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CHARACTER CLUSTERS FOR LIE ALGEBRA MODULES OVER A FIELD OF NONZERO CHARACTERISTIC

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Abstract

For a Lie algebra L over an algebraically closed field F of nonzero characteristic, every finite dimensional L-module can be decomposed into a direct sum of submodules such that all composition factors of a summand have the same character. Using the concept of a character cluster, this result is generalised to fields which are not algebraically closed. Also, it is shown that if the soluble Lie algebra L is in the saturated formation $\mathfrak{F}$ and if V, W are irreducible L-modules with the same cluster and the p-operation vanishes on the centre of the p-envelope used, then V, W are either both $\mathfrak{F}$-central or both $\mathfrak{F}$-eccentric. Clusters are used to generalise the construction of induced modules.

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1. Introduction

Lie algebras are very important and have been actively investigated by many authors. See [3–6] for examples of recent results.

Throughout this paper, L is a finite dimensional Lie algebra over the field F of characteristic $p \neq 0$. Let V be a finite dimensional L-module. To define a character for V, we must embed L in a p-envelope $(L^p, [p])$. The action ρ of L on V can be extended to L^p. (See Strade and Farnsteiner [7, Theorem 5.1.1].)

Definition 1.1. A character for V is a linear map $c : L^p \to F$ such that for all $x \in L^p$,

$$\rho(x)^p - \rho(x^{[p]}) = c(x)^p 1.$$

Not every module has a character, but if F is algebraically closed and V is irreducible, then V has a character. (See Strade and Farnsteiner [7, Theorem 5.2.5].) The following is Strade and Farnsteiner [7, Theorem 5.2.6].

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THEOREM 1.2. *Suppose that F is algebraically closed and let $(L, [p])$ be a restricted Lie algebra over F. Let V be a finite dimensional L-module. Then there exist $c_i : L \to F$ and submodules V_i such that $V = \bigoplus_i V_i$ and every composition factor of V_i has character c_i.*

This decomposition in terms of characters is functorial and is clearly useful. In this note, the concept of a character cluster is used to obtain a similar result which does not require the field to be algebraically closed. As a further application, it is shown that, if the soluble Lie algebra L is in the saturated formation $\mathfrak{F}$ and V, W are irreducible L-modules with the same cluster and the p-operation vanishes on the centre of the p-envelope used, then either both V, W are $\mathfrak{F}$-central or both are $\mathfrak{F}$-eccentric. Over a perfect field, clusters are used to generalise the construction of induced modules.

To simplify the exposition, we work with a restricted Lie algebra $(L, [p])$. To apply the results to a general Lie algebra, as is the case for characters, we have to embed the algebra in a p-envelope, and the clusters obtained depend on that embedding.

2. Preliminaries

In the following, $(L, [p])$ is a restricted Lie algebra over the field F, $\bar{F}$ is the algebraic closure of F and $\bar{L} = \bar{F} \otimes_F L$ is the algebra obtained by extension of the field. A character of L is an F-linear map $c : L \to \bar{F}$. If $\{e_1, \ldots, e_n\}$ is a basis of L, then c can be expressed as a linear form $c(x) = \sum a_i x_i$ for $x = \sum x_i e_i$, where $a_i \in \bar{F}$. If α is an automorphism of $\bar{F}/F$, that is, an automorphism of $\bar{F}$ which fixes all elements of F, then c^α is the character $c^\alpha(x) = \sum a_i^\alpha x_i$ and is called a conjugate of c. We do not distinguish in notation between $c : L \to \bar{F}$ and its linear extension $\bar{L} \to \bar{F}$. We denote by $F[c]$ the field $F[a_1, \ldots, a_n]$ generated by the coefficients a_i. It is the field generated by the $c(x)$ for all $x \in L$ and is independent of the choice of basis.

If V is an L-module, then $\bar{V}$ is the $\bar{L}$-module $\bar{F} \otimes_F V$. The action of $x \in L$ on V is denoted by $\rho(x)$. The module V has character c if $(\rho(x)^p - \rho(x^{[p]}))v = c(x)^p v$ for all $x \in L$ and all $v \in V$.

