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SPLITTING THEOREMS IN ABELIAN-BY-HYPERCYCLIC GROUPS

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Abstract

If $\mathfrak{F}$ is a saturated formation of finite soluble groups and G is a finite group whose $\mathfrak{F}$-residual A is abelian then it is well known that G splits over A and the complements are conjugate. Hartley and Tomkinson (1975) considered the special case of this result in which $\mathfrak{F}$ is the class of nilpotent groups and obtained similar results for abelian-by-hypercentral groups with rank restrictions on the abelian normal subgroup. Here we consider the supersoluble case, obtaining corresponding results for abelian-by-hypercyclic groups.

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1. Introduction

In Carter and Hawkes (1967), the following result was proved: if $\mathfrak{F}$ is a saturated formation of finite soluble groups and G is a finite group whose $\mathfrak{F}$-residual A is abelian, then G splits over A and the complements to A in G are conjugate. This result has already been extended to classes of locally finite groups in which a formation theory can be developed (for example, Gardiner, Hartley and Tomkinson (1971), Theorem 4.12). Hartley and Tomkinson (1975) considered the special case of the result given above in which $\mathfrak{F}$ is the class of nilpotent groups and obtained similar results for infinite groups with rank restrictions on the abelian sections. Results for the supersoluble case have been obtained by Newell (1975) and in this paper we also consider this case aiming at results of the same type as those in Hartley and Tomkinson (1975) and we do not use the sets $L_G(\mu)$, $S_G(\mu)$ introduced by Newell (1975).

We recall some definitions. A group is called *hypercyclic* if it has an ascending normal series with cyclic factors. G is *parasoluble* if it has a finite normal series

$$(1) \qquad 1 = G_0 \lhd G_1 \lhd \dots \lhd G_n = G$$

such that, for each $i = 1, \dots, n$, G_i/G_{i-1} is abelian and each subgroup of G_i/G_{i-1} is normal in G/G_{i-1}. These definitions are usually considered to be the supersoluble analogues of groups having an ascending or finite central series. However, the results which we obtain here, which correspond to the results concerning nilpotent groups obtained in Hartley and Tomkinson (1975), require a stronger condition than G being parasoluble. We define G to be *pretersoluble* if it has a finite normal series (1) such that G_i/G_{i-1} is abelian, each subgroup of G_i/G_{i-1} is normal in G/G_{i-1} and, if X_p is the p-subgroup of G_i/G_{i-1} then $G/C_G(X_p)$ is cyclic of order dividing $p-1$ for p odd and G either induces the involution automorphism on X_2 or acts trivially on X_2. This last condition implies that whenever there is a factor isomorphic to C_{p^∞}, G induces a finite group of automorphisms.

We consider a group G and a $\mathbf{Z}G$-module A. The rank conditions on A are conditions on A as an abelian group. In Robinson (1972b), an abelian group A is called an $\mathfrak{S}_0$-group if it has finite 0-rank and finite p-rank for each prime p; A is an $\mathfrak{S}_1$-group if it is an $\mathfrak{S}_0$-group and its torsion group is a Černikov group. An abelian $\mathfrak{S}_1$-group with finite torsion subgroup will be called an $\mathfrak{S}_1^*$-group as in Hartley and Tomkinson (1975). A group G is an $\mathfrak{S}_0$-group ($\mathfrak{S}_1$-group) if it has a finite series whose factors are abelian $\mathfrak{S}_0$-groups ($\mathfrak{S}_1$-groups).

The condition that A should be a residual of the extension E of A by G is interpreted as A having no images of a certain type. As usual we say that A has no images satisfying a certain property to mean that A has no non-zero images with that property. A $\mathbf{Z}G$-module M is called *G-hypercyclic* if it has an ascending series of submodules in which each factor is a cyclic group. M is called *G-pretersoluble* if it has a finite series of submodules

$$0 = M_0 < M_1 < \ldots < M_n = M$$

such that each subgroup of M_i/M_{i-1} is a $\mathbf{Z}G$-submodule and if X_p is the p-subgroup of M_i/M_{i-1} then $G/C_G(X_p)$ is cyclic of order dividing $p-1$ for p odd and G either induces the involution automorphism or acts trivially on X_2. It is clear that a G-pretersoluble module M has an image M/N which, as an additive group, is either cyclic of order p or is a group of type C_{p^∞} and $G/C_G(M/N)$ is finite.

The main result we prove is:

THEOREM A. *Let G be a hypercyclic group and A a $\mathbf{Z}G$-module. If*

(i) *A is an $\mathfrak{S}_1$-group and has no G-hypercyclic image, or*

(ii) *A is an $\mathfrak{S}_1^*$-group and has no G-pretersoluble image,*

then every extension of A by G splits conjugately over A.

The condition that A has no G-pretersoluble image will not be sufficient to obtain Theorem A if A has an infinite torsion subgroup. But again a near-splitting result similar to Hartley and Tomkinson (1975), Theorem C, can be obtained, corresponding to the situation in which G is pretersoluble and A has no G-pretersoluble image.

THEOREM B. *Let E be a group with pretersoluble residual A. Suppose that E/A is pretersoluble and A is an abelian $\mathfrak{S}_1$-group. Then there is a finite normal subgroup F of E contained in A and such that E/F splits over A/F. There is a finite normal subgroup $F^* \geqslant F$ of E, contained in A and such that the complements to A/F in E/F are conjugate modulo F^*.*

The proofs of Theorems A and B break down into fairly distinct stages. First, we give reduction theorems which show that the conditions of having no G-hypercyclic or G-pretersoluble image pass down to submodules. These results mean that in later stages we can use induction on a series in A and usually assume that A is either torsion-free and rationally irreducible, finite and irreducible, or a divisible abelian p-group in which each submodule is finite. As in Newell (1975), we then consider the two different cases in which $[A, G'] = 1$ and $[A, G'] \neq 1$. The second of these is dealt with using the results of Hartley and Tomkinson (1975) and the first case is reduced to the situation in which $E = AG$ is metabelian. The results required for the metabelian case are given in Section 4 in the form of splitting theorems for modules which are deduced from the results of Hartley and Tomkinson (1975).

The case in which A is finite can be given in the much more general form which would be necessary if these results were to be put in a formation-theoretic setting and is given in this form in Section 3.

Some of the steps in our argument are clearly based on methods used in formation theory. For example, if U/V is a factor of the $\mathbf{Z}G$-module A, then $|U/V| = p$ if and only if $G/C_G(U/V) \in \mathfrak{A}_{p-1}$, the class of abelian groups with exponent dividing $p-1$. Also we frequently consider $G^* = G'G^{p(p-1)}$, the $\mathfrak{A}_{p(p-1)}$-residual of G. Our notation for group classes is usually standard but as in Gardiner, Hartley and Tomkinson (1971) we use $\mathfrak{X}^*$ to denote the class of finite $\mathfrak{X}$-groups [except for $\mathfrak{S}_1^*$ defined above] so that $G \in \mathfrak{A}\mathfrak{S}_p^* \mathfrak{A}_{p-1}^*$ means that G is abelian-by-(finite p)-by-(finite abelian of exponent dividing $p-1$). This class is also derived from the formation definition of supersoluble groups.

However, some of the steps seem to depend heavily on the cyclicity of normal subgroups. For example, we frequently use the fact that if $\langle g \rangle \lhd G$, then $[A, \langle g \rangle] = [A, g]$. Also if $\langle g \rangle$ is an infinite cyclic normal subgroup then $|G/C_G\langle g \rangle| = 1$ or 2 and it is often important to note that $2 \mid p(p-1)$ for all primes p.

Many of the results concerning abelian-by-hypercentral groups have been given a cohomological setting by Robinson (1976). The condition that A has no G-nilpotent image is equivalent to $H_0(G, A) = 0$. If A has no G-hypercentral image then it can be shown that $C_A(G) = 0$; that is, $H^0(G, A) = 0$. It is not easy to see how the present results could be given a similar formulation.

Our notation and terminology will be consistent with Hartley and Tomkinson (1975) unless otherwise explained.

2. Reduction theorems

We begin by considering the action of a locally supersoluble group on a module A which is a periodic abelian group. To do this we require a special case of the following result of Baer (1972) for finite soluble groups.

