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THE SIMILARITY DEGREE OF SOME C^*-ALGEBRAS

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Abstract

We define the class of weakly approximately divisible unital C^*-algebras and show that this class is closed under direct sums, direct limits, any tensor product with any C^*-algebra, and quotients. A nuclear C^*-algebra is weakly approximately divisible if and only if it has no finite-dimensional representations. We also show that Pisier's similarity degree of a weakly approximately divisible C^*-algebra is at most five.

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1. Introduction

One of the most famous and oldest open problems in the theory of C^*-algebras is Kadison's similarity problem [12], which asks whether every bounded unital homomorphism ρ from a C^*-algebra $\mathcal{A}$ into the algebra $B(H)$ of operators on a Hilbert space H must be similar to a $*$-homomorphism, that is, does there exist an invertible $S \in B(H)$ such that $\pi(A) = S\rho(A)S^{-1}$ defines a $*$-homomorphism? One measure of the quality of a good problem is the number of interesting equivalent formulations. In this regard Kadison's problem gets high marks.

(1) Inner derivation problem [4, 13]: if $\mathcal{M} \subseteq B(H)$ is a von Neumann algebra and $\delta : \mathcal{M} \to B(H)$ is a derivation, does there exist a $T \in B(H)$ such that, for every $A \in \mathcal{M}$,

$$\delta(A) = AT - TA?$$

(2) Hyperreflexivity problem [4, 13]: if $\mathcal{M} \subseteq B(H)$ is a von Neumann algebra, does there exist a K, $1 \leq K < \infty$, such that, for every $T \in B(H)$,

$$\mathrm{dist}(T, \mathcal{M}) \leq K \sup\{\|PT - TP\| : P \in \mathcal{M}',\ P = P^* = P^2\}?$$

(3) Dixmier's invariant operator range problem [6] (Foiaş [7], Pisier [21, Theorem 10.5], see also [10]): if $\mathcal{M} \subseteq B(H)$ is a von Neumann algebra, $A \in B(H)$ and $T(A(H)) \subseteq A(H)$ for every $T \in \mathcal{M}$, then does there exist $D \in \mathcal{M}'$ such

that $A(H) = D(H)$? Paulsen [16] proved that an affirmative answer is equivalent to the assertion that the range of $A \oplus A \oplus \cdots$ is invariant for $\mathcal{M} \otimes \mathcal{K}(\ell^2)$.

In [8] Haagerup proved that Kadison's question has an affirmative answer whenever the representation ρ has a cyclic vector, a result that is independent of the structure of the algebra $\mathcal{A}$. Haagerup [8] also showed that a homomorphism ρ is similar to a $*$-homomorphism if and only if ρ is completely bounded. (See also [3]; see the union of [9] and [26] for another proof; see [16, 17] for a lovely exposition of these ideas.) In [18] Pisier proved that, for a fixed C^*-algebra $\mathcal{A}$, every bounded homomorphism of $\mathcal{A}$ is similar to a $*$-homomorphism if and only if $\mathcal{A}$ satisfies a certain factorisation property. It was shown in [10] that Kadison's similarity property is universally true if and only if there is a Pisier-like factorisation in terms of scalar matrices and noncommutative polynomials that is independent of the C^*-algebra. It was also shown in [10] that if $H = \ell^2 \oplus \ell^2 \oplus \cdots$ and $D = 1 \oplus \frac{1}{2} \oplus \frac{1}{2^2} \oplus \cdots$ and $\mathcal{S}$ is the unital algebra of all operators $T \in B(H)$ with an operator matrix $T = (A_{ij})$ such that $\rho(T) = D^{-1}TD = (2^{j-i}A_{ij})$ is bounded, then Kadison's similarity problem has an affirmative answer if and only if, for every unital C^*-subalgebra $\mathcal{A}$ of $\mathcal{S}$, the homomorphism $\rho|_{\mathcal{A}}$ is similar to a $*$-homomorphism.

Our main focus in this paper is another amazing result of Pisier [18] where he shows that, for a unital C^*-algebra $\mathcal{A}$, Kadison's similarity property holds for $\mathcal{A}$ if and only if there is a positive number d for which there is a positive number K such that

$$\|\rho\|_{cb} \le K\|\rho\|^d$$

for every bounded unital homomorphism ρ on $\mathcal{A}$. Pisier proved that the smallest such d is an integer which he calls the *similarity degree* $d(\mathcal{A})$ of $\mathcal{A}$. Here are a few results on the similarity degree.

