Canad. Math. Bull. Vol. **54** (2), 2011 pp. 381–384
doi:10.4153/CMB-2010-095-0

A Short Note on the Higher Level Version of the Krull–Baer Theorem

Dejan Velušček

Abstract. Klep and Velušček generalized the Krull–Baer theorem for higher level preorderings to the non-commutative setting. A n-real valuation v on a skew field D induces a group homomorphism $\overline{v}$. A section of $\overline{v}$ is a crucial ingredient of the construction of a complete preordering on the base field D such that its projection on the residue skew field k_v equals the given level 1 ordering on k_v. In the article we give a proof of the existence of the section of $\overline{v}$, which was left as an open problem by Klep and Velušček, and thus complete the generalization of the Krull–Baer theorem for preorderings.

1 Introduction

In the breakthrough article on orderings of higher level [1], Becker proved a higher level version of the Krull-Baer theorem regarding complete preorderings [1, Satz 2.4] that explicitely describes all complete preorderings that are lifts of a given level 1 ordering on the residue field. One of the crucial ingredients of the construction of preorderings is a section of a surjective homomorphism $\overline{v}\colon D^\times/\prod_n D^\times \to \Gamma/\prod_n \Gamma$ induced by a valuation $v\colon D \to \Gamma \cup \{\infty\}$. By introducing the notion of n-real valuations, Klep and Velušček [2] generalized Becker's version of the Krull–Baer theorem to skew fields [2, Theorem 6] modulo the existence of the section of $\overline{v}$, which was left as an open problem.

In the following sections we use the theory of abelian groups of finite exponent to construct a section of the homomorphism $\overline{v}$, which, together with [2, Theorem 6], gives us a generalization of Becker's result for skew fields.

2 Preliminaries

In this section we give all the information about valuations, orderings and complete preorderings of skew fields that is needed in this paper. We restrict our comments to the definitions and theorems that are actually used and presuppose basic knowledge of commutative valuations as well as commutative orderings. Throughout this paper D will denote a skew field.

Let G be an arbitrary group. Throughout the paper $\prod_n G$ will denote the subgroup of G generated by n-th powers and multiplicative commutators. The elements of $\prod_n G$ are called *the permuted products of n-th powers* of elements of G. It is clear

Received by the editors April 8, 2008.
Published electronically August 19, 2010.
Author was partially supported by the Ministry of Higher Education, Science and Technology of Slovenia.
AMS subject classification: **14P99**, 06Fxx.
Keywords: orderings of higher level, division rings, valuations.

that every element of $\prod_n G$ can be written as a product of terms that are a result of applying a permutation on a certain product of n-th powers, and vice versa. For example: $x^2yz^3yx^2yzy$ is a permuted product of 4-th powers.

Definition Let $n \in \mathbb{N}$. A subset $P \subsetneq D$ is called an *ordering of level* $n/2$ (*of exponent* n) if

$$\prod_n D^\times \subseteq P, \quad P + P \subseteq P, \quad P \cdot P \subseteq P$$

and $D^\times / P^\times$ is a cyclic group.

Definition A set $P \subseteq D$ is a *complete preordering of exponent* n or *level* $n/2$ if

$$\prod_n D^\times \subseteq P, \quad -1 \notin P, \quad P + P \subseteq P, \quad P \cdot P \subseteq P, \quad a^2 \in P \ \Rightarrow\ a \in P \cup -P.$$

Note that an ordering of level n is also a complete preordering of level n, and every complete preordering of level n can be extended to an ordering of level n (see *e.g.,* [5, Theorem 3.13]).

All valuations considered will be invariant. A valuation with (not necessarily commutative) value group Γ_v will be denoted by $v\colon D \longrightarrow \Gamma_v \cup \{\infty\}$ and $\mathcal{O}_v, \mathfrak{m}_v, k_v, \Gamma_v$ will represent its valuation ring, its maximal ideal, its residue skew field and its value group, respectively (see Schilling [4] for more details). For $a \in \mathcal{O}_v$, $\overline{a} := a + \mathfrak{m}_v$ will denote its image in k_v.