LEMMA 2.1. *Let $\mathfrak{F}$ be a saturated formation of finite soluble groups. If N is a normal subgroup of the finite soluble group G such that $G/C_G(N)\in\mathfrak{F}$, then $N = X\times X^*$, where X and X^* are normal subgroups of G, each G-chief factor of X is $\mathfrak{F}$-central and each G-chief factor of X^* is $\mathfrak{F}$-eccentric.*

If $\mathfrak{F}$ is the class of finite supersoluble groups, then a chief factor is $\mathfrak{F}$-central if and only if it is cyclic of prime order. This is the case which is used in the following result.

LEMMA 2.2. *Let G be a locally supersoluble group and let A be a $\mathbf{Z}G$-module which is a periodic group of finite rank. Then $A = C\oplus C^*$, where C is a G-hypercyclic module and C^* has no factors which are cyclic groups. This decomposition is unique.*

PROOF. It is sufficient to consider the case in which A is a p-group. Let $A_i = \Omega_i(A)$ and $G_i = C_G(A_i)$. A_i is finite and G/G_i is a finite supersoluble group. Applying Lemma 2.1 to the extension of A_i by G/G_i, we see that $A_i = C_i\oplus C_i^*$, where C_i is G-hypercyclic and C_i^* has no factors which are cyclic groups.

It is clear that $C_i\leqslant C_{i+1}$ and $C_i^*\leqslant C_{i+1}^*$. If we let $C = \bigcup_{i=1}^{\infty} C_i$ and $C^* = \bigcup_{i=1}^{\infty} C_i^*$, then C and C^* are $\mathbf{Z}G$-submodules of A.

$$C+C^* = \bigcup_{i=1}^{\infty}(C_i+C_i^*) = \bigcup_{i=1}^{\infty} A_i = A$$

and

$$C\cap C^* = \left(\bigcup_{i=1}^{\infty} A_i\right)\cap C\cap C^* = \bigcup_{i=1}^{\infty}(A_i\cap C\cap C^*) = \bigcup_{i=1}^{\infty}(C_i\cap C_i^*) = 0.$$

Thus $A = C\oplus C^*$.

C is clearly G-hypercyclic. If U/V is an irreducible factor of C^*, then there is an integer i such that $U\cap C_i\nleqslant V$ and so $U/V\cong(U\cap C_i^*)/(V\cap C_i^*)$ and so U/V is not a cyclic group.

The uniqueness is clear for if X is a G-hypercyclic submodule of A, then $(X+C)/C$ is G-hypercyclic and is isomorphic to $(X+C)\cap C^*$. Thus $X+C = C$ and $X\leqslant C$. Similarly every submodule without cyclic factors is contained in C^*.

In showing that the conditions used in Theorems A and B pass down to submodules we begin with the slightly simpler pretersoluble case.

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LEMMA 2.3. *Let G be a hypercyclic group and A a $\mathbb{Z}G$-module with a submodule B such that A/B is an $\mathfrak{S}_1^*$-group. If A has no G-pretersoluble image, then B has no G-pretersoluble image.*

PROOF. If there is a counterexample to the lemma, then by the remarks prior to the statement of Theorem A, we may suppose that B is either a group of order p or a group of type C_{p^∞} with $G/C_G(B)$ finite.

Let T/B be the torsion part of A/B so that T/B is finite. If B is cyclic of order p, then it follows from Lemma 2.2 that T has an image which is cyclic of order p. If B is of type C_{p^∞}, then there is a finite submodule F of T such that $T = FB$ and so T/F is G-pretersoluble. Thus T has a G-pretersoluble image and we may assume that T satisfies the conditions given for B above.

Our counterexample may be chosen so that the rank of A/T is minimal and clearly this rank is non-zero. If A/T is not rationally irreducible, then it has a submodule C/T such that A/C is torsion-free. By induction, C has a G-pretersoluble image C/C_1, say, and then considering A/C_1, we see that A has a G-pretersoluble image. Thus A/T is rationally irreducible.

Suppose $C_G(A/T) \neq 1$; then there is an element $x \in C_G(A/T)$ such that $\langle x \rangle$ is a non-trivial normal subgroup of G. Let $K = C_G(\langle x \rangle)$ so that G/K is finite. The mapping $a \to a(x-1)$ is a non-zero K-homomorphism of A into T and so A has a K-image A/D such that $|A/D| = p$ or A/D is a group of type C_{p^∞} and $C_K(A/D) = C_K(T)$. But $D = C_A(x) = C_A(\langle x \rangle)$ is a G-submodule of A and $C_G(A/D) \geqslant C_K(T)$. Thus $G/C_G(A/D)$ is finite and A/D is a G-pretersoluble image of A.

We may therefore assume that A/T is faithful for G. If $C_G(T) \neq 1$, then there is an element $y \in C_G(T)$ such that $\langle y \rangle$ is a non-trivial normal subgroup of G. Since A/T is faithful, $C_{G/T}(y) \neq A/T$ and so $C_{A/T}(\langle y \rangle) = 0$, since A/T is rationally irreducible. The mapping $\sigma\colon a \to a(y-1)$ is a $C_G(y)$-homomorphism of A and, since $C_{A/T}(y) = 0$, it induces a $C_G(y)$-monomorphism of A/T. By Ex. 5 on p. 153 of Fuchs (1973), $|A/T\colon (A/T)\sigma|$ is finite, that is, $|A\colon A\sigma + T|$ is finite. If $a(y-1) \in T$, then $a \in T$ (since $C_{A/T}(y) = 0$) and so $a(y-1) = 0$, that is, $A\sigma \cap T = 0$. Thus $A/A\sigma$ is a torsion abelian group. Also $A\sigma = [A, y] = [A, \langle y \rangle]$ is a $\mathbb{Z}G$-submodule of A. If T is finite, then $A/A\sigma$ is finite and, by Lemma 2.2, has an image of order p. If $T \cong C_{p^\infty}$ then $A/A\sigma$ has a finite submodule $F/A\sigma$ such that $F + T = A$. Then $A/F \cong T/(T \cap F)$ is a G-pretersoluble image of A.

If $C_G(T) = 1$, then G is a finite cyclic group. Let $x \in A - T$ and $R = \mathbb{Z}G$; then $xR = X \geqslant \Omega_1(T)$ and so there is an $r \in R$ such that xr is a non-zero element of $\Omega_1(T)$. Since R is commutative, multiplication by r is an R-homomorphism of A into itself. $Xr \leqslant \Omega_1(T)$ and so $|X/(X \cap \operatorname{Ann} r)| = p$. Also $A/(X+T)$ is a torsion group, since A/T is rationally irreducible, and $(X+T)/X$ is a torsion group. Thus

$A/(X \cap \operatorname{Ann} r)$ and hence $A/\operatorname{Ann} r$ is a torsion group. Thus $Ar \leqslant T$ and so $A/\operatorname{Ann} r$ is a G-pretersoluble image of A.

The result that will be required for Theorem B is the following which is based on part of Hartley and Tomkinson (1975), Lemma 2.3.

LEMMA 2.4. *Let G be a hypercyclic group and let A be a* $\mathbb{Z}G$*-module which is a periodic group of finite rank. If B is a submodule of A which is a divisible group and if A has no G-pretersoluble image, then B has no G-pretersoluble image.*

PROOF. We may suppose that we have a counterexample in which the rank of A is minimal. It is clear that we may assume B to be of type C_{p^∞} with $G/C_G(B)$ finite and A to be a p-group. There is an integer $n \geqslant 1$ such that $A/\Omega_n(A)$ is divisible and, by considering this in place of A, we may assume that A is divisible.

Let $x \in C_G(B)$ such that $\langle x \rangle$ is a non-trivial normal subgroup of G. The mapping $\sigma: a \to a(x-1)$ is a $C_G(x)$-endomorphism of A and $\operatorname{Ker} \sigma \geqslant B$. Thus $[A,x] = \operatorname{Im} \sigma < A$. Also $[A,x]$ is divisible and so the rank of $A/[A,x]$ is less than that of A. $[A,x] = [A, \langle x \rangle]$ is a $\mathbb{Z}G$-submodule of A. If $B \nleqslant [A,x]$, then by induction $A/[A,x]$ has a G-pretersoluble image. Thus $B \leqslant [A,x]$ and by induction again, $[A,x]$ has a G-pretersoluble image. We may therefore assume that $[A,x] = B$ and so $A/\operatorname{Ker} \sigma$ is of type C_{p^∞}. $\operatorname{Ker} \sigma = C_A(x) = C_A(\langle x \rangle)$ is a $\mathbb{Z}G$-submodule and

$$C_G(A/\operatorname{Ker} \sigma) \geqslant C_G(x) \cap C_G(A/\operatorname{Ker} \sigma) = C_G(x) \cap C_G(B).$$

It follows that $G/C_G(A/\operatorname{Ker} \sigma)$ is finite and so $A/\operatorname{Ker} \sigma$ is a G-pretersoluble image of A.