In the universal enveloping algebra $U(L)$, the element $x^p - x^{[p]}$ is central. (See Strade and Farnsteiner [7, page 203].) For the module V giving the representation ρ, we put $\phi_x = \rho(x)^p - \rho(x^{[p]})$. We then have $[\phi_x, \rho(y)] = 0$ for all $x, y \in L$.

LEMMA 2.1. *The map $\phi : L \to \mathrm{End}(V)$ defined by $\phi_x(v) = (\rho(x^p) - \rho(x^{[p]}))v$ is p-semilinear.*

PROOF. In the universal enveloping algebra $U(L)$,

$$(a + b)^p = a^p + b^p + \sum_{i=1}^{p-1} s_i(a, b)$$

(see Strade and Farnsteiner [7, page 62, Equation (3)]) and

$$(a + b)^{[p]} = a^{[p]} + b^{[p]} + \sum_{i=1}^{p-1} s_i(a, b)$$

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(see Strade and Farnsteiner [7, page 64, Property (3)]). Putting these together,

$$(a+b)^p - (a+b)^{[p]} = a^p + b^p - a^{[p]} - b^{[p]}.$$

It follows that $\phi_{a+b} = \phi_a + \phi_b$. Clearly, $\phi_{\lambda a} = \lambda^p \phi_a$. □

Remark 2.2. In the decomposition of $\bar{V}$ given by Theorem 1.2, the summand corresponding to the character c is

$$\{v \in \bar{V} \mid (\phi_x - c(x)^p 1)^r v = 0 \text{ for some } r \text{ and all } x \in \bar{L}\}.$$

By Lemma 2.1, we need only consider those $x \in L$, or indeed, in some chosen basis of L.

3. Clusters

Definition 3.1. The cluster $\mathrm{Cl}(V)$ of an L-module V is the set of characters of the composition factors of the $\bar{L}$-module $\bar{V} = \bar{F} \otimes_F V$.

Lemma 3.2. *Suppose $c \in \mathrm{Cl}(V)$. Then the conjugates c^α of c are in* $\mathrm{Cl}(V)$.

Proof. Let A/B be a composition factor of $\bar{V}$ and let $\{v_1, \ldots, v_k\}$ be a basis of V. The action $\rho(x)$ of $x \in L$ on V and so also on $\bar{V}$ is given in respect to this basis by a matrix X with coefficients in F. An automorphism α maps $v = \lambda_1 v_1 + \cdots + \lambda_k v_k$ to $v^\alpha = \lambda_1^\alpha v_1 + \cdots + \lambda_k^\alpha v_k$. Since $X^\alpha = X$, we have that $(xv)^\alpha = xv^\alpha$. Thus A^α, B^α are submodules of $\bar{V}$ and A^α/B^α is a composition factor. The linear map $\phi_x = \rho(x)^p - \rho(x^{[p]})$ also commutes with α. Thus from $\phi_x(a+B) = c(x)^p a + B$, it follows that $\phi_x(a^\alpha) + B^\alpha = c^\alpha(x)^p a^\alpha + B^\alpha$. Thus $c^\alpha \in \mathrm{Cl}(V)$. □

The statement $(xv)^\alpha = xv^\alpha$ may suggest that A/B and A^α/B^α are isomorphic. They are not. The map $v \mapsto v^\alpha$ is not linear, as $(\lambda v)^\alpha = \lambda^\alpha v^\alpha$.

By Lemma 3.2, a cluster $\mathrm{Cl}(V)$ is a union of conjugacy classes of characters.

Definition 3.3. A cluster $\mathrm{Cl}(V)$ is called simple if it consists of a single conjugacy class of characters.