Definition A valuation v is said to be *compatible* with a complete preordering P if $1 + \mathfrak{m}_v \subseteq P$.

See *e.g.,* [5] for more details.

Definition The valuation v is *n-real* if k_v admits an ordering $\overline{P}$ of level 1 with the property $\overline{\mathcal{O}_v^\times \cap \prod_{2n} D^\times} \subseteq \overline{P}$. An ordering $\overline{P}$ of k_v will be called *n-compatible with v* if $\overline{\mathcal{O}_v^\times \cap \prod_{2n} D^\times} \subseteq \overline{P}$.

For more results on n-real valuations, see [2].

3 Lifting Preorderings

The main ingredient of the proof of [1, Satz 2.4] and the proof of [2, Theorem 6] is a section of a surjective homomorphism $\overline{v}\colon D^\times / \prod_n D^\times \to \Gamma / \prod_n \Gamma$ that is induced by a valuation $v\colon D \twoheadrightarrow \Gamma \cup \{\infty\}$. Its existence in the noncommutative case was left as an open problem in [2]. In the following section we present a construction of a section of the homomorphism $\overline{v}$ that enables us to generalize the theorem [1, Satz 2.4] to the noncommutative setting.

Lemma 3.1 *Let $v\colon D \twoheadrightarrow \Gamma \cup \{\infty\}$ be a valuation on D. Then v induces a surjective homomorphism $\overline{v}\colon D^\times / \prod_n D^\times \to \Gamma / \prod_n \Gamma$ with a section $\mu\colon \Gamma / \prod_n \Gamma \to D^\times / \prod_n D^\times$,* i.e., $\overline{v} \circ \mu = \mathrm{id}$.

https://doi.org/10.4153/CMB-2010-095-0 Published online by Cambridge University Press

Proof The existence of $\overline{\nu}$ is clear. For an arbitrary group G let us denote by $[G,G]$ the commutator subgroup of G. Note that $[G,G]$ is a normal subgroup of G and $\prod_m G$ for all $m \in \mathbb{N}$. Let us denote $\Gamma^{ab} = \Gamma/[\Gamma,\Gamma]$. Hence, by isomorphism theorem

$$\frac{\Gamma}{\prod_n \Gamma} \cong \frac{\Gamma^{ab}}{n\Gamma^{ab}}.$$

Without loss of generality we can take the canonical isomorphism to be the identity and let $n = \prod_{\text{prime } p} p^{\alpha_p}$ be the prime decomposition of n. By [3, Chapter 8, Exercise 1] we have

$$\frac{\Gamma^{ab}}{n\Gamma^{ab}} = \bigoplus_p \frac{\Gamma^{ab}}{p^{\alpha_p}\Gamma^{ab}}.$$

Thus, it is enough to find $\mu_p \colon \Gamma^{ab}/(p^{\alpha_p}\Gamma^{ab}) \to D^\times/\prod_n D^\times$ such that $\overline{\nu}\circ\mu_p = id|_{\Gamma^{ab}/(p^{\alpha_p}\Gamma^{ab})}$ for every prime p from the prime decomposition of n.

The group $\Gamma^{ab}/\left(p^{\alpha_p}\Gamma^{ab}\right)$ is a bounded p-group, thus by [3, Theorem 17.2]

$$\frac{\Gamma^{ab}}{p^{\alpha_p}\Gamma^{ab}} = \bigoplus_{i\in I}\langle a_i\rangle$$

for some $a_i \in \Gamma^{ab}/\left(p^{\alpha_p}\Gamma^{ab}\right)$. Since $\overline{\nu}$ is surjective, we can pick a $b_i \in D^\times/\prod_n D^\times$ such that $\overline{\nu}(b_i) = a_i$ for every $i \in I$. Take an arbitrary $\sum_{i\in I} k_i a_i \in \Gamma^{ab}/\left(p^{\alpha_p}\Gamma^{ab}\right)$, $k_i \in \mathbb{Z}$ and define

$$\mu_p\Big(\sum_{i\in I} k_i a_i\Big) = \prod_{i\in I} b_i^{k_i}.$$

Clearly μ_p is a homomorphism and $\overline{\nu}\circ\mu_p = id|_{\Gamma^{ab}/(p^{\alpha_p}\Gamma^{ab})}$, which finishes the proof. ∎