For our main theorem we require the following:

LEMMA 2.5. *Let G be a hypercyclic group and A a* $\mathbb{Z}G$*-module which is an* $\mathfrak{S}_0$*-group. If B is a submodule of A and if A has no G-hypercyclic image then B has no G-hypercyclic image.*

This lemma is an immediate consequence of the following characterization of modules without G-hypercyclic images.

LEMMA 2.6. *Let G be a hypercyclic group and A a* $\mathbb{Z}G$*-module which is an* $\mathfrak{S}_0$*-group. Then A has no G-hypercyclic image if and only if no G-factor of A is cyclic.*

PROOF. It is clear that if A has no cyclic G-factor then it has no G-hypercyclic image. So we assume that A has a cyclic G-factor U/V of order p and show that A has a G-hypercyclic p-image.

Choose a submodule X of A maximal subject to $U \cap X = V$. Then A/X is monolithic with monolith $(U+X)/X$ of order p. It follows from Lemma 2.2 that the torsion subgroup of A/X is G-hypercyclic and is a p-group.

Thus if the lemma is false, there is a counterexample A with A monolithic, the torsion subgroup T is a G-hypercyclic p-group with monolith U and the rank of A/T is minimal. A/T is non-zero and if A/T is not rationally irreducible then there is a submodule B/T such that A/B is torsion-free. By induction B has a G-hypercyclic p-image B/B_1. Considering A/B_1, the induction hypothesis again shows that A/B_1 has a G-hypercyclic p-image. Therefore we may assume

(2) *The torsion subgroup T of A is a G-hypercyclic p-group with monolith U and A/T is rationally irreducible.*

The next step of the proof is to show

(3) *A/T is faithful for G.*

Otherwise there is an element $x \in C_G(A/T)$ such that $\langle x \rangle$ is a normal subgroup of G which either is infinite or has prime order. Let $K = C_G(x)$ so that G/K is finite. The mapping $a \to a(x-1)$ is a $\mathbf{Z}K$-homomorphism from A into T and is non-zero since A is faithful for G. Thus A has a non-zero K-hypercyclic image A/C, where $C = C_A(\langle x \rangle)$, the kernel of the map $a \to a(x-1)$. C is a $\mathbf{Z}G$-submodule of A and if $C \not\geqslant T$, then A/C contains G-hypercyclic p-factors. By Lemma 2.2, A has a G-hypercyclic p-image.

Therefore we may assume that $C \geqslant T$; that is, $x \in C_G(A/T) \cap C_G(T)$. If x has prime order $q \neq p$, then $[a,x]^q = [a,x^q] = 1$, since $A\langle x \rangle \in \mathfrak{N}_2$, and so $[a,x] = 1$ and $x \in C_G(A)$, contrary to A being faithful for G. Thus x has order p or infinite order.

In either case G/K is a finite abelian group of exponent dividing $p(p-1)$. As above $C = C_A(\langle x \rangle)$ is a $\mathbf{Z}G$-submodule of A and A/C is K-isomorphic to a $\mathbf{Z}K$-submodule of T. Thus every irreducible K-factor in A/C is centralized by G^*K (where $G^* = G' G^{p(p-1)}$). Let D/C be a minimal $\mathbf{Z}G$-submodule of A/C and let $L = C_G(D/C)$. G^*K stabilizes a series of the finite p-group D/C and so G^*KL/L is a finite p-group. But $O_p(G/L)$ is trivial and so $L \geqslant G^*K$ and hence G/L is abelian with exponent dividing $p-1$. Hence D/C is cyclic of order p. Applying Lemma 2.2 to the torsion group A/C, we see that A/C has a G-hypercyclic p-image. This contradiction establishes (3).

(4) $G \in \mathfrak{A}\mathfrak{S}_p^* \mathfrak{A}_{p-1}^*$.

G acts faithfully and irreducibly on the $\mathbf{Q}$-space $\mathbf{Q} \otimes_{\mathbf{Z}} (A/T)$ and so has an abelian normal subgroup H of finite index (Robinson (1972a), Theorem 3.24). By Clifford's Theorem, H is a completely reducible $\mathbf{Q}$-linear group and so is a subgroup of the direct product of a free abelian group and a finite group (Robinson (1968a), Lemma 3.12). Therefore G has a torsion-free abelian normal subgroup L of finite index. L is contained in the hypercentre of $G^2 L$ and hence in that of $G^* L$.

Choose $M \leqslant G^* L$ maximal with respect to $M \lhd G$, $M \cap L = 1$. Then G^*L/M is torsion-free abelian. For, let $X/M \geqslant ML/M$ be a maximal torsion-free abelian

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subgroup of G^*L/M normal in G/M. If $X \neq G^*L/M$, then there is a subgroup $Y \leqslant G^*L$ such that $Y \lhd G$ and Y/X is cyclic of prime order q, say. X/M is contained in the hypercentre of G^*L/M and hence Y/M is a hypercentral group. If Y/M is torsion-free then it is abelian (being abelian-by-finite, torsion-free and locally nilpotent, see Robinson (1972b), Lemma 6.37) contrary to the maximality of X. If Y/M is not torsion-free then its periodic subgroup P/M is non-trivial. But then $P \cap L = 1$ contrary to the maximality of M.

$M \cap G^*$ stabilizes a series of T and is finite so $(M \cap G^*)/C_{(M \cap G^*)}(T)$ is a p-group. Also each p-chief factor of M is centralized by G^* and so $M \cap G^*$ is p-nilpotent.

If $M \cap G^*$ is not a p-group, then there is an element $y \in C_{(M \cap G^*)}(T)$ such that $\langle y \rangle$ is a normal subgroup of G of prime order $q \neq p$. $[A, y] = [A, \langle y \rangle]$ is a $\mathbb{Z}G$-submodule of A. Since G acts faithfully on A/T, $[A, y] \nleqslant T$ and so $A/([A, y]+T)$ is a torsion group, using (2). Thus $A/[A, y]$ is a torsion group. If $a(y-1) \in T$, then $a \in T$ and so $a(y-1) = 0$. Thus $[A, y] \cap T = 0$. Applying Lemma 2.2 to $A/[A, y]$, we see that A has a G-hypercyclic p-image. Thus we may assume that $M \cap G^*$ is a p-group.

G/G^* is an abelian group of finite exponent and so, in particular, G^*L/G^* is residually finite and there is a subgroup $G_1 \geqslant G^*$ such that G^*L/G_1 is finite and $G_1 \cap G^*M = G^*$ and hence $G_1 \cap M = G^* \cap M$. Let $L_1 = L \cap G_1$ and $M_1 = M \cap G_1$; then L_1 is a torsion-free abelian normal subgroup of G, M_1 is a finite normal p-subgroup, G_1/M_1 is torsion-free abelian and $G/G_1 \in \mathfrak{A}^*_{p(p-1)}$.

G/L is a finite supersoluble group and so each p-factor is centralized by G^*L. Thus G^*L/L and hence G_1/L_1 is p-nilpotent. Let Q/L_1 be the unique Sylow p'-subgroup of G_1/L_1. Since $M_1 = M \cap G^*$ is a p-group, $Q \cap M_1 = 1$ and so Q is a torsion-free abelian normal subgroup of G. G_1/Q is a finite p-group and $G/G_1 \in \mathfrak{A}^*_{p(p-1)}$. This completes the proof of (4).

Now let $S = C_G(U)$ so that $G/S \in \mathfrak{A}_{p-1}$. Then $R = S \cap Q$ is an abelian normal subgroup of G and $G/R \in \mathfrak{S}^*_p \mathfrak{A}^*_{p-1}$. By the hypercentral case (Hartley and Tomkinson (1975), Lemma 2.8), there is a non-trivial R-hypercentral p-image A/B of A. Let $\{s_1, \ldots, s_n\}$ be a transversal to R in G. Then A/Bs_i is also an R-hypercentral p-image and hence so is $D = A/\bigcap_{i=1}^n Bs_i$. Thus $D_1 = C_D(R)$ is a non-zero submodule of D. But D is a $\mathbb{Z}G$-module and hence D_1 is a $\mathbb{Z}G$-module. Thus D_1 is a non-zero module with $G/C_G(D_1) \in \mathfrak{S}^*_p \mathfrak{A}^*_{p-1}$. Thus D_1 is G-hypercyclic. Applying Lemma 2.2 to D, we see that A has a G-hypercyclic p-image.