(1) $\mathcal{A}$ is nuclear if and only if $d(\mathcal{A}) = 2$ [2, 4, 22];
(2) if $\mathcal{A} = \mathcal{B}(\mathcal{H})$, then $d(\mathcal{A}) = 3$ [20];
(3) $d(\mathcal{A} \otimes \mathcal{K}(\mathcal{H})) \le 3$ for any C^*-algebra $\mathcal{A}$ [8, 19];
(4) if $\mathcal{M}$ is a factor of type II_1 with property Γ, then $d(\mathcal{M}) = 3$ [5];
(5) if $\mathcal{A}$ is an approximately divisible C^*-algebra [1], then $d(\mathcal{A}) \le 5$ [14, 15];
(6) if $\mathcal{A}$ is nuclear and contains unital matrix algebras of any order, then $d(\mathcal{A} \otimes \mathcal{B}) \le 5$ for any unital C^*-algebra $\mathcal{B}$ [23];
(7) if $\mathcal{A}$ is nuclear and contains finite-dimensional C^*-subalgebras of arbitrarily large subrank (see the definition below), then $d(\mathcal{A} \otimes \mathcal{B}) \le 5$ for any unital C^*-algebra $\mathcal{B}$ [14];
(8) if $\mathcal{A}$ is nuclear and contains homomorphic images of certain dimension-drop C^*-algebras $\mathcal{Z}_{p,q}$ for all relatively prime integers p, q (for example, $\mathcal{A}$ contains a copy of the Jiang–Su algebra), then $d(\mathcal{A} \otimes \mathcal{B}) \le 5$ for any unital C^*-algebra $\mathcal{B}$ [11].

In this paper we define the class of weakly approximately divisible C^*-algebras and show that this class is closed under unital $*$-homomorphisms, arbitrary tensor products and direct limits. We also define the class of tracially nuclear C^*-algebras that properly contains the class of nuclear C^*-algebras, and we show that a tracially nuclear C^*-algebra is weakly approximately divisible if and only if it has no finite-dimensional representations. We prove that if $\mathcal{A}$ is weakly approximately divisible, then $d(\mathcal{A}) \le 5$. We extend the results (6)–(8) above to the case when $\mathcal{A}$ is tracially nuclear and has no finite-dimensional representations, and the tensor product is with respect to any C^*-crossnorm.

2. Weakly approximately divisible algebras

If τ is a tracial state on $\mathcal{M}$, we let $\|\cdot\|_\tau$ denote the seminorm on $\mathcal{M}$ defined in the Gelfand–Naimark–Segal (GNS) construction by

$$\|a\|_\tau^2 = \tau(a^* a).$$

Let $\mathcal{B}$ be a finite-dimensional unital C^*-subalgebra of a unital C^*-algebra $\mathcal{A}$. First, we know that $\mathcal{B}$ is $*$-isomorphic to $\mathcal{M}_{k_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{k_m}(\mathbb{C})$ and its *subrank*, subrank($\mathcal{B}$), is defined to be $\min(k_1, \ldots, k_m)$. Note that if $\pi : \mathcal{B} \to \mathcal{D}$ is a unital $*$-homomorphism, then

$$\text{subrank}(\mathcal{B}) \le \text{subrank}(\pi(\mathcal{B})).$$

If $P_1 = 1 \oplus 0 \oplus \cdots \oplus 0$, $P_2 = 0 \oplus 1 \oplus 0 \oplus \cdots \oplus 0, \ldots, P_m = 0 \oplus \cdots \oplus 1$ are the minimal central projections of $\mathcal{B}$, then, for $1 \le s \le m$, we have $P_s \mathcal{A} P_s$ is isomorphic to $\mathcal{M}_{k_s}(\mathbb{C}) \otimes \mathcal{A}_s = \mathcal{M}_{k_s}(\mathcal{A}_s)$ for some algebra $\mathcal{A}_s$. The relative commutant of $\mathcal{M}_{k_s}(\mathbb{C})$ in $\mathcal{M}_{k_s}(\mathcal{A}_s)$ is

$$\mathcal{D}_s = \left\{ \begin{pmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{pmatrix} : A \in \mathcal{A}_s \right\},$$

and the relative commutant of $\mathcal{B}$ in $\mathcal{A}$ is $\mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_m$. Suppose that $T \in \mathcal{A}$, and $P_s T P_s = (a_{ijs})_{1 \le i,j \le k_s}$. Let $D_s = \text{diag}(c, \ldots, c)$ where $c = (1/k) k_s (a_{11s} + \cdots + a_{k_s k_s s})$. The map $E_{\mathcal{B}} : \mathcal{A} \to \mathcal{B}' \cap \mathcal{A}$ sending T to $D_1 \oplus \cdots \oplus D_m$ is called the conditional expectation from $\mathcal{A}$ to $\mathcal{B}' \cap \mathcal{A}$ and is a completely positive unital idempotent. For $1 \le s \le m$, let $\mathcal{G}_s$ be the group of all matrices in $\mathcal{M}_{k_s}(\mathbb{C})$ such that the only nonzero entry in each row and each column is 1 or -1, and let $\mathcal{G} = \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_m \subseteq \mathcal{B}$. Then

$$E_{\mathcal{B}}(T) = \frac{1}{\text{Card}\, \mathcal{G}} \sum_{U \in \mathcal{G}} U T U^*. \tag{$*$}$$

