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# ON REAL FORMS OF A COMPLEX ALGEBRAIC CURVE

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**Abstract**

Two projective nonsingular complex algebraic curves $X$ and $Y$ defined over the field $\mathbb{R}$ of real numbers can be isomorphic while their sets $X(\mathbb{R})$ and $Y(\mathbb{R})$ of $\mathbb{R}$-rational points could be even non homeomorphic. This leads to the count of the number of real forms of a complex algebraic curve $X$, that is, those nonisomorphic real algebraic curves whose complexifications are isomorphic to $X$. In this paper we compute, as a function of genus, the maximum number of such real forms that a complex algebraic curve admits.

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## 1. Introduction

In [6] Natanzon proved, using topological methods, that a complex algebraic curve of genus $g \geq 2$ has at most $2(\sqrt{g}+1)$ real forms with real points. He also showed that this bound is attained for infinitely many values of $g$, these being of the form $(2^n-1)^2$. A combinatorial proof of this appears in [3], where it is also proved that these are the only values for which the Natanzon bound is sharp.

Here we find the maximal number $\omega(g)$ of real forms with real points that a complex algebraic curve of genus $g \geq 2$ can admit. First we prove that a bound for $\omega(g)$ depend only on the parity structure of $g-1$. To be precise, consider the set of all positive integers to be divided in strata, the $r$ th of which $N_r$ consists of all integers of the form $g = 2^{r-1}u + 1$ for some odd $u$. We show that $\omega(g) \leq 2^{r+1}$ for $g \in N_r$, the equality

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holds if and only if $u \geq 2^{r+1} - 3$, and then we find exact values of $\omega$ for the remaining values of $g$ in $N_r$. This result generalizes the result obtained in [4] for even genera, where it was proved that a complex algebraic curve of even genus has at most 4 real forms with real points and that this bound is sharp for all even $g$.

## 2. Preliminaries

The above results will be proved using the language of Riemann and Klein surfaces. We shall employ Fuchsian and NEC groups using combinatorial methods. Most of the results that we use here are dispersed in original papers [1, 5, 8]. So for reader's convenience we refer him to the monograph [2] where he can find all of them.

It is well known [1, 2, 7] that the categories of real algebraic curves and compact Klein surfaces are equivalent in the same way as the categories of complex algebraic curves and Riemann surfaces are equivalent. Under this equivalence, a real curve with real points corresponds to a bordered Klein surface. Besides, the complexifications of real curves correspond to complex doubles of Klein surfaces.

A real form of a complex algebraic curve $\mathscr{C}$ is an equivalence class of all isomorphic real curves which have $\mathscr{C}$ as their complexification. So there exists a one-to-one correspondence between the real forms of a complex algebraic curve $\mathscr{C}$ and the conjugacy classes of anticonformal involutions of the Riemann surface $X(\mathscr{C})$, which will be referred to as *symmetries* throughout the paper.

Arbitrary compact Riemann surfaces of genus $g \geq 2$ can be represented as the orbit space $\mathscr{H}/\Gamma$ of the hyperbolic plane $\mathscr{H}$ with respect to the action of a Fuchsian surface group $\Gamma$, a discrete subgroup of $\mathrm{Aut}^+(\mathscr{H}) = \mathrm{PSL}(2, \mathbb{R})$ without elliptic elements. A discrete subgroup $\Lambda$ of $\mathrm{Aut}^\pm(\mathscr{H})$ with compact orbit space is called an *NEC* (*non-Euclidean crystallographic*) group. The algebraic structure of an NEC group $\Lambda$ is determined by the signature

$$s(\Lambda) = \big(h; \pm; [m_1, \dots, m_v]; \{C_1, \dots, C_n\}\big), \tag{1}$$

where $C_i = (n_{i1}, \dots, n_{is_i})$ are called *the period cycles*, the integers $n_{ij}$ are the *link periods*, $m_i$ *proper periods* and finally $h$ the *orbit genus* of $\Lambda$.