3. Finite modules

Our aim in this section is to prove

THEOREM 3.1. *Let G be a hypercyclic group and A a finite $\mathbb{Z}G$-module. If A has no cyclic G-image, then every extension of A by G splits conjugately over A.*

Rather than prove this directly, we obtain a more general result with a formation-theoretic flavour. We work in the class $\acute{\mathfrak{S}}$ of "hyper-$\mathfrak{S}_0$"-groups; that is, groups with an ascending normal series in which the factors are abelian of finite rank. It is clear that $\acute{\mathfrak{S}}$ contains all $\mathfrak{S}_0$-groups and all hypercyclic groups.

In general, an $\acute{\mathfrak{S}}$-group may have infinite chief factors; for example, the extension of the additive group of rationals by its group of automorphisms. But in the following result, from which Theorem 3.1 is easily deduced, it is only necessary to consider the finite chief factors of G.

LEMMA 3.2. *Let G be an $\acute{\mathfrak{S}}$-group and A a finite irreducible $\mathbf{Z}_p G$-module. If M is a normal subgroup of G such that each p-chief factor of G is centralized by M and $[A, M] = A$, then every extension of A by G splits conjugately over A.*

PROOF. Let E be an extension of A by G. Factoring out a suitable normal subgroup of E we may assume

(5) *every non-trivial normal subgroup of E contains A.*

We consider the three different types of abelian normal subgroup which G may possess.

(A) *G has a finite normal p'-subgroup N.*

In this case, if Q is a Sylow p'-subgroup of AN, then $N_E(Q)$ is a complement to A in E. If X is any other complement, then $X = N_E(X \cap AN) = N_E(Q^a)$, for some $a \in A$.

(B) *G has a finite minimal normal p-subgroup N.*

Let $K = C_E(AN)$ and consider E_1, the split extension of AN by G/K. Let M_1 be the split extension of AN by MK/K and let Q be a Sylow p'-subgroup of M_1. M_1/A is p-nilpotent and so, by a Frattini argument, $N_{E_1}(Q)$ is a complement to A in E_1. Since AN is a p-group, $A \cap \zeta(AN) \neq 1$. But $A \cap \zeta(AN) \lhd E$ and so $A \leqslant \zeta(AN)$. Therefore $AN \cap N_{E_1}(Q) \lhd AN_{E_1}(Q) = E_1$ and so $AN \cap N_{E_1}(Q)$ is a normal subgroup of E, contrary to (5).

(C) *G has a torsion-free rationally irreducible abelian normal subgroup N of finite rank.*

Let $L = C_{AN}(A)$; then AN/L is finite and $L \in \mathfrak{N}_2$. If $x, y \in L$, then

$$[x^p, y] = [x, y]^p = 1$$

and so L^p is abelian. The torsion subgroup of L^p is finite with exponent p and so L^{p^2} is torsion-free. But then L^{p^2} is a non-trivial normal subgroup of E intersecting A trivially, contrary to (5). This completes the proof of the lemma.

In a hypercyclic group G each p-chief factor is cyclic of order p and so is centralized by $M = G' G^{p-1}$. Together with a simple induction argument this yields Theorem 3.1. The corresponding result for hypercentral groups can be obtained by taking $M = G$.

4. Splitting theorems for modules

The results of this section are special cases of Theorems A and B and will be used in the proof of those theorems after reducing to the metabelian case. We note first the corresponding special case of Hartley and Tomkinson (1975), Theorem B.

LEMMA 4.1. *Let G be a hypercentral group and let A be a $\mathbb{Z}G$-module. Suppose that A has a submodule B such that A/B is G-hypertrivial and either*

(i) *B is an $\mathfrak{S}_1^*$-group and $[B, G] = B$,*

or

(ii) *B is an $\mathfrak{S}_1$-group and has no G-hypertrivial image.*

Then there is a unique submodule C such that $A = B \oplus C$.

PROOF. Let $E = AG$; then E/B is hypercentral and E splits conjugately over B. If M is a complement to B in E then $A = B \oplus (M \cap A)$. Since $M \cap A$ is G-hypertrivial and B has no G-trivial factors it is clear that $M \cap A$ is the unique complement.

The following simple lemma will be used when we obtain a module which is K-trivial, where K is the centralizer of an infinite cyclic normal subgroup of G.

LEMMA 4.2. *Let G be a cyclic group of order 2 and let A be a $\mathbb{Z}G$-module which is the direct sum of two groups of type C_{p^∞}. Then A has a proper infinite submodule B and hence has a G-pretersoluble image.*

PROOF. Let $G = \langle g \rangle$ and let $X = \langle x_n ; px_1 = 0, px_{n+1} = x_n \rangle$ be a subgroup of A of type C_{p^∞}. Then $B = \langle x_n(g-1); n = 1, 2, \ldots \rangle$ is a proper infinite submodule of A.

THEOREM 4.3. *Let G be a hypercentral group and let A be a $\mathbb{Z}G$-module. If A has a submodule B which is an $\mathfrak{S}_1^*$-group such that A/B is G-hypercyclic and B has no G-pretersoluble image, then there is a unique submodule C such that $A = B \oplus C$.*

PROOF. Since the torsion subgroup of B is finite, there is a finite series of submodules

$$0 \leqslant T = B_0 < B_1 < \ldots < B_n = B$$

such that T is finite and each B_i/B_{i-1} is torsion-free of finite rank and rationally irreducible. By Lemma 2.3, none of the submodules B_i $(i = 0, \ldots, n)$ has a G-pretersoluble image. It is sufficient, using an induction argument and Theorem 3.1, to prove the theorem for the case in which B is torsion-free of finite rank and rationally irreducible.

Let U be a submodule of A maximal with respect to $U \cap B = 0$. By considering A/U, we may assume that

(6) *every non-trivial submodule of A intersects B non-trivially.*

We show that with this restriction B must be equal to A. If $B < A$, then since A/B is G-hypercyclic there is a submodule S such that S/B is a cyclic group of either infinite or prime order. If $|S/B| = p$, then consider the two cases $B^p = B$ and $B^p < B$. If $B^p < B$, then the finite module S/B^p is a direct sum $\langle x \rangle B^p/B^p \oplus B/B^p$, by Lemma 2.1. In both cases, there is an element $x \in S$ such that $S = B\langle x \rangle$ and $x^p \in B^p$. Thus there is an element $b \in B$ such that $x^p = b^p$ and so $(xb^{-1})^p = 1$. $S = B \oplus \langle xb^{-1} \rangle$ and so $\langle xb^{-1} \rangle$ is the (non-trivial) torsion subgroup of S and intersects B trivially contrary to (6).

Thus S/B is an infinite cyclic group. Let $K = C_G(S/B)$ so that $G/K = 1$ or 2. If B has a K-trivial image B/D, then we may assume that B/D is either cyclic of order p or is of type C_{p^∞}. If $G = K \cup Kg$, then $E = D \cap Dg$ is a $\mathbf{Z}G$-submodule and $B/E[B,K]$ is a K-trivial G-image of B. If B/D is finite, then $B/E[B,K]$ is finite and since each irreducible factor is K-trivial it must be G-pretersoluble, since $2 | p(p-1)$, for all primes p. If B/D is of type C_{p^∞}, then we may assume that B/E is a divisible p-group of rank at most 2 and is K-trivial. Applying Lemma 4.2 to the $\mathbf{Z}(G/K)$-module B/E we see that B has a G-pretersoluble image.

Therefore we may assume that B has no K-trivial image and so, by Lemma 4.1, B has a unique K-invariant complement $\langle x \rangle$ in S. It is clear that $\langle x \rangle$ is a $\mathbf{Z}G$-submodule contrary to (6). Thus $B = A$ and the module U which was factored out is a complement to B in A. Since U is G-hypercyclic and B has no cyclic G-factors (Lemma 2.3), the uniqueness of the complement is clear.