Moreover, if $S \in \mathcal{B}' \cap \mathcal{A}$ and $T \in \mathcal{A}$, then

$$E_{\mathcal{B}}(ST) = S E_{\mathcal{B}}(T) \quad \text{and} \quad E_{\mathcal{B}}(TS) = E_{\mathcal{B}}(T) S.$$

Furthermore, if τ is a tracial state on $\mathcal{A}$, then, for every $A \in \mathcal{A}$,

$$\|E_{\mathcal{B}}(A)\|_\tau \le \|A\|_\tau.$$

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Suppose that $\mathcal{M}$ is a von Neumann algebra and $\{v_i : i \in I\} \subseteq \mathcal{M}$ is a family satisfying $\sum_{i\in I} v_i^* v_i = 1$ (convergence is in the weak* topology). Then $\varphi(T) = \sum_{i\in I} v_i^* T v_i$ defines a unital completely positive map from $\mathcal{M}$ to $\mathcal{M}$. Let us call such a map *internally spatial*, and call a unital completely positive map *internal* if it is a convex combination of internally spatial maps on $\mathcal{M}$.

Remark 2.1. There are two key properties of internal maps.

(1) They can be pushed forward through normal unital $*$-homomorphisms between von Neumann algebras. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are von Neumann algebras and $\rho : \mathcal{M} \to \mathcal{N}$ is a unital weak*–weak*-continuous unital $*$-homomorphism, and suppose that $\{v_i : i \in I\} \subseteq \mathcal{M}$ with $\sum_{i\in I} v_i^* v_i = 1$ and $\varphi(T) = \sum_{i\in I} v_i^* T v_i$. Then $\{\pi(v_i) : i \in I\} \subseteq \mathcal{N}$ and

$$1 = \pi(1) = \pi\Big(\sum_{i\in I} v_i^* v_i\Big) = \sum_{i\in I} \pi(v_i)^* \pi(v_i).$$

We define $\varphi^\pi(S) = \sum_{i\in I} \pi(v_i)^* S \pi(v_i)$, and we have, for every $a \in \mathcal{M}$,

$$\varphi^\pi(\pi(a)) = \pi(\varphi(a)).$$

So if $b \in \pi(\mathcal{A})$ and $b = \pi(a)$, then $\varphi^\pi(b) = \pi(\varphi(a))$, which is independent of a. For a general φ this only makes sense when $\varphi(\ker \pi) \subseteq \ker \pi$. It follows that φ^π makes sense when φ is an internal map, and in this case, φ^π is an internal map on $\mathcal{N}$.

(2) If $\varphi(T) = \sum_{i\in I} v_i^* T v_i$ and T commutes with each v_i, then, for every S,

$$\varphi(ST) = \varphi(S)T.$$

Hence if ψ is a convex combination of spatially internal maps defined in terms of elements commuting with an operator T, we have $\psi(ST) = \psi(S)T$.

Definition 2.2. We say that a unital C^*-algebra $\mathcal{A}$ is *weakly approximately divisible* if and only if, for every finite subset $\mathcal{F}$ of $\mathcal{A}$, there is a net $\{(\mathcal{B}_\lambda, \varphi_\lambda)\}_{\lambda\in\Lambda}$ where each $\mathcal{B}_\lambda$ is a finite-dimensional unital C^*-subalgebra of $\mathcal{A}^{\#\#}$ and φ_λ is an internal completely positive map such that:

(1) $\lim_\lambda \mathrm{subrank}(\mathcal{B}_\lambda) = \infty$;
(2) $\varphi_\lambda : \mathcal{A} \to \mathcal{B}_\lambda' \cap \mathcal{A}^{\#\#}$;
(3) for every $a \in \mathcal{F}$, $\varphi_\lambda(a) \to a$ in the weak* topology on $\mathcal{A}^{\#\#}$.