A group $\Lambda$ with signature (1) has the presentation with generators: $x_1, \dots, x_v$, $e_1, \dots, e_n$, $c_{ij}$, $1 \leq i \leq n$, $0 \leq j \leq s_i$ and $a_1, b_1, \dots, a_h, b_h$ if the sign is $+$ or $d_1, \dots, d_h$ otherwise, and relators: $x_i^{m_i}$, $i = 1, \dots, v$, $c_{ij-1}^2$, $c_{ij}^2$, $(c_{ij-1}c_{ij})^{n_{ij}}$, $c_{i0}e_i^{-1}c_{is_i}e_i$, $i = 1, \dots, n$, $j = 0, \dots, s_i$ and $x_1 \cdots x_v e_1 \cdots e_n a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_h b_h a_h^{-1} b_h^{-1}$ or $x_1 \cdots x_v e_1 \cdots e_n d_1^2 \cdots d_h^2$, according to whether the sign is $+$ or $-$.

The hyperbolic area of $\mathscr{H}/\Lambda$ coincides with the hyperbolic area of an arbitrary fundamental region of $\Lambda$ and equals

$$(2)\qquad \mu(\Lambda) = 2\pi\left(\varepsilon h - 2 + n + \sum_{i=1}^{v}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{ij}}\right)\right),$$

where $\varepsilon = 2$ if there is a '+' sign and $\varepsilon = 1$ otherwise. If $\Lambda'$ is a subgroup of $\Lambda$ of finite index, then it is itself an NEC group and the following Riemann-Hurwitz formula holds

$$(3)\qquad [\Lambda : \Lambda'] = \mu(\Lambda')/\mu(\Lambda).$$

Given an NEC group $\Lambda$, the subgroup $\Lambda^+$ of $\Lambda$ consisting of the orientation-preserving elements is called the *canonical Fuchsian subgroup of* $\Lambda$.

An NEC group $\Gamma$ without orientation preserving automorphisms of finite order is called a *surface group* and it has signature $(h; \pm; [-], \{(-), \overset{n}{\ldots}, (-)\})$. In such case $\mathscr{H}/\Gamma$ is a *Klein surface*, that is, a surface with a dianalytic structure of topological genus $h$, orientable or not according to whether the sign is '+' or '−' and having $n$ boundary components. Conversely, a Klein surface whose complex double has genus greater than one can be expressed as $\mathscr{H}/\Gamma$ for some NEC surface group $\Gamma$. Furthermore, given a Riemann (respectively Klein) surface represented as the orbit space $X = \mathscr{H}/\Gamma$, modulo a surface group $\Gamma$, a finite group $G$ is a group of automorphisms of $X$ if and only if $G = \Lambda/\Gamma$ for some NEC group $\Lambda$.

## 3. On conjugacy classes of involutions in 2-groups

In this section we study conjugacy classes in 2-groups. A group $G$ is said to be *abstractly oriented* if there is an epimorphism $\alpha : G \to Z_2 = \{\pm 1\}$ and an element $g$ of $G$ to be *orientation preserving* (*orientation reversing*) if $\alpha(g) = +1$ ($\alpha(g) = -1$).

LEMMA 3.1. *A 2-group $G$ containing $Z_N$ as a subgroup (not necessarily normal) of index $2^r$ has at most $2^{r+1} - 1$ conjugacy classes of elements of order 2. Furthermore, if $G$ is abstractly oriented and a generator $x$ of $Z_N$ preserves the orientation, then $G$ has at most $2^r$ conjugacy classes of orientation reversing elements of order 2.*

PROOF. Let

$$Z_N = H_0 \overset{2}{\le} H_1 \overset{2}{\le} H_2 \overset{2}{\le} \cdots \overset{2}{\le} H_{r-1} \overset{2}{\le} H_r = G$$

be a subnormal sequence and let $x_i \in H_i \setminus H_{i-1}$, for $i = 1, \ldots r$.

Then each element $g$ of $G$ can be represented as $g = x^{\varepsilon} x_1^{\varepsilon_1} \cdots x_r^{\varepsilon_r}$ for some $0 \le \varepsilon < N$ and $\varepsilon_i = 0$ or $\varepsilon_i = 1$ uniquely determined. Denote $x_1^{\varepsilon_1} \cdots x_r^{\varepsilon_r}$ by $w$ and observe that there are $2^r$ elements of this form. So the proof will be complete if we show that for any such element $w$ there are at most 2 conjugacy classes of elements of order 2 among

$$w, xw, x^2w, \ldots, x^{N-1}w. \tag{4}$$

Observe that $w$ may not be an element of order 2, and furthermore that among these elements there may not even be elements of order 2. Assume then that there are at least two elements $x^k w$ and $x^l w$ of order 2, and assume that $k$ and $l$ are chosen so that $k > l$ and $m = k - l$ is as small as possible. We shall show that each element $x^n w$ which has order 2 is conjugate to $x^k w$ or to $x^l w$. We have $1 = (x^k w)^2 = x^m (x^l w)^2 w^{-1} x^m w = x^m w^{-1} x^m w$. So $wx^{-m} = x^m w$ and therefore $x^{l+\alpha m} w$ has order 2 and

$$x^{\alpha m}(x^l w)x^{-\alpha m} = x^{l+2\alpha m} w, \tag{5}$$

$$x^{\alpha m}(x^k w)x^{-\alpha m} = x^{k+2\alpha m} w \tag{6}$$

for arbitrary $\alpha$.