Theorem 4.4. *Let G be a hypercentral group and let A be a $\mathbf{Z}G$-module. If A has a submodule B which is an $\mathfrak{S}_1$-group such that A/B is G-hypercyclic and B has no G-hypercyclic image, then there is a unique submodule C such that $A = B \oplus C$.*

Proof. By induction, using Lemma 2.5 and Theorem 4.3, we may assume that

(7) *B is a divisible abelian p-group and every submodule of B is finite.*

Again, by factoring out a suitable submodule, we may also assume

(8) *every non-trivial submodule of A intersects B non-trivially.*

If $B < A$, then there is a submodule S such that S/B is a cyclic group of infinite or prime order. If S/B is finite, then S is a torsion group and so, by Lemma 2.2, $S = B \oplus B^*$, where B^* is a cyclic group, contrary to (8).

Thus S/B is an infinite cyclic group. Let $K = C_G(S/B)$ so that $|G/K| = 1$ or 2. If B has a K-hypertrivial p-image B/D, $G = K \cup Kg$ and $E = D \cap Dg$, then B/E is a K-hypertrivial G-image of B. Thus $C_{B/E}(K)$ is a non-zero K-trivial $\mathbf{Z}G$-module and so is G-pretersoluble (since $2 | p(p-1)$, for all primes p). This is contrary to Lemma 2.5 and so B has no K-hypertrivial image. Therefore, by Lemma 4.1, B has a unique K-invariant complement $\langle x \rangle$ in S. It is clear that $\langle x \rangle$ is a $\mathbf{Z}G$-submodule contrary to (8). Thus $B = A$ and the module factored out to obtain (8) is a complement to B in A. Since C is G-hypercyclic and B has no cyclic G-factors (Lemma 2.6), the uniqueness of C is clear.

The corresponding result required for Theorem B is slightly more complicated to state. We begin with the corresponding result for the nilpotent case.

LEMMA 4.5. *Let G be a nilpotent group and A a $\mathbf{Z}G$-module. If A has a submodule B which is an $\mathfrak{S}_1$-group such that A/B is G-polytrivial and $[B, G] = B$, then there is a finite submodule $F \leqslant B$ such that B/F has a complement in A/F. There is a finite submodule $F^* \geqslant F$ of B such that the complements to B/F in A/F are unique modulo F^*, that is, $C_1 F^* = C_2 F^*$.*

PROOF. Let $E = AG$; then E/B is nilpotent and $[B, E] = B$. By Hartley and Tomkinson (1975), Theorem C, there are finite submodules F and F^*, $F \leqslant F^* \leqslant B$, such that E/F splits over B/F and the complements are conjugate modulo F^*. If M/F is a complement to B/F in E/F, then $A/F = B/F \oplus (A \cap M)/F$. Let C/F be any complement to B/F in A/F. Then CG/F is a complement to B/F in E/F and so there is an element $x \in E$ such that $(CG)^x \leqslant MF^*$. Thus

$$C \leqslant C \cap MF^* = (A \cap M) F^*,$$

as required.

THEOREM 4.6. *Let G be a nilpotent group and A a $\mathbf{Z}G$-module. If A has a submodule B which is an $\mathfrak{S}_1$-group such that A/B is G-pretersoluble and B has no G-pretersoluble image, then there is a finite submodule $F \leqslant B$ such that B/F has a complement in A/F. There is a finite submodule $F^* \geqslant F$ of B such that the complements to B/F in A/F are unique modulo F^*.*

PROOF. By induction, using Lemma 2.3 and Theorem 4.3, we may assume that

(9) *B is a divisible p-group and every proper submodule of B is finite.*

Let S/B be a submodule of A/B such that each subgroup of S/B is a submodule and if X_p is the p-subgroup of S/B then $G/C_G(X_p)$ is cyclic of order dividing $p-1$ if p is odd and G either induces the involution automorphism or acts trivially on X_2. If the p'-subgroup of S/B is Q/B then Q splits over B and so we need only consider the cases in which S/B is a p-group and S/B is torsion-free.

(A) *S/B is torsion-free.*

Let $K = C_G(S/B)$; then $|G/K| = 1$ or 2. If B has a K-trivial image, then as in the proof of Theorem 4.3, B has a G-pretersoluble image. Thus by Lemma 4.5, there are finite $\mathbf{Z}K$-submodules $F_1 \leqslant F_1^* \leqslant B$ such that B/F_1 has a complement in S/F_1 and the complements are unique modulo F_1^*. If D/F_1 is a complement to B/F_1 in S/F_1 and $G = K \cup Kg$, then Dg/F_1 is also a complement and $D+Dg$ is a $\mathbf{Z}G$-submodule contained in DF_1^*. If $F = B \cap (D+Dg)$ then $D+Dg = DF$ and DF/F is a $\mathbf{Z}G$-submodule complementing B/F. Applying Lemma 4.5 to the module S/F, there is a finite submodule F^* of B such that the complements to B/F in S/F are unique modulo F^*.

(B) *S/B is a p-group.*

If $p = 2$, then $|G/C_G(S/B)| = 1$ or 2 and we can repeat the argument of (A). If p is odd and $K = C_G(S/B)$ then G/K is a finite p'-group. If B has a K-trivial image B/D and if $\{g_1, \ldots, g_n\}$ is a transversal to K in G, then $B/\bigcap_{i=1}^n Dg_i$ is a K-trivial G-image of B and is G-hypercyclic. It follows from Wehrfritz (1973), Lemma 11.7 that B has a G-pretersoluble image. Thus by Lemma 4.5, there are finite $\mathbf{Z}K$-submodules $F_1 \leqslant F_1^* \leqslant B$ such that B/F_1 has a complement in S/F_1 and the complements are unique modulo F_1^*. If D/F_1 is a complement to B/F_1 in S/F_1 then Dg_i/F_1 $(i = 1, \ldots, n)$ is also a complement and $\sum_{i=1}^n Dg_i$ is a $\mathbf{Z}G$-submodule contained in DF_1^*. If $F = B \cap (\sum Dg_i)$, then $\sum Dg_i = DF$ and DF/F is a $\mathbf{Z}G$-submodule complementing B/F. As in (A) we obtain the required submodule F^*.

5. Variations on results of P. Hall and D. Robinson

The results referred to in the heading to this section concern a normal nilpotent subgroup N of a group G. Hall (1958) proved that if G/N' is nilpotent then so is G. This theorem was extended by Robinson (1968b) who showed that the nilpotency of G/N' could be replaced by property $\mathscr{P}$, where $\mathscr{P}$ can represent supersoluble, hypercentral or hypercyclic (among other examples).

The proof of this result depends largely on the fact that N_{i+1}/N_{i+2} is a $\mathbf{Z}G$-homomorphic image of $(N_i/N_{i+1}) \otimes_{\mathbf{Z}} (N/N')$, where N_i denotes the ith term of the lower central series of N and N_i/N_{i+1} is considered as a $\mathbf{Z}G$-module in the usual way. The homomorphism is given by

$$\bar{a} \otimes \bar{n} \to [a, n]\, N_{i+2}$$

and it is clear that instead of taking $\bar{a} \in N_i/N_{i+1}$ and $\bar{n} \in N/N'$ we could take $\bar{a} \in N_i/N_{i+1}(N_i \cap \zeta(N))$ and $\bar{n} \in N/N'\,\zeta(N)$. Thus N_{i+1}/N_{i+2} is a $\mathbf{Z}G$-homomorphic image of $(N_i/N_{i+1}(N_i \cap \zeta(N))) \otimes_{\mathbf{Z}} (N/N'\,\zeta(N))$ and so a similar result can be obtained by assuming conditions on $N/N'\,\zeta(N)$.

To state the required result more explicitly we let $\mathfrak{C}$ denote a class of $\mathbf{Z}G$-modules satisfying

(C1) *if* $B \leqslant A \in \mathfrak{C}$, *then* $B \in \mathfrak{C}$ *and* $A/B \in \mathfrak{C}$;

(C2) *if* A *and* B *are in* $\mathfrak{C}$, *then* $A \otimes_{\mathbf{Z}} B \in \mathfrak{C}$.

The properties which will be considered are then defined as follows. Let N be a group admitting G as a group of operators. N is said to be a $\mathrm{P}_G\mathfrak{C}$-group ($\acute{\mathrm{P}}_G\mathfrak{C}$-group) if it has a finite normal series (ascending normal series) of G-admissible subgroups such that each factor is abelian and, when regarded as a $\mathbf{Z}G$-module, is in the class $\mathfrak{C}$. The proof given in Robinson (1968b) then gives

THEOREM 5.1. *If* N *is nilpotent and* $N/N' \ \zeta(N) \in \mathrm{P}_G\mathfrak{C}$ ($\acute{\mathrm{P}}_G\mathfrak{C}$), *then* $N' \in \mathrm{P}_G\mathfrak{C}$ ($\acute{\mathrm{P}}_G\mathfrak{C}$).