Remark 2.3. Suppose that n is a positive integer and let $\mathcal{V}_n$ be the set of n-tuples $(a_1, \ldots, a_n)$ of elements in $\mathcal{A}$ such that the conditions in Definition 2.2 hold when $\mathcal{F} = \{a_1, \ldots, a_n\}$. Suppose that U_k is a weak* neighbourhood of a_k in $\mathcal{A}^{\#\#}$ for $1 \le k \le n$. Since addition on $\mathcal{A}^{\#\#}$ is weak*-continuous, there is a weak* neighbourhood V_k of a_k and a weak* neighbourhood E of 0 such that

$$V_k + E \subseteq U_k$$

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for $1 \le k \le n$. Suppose that $(b_1, \ldots, b_n)$ is in the norm closure of $\mathcal{V}_n$ and that U_k is a weak* neighbourhood of b_k in $\mathcal{A}^{\#\#}$ for $1 \le k \le n$. Since addition on $\mathcal{A}^{\#\#}$ is weak*-continuous, there is a weak* neighbourhood V_k of b_k and a weak* neighbourhood E of 0 such that

$$V_k + E \subseteq U_k$$

for $1 \le k \le n$. Since $0 \in E$ and E is weak*-open, there is an $\varepsilon > 0$ such that $\{x \in \mathcal{A}^{\#\#} : \|x\| < \varepsilon\} \subseteq E$. Now choose $(a_1, \ldots, a_n) \in \mathcal{V}_n$ so that $a_k \in V_k$ and $\|a_k - b_k\| < \varepsilon$ for $1 \le k \le n$. Next suppose that m is a positive integer. It follows from the definition of $\mathcal{V}_n$ that there is a finite-dimensional C^*-subalgebra $\mathcal{B}$ of $\mathcal{A}^{\#\#}$ and a completely positive unital map $\varphi : \mathcal{A} \to \mathcal{B}' \cap \mathcal{A}^{\#\#}$ such that $\operatorname{subrank}(\mathcal{B}) \ge m$ and such that $\varphi(a_k) \in V_k$ for $1 \le k \le n$. It follows that $\varphi(b_k) - \varphi(a_k) = \varphi(b_k - a_k) \in E$ for $1 \le k \le n$, so

$$\varphi(b_k) \in V_k + E \subseteq U_k$$

for $1 \le k \le n$. Hence $(b_1, \ldots, b_n) \in \mathcal{V}_n$. Thus $\mathcal{V}_n$ is norm closed. It is also clear that $\mathcal{V}_n$ is a linear space. Hence, to verify that $\mathcal{A}$ is weakly approximately divisible, it is sufficient to show that the conditions of Definition 2.2 hold for all finite subsets $\mathcal{F}$ of a set W whose norm closed linear span $\overline{\mathrm{sp}}(W)$ is $\mathcal{A}$.

Recall [25] that a C^*-algebra $\mathcal{A}$ is *nuclear* if, for every Hilbert space H and every unital $*$-homomorphism $\pi : \mathcal{A} \to B(H)$, we have that $\pi(\mathcal{A})''$ is a hyperfinite von Neumann algebra. We say that $\mathcal{A}$ is *tracially nuclear* if, for every tracial state τ on $\mathcal{A}$ with GNS representation π_τ, we have that $\pi_\tau(\mathcal{A})''$ is a hyperfinite von Neumann algebra. As a flip side of the notion of residually finite-dimensional (RFD) C^*-algebras, we say that a unital C^*-algebra $\mathcal{A}$ is *NFD* if $\mathcal{A}$ has no unital finite-dimensional representations.

Theorem 2.4. *Suppose that $\mathcal{A}$ and $\mathcal{D}$ are unital C^*-algebras. Then the following statements hold.*

(1) *If $\mathcal{A}$ is approximately divisible, then $\mathcal{A}$ is weakly approximately divisible.*
(2) *If $\mathcal{A}$ is weakly approximately divisible and $\pi : \mathcal{A} \to \mathcal{D}$ is a surjective unital $*$-homomorphism, then $\mathcal{D}$ is weakly approximately divisible.*
(3) *If $\mathcal{A}$ is weakly approximately divisible, then $\mathcal{A}$ has no finite-dimensional representations.*
(4) *If $\mathcal{A}$ is weakly approximately divisible, then $\mathcal{A} \otimes_{\max} \mathcal{D}$ is weakly approximately divisible.*
(5) *A finite direct sum $\sum^{\oplus}_{1 \le k \le n} \mathcal{A}_k$ of unital C^*-algebras is weakly approximately divisible if and only if each summand $\mathcal{A}_k$ is weakly approximately divisible.*
(6) *If n is a positive integer, then $\mathcal{A} \otimes \mathcal{M}_n(\mathbb{C})$ is weakly approximately divisible if and only if $\mathcal{A}$ is.*
(7) *A direct limit of weakly approximately divisible C^*-algebras is weakly approximately divisible.*
(8) *If $\mathcal{A}$ is an NFD C^*-algebra and $\mathcal{M}$ is the type II_1 direct summand of $\mathcal{A}^{\#\#}$ and $\gamma : \mathcal{A} \to \mathcal{M}$ is the inclusion into $\mathcal{A}^{\#\#}$ followed by the projection map, then $\mathcal{A}$ is*