Now let $x^n w$ be an arbitrary element of order 2. Then $n = l + \alpha m + j$ for some $\alpha, j$, where $0 \le j < m$, and since both $x^n w$ and $x^{l+\alpha m} w$ have order 2, it follows by minimality of $m$ that $j = 0$. Thus $x^n w = x^{l+\alpha m} w$, which is a conjugate of $x^l w$ if $\alpha$ is even, or a conjugate of $x^k w$ if $\alpha$ is odd by (5) and (6) respectively.

Now let $\alpha$ be an orientation of $G$ and let $x_{i_1}, \ldots, x_{i_s}$ be all orientation reversing elements among $x_1, \ldots, x_r$. Then at most $2^r$ of the elements $w$ have odd exponent sum in $x_{i_1}, \ldots, x_{i_s}$ and so the second part of lemma follows. □

Notice that a dihedral group $\mathrm{D}_N$ generated by two involutions $x_0$ and $y_0$ can be viewed as a semidirect product $\mathrm{Z}_N \ltimes \mathrm{Z}_2 = \langle x_0 y_0 \rangle \ltimes \langle x_0 \rangle$. So we have as a consequence the following result.

COROLLARY 3.2. *A 2-group $G$ containing a dihedral group $\mathrm{D}_N$ as a subgroup of index $2^r$ has at most $2^{r+2} - 1$ conjugacy classes of elements of order 2. Furthermore if $G$ is abstractly oriented and the generators $x_0, y_0$ reverse the orientation, then $G$ has at most $2^{r+1}$ conjugacy classes of orientation reversing elements of order 2.*

## 4. Real forms of complex algebraic curve

Here we shall look for bounds for the number of conjugacy classes of symmetries of a Riemann surface $X$ of genus $g \ge 2$. Recall that by a symmetry we mean anticonformal involution of $X$. Assume that $\sigma_1, \ldots, \sigma_k$ are representatives of such classes. By the Sylow theorem all Sylow 2-subgroups are conjugate. So we can assume that all these symmetries generate a 2-group $G$, say of order $K = 2^s$. We start with the following technical observation.

LEMMA 4.1. *Let $X$ be a Riemann surface of genus $g$, and suppose that $2^{r-1}$ is the maximum power of 2 dividing $g-1$. Let $G$ be a 2-group of automorphisms of $X$ of order $2^s$, where $s \geq r+1$, such that its subgroup $G^+$ of orientation preserving automorphisms acts on $X$ with fixed points. Then $G$ contains a cyclic or dihedral subgroup of index $2^r$.*

PROOF. Let $X = \mathscr{H}/\Gamma$ for some surface NEC group $\Gamma$ and let $G = \Lambda/\Gamma$. Assume that $\Lambda$ has a signature of the general form (1). Since the action of $G^+$ is not fixed point free, $\Lambda$ has a proper period or nonempty period cycle.

We claim that $G$ contains a cyclic or dihedral group as a subgroup of index $2^r$ if $\Lambda$ has a proper period $m \geq 2^{s-r}$ or a link period $n \geq 2^{s-r-1}$. Indeed in the first case the image $x$ in $G$ of an elliptic generator of $\Lambda$ corresponding to $m$ is still an element of order $m$ in $G$ since otherwise $\Gamma$ would have elements of finite order, and so for $m' = 2^{r-s}m$, $x^{m'}$ generates a cyclic subgroup of $G$ of index $2^r$. In the second case the images $c$ and $c'$ in $G$ of reflections with product of order $n$ are involutions since otherwise $\Gamma$ would be a proper NEC group. Furthermore their product has order $n$, since otherwise $\Gamma$ would contain an element of finite order. So for $n' = 2^{r-s+1}n$, the elements $c'$ and $c'(c'c)^{n'}$ generate a dihedral subgroup of $G$ of index $2^r$.

