phi_{\alpha,\beta}$ *be a homomorphism from* B_α *into* B_β, *and suppose that if* $\alpha > \beta > \gamma$ *then*

$$\phi_{\alpha,\gamma} = \phi_{\alpha,\beta}\phi_{\beta,\gamma}. \tag{2}$$

Let $\phi_{\alpha,\alpha}$ *be the identical automorphism of* B_α. *Let* S *be the union of the rectangular bands* B_α *and define the product of two elements* e_α *and* f_β *of* S (*in* B_α *and* B_β *respectively*) *by*

$$e_\alpha f_\beta = (e_\alpha \phi_{\alpha,\gamma})(f_\beta \phi_{\beta,\gamma}), \tag{3}$$

where $\gamma = \alpha\beta$, *the product of* α *and* β *in the semilattice* Y, *and where the right-hand product is evaluated in the rectangular band* B_γ.

Then S *is a naturally ordered band. Conversely, any naturally ordered band can be constructed in this way.*

Proof. First, since it is clear that the transitivity condition (2) also holds under the weaker assumption that $\alpha \geqq \beta \geqq \gamma$, the groupoid S whose construction is described in the statement of the theorem is an example of what Fantham [**3**] calls a mapping semigroup of an array of semigroups over the semilattice Y. Hence, by [**3**, Proposition 3], S is a semigroup. Clearly S is a band, since every element of S belongs to some B_α.

If $e_\alpha \in B_\alpha$ and $f_\beta \in B_\beta$, then $f_\beta \leqq e_\alpha$ if and only if $f_\beta = e_\alpha f_\beta e_\alpha$. In fact, we can show that

$$f_\beta \leqq e_\alpha \quad \text{if and only if} \quad \beta \leqq \alpha \quad \text{and} \quad f_\beta = e_\alpha \phi_{\alpha,\beta}. \tag{4}$$

For if $f_\beta \leqq e_\alpha$ then the multiplication rule (3) implies that $\alpha\beta\alpha = \beta$, from which we deduce that $\beta \leqq \alpha$. Again by (3), we have that

$$f_\beta = (e_\alpha \phi_{\alpha,\beta}) f_\beta (e_\alpha \phi_{\alpha,\beta}) = e_\alpha \phi_{\alpha,\beta},$$

since B_β is a rectangular band. Conversely, if $\beta \leqq \alpha$ and $f_\beta = e_\alpha \phi_{\alpha,\beta}$, then

$$e_\alpha f_\beta e_\alpha = (e_\alpha \phi_{\alpha,\beta}) f_\beta (e_\alpha \phi_{\alpha,\beta}) = f_\beta^3 = f_\beta,$$

and so $f_\beta \leqq e_\alpha$.

To show that the band S is naturally ordered, suppose that $f_\beta \leqq e_\alpha$ and $h_\delta \leqq g_\gamma$, where $g_\gamma \in B_\gamma$ and $h_\delta \in B_\delta$; we must show that $f_\beta h_\delta \leqq e_\alpha g_\gamma$. By (4), we have that

$$\beta \leqq \alpha, \quad \delta \leqq \gamma, \quad f_\beta = e_\alpha \phi_{\alpha,\beta}, \quad h_\delta = g_\gamma \phi_{\gamma,\delta}.$$

Now
$$e_\alpha g_\gamma = (e_\alpha \phi_{\alpha,\alpha\gamma})(g_\gamma \phi_{\gamma,\alpha\gamma}) = p_{\alpha\gamma},$$
say, and
$$f_\beta h_\delta = (f_\beta \phi_{\beta,\beta\delta})(h_\delta \phi_{\delta,\beta\delta}) = q_{\beta\delta}.$$

The natural order relation in the semilattice Y is compatible with the multiplication in Y, and so certainly $\beta\delta \leqq \alpha\gamma$. Also

$$q_{\beta\delta} = (f_\beta \phi_{\beta,\beta\delta})(h_\delta \phi_{\delta,\beta\delta}) = (e_\alpha \phi_{\alpha,\beta}\phi_{\beta,\beta\delta})(g_\gamma \phi_{\gamma,\delta}\phi_{\delta,\beta\delta}) = (e_\alpha \phi_{\alpha,\alpha\gamma}\phi_{\alpha\gamma,\beta\delta})(g_\gamma \phi_{\gamma,\alpha\gamma}\phi_{\alpha\gamma,\beta\delta})$$
$$= [(e_\alpha \phi_{\alpha,\alpha\gamma})(g_\gamma \phi_{\gamma,\alpha\gamma})]\, \phi_{\alpha\gamma,\beta\delta} = p_{\alpha\gamma}\phi_{\alpha\gamma,\beta\delta},$$

and so $f_\beta h_\delta \leqq e_\alpha g_\gamma$ as required.