We shall mainly be concerned with supersolubility conditions and will use this result in a similar way to Newell (1975), Lemma 4. If $\mathfrak{C}$ is the class of $\mathbf{Z}G$-modules whose underlying additive group is cyclic, then $\mathrm{P}_G\mathfrak{C}$ is the class of G-supersoluble groups. To obtain the class of G-pretersoluble groups we consider $\mathrm{P}_G\mathfrak{P}$ where $\mathfrak{P}$ is the class of $\mathbf{Z}G$-modules X in which the underlying additive group is either

(P1) *torsion-free and each additive subgroup is a submodule,*

or

(P2) *a torsion group in which each additive subgroup is a submodule and if* X_p *is the* p*-subgroup then* $G/C_G(X_p)$ *is cyclic of order dividing* $p-1$, *for* p *odd, and* G *either acts trivially or induces the involution automorphism on* X_2.

We need to prove that $\mathfrak{P}$ satisfies the conditions (C1) and (C2). If X satisfies condition (P1) then G either acts trivially or induces the involution automorphism on X and so it follows easily that $\mathfrak{P}$ satisfies condition (C1).

LEMMA 5.2. *If* X *and* Y *are* $\mathbf{Z}G$*-modules in the class* $\mathfrak{P}$, *then* $X \otimes_{\mathbf{Z}} Y$ *is also in* $\mathfrak{P}$.

PROOF. (i) If X and Y satisfy (P1), then $X \otimes_{\mathbf{Z}} Y$ is a torsion-free group (Fuchs (1973), Theorem 61.5). Let $g \in G$; then $(x \otimes y)g = xg \otimes yg$. For all $x \in X$, $xg = \varepsilon_1 x$ and for all $y \in Y$, $yg = \varepsilon_2 y$, where $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$. Thus for all $x \otimes y \in X \otimes Y$, $(x \otimes y)g = \varepsilon_1 x \otimes \varepsilon_2 y = \varepsilon_1 \varepsilon_2 (x \otimes y)$. Hence $X \otimes Y$ satisfies condition (P1).

(ii) If X satisfies (P1) and Y satisfies (P2), then $X \otimes Y$ is a torsion group and $(X \otimes Y)_p = X \otimes Y_p$ (Fuchs (1973), Theorem 61.5). If $p = 2$, then the argument of (i) shows that G acts trivially or induces the involution automorphism on $(X \otimes Y)_2$. If p is odd, then $G/C_G(X)$ is cyclic of order 1 or 2 and $G/C_G(Y_p)$ is cyclic of order n, where $n \mid (p-1)$. If $2 \nmid n$, then $G/C_G(X) \cap C_G(Y_p)$ is cyclic of order $2n$ and $2n \mid (p-1)$. So we may assume that $2 \mid n$ and G induces the involution automorphism on X. $G/C_G(Y_p)$ contains a unique element $\bar{g}$ of order 2. If $C_G(X) \geqslant C_G(Y_p)$ then

$C_G(X \otimes Y_p) = C_G(Y_p)$ and we have the required property. Otherwise we may choose $g \in C_G(X)$ such that $gC_G(Y_p) = \bar{g}$ and $h \in C_G(Y_p) - C_G(X)$. Then for all $x \otimes y \in X \otimes Y_p$, we have

$$(x \otimes y)gh = xh \otimes yg = (-x) \otimes (-y) = x \otimes y$$

and so $\langle gh \rangle (C_G(X) \cap C_G(Y_p)) \leqslant C_G(X \otimes Y_p)$. But $G/\langle gh \rangle (C_G(X) \cap C_G(Y_p))$ is cyclic of order dividing $p-1$.

(iii) If X and Y both satisfy (P2) then $X \otimes Y$ is a torsion group and $(X \otimes Y)_p = X_p \otimes Y_p$. If $p = 2$, then the argument of (i) shows that G has the required action on $(X \otimes Y)_2$. If p is odd, then $G/C_G((X \otimes Y)_p)$ is abelian with exponent dividing $p-1$. Also each element of $X_p \otimes Y_p$ is mapped to a power of itself by each element of G and so $G/C_G((X \otimes Y)_p)$ is cyclic, as required.

6. Proof of Theorem A

We begin with a simple lemma which will allow us to reduce to the case in which G is finitely generated and hence supersoluble.

LEMMA 6.1. *Let A be a normal subgroup of E. Suppose E/A has a subgroup H_0/A and a local system of subgroups H_i/A, $i \in I$, each containing H_0/A such that $C_A(H_0) = 1$ and each H_i, $i \in I \cup \{0\}$, splits conjugately over A. Then E splits conjugately over A.*

PROOF. Let L_0 be a complement to A in H_0. If M_i is any complement to A in H_i, then $A(H_0 \cap M_i) = H_0$ and so $H_0 \cap M_i$ is a complement to A in H_0. Thus there is an element $a \in A$ such that $L_0 = (H_0 \cap M_i)^a \leqslant M_i^a$ and so L_0 is contained in a complement L_i $(= M_i^a)$ to A in H_i. If X and Y are two complements to A in H_i containing L_0, then $Y = X^a$ for some $a \in A$ and $L_0 = X \cap H_0 = X^a \cap H_0 = (X \cap H_0)^a$. Thus $a \in N_A(L_0) = C_A(L_0) = C_A(H_0) = 1$ and so $Y = X$. Thus there is a unique complement L_i to A in H_i containing L_0.

Let $L = \bigcup_{i \in I} L_i$. Then L is a complement to A in E. If M is any other complement, then $M \cap H_0 = L_0^a$, for some $a \in A$, and so $M \cap H_i = L_i^a$ for all $i \in I$. Hence $M = L^a$, as required.

In proving Theorem A, we let E be any extension of A by G so that E/A is a hypercyclic group and A is an abelian normal subgroup of E. Using Lemmas 2.3 and 2.5 we can use induction on a series for A and consider the three cases:

(I) *A is finite with no cyclic factors.*

(II) *A is a divisible p-group with each proper submodule finite and A has no cyclic factors.*

(III) *A is torsion-free rationally irreducible of finite rank and A has no G-pretersoluble image.*

Case (I) This is included in Theorem 3.1.

Case (II) There is a finitely generated subgroup H_0/A of E/A such that $H_0 C_E(\Omega_1(A)) = E$. By taking a local system of finitely generated subgroups H_i/A of E/A containing H_0/A and using Lemma 6.1 it is sufficient to prove:

(10) *if H/A is a finitely generated subgroup of E/A such that $HC_E(\Omega_1(A)) = E$, then H splits conjugately over A.*

Since $\Omega_1(A)$ has no cyclic H-factors, it follows that A has no cyclic H-factors. By induction and using Theorem 3.1, we may assume that A satisfies condition (II) as a $\mathbf{Z}H$-module.

If $[A, H'] = A$, then since H'/A is nilpotent, we may apply Hartley and Tomkinson (1975), Theorem C to see that H' splits over A mod f (see Hartley and Tomkinson (1975), p. 226 for notation). Also $C_A(H')$ is finite and so, by Hartley and Tomkinson (1975), Lemma 3.4, H splits over A mod f. Let $M/\Omega_m(A)$ be a complement to $A/\Omega_m(A)$ and suppose all such complements are conjugate modulo $\Omega_n(A)$. $\Omega_m(A)$ is finite and has no cyclic M-factors and so, using Theorem 3.1, we can deduce that H splits over A. Let X and Y be two complements. Then $X\Omega_m(A)$ and $Y\Omega_m(A)$ are conjugate modulo $\Omega_n(A)$ and so we may assume that $X\Omega_n(A) = Y\Omega_n(A)$. X and Y are complements to $\Omega_n(A)$ in $X\Omega_n(A)$ and so, by Theorem 3.1, are conjugate.