We shall show that there exists either a proper period $m$ or a link period $n$ as above. By the Hurwitz Riemann formula

$$\frac{g-1}{2^{r-1}} = 2^{s-r}\left(\alpha h - 2 + n + \sum_{i=1}^{v}\frac{m_i - 1}{m_i} + \sum_{i=1}^{n}\sum_{j=1}^{s_i}\frac{n_{ij}-1}{2n_{ij}}\right).$$

Thus, since $(g-1)/2^{r-1}$ is odd and $s - r \geq 1$, we see that $m_i = 2^{s-r}$ for some $i$ or $n_{ij} = 2^{s-r-1}$ for some $i, j$. In the first case $G$ contains $Z_{2^{s-r}}$ as a subgroup of index $2^r$, whilst in the second one $G$ contains $D_{2^{s-r-1}}$. This completes the proof. □

THEOREM 4.2. *Let $X$ be a Riemann surface of genus $g$. Assume that $2^{r-1}$ is the maximal power of 2 dividing $g-1$, that is, $g = 2^{r-1}u + 1$ with odd $u$. Suppose there are $k$ nonconjugate symmetries of $X$ with fixed points. Then $k \leq 2^{r+1}$, and furthermore this bound is attained if and only if $u \geq 2^{r+1} - 3$.*

PROOF. The first part is a direct consequence of Lemma 4.1, Lemma 3.1 and Corollary 3.2. Assume then that a Riemann surface $X = \mathscr{H}/\Gamma$ has $2^{r+1}$ nonconjugate symmetries with fixed points. As we remarked at the beginning of the section we can assume that these symmetries generate a 2-group $G$. Let $G = \Lambda/\Gamma$ for some NEC group $\Lambda$, and assume that $\Lambda$ has signature of the general form (1). Let $C_1, \dots, C_n$ be

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all different period cycles of $\Lambda$ involving these reflections, and assume that $C_1, \dots, C_t$ are non-empty and $C_{t+1}, \dots, C_n$ are empty. As each empty period cycle involves at most one reflection we see that $C_1, \dots, C_t$ involve at least $2^{r+1} - (n-t)$ reflections. So if $C_i$ has length $s_i$, then

$$s_1 + \cdots + s_t \geq 2^{r+1} - n + t.$$

We shall show that

$$\mu(\Lambda) \geq 2\pi\left(\frac{2^{r+1}-3}{4} - \frac{2^r}{|G|}\right).$$

If $t = 0$, then $n \geq 2^{r+1}$ and so $\mu(\Lambda) \geq 2\pi(2^{r+1} - 2)$. So assume that $t \geq 1$. Then by Lemma 4.1, $\Lambda$ has a proper period $\geq |G|/2^r$ or a link period $\geq |G|/2^{r+1}$. In the first case

$$\begin{aligned}\mu(\Lambda) &\geq 2\pi\left(-2 + n + \frac{s_1 + \cdots + s_t}{4} + 1 - \frac{2^r}{|G|}\right)\\ &\geq 2\pi\left(\frac{2^{r+1} + 3n + t - 4}{4} - \frac{2^r}{|G|}\right) > \left(\frac{2^{r+1}-3}{4} - \frac{2^r}{|G|}\right).\end{aligned}$$

Now assume that $\Lambda$ has a link period $\geq |G|/2^{r+1}$. Then

$$\begin{aligned}\mu(\Lambda) &\geq 2\pi\left(-2 + n + \frac{s_1 + \cdots + s_t - 1}{4} + \frac{1}{2} - \frac{2^r}{|G|}\right)\\ &\geq 2\pi\left(\frac{2^{r+1} + 3n + t - 7}{4} - \frac{2^r}{|G|}\right) \geq \left(\frac{2^{r+1}-3}{4} - \frac{2^r}{|G|}\right).\end{aligned}$$

So

$$4\pi(g-1) = \mu(\Gamma) = |G|\mu(\Lambda) \geq 2\pi\left(\frac{2^{r+1}-3}{4} - \frac{2^r}{|G|}\right)|G|$$

which gives

$$8(g-1) \geq (2^{r+1} - 3)|G| - 2^{r+2}.$$

Now if $G$ contains $D_N$ as a subgroup of index $2^r$ we obtain that

$$g - 1 \geq 2^{r-2}(2^{r+1} - 3)N - 2^{r-1} = 2^{r-1}\left(\frac{N}{2}\left(2^{r+1} - 3\right) - 1\right).$$

So we see that indeed $g \geq 2^{r-1}u + 1$, where $u = (2^{r+1} - 3)N/2 - 1 \geq 2^{r+1} - 4$. However $u$ is odd by assumption, and so, in particular, $u \geq 2^{r+1} - 3$.