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weakly approximately divisible if and only if, for every finite subset $\mathcal{F} \subseteq \mathcal{A}$ there is a net $\{(\mathcal{B}_\lambda, \varphi_\lambda)\}$ where $\mathcal{B}_\lambda$ is a finite-dimensional C^*-subalgebra of $\mathcal{M}$, φ_λ is an internal map on $\mathcal{M}$ and

$$\varphi_\lambda(\pi(a)) \to \gamma(a)$$

in the weak topology for every $a \in \mathcal{F}$.*

(9) *If $\mathcal{A}$ is tracially nuclear, then $\mathcal{A}$ is weakly approximately divisible if and only if $\mathcal{A}$ is NFD.*

(10) *If $\mathcal{A}$ is nuclear, then $\mathcal{A}$ is weakly approximately divisible if and only if $\mathcal{A}$ is NFD.*

Proof. (1) This follows immediately from the definitions.

(2) If $\pi : \mathcal{A} \to \mathcal{D}$ is a surjective unital $*$-homomorphism, then π extends to a weak*–weak*-continuous surjective unital $*$-homomorphism $\rho : \mathcal{A}^{\#\#} \to \mathcal{D}^{\#\#}$. Given $d_1, \ldots, d_n \in \mathcal{D}$, choose $a_1, \ldots, a_n \in \mathcal{A}$ so that $\pi(a_k) = d_k$ for $1 \le k \le n$. Choose a net $\{(\mathcal{B}_\lambda, \varphi_\lambda)\}$ according to Definition 2.2 with $\mathcal{F} = \{a_1, \ldots, a_n\}$. It follows that φ_λ^ρ is an internal completely positive map on $\mathcal{D}^{\#\#}$ and

$$\varphi_\lambda^\rho(\mathcal{D}) = \varphi_\lambda^\rho(\rho(\mathcal{A})) = \rho(\varphi_\lambda(\mathcal{A})) \subseteq \rho(\mathcal{B}_\lambda' \cap \mathcal{A}^{\#\#}) \subseteq \rho(\mathcal{B}_\lambda)' \cap \mathcal{D}^{\#\#}.$$

Further, for each d_k,

$$\text{w*-}\lim_\lambda \varphi_\lambda^\rho(d_k) = \text{w*-}\lim_\lambda \rho(\varphi_\lambda(a_k)) = \rho(a_k) = d_k,$$

since ρ is weak*–weak*-continuous. Since $\operatorname{subrank}(\mathcal{B}_\lambda) \le \operatorname{subrank}(\rho(\mathcal{B}_\lambda))$, we conclude that $\mathcal{D}$ is weakly approximately divisible.

(3) This follows from (2) and the obvious fact that no finite-dimensional C^*-algebra is weakly approximately divisible.

(4) Let $\rho : \mathcal{A} \otimes_{\max} \mathcal{D} \to (\mathcal{A} \otimes_{\max} \mathcal{D})^{\#\#}$ be the natural inclusion map. We can assume $(\mathcal{A} \otimes_{\max} \mathcal{D})^{\#\#} \subseteq B(H)$ for some Hilbert space H so that, on bounded subsets of $(\mathcal{A} \otimes_{\max} \mathcal{D})^{\#\#}$, the weak* topology coincides with the weak-operator topology. If $\rho : \mathcal{A} \to \mathcal{A} \otimes 1 \subseteq \mathcal{A} \otimes_{\max} \mathcal{D}$ is the inclusion map, then there is a weak*–weak*-continuous unital $*$-homomorphism $\sigma : \mathcal{A}^{\#\#} \to (\mathcal{A} \otimes_{\max} \mathcal{D})^{\#\#}$ such that the restriction of σ to $\mathcal{A}$ is ρ. Let $W = \{a \otimes b : a \in \mathcal{A}, b \in \mathcal{B}\}$. Clearly, $\overline{\mathrm{sp}} W = \mathcal{A} \otimes_{\max} \mathcal{B}$ (where the closure is with respect to $\| \|_{\max}$). Suppose that $a_1 \otimes b_1, \ldots, a_n \otimes b_n \in W$. Since $\mathcal{A}$ is weakly approximately divisible, we can choose a net $\{(\mathcal{B}_\lambda, \varphi_\lambda)\}$ as in Definition 2.2. We know that $\{\varphi_\lambda^\sigma\}$ is a net of internal maps on $(\mathcal{A} \otimes_{\max} \mathcal{D})^{\#\#}$ and