If $[A, H'] < A$, then $[A, H']$ is finite and, using Theorem 3.1 as above, we may assume that $[A, H'] = 1$. H'/A is nilpotent and A is contained in the centre of H' which is therefore nilpotent. Since H/A is supersoluble, we have $H/\zeta(H')$ is supersoluble and so, by Theorem 5.1, H'' is H-supersoluble. Since A has no cyclic H-factors it follows that $A \cap H'' = 1$. Now consider H'/H'' as a $\mathbf{Z}(H/H')$-module. AH''/H'' has no H-hypercyclic image and H'/AH'' is H-hypercyclic. By Theorem 4.4, there is a unique normal subgroup C of H containing H'' such that $CA = H'$ and $C \cap AH'' = H''$. H'/C is a $\mathbf{Z}(H/H')$-module with no H-hypercyclic image and H/H' is abelian. By Hartley and Tomkinson (1975), Theorem B, H/C splits conjugately over H'/C and hence H/H'' splits conjugately over AH''/H''. If X/H'' is a complement to AH''/H'' in H/H'', then X is a complement to A in H. If Y is any other complement, then $H' \leqslant C_H(A) = AC_Y(A)$. Thus $H'' \leqslant Y$ and so Y/H'' is a complement to AH''/H'' and is therefore conjugate to X/H''.

Case (III) (a) $[A, E'] = 1$.

Suppose $A \cap E'' \neq 1$. Let $x \in E - C_E(A)$ and let H/A be a finitely generated subgroup of E/A such that $x \in H$ and $A \cap H'' \neq 1$. H/A is supersoluble and so $H'A/A$ is nilpotent. A is contained in the centre of $H'A$ which is therefore a nilpotent normal subgroup of H. Since H/A is supersoluble, we have $H/\zeta(H'A)$ is supersoluble and so, by Theorem 5.1, $(H'A)'$ is H-supersoluble. Thus $A \cap H''$ is

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H-supersoluble and torsion-free. Hence $A \cap H''$ contains an infinite cyclic subgroup C which is normal in H. Since $x \notin C_E(A)$, we have $C_A(x) = C_A(E'\langle x \rangle) = 0$ and so x induces the involution automorphism on C. The mapping σ: $a \to a^2[a, x]$ is a $\mathbf{Z}E$-endomorphism of A (since $E/C_E(A)$ is abelian) and $C \leqslant \operatorname{Ker} \sigma$. $A/\operatorname{Ker} \sigma$ is a torsion group and $\operatorname{Im} \sigma$ is torsion-free, so $\operatorname{Ker} \sigma = A$ and hence x induces the involution automorphism on the whole of A. Since this is true for any element of E not in $C_E(A)$ we must have $E/C_E(A) = \langle x C_E(A) \rangle$ has order 2. Thus A is an E-pretersoluble module contrary to hypothesis.

Thus $A \cap E'' = 1$. The argument of Case II can now be repeated to show that E splits conjugately over A.

(b) $[A, E'] \neq 1$.

Let $K = C_E(A)$. $E'K/K$ is hypercentral and we can choose $xK \in E/K$ such that $\langle xK \rangle \lhd E/K$ and $xK \in \zeta(E'K/K)$. Since A is rationally irreducible,

$$C_A(x) = C_A(\langle x \rangle K) = 0$$

and so the $C_E(\langle x \rangle)$-homomorphism $a \to a(x-1)$ is a monomorphism. By Fuchs (1973), *p*. 153, $|A : [A, x]|$ is finite and so E/A contains a finitely generated subgroup H_0/A such that $x \in H_0'$, $[A, E'] = [A, H_0'] \geqslant [A, x]$, and $H_0 C_E(A/[A, x]) = E$. By taking a local system of finitely generated subgroups H_i/A of E/A containing H_0/A and using Lemma 6.1 it is sufficient to prove:

(11) *if H/A is a finitely generated subgroup of E/A such that $x \in H'$, $[A, E'] = [A, H']$ and $HC_E(A/[A, x]) = E$, then H splits conjugately over A.*

Since H covers $E/C_E(A/[A, x])$, it follows that the finite module $A/[A, x]$ has no cyclic H-image. We can therefore apply Case I to deduce that $H/[A, x]$ splits conjugately over $A/[A, x]$. Let $M/[A, x]$ be a complement; we may clearly assume that $x \in M$ so that $[M, x] \leqslant M \cap \langle x \rangle K$.

$(K \cap H)/A$ is supersoluble and so there is a finite series

$$A = K_0 < \ldots < K_r = K \cap H$$

with each factor K_i/K_{i-1} cyclic.

If $r = 0$, then $[M, x] \leqslant M \cap \langle x \rangle K = M \cap \langle x \rangle A = \langle x \rangle (M \cap A) \leqslant \langle x \rangle [A, x]$. If $m \in M$, then $[m, x] = x^n[a, x]$ for some $a \in A$, $n \in \mathbf{Z}$. Therefore $[ma^{-1}, x] = x^n$ and so $ma^{-1} \in N_H(\langle x \rangle) = L$, say. Thus $M \leqslant AL$ and so $AL = H$. Also

$$A \cap L = N_A(\langle x \rangle) = C_A(x) = 1$$

and so L is a complement to A in H.

If L_1 is any other complement, then we may assume that $L_1[A, x] = L[A, x]$. Let y be the element of L_1 congruent to x mod $[A, x]$. Then there is an element $b \in A$,

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such that

$$y = x[b,x] = xb^{-1}b^{x} = xb^{x}b^{-1} = bxb^{-1}$$

and so $L_1^b \geqslant N_H(\langle y^b \rangle) = N_H(\langle x \rangle) = L$ and so $L_1^b = L$.

Now suppose that $r > 0$ and that the result holds for smaller values of r. K_1/A is centralized by H' and hence by x. Thus $[M \cap K_1, x] \leqslant M \cap A = [A, x]$. If $t \in M \cap K_1$, then $[t,x] = [a,x]$ for some $a \in A$ and so $ta^{-1} \in C_{K_1}(x) = S$, say. Therefore $M \cap K_1 \leqslant AS$ and so $K_1 = A \times S$.

There is a finite central series

$$A = A_0 \lhd \ldots \lhd A_n = H'.$$

We show by induction on n that $[H', S] = 1$. Suppose $[A_{n-1}, S] = 1$; then $S = C_{K_1}(A_{n-1})$ is normalized by A_n and so $[A_n, S] \leqslant A \cap S = 1$. Thus $[H', S] = 1$ and $S = \zeta(H') \cap K_1 \lhd H$.

By the induction hypothesis, H/S splits conjugately over AS/S and hence H splits over A, since $A \cap S = 1$. If L_1 and L_2 are two complements, then for $i = 1, 2$, we have $K_1 = A \times (L_i \cap K_1)$ and $L_i \cap K_1 \lhd AL_i = H$. Therefore

$$[H', L_i \cap K_1] \leqslant A \cap L_i \cap K_1 = 1.$$

Thus $L_i \cap K_1 = S$. By induction, L_1/S and L_2/S are conjugate and hence L_1 and L_2 are conjugate.

This completes the proof of (11) and hence that of Theorem A.

7. Proof of Theorem B

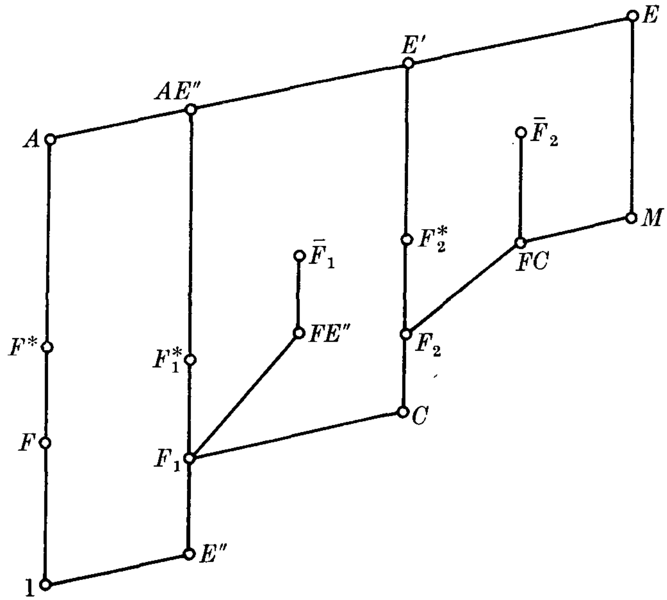
Let E be a group containing a normal abelian $\mathfrak{S}_1$-subgroup A such that E/A is pretersoluble and A has no E-pretersoluble image. We require finite subgroups $F \leqslant F^* \leqslant A$, normal in E and such that E/F splits over A/F, and the complements to A/F in E/F are conjugate modulo F^*.