Conversely, let $g = 2^{r-1}u + 1$, where $u \geq 2^{r+1} - 4$ is any odd integer and let $s = 2(u+1)/N + 2$, where $N$ is a power of 2 such that $s \geq 2^{r+1}$. Consider a maximal NEC group $\Lambda$ with signature

$$(0; +; [-]; \{(N, 2, \overset{s}{\ldots}, 2)\}),$$

and let $G = D_N \oplus Z_2^r = \langle y_0, x_0\rangle \oplus \langle x_1\rangle \oplus \cdots \oplus \langle x_r\rangle$. Also let $\{a_1, \ldots, a_{2^{r+1}}\}$ be representatives of all conjugacy classes of elements of order 2 in $G$ which have odd length in $y_0, x_0, \ldots, x_r$, and assume that they are so chosen that $a_1 = y_0$ and $a_2 = x_0$. Define $\theta(e) = 1$ and $\theta(c_0) = \theta(c_{s+1}) = a_1$, $\theta(c_1) = a_2$, and suppose $\theta(c_i) \in \{a_3, \ldots, a_{2^{r+1}}\}$ are chosen for $2 \le i \le s$ so that $\theta(c_i) \ne \theta(c_{i+1})$ and the induced map $\theta : \Lambda \to G$ is an epimorphism; observe that this is indeed possible as $s \ge 2^{r+1}$. Then for $\Gamma = \operatorname{Ker}\theta$, $X = \mathscr{H}/\Gamma$ is a Riemann surface of genus $g$ having $2^{r+1}$ nonconjugate symmetries with fixed points. □

Every even value of $g$ can be written as $2^{r-1}u + 1$ for $r = 1$ and some odd $u$. As a result we obtain as a corollary the principal result of [4].

COROLLARY 4.3. *A Riemann surface of even genus $g$ has at most 4 nonconjugate symmetries with fixed points. Furthermore, this bound is attained for every even genus $g$.*

REMARK 4.4. Given an arbitrary $g \ge 2$, there is an integer $r \ge 1$ and an odd $u$ such that $g = 2^{r-1}u + 1$. Fix $r \ge 1$ and consider all values of $g$ of this form. Observe that these are just solutions of the congruence $x \equiv 2^{r-1} \mod 2^r$. If $u \ge 2^{r+1} - 3$, then $g \ge 2^{r-1}(2^{r+1} - 3) + 1 = 2^{2r} - 3 \times 2^{r-1} + 1 > 2^{2r} - 4 \times 2^{r-1} + 1 = (2^r - 1)^2$. So $2(\sqrt{g} + 1) > 2^{r+1}$ and thus we see that the Natanzon bound is worse than ours for almost all values of $g$ of this form. If $u \le 2^{r+1} - 5$ then $g \le 2^{r-1}(2^{r+1} - 5) + 1 = 2^{2r} - 5 \times 2^{r-1} + 1 < 2^{2r} - 4 \times 2^{r-1} + 1 = (2^r - 1)^2$. So $2(\sqrt{g} + 1) < 2^{r+1}$. Notice, however, that Natanzon's bound is sharp only for $u = 2^{r-3} - 1$.

For arbitrary $g \ge 2$, we define $\nu(g)$ as the maximal number of conjugacy classes with fixed points that can be admitted by a Riemann surface $X$ of genus $g$. The previous theorem can be seen as a determination of the value of $\nu$ for $g = 2^{r-1}u + 1$ where $u \ge 2^{r+1} - 3$; indeed $\nu(g) = 2^{r+1}$ for such $g$. Now we shall calculate the remaining values of this function. Observe that given an odd $u \le 2^{r+2} - 7$ there is a unique $s \le r + 2$ for which $(2^{s-2} - 4)/2^{r-s+2} \le u \le (2^{s-1} - 4)/2^{r-s+1}$.