$$\varphi_\lambda^\sigma(a_k \otimes 1) = \varphi_\lambda^\sigma(\sigma(a_k)) = \sigma(\varphi_\lambda(a_k)) \to \sigma(a_k) = a_k \otimes 1$$

in the weak* topology for $1 \le k \le n$. On the other hand, each φ_λ is a convex combination of spatially internal maps defined by partial isometries in $\mathcal{A}^{\#\#}$, so each φ_λ^σ is a convex combination of spatially internal maps defined by partial isometries in $\sigma(\mathcal{A}^{\#\#})$ which is contained in $(\mathcal{A} \otimes_{\max} \mathcal{D})^{\#\#} \cap (1 \otimes \mathcal{D})'$. Hence, for every $S \in (\mathcal{A} \otimes_{\max} \mathcal{D})^{\#\#}$ and every $d \in \mathcal{D}$,

$$\varphi_\lambda^\sigma(S(1 \otimes d)) = \varphi_\lambda^\sigma(S)(1 \otimes d).$$

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Hence, for $1 \le k \le n$,

$$\varphi_\lambda^\sigma(a_k \otimes d_k) = \varphi_\lambda^\sigma((a_k \otimes 1)(1 \otimes d_k)) = \varphi_\lambda^\sigma(a_k \otimes 1)(1 \otimes d_k).$$

But $\varphi_\lambda^\sigma(a_k \otimes 1) \to a_k \otimes 1$ in the weak* topology. Hence

$$\varphi_\lambda^\sigma(a_k \otimes d_k) \to a_k \otimes d_k$$

in the weak* topology on $(\mathcal{A} \otimes_{\max} \mathcal{B})^{\#\#}$ for $1 \le k \le n$. Since, for every λ,

$$\mathrm{subrank}(\mathcal{B}_\lambda) \le \mathrm{subrank}(\sigma(\mathcal{B}_\lambda)),$$

we see that $\mathcal{A} \otimes_{\max} \mathcal{B}$ is weakly approximately divisible.

(5) This easily follows from the fact that $(\sum_{1\le k\le n}^{\oplus} \mathcal{A}_k)^{\#\#} = \sum_{1\le k\le n}^{\oplus} \mathcal{A}_k^{\#\#}$.

(6) This is clear, since $(\mathcal{A} \otimes \mathcal{M}_n(\mathbb{C}))^{\#\#}$ is isomorphic to $\mathcal{A}^{\#\#} \otimes \mathcal{M}_n(\mathbb{C})$.

(7) Suppose that $\{\mathcal{A}_i : i \in I\}$ is an increasingly directed family of C^*-subalgebras of $\mathcal{A}$ such that $W = \bigcup_{i\in I} \mathcal{A}_i$ is dense in $\mathcal{A}$. Suppose that $\mathcal{F} \subseteq W$ is finite. Then there is an $i \in I$ such that $\mathcal{F} \subseteq \mathcal{A}_i$. If $\rho : \mathcal{A}_i \to \mathcal{A}$ is the inclusion map, there is a unital weak*–weak*-continuous unital $*$-homomorphism $\sigma : \mathcal{A}_i^{\#\#} \to \mathcal{A}^{\#\#}$ whose restriction to $\mathcal{A}_i$ is ρ. The rest follows as in the proof of (2).

(8) If $\mathcal{A}$ is weakly approximately divisible, then for a finite subset $\mathcal{F} \subseteq \mathcal{A}$ we can find a net $\{(\mathcal{B}_\lambda, \varphi_\lambda)\}$ as in Definition 2.2 that works in $\mathcal{A}^{\#\#}$, and if we project all of this onto $\mathcal{M}$, we get the desired net. Now suppose that $\mathcal{A}$ satisfies the condition in (8). We can write $\mathcal{A}^{\#\#} = \mathcal{M} \oplus \mathcal{N}$, and since $\mathcal{A}$ has no finite-dimensional representations, $\mathcal{N}$ is the direct sum of a type I_∞ algebra, a II_∞ and a type III algebra. In particular, this means that there is an orthogonal sequence $\{P_n\}$ of pairwise Murray–von Neumann equivalent projections whose sum is 1. Suppose that N is a positive integer, and let $Q_k = \sum_{j=(k-1)N+1}^{kN} P_j$. Then $\{Q_n\}$ is an orthogonal sequence of pairwise equivalent projections whose sum is 1. We can construct a system of matrix units $\{E_{ij}\}_{1\le i,j<\infty}$ so that $E_{kk} = Q_k$ for all $k \ge 1$. Then every $T \in \mathcal{N}$ has an infinite operator matrix $T = (T_{ij})$. The map

$$\psi_N(T) = \mathrm{diag}(T_{11}, T_{11}, \ldots) = \sum_{j=1}^{\infty} E_{j1} T E_{j1}^*$$