Using Lemma 3.1 and Theorem 6 of [2] we get the constructive version of the Baer–Krull-like theorem for complete preorderings that generalizes the commutative theorem [1, Satz 2.4].

Corollary 3.2 (c.f. [1, Satz 2.4]) *Assume $n \in \mathbb{N}$ is even, $v\colon D \twoheadrightarrow \Gamma\cup\{\infty\}$ is $\frac{n}{2}$-real and let $\overline{\nu}\colon D^\times/\prod_n D^\times \to \Gamma/\prod_n\Gamma$ be the induced group homomorphism. Denote by $\mu\colon \Gamma/\prod_n\Gamma \to D^\times/\prod_n D^\times$ a section of $\overline{\nu}$ as in Lemma 3.1. Fix a set of representatives $\mathfrak{U} \subseteq D^\times$ for $\mu(\Gamma/\prod_n\Gamma)$ with $1\in\mathfrak{U}$.*

Let $\overline{P}$ be a level 1 ordering of k_v that is n-compatible with v. Suppose Γ_0 is a subgroup of Γ containing $\prod_n\Gamma$ such that the Sylow 2-subgroup of Γ/Γ_0 is cyclic of order 2^r, $r \geqslant 0$ and $\chi\colon \Gamma_0 \to k_v^\times/\overline{P}^\times$ is a character satisfying $\chi(\prod_n\Gamma) = 1$ and $\chi(\Gamma_0\cap\prod_{2^r}\Gamma) \neq 1$ if $r\geqslant 1$.

Define $\mathfrak{U}_0 \subseteq \mathfrak{U}$ to be the system of representatives of $\mu(\Gamma_0/\prod_n\Gamma)$ and for every $a\in\mathfrak{U}_0$ denote $M_a := \{\varepsilon\in\mathcal{O}_v^\times \mid \chi(v(a)) = \overline{\varepsilon}\overline{P}\}$. Then:

(a) *$P := \bigcup_{a\in\mathfrak{U}_0} aM_a\prod_n D^\times$ is a complete preordering of exponent n compatible with v and induces the given ordering $\overline{P}$ of k_v.*
(b) *Every complete preordering of exponent n compatible with v that induces the ordering $\overline{P}$ of k_v is obtained in this way.*

https://doi.org/10.4153/CMB-2010-095-0 Published online by Cambridge University Press

References

[1] E. Becker, *Summen n-ter Potenzen in Körpern.* J. Reine Angew. Math. **307/308**(1979), 8–30.
[2] I. Klep, D. Velušček, *n-real valuations and the higher level version of the Krull–Baer theorem.* J. Algebra **279**(2004), no. 1, 345–361. doi:10.1016/j.jalgebra.2004.05.012
[3] L. Fuchs, *Infinite abelian groups. Vol. I.* Pure and Applied Mathematics, 36, Academic Press, New York-London, 1970.
[4] O. F. G. Schilling, *The theory of valuations*, Mathematical Surveys, 4, American Mathematical Society, New York, 1950.
[5] V. Powers, *Holomorphy rings and higher level orders on skew fields.* J. Algebra **136**(1991), no. 1, 51–59. doi:10.1016/0021-8693(91)90063-E

University of Ljubljana, Faculty of Mathematics and Physics, Department of Mathematics, Ljubljana, Slovenia
e-mail: dejan.veluscek@fmf.uni-lj.si