THEOREM 4.5. *Let $g = 2^{r-1}u + 1$, where $r \ge 1$ and $u \le 2^{r+2} - 7$ is odd. Then*

$$\nu(g) = \begin{cases} 2^{s-1} & \text{if } (2^{s-1} - 4)/2^{r-s+2} \le u \le (2^{s-1} - 4)/2^{r-s+1} \\ 2^{r-s+2}u + 4 & \text{if } (2^{s-2} - 4)/2^{r-s+2} \le u \le (2^{s-1} - 4)/2^{r-s+2}, \end{cases}$$

*where $s$ is as above.*

PROOF. Let $X$ be a Riemann surface of genus $g = 2^{r-1}u + 1$, where $u \le 2^{r+2} - 7$, such that $X$ has $k$ nonconjugate symmetries $\sigma_1, \ldots, \sigma_k$ with fixed points. First we shall show that $k \le \nu(g)$. As before, Sylow theory, we can assume that $\sigma_1, \ldots, \sigma_k$

generate a 2-group $G$, say of order $2^t$. Let $X = \mathscr{H}/\Gamma$ and $G = \Lambda/\Gamma$ for some Fuchsian surface group $\Gamma$ and a proper NEC-group $\Lambda$.

Now it is easy to see that $\mu(\Lambda) \geq \pi(k-4)/2$. So by the Hurwitz Riemann formula $4(g-1) = |G|\mu(\Lambda) \geq 2^{t-1}(k-4)$, which gives

$$k \leq 2^{r-t+2}u + 4. \tag{7}$$

First suppose $(2^{s-1}-4)/2^{r-s+2} \leq u \leq (2^{s-1}-4)/2^{r-s+1}$. Then for $t > s$ we have $k \leq 2^{r-s+1}u + 4 \leq 2^{s-1}$ by (7), whilst for $t \leq s$, we have $k \leq 2^{t-1} \leq 2^{s-1}$ by Lemma 3.1.

Now suppose $(2^{s-2}-4)/2^{r-s+2} \leq u \leq (2^{s-1}-4)/2^{r-s+2}$. Then for $t \geq s$ we have $k < 2^{r-s+2}u + 4$ by (7), whilst for $t < s$, we have $k \leq 2^{t-1} \leq 2^{s-2}$ by Lemma 3.1, and so $k \leq 2^{r-s+2}u + 4$, since $2^{r-s+2}u + 4 \geq 2^{s-2}$ for $u \geq (2^{s-2}-4)/2^{r-s+2}$.

In order to finish the proof we consider arbitrary $s \leq r+2$ and arbitrary $u$ in the range $(2^{s-2}-4)/2^{r-s+2} \leq u \leq (2^{s-1}-4)/2^{r-s+1}$. Let $G = Z_2^s = \langle x_1 \rangle \oplus \ldots \oplus \langle x_s \rangle$. Let $A = \{a_1, \ldots, a_{2^{s-1}}\}$ be the set of all elements of order 2 which can be written as words of odd length in $x_1, \ldots, x_s$. Finally let $k = 2^{r-s+2}u + 4$ and let $\Lambda$ be a maximal NEC group with the signature $(0; +; [-]; \{(2, \overset{k}{\ldots}, 2)\})$. Then there exists an epimorphism $\theta : \Lambda \to G$ such that $\theta(c_i) \neq \theta(c_{i+1})$ and such that for $k \leq 2^{s-1}$, the $k$ reflections $c_1, \ldots, c_k$ are mapped into distinct elements of $A$ (observe that the last condition holds if and only if $u \leq (2^{s-1}-4)/2^{r-s+2}$). □

Given an arbitrary $g \geq 2$, let $\omega(g)$ denote the maximal number of real forms with real points that some complex algebraic curve of genus $g$ can admit. Observe that $\omega(g) = \nu(g)$ for arbitrary $g$. So we can express Theorem 4.2 and Theorem 4.5 in terms of algebraic curves as follows

COROLLARY 4.6. *Let $g = 2^{r-1}u + 1$, where $r \geq 1$ and $u$ is odd. Then*

$$\omega(g) = \begin{cases} 2^{r+1} & \text{if } u \geq 2^{r+2} - 5 \\ 2^{s-1} & \text{if } (2^{s-1}-4)/2^{r-s+2} \leq u \leq (2^{s-1}-4)/2^{r-s+1} \\ 2^{r-s+2}u + 4 & \text{if } (2^{s-2}-4)/2^{r-s+2} \leq u \leq (2^{s-1}-4)/2^{r-s+2}, \end{cases}$$

*where $s \leq r+2$ is the unique integer for which*

$$(2^{s-2}-4)/2^{r-s+2} \leq u \leq (2^{s-1}-4)/2^{r-s+1}.$$

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