Let T be the torsion subgroup of A. By Lemma 2.2, $T = C \times C^*$, where C is E-hypercyclic and C^* contains no cyclic E-factors. Applying Theorem A to the modules A/T and T/C, we see that E/C splits conjugately over A/C. Therefore we may assume that A is a divisible torsion group and is E-hypercyclic. A simple induction argument using Lemma 2.4 allows us to assume that A is a divisible p-group and contains no proper infinite normal subgroup of E.

If $[A, E'] = A$, then by Hartley and Tomkinson (1975), Theorem C, E' splits over A mod f. Also $C_A(E')$ is finite and so, by Hartley and Tomkinson (1975), Lemma 3.4, E splits over A mod f, as required.

We may now assume that $[A, E'] < A$. Since $[A, E']$ is finite, we may factor it out and assume that $[A, E'] = 1$. It follows that E' is nilpotent. If $A \leqslant E''$, then E/E'' is pretersoluble and hence E is pretersoluble by the results of Section 5. Thus we may assume that $A \cap E''$ is finite and by factoring out this subgroup we have

$A \cap E'' = 1$. By considering E'/E'' as a $\mathbb{Z}(E/E')$-module we may apply Theorem 4.6 to see that E has normal subgroups F_1 and F_1^* with $E'' \leqslant F_1 \leqslant F_1^* \leqslant AE''$ and F_1^*/E'' finite such that AE''/F_1 has a complement in E'/F_1 and the complements are unique modulo F_1^*.



Let C/F_1 be a complement to AE''/F_1 in E'/F_1. Then E'/C is a $\mathbb{Z}(E/E')$-module with no E-pretersoluble image. Applying Hartley and Tomkinson (1975), Theorem C, E has normal subgroups F_2 and F_2^* with $C \leqslant F_2 \leqslant F_2^* \leqslant E'$ and F_2^*/C finite such that E/F_2 splits over E'/F_2 and the complements are conjugate modulo F_2^*.

There is a finite normal subgroup F of E contained in A such that $FE'' \geqslant F_1$ and $FC \geqslant F_2$. Let M/FC be a complement to E'/FC in E/FC. Then

$$AM = ACM = E'M = E$$

and

$$A \cap M = A \cap FC = F(A \cap C) = F(A \cap AE'' \cap C) = F(A \cap F_1) = F(A \cap FE'') = F.$$

So M/F is a complement to A/F in E/F. Let L/F be any other complement; then $E' = A(L \cap E') = AC_L(A)$ and so $E'' \leqslant L$. Therefore L/FE'' is a complement to AE''/FE'' in E/FE''. $(L \cap E')/FE''$ and $(M \cap E')/FE''$ are both complements to AE''/FE'' in E'/FE''. Applying Theorem 4.6 to the module E'/FE'', there is a finite submodule $\bar{F}_1/FE''$ such that $\bar{F}_1(L \cap E') = \bar{F}_1(M \cap E')$. $L\bar{F}_1/\bar{F}_1 C$ and $M\bar{F}_1/\bar{F}_1 C$ are complements to $E'/\bar{F}_1 C$ in $E/\bar{F}_1 C$. Applying Hartley and Tomkinson (1975), Theorem C to the group $E/\bar{F}_1 C$, there is a finite normal subgroup $\bar{F}_2/\bar{F}_1 C$ and an element $x \in E$ such that $(L\bar{F}_2)^x = M\bar{F}_2$. There is a finite normal subgroup F^* of

E contained in A such that $F^*E'' \geqslant F_1$ and $F^*C \geqslant F_2$. It follows that

$$(LF^*)^x = L^x F^* = MF^*,$$

as required.

We conclude by showing that pretersoluble cannot be replaced by parasoluble in Theorem B. Let A be the direct product of two C_{2^∞}-groups,

$$A = \langle x_1, x_2, \ldots \rangle \times \langle y_1, y_2, \ldots \rangle$$

where $x_1^2 = y_1^2 = 1$, $x_{n+1}^2 = x_n$, $y_{n+1}^2 = y_n$, and let $B = \mathrm{Dr}_{n=1}^{\infty}\langle b_n \rangle$, where $\langle b_n \rangle$ is cyclic of order 2^n. Let $H = A \times B$; then H has automorphisms t_k, $k = 1, 2, \ldots$, defined as follows (writing H additively)

$$x_n t_k = (2^k - 1)x_n + 2^k y_n,$$

$$y_n t_k = 2^k x_n + y_n,$$

$$b_n t_k = b_n, \quad \text{if } n \neq k,$$

$$b_k t_k = x_k - b_k.$$

Then $T = \langle t_1, t_2, \ldots \rangle$ is free abelian of countable rank and A has no infinite T-invariant subgroups. If E is the split extension of H by T, then T/A is parasoluble but not pretersoluble and A has no E-parasoluble image.

If A has a proper supplement in E then there is a proper $\mathbf{Z}E$-submodule C of H such that $A + C = H$ and $A \cap C$ is finite. For each k, C contains elements of the form

$$c_k = \alpha x_m + \beta y_m + b_k.$$

Now $c_k t_k = \{(2^k - 1)\alpha + 2^k \beta\} x_m + (2^k \alpha + \beta) y_m + x_k - b_k$. Since C is $\mathbf{Z}E$-parasoluble $c_k t_k$ must be equal to some power of c_k, say $c_k t_k = sc_k$. By considering the b_k component we see that $-b_k = sb_k$ and so $s \equiv -1 \pmod{2^k}$, that is, $s = n2^k - 1$. Considering the x components we have

$$x_k + \{(2^k - 1)\alpha + 2^k \beta\} x_m = s\alpha x_m.$$

Therefore $x_k = 2^k\{(n-1)\alpha - \beta\} x_m$ and so $m \geqslant 2k$. Thus $2^k c_k$ is an element of A outside $\Omega_{k-1}(A)$ and this is contrary to $A \cap C$ being finite.

In this example, the parasoluble group E/A satisfies no reasonable rank conditions and it seems possible that parasoluble could be used in place of pretersoluble in Theorem B if we insisted that the whole of E were an $\mathfrak{S}_1$-group.

References

R. Baer (1972), "Durch Formationen bestimmte Zerlegungen von Normalteilern endlicher Gruppen", *J. Algebra* **20**, 38–56.

R. Carter and T. Hawkes (1967), "The $\mathfrak{F}$-normalizers of a finite soluble group", *J. Algebra* **5**, 175–202.

L. Fuchs (1973), *Infinite Abelian Groups*, II, Pure and Applied Mathematics, **36–II** (Academic Press, New York and London).

A. D. Gardiner, B. Hartley and M. J. Tomkinson (1971), "Saturated formations and Sylow structure in locally finite groups", *J. Algebra* **17**, 177–211.

P. Hall (1958), "Some sufficient conditions for a group to be nilpotent", *Illinois J. Math.* **2**, 787–801.

B. Hartley and M. J. Tomkinson (1975), "Splitting over nilpotent and hypercentral residuals", *Math. Proc. Cambridge Philos. Soc.* **78**, 215–226.

M. L. Newell (1975), "Some splitting theorems for infinite supersoluble groups", *Math. Z.* **144**, 265–275.

D. J. S. Robinson (1968a), "Residual properties of some classes of infinite soluble groups", *Proc. London Math. Soc.* (3) **18**, 495–520.

D. J. S. Robinson (1968b), "A property of the lower central series of a group", *Math. Z.* **107**, 225–231.

D. J. S. Robinson (1972a), *Finiteness Conditions and Generalized Soluble Groups, Part* 1, Ergebnisse der Mathematik und ihrer Grenzgebiete, **62** (Springer-Verlag, Berlin, Heidelberg and New York).

D. J. S. Robinson (1972b), *Finiteness Conditions and Generalized Soluble Groups, Part* 2, Ergebnisse der Mathematik und ihrer Grenzgebiete, **63** (Springer-Verlag, Berlin, Heidelberg and New York).

D. J. S. Robinson (1976), "The vanishing of certain homology and cohomology groups", *J. Pure Appl. Algebra* **7**, 145–167.

B. A. F. Wehrfritz (1973), *Infinite Linear Groups. An Account of the Group-theoretic Properties of Infinite Groups of Matrices*, Ergebnisse der Mathematik und ihrer Grenzgebiete, **76** (Springer-Verlag, Berlin, Heidelberg and New York).

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