Conversely, if S is a naturally ordered band, then S is, by virtue of the theorem of Clifford [1] and McLean [5], a semilattice Y of rectangular bands $\{B_\alpha : \alpha \in Y\}$. The rectangular bands B_α are the $\mathscr{J}$-classes of S, and $B_\alpha \leqq B_\beta$ in the natural order among the $\mathscr{J}$-classes (see [**2**, §2.1)] if and only if $\alpha \leqq \beta$ in the semilattice Y.

LEMMA. *Let α, β be elements of the semilattice Y such that $\beta \leqq \alpha$, and let e_α be an arbitrary element of B_α. Then there exists one and only one element f_β of B_β such that $f_\beta \leqq e_\alpha$.*

Proof. If b_β is an arbitrary element of B_β, then $e_\alpha b_\beta e_\alpha \in B_{\alpha\beta\alpha} = B_\beta$, since $\beta \leqq \alpha$. Also $e_\alpha b_\beta e_\alpha \leqq e_\alpha$, since

$$e_\alpha \,.\, e_\alpha b_\beta e_\alpha = e_\alpha b_\beta e_\alpha \,.\, e_\alpha = e_\alpha b_\beta e_\alpha.$$

Suppose now that f_β and g_β are two elements of B_β such that $f_\beta \leqq e_\alpha$, $g_\beta \leqq e_\alpha$. Then

$$f_\beta e_\alpha = e_\alpha f_\beta = f_\beta, \quad g_\beta e_\alpha = e_\alpha g_\beta = g_\beta$$

and, since B_β is a rectangular band,

$$f_\beta g_\beta f_\beta = f_\beta, \quad g_\beta f_\beta g_\beta = g_\beta.$$

Since $\leqq$ is by assumption compatible,

$$f_\beta g_\beta \leqq f_\beta e_\alpha = f_\beta, \quad f_\beta g_\beta f_\beta \leqq f_\beta g_\beta e_\alpha = f_\beta g_\beta.$$

Hence

$$f_\beta = f_\beta g_\beta f_\beta \leqq f_\beta g_\beta \leqq f_\beta,$$

from which we deduce that $f_\beta g_\beta = f_\beta$. But, by a similar argument, $f_\beta g_\beta = g_\beta$, and so $f_\beta = g_\beta$. This completes the proof of the lemma.

Returning now to the proof of Theorem 2, we can, by virtue of the lemma, define a mapping $\phi_{\alpha,\beta} : B_\alpha \to B_\beta$ (if $\beta \leqq \alpha$) by taking $e_\alpha \phi_{\alpha,\beta}$ to be the unique element f_β of B_β such that $f_\beta \leqq e_\alpha$. The compatibility of the order ensures that $\phi_{\alpha,\beta}$ is a homomorphism, while if $\beta = \alpha$ the mapping is the identical automorphism of B_α. The condition (2) is a direct result of the transitivity of the order.

To verify (3), first notice that if $\gamma \leqq \alpha$ and $e_\alpha \in B_\alpha$, then $e_\alpha b_\gamma e_\alpha \leqq e_\alpha$ for *any* b_γ in B_γ, and so $e_\alpha \phi_{\alpha,\gamma} = e_\alpha b_\gamma e_\alpha$. If $f_\beta \in B_\beta$ and if we take γ as $\alpha\beta$, we know that $e_\alpha f_\beta \in B_\gamma$. Hence

$$e_\alpha \phi_{\alpha,\gamma} = e_\alpha (e_\alpha f_\beta) e_\alpha,$$

and similarly $f_\beta \phi_{\beta,\gamma} = f_\beta (e_\alpha f_\beta) f_\beta$. Hence

$$(e_\alpha \phi_{\alpha,\gamma})(f_\beta \phi_{\beta,\gamma}) = e_\alpha (e_\alpha f_\beta) e_\alpha f_\beta (e_\alpha f_\beta) f_\beta = (e_\alpha f_\beta)^3 = e_\alpha f_\beta.$$

This completes the proof.

Remark. As can easily be verified, any homomorphism ϕ from a rectangular band $I \times J$ into a rectangular band $K \times L$ determines two mappings $\lambda : I \to K$, $\mu : J \to L$ such that

$$(i,j)\phi = (i\lambda, j\mu) \tag{5}$$

for every (i,j) in $I \times J$. Conversely, if $\lambda : I \to K$ and $\mu : J \to L$ are arbitrary mappings, then (5) defines a homomorphism $\phi : I \times J \to K \times L$. These statements can alternatively be deduced from a general theorem due to Munn [6] and quoted by Clifford and Preston [2, Theorem 3.11].

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