is spatially internal and, for every T,

$$\Big(\sum_{k=1}^{N} P_k\Big)\psi_N(T)\Big(\sum_{k=1}^{N} P_k\Big) = \Big(\sum_{k=1}^{N} P_k\Big)T\Big(\sum_{k=1}^{N} P_k\Big) \to T$$

in the weak* topology. Hence $\psi_N(T) \to T$ in the weak* topology. Moreover, $\mathcal{N} \cap \psi_N(\mathcal{N})'$ contains full matrix algebras of all orders. Next suppose that $\mathcal{F} \subseteq \mathcal{A}$ is finite. For each $A \in \mathcal{F}$ we write $A = \gamma(A) \oplus T_A$ relative to $\mathcal{A}^{\#\#} = \mathcal{M} \oplus \mathcal{N}$. Given the net $\{(\mathcal{B}_\lambda, \varphi_\lambda)\}$ in $\mathcal{M}$ based on our assumption on $\mathcal{A}$, we let $N_\lambda = \mathrm{subrank}(\mathcal{B}_\lambda)$ and choose a full $N_\lambda \times N_\lambda$ matrix algebra C_λ in $\mathcal{N} \cap \psi_N(\mathcal{N})'$. Then $\tau_\lambda(S \oplus T) = \varphi_\lambda(S) \oplus \psi_{N_\lambda}(T)$ is an internal map on $\mathcal{A}^{\#\#}$ whose range is in $(\mathcal{B}_\lambda \oplus C_\lambda)' \cap \mathcal{A}^{\#\#}$ such that

$$\tau_\lambda(A) \to A$$

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in the weak* topology for every $A \in \mathcal{F}$. Hence $\mathcal{A}$ is weakly approximately divisible.

(9) Let $\mathcal{M}$ and γ be as in (8). Let Λ be the set of all triples $\lambda = (\mathcal{F}_\lambda, \mathcal{T}_\lambda, k_\lambda)$ where $\mathcal{F}_\lambda \subseteq \mathcal{A}$ is finite, $\mathcal{T}_\lambda$ is a finite set of normal tracial states on $\mathcal{M}$, and $k_\lambda \in \mathbb{N}$. With the ordering $(\subseteq, \subseteq, \leq)$ we see that Λ is a directed set. If τ is a tracial state on $\mathcal{M}$, we let $\|\cdot\|_\tau$ denote the seminorm on $\mathcal{M}$ defined by

$$\|A\|_\tau = \tau(A^*A)^{1/2}.$$

Suppose that $\lambda \in \Lambda$. There is a central projection $P \in \mathcal{M}$ so that $\mathcal{M} = \mathcal{M}_a \oplus \mathcal{M}_s$ ($\mathcal{M}_a = P\mathcal{M}$) and so that $\gamma = \gamma_a \oplus \gamma_s$ and such that $\gamma_a \ll \sum_{\tau\in\mathcal{T}_\lambda}^{\oplus} \pi_\tau$ and γ_s is disjoint from $\sum_{\tau\in\mathcal{T}_\lambda}^{\oplus} \pi_\tau$. Also, by assumption, $(\sum_{\tau\in\mathcal{T}_\lambda}^{\oplus} \pi_\tau)(\mathcal{A})'' = \mathcal{M}_a$ is hyperfinite. Hence, there is a finite-dimensional unital subalgebra $\mathcal{D}_\lambda$ of $\mathcal{M}_a$ and a contractive map $\eta : \mathcal{F}_\gamma \to \mathcal{D}_\lambda$ such that

$$\max_{\tau\in\mathcal{T}_\lambda, A\in\mathcal{F}_\lambda} \|P\gamma(A) - \eta(A)\|_\tau < \frac{1}{k}.$$

Note that $\|T\|_\tau = \|PT\|_\tau$ for every $T \in \mathcal{M}$ and every $\tau \in \mathcal{T}_\lambda$. The relative commutant $\mathcal{D}'_\lambda \cap \mathcal{M}_a$ is also a II_1 von Neumann algebra, so there are k_λ mutually orthogonal unitarily equivalent projections in $\mathcal{D}'_\lambda \cap \mathcal{M}_a$ whose sum is 1. Hence $\mathcal{D}'_\lambda \cap \mathcal{M}_a$ contains a unital subalgebra $\mathcal{E}_\lambda$ that is isomorphic to $\mathcal{M}_k(\mathbb{C})$. Similarly, $\mathcal{M}_s$ (if it is not 0) is a II_1 von Neumann algebra and contains an isomorphic copy $\mathcal{G}_\lambda$ of $\mathcal{M}_{k_\lambda}(\mathbb{C})$. Then $\mathcal{B}_\lambda = \mathcal{E}_\lambda \oplus \mathcal{G}_\lambda$ is finite-dimensional and $\operatorname{subrank}(\mathcal{B}_\lambda) = k_\lambda$. Define $\varphi_\lambda = E_{\mathcal{B}_\lambda}$. For every $A \in \mathcal{F}_\lambda$ and $\tau \in \mathcal{T}_\lambda$,

$$\begin{aligned}
\|A - \varphi_\lambda(A)\|_\tau &= \|PA - P\varphi_\lambda(A)\|_\tau \leq \|PA - \eta(A)\|_\tau + \|\eta(A) - E_{\mathcal{E}_\lambda}(PA)\|_\tau \\
&= \|PA - \eta(A)\|_\tau + \|E_{\mathcal{E}_\lambda}(\eta(A)) - E_{\mathcal{E}_\lambda}(PA)\|_\tau \\
&\leq 2\|PA - \eta(A)\|_\tau \leq \frac{2}{k_\lambda}.
\end{aligned}$$

Clearly,

$$\lim_\lambda \operatorname{subrank}(\mathcal{B}_\lambda) = \infty,$$

and, since there are sufficiently many tracial states on $\mathcal{M}$ [24], we have, for every $A \in \mathcal{A}$,

$$\varphi_\lambda(a) \to A$$

in the ultrastrong topology on $\mathcal{M}$. By assumption $\mathcal{A}$ has no finite-dimensional representations, so it follows from (8) that $\mathcal{A}$ is weakly approximately divisible.

(10) This follows immediately from (9) since the nuclearity of $\mathcal{A}$ is equivalent to the hyperfiniteness of $\pi(\mathcal{A})''$ for every representation π of $\mathcal{A}$. □

3. Similarity degree

THEOREM 3.1. *If $\mathcal{A}$ is weakly approximately divisible, then the similarity degree of $\mathcal{A}$ is at most five.*

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Proof. Suppose that H is a Hilbert space and $\rho : \mathcal{A} \to B(H)$ is a bounded unital homomorphism. Then ρ extends uniquely to a normal homomorphism $\bar{\rho} : \mathcal{A}^{\#\#} \to B(H)$. Suppose that $A = (a_{ij}) \in \mathcal{M}_n(\mathcal{A})$. Since $\mathcal{A}$ is weakly approximately divisible, we can choose a net $\{(\mathcal{B}_\lambda, \varphi_\lambda)\}_{\lambda\in\Lambda}$ as in Definition 2.2 corresponding to $\mathcal{F} = \{a_{ij} : 1 \le i, j \le n\}$. We know that

$$\bar{\rho}_n(\varphi_\lambda(a_{ij})) = (\bar{\rho}(\varphi_\lambda(a_{ij}))) \to (\bar{\rho}(a_{ij})) = \rho_n(A),$$

where the convergence is in the weak* topology. Moreover, since φ_λ is completely contractive,

$$\|(\varphi_\lambda(a_{ij}))\| \le \|A\|,$$

so

$$\lim_\lambda \|(\varphi_\lambda(a_{ij}))\| = \|A\|,$$

and

$$\|\rho_n(A)\| \le \limsup_\lambda \|\bar{\rho}_n(\varphi_\lambda(a_{ij}))\|.$$

However, $\varphi_\lambda(a_{ij}) \in \mathcal{B}_\lambda'$ for $1 \le i, j \le n$ and $\lim_\lambda \operatorname{subrank}(\mathcal{B}_\lambda) = \infty$. So the remainder of the proof follows from [14, Lemma 3.1]. □

In [23] Pop proved that if $\mathcal{A}$ is a nuclear C^*-algebra containing copies of $\mathcal{M}_n(\mathbb{C})$ for arbitrarily large values of n, then the similarity degree of $\mathcal{A} \otimes \mathcal{B}$ is at most five for every unital C^*-algebra $\mathcal{B}$. In [14] the second author showed that this result remains true if $\mathcal{A}$ is nuclear and contains finite-dimensional algebras with arbitrarily large subrank. It was shown by [11] that if $\mathcal{A}$ is nuclear and contains homomorphic images of certain dimension-drop C^*-algebras $\mathcal{Z}_{p,q}$ for all relatively prime integers p, q (for example, $\mathcal{A}$ contains a copy of the Jiang–Su algebra), then, for every unital C^*-algebra $\mathcal{B}$, the similarity degree of $\mathcal{A} \otimes \mathcal{B}$ is at most five. The following corollary includes all of these results.

Corollary 3.2. *If $\mathcal{A}$ is a unital tracially nuclear NFD C^*-algebra, then, for every unital C^*-algebra $\mathcal{B}$, the similarity degree of $\mathcal{A} \otimes \mathcal{B}$ is at most five.*

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