Theorem 3.4. *Let V be an irreducible L-module. Then* $\mathrm{Cl}(V)$ *is simple.*

Proof. Notice that $\bar{V} = \bar{F} \otimes_F V$ has a direct decomposition $\bar{V} = \sum_c \bar{V}_c$, where the component $\bar{V}_c$ is, by Remark 2.2, the space

$$\{v \in \bar{V} \mid (\phi_x - c(x)^p 1)^r v = 0 \text{ for all } x \in L \text{ and some } r\}.$$

Here, we may take for r the length of a composition series of $\bar{V}_c$, which is independent of x. Let $c \in \mathrm{Cl}(V)$. Let $\bar{V}_0 = \sum_\alpha \bar{V}_{c^\alpha}$, where the sum is over the distinct conjugates c^α. Let $f_x(t) = \Pi_\alpha (t - c^\alpha(x)^p)$. The coefficients of $f_x(t)$ are invariant under the automorphisms of $\bar{F}/F$. Therefore for some k, we have that $f_x(t)^{p^k}$ is a polynomial over F. As the field is not assumed to be perfect, this may require $k > 0$. Let $m_x(t)$

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be the least power of $f_x(t)$ which is a polynomial over F. Then, with r the length of a composition series of $\bar{V}_c$,

$$\bar{V}_0 = \{v \in \bar{V} \mid m_x(\phi_x)^r v = 0 \text{ for all } x \in L\}.$$

The condition $m_x(\phi_x)^r v = 0$ for all $x \in L$ may be regarded as a set of linear equations over F in the coordinates of v. These equations have a nonzero solution over $\bar{F}$ since $\bar{V}_c \neq 0$. Therefore, they have a nonzero solution over F, that is,

$$V_0 = \{v \in V \mid m_x(\phi_x)^r v = 0 \text{ for all } x \in L\} \neq 0.$$

Since the ρ_y commute with the ϕ_x, V_0 is a submodule of V. Therefore, $V_0 = V$ and it follows that the set of conjugates of c is the whole of $\mathrm{Cl}(V)$. □

4. The cluster decomposition

We have seen that if $c \in \mathrm{Cl}(V)$, then every conjugate c^α of c is in $\mathrm{Cl}(V)$. It is convenient to expand our terminology and call any finite set C of linear maps $c : L \to \bar{F}$ a cluster if, for each $c \in C$, all conjugates of c are in C. With this expansion of our terminology, every cluster is a union of simple clusters.

Theorem 4.1. *Let $(L, [p])$ be a restricted Lie algebra and let V be an L-module. Suppose that $\mathrm{Cl}(V)$ is the union $C_1 \cup \cdots \cup C_k$ of the distinct simple clusters C_i. Then $V = V_1 \oplus \cdots \oplus V_k$ with submodules V_i such that $\mathrm{Cl}(V_i) = C_i$.*

Proof. By Theorem 1.2, $\bar{V}$ is the direct sum over the set of characters c of submodules $\bar{V}_c$ whose composition factors all have character c. By Remark 2.2, $\bar{V}_c$ is the space annihilated by some sufficiently high power of $(\phi_x - c(x)^p 1)$ for all $x \in L$.

Suppose $c \in C_i$. Put $\bar{V}_i = \sum_\alpha \bar{V}_{c^\alpha}$. Some power $m_x(t)$ of $\Pi_\alpha(t - c^\alpha(x)^p)$ is a polynomial over F, and $\bar{V}_i$ is the space annihilated by $m_x(\phi_x)^r$ for all $x \in L$ and some sufficiently large r. Put

$$V_i = \{v \in V \mid m_x(\phi_x)^r v = 0 \text{ for all } x \in L\}.$$

The set of conditions $m_x(\phi_x)^r v = 0$ for all $x \in L$ may be regarded as a set of linear equations over F in the coordinates of v, so the F-dimension of its solution space V_i in V is equal to the $\bar{F}$-dimension of its solution space $\bar{V}_i$ in $\bar{V}$. It follows that $V = \bigoplus_i V_i$. Clearly, V_i is a submodule of V and $\mathrm{Cl}(V_i) = C_i$. □

Theorem 4.2. *Suppose that $S \lhd\!\lhd L$ and let V be an L-module. Then the components of the cluster decomposition $V = \bigoplus_C V_C$ with respect to S are L-submodules.*

Proof. Although S need not be a restricted algebra, it is embedded in the restricted algebra $(L, [p])$ and the components are defined using the operation $[p]$. There exists a series $S = S_0 \lhd S_1 \lhd \cdots \lhd S_n = L$. We use induction over i to prove that V_C is an S_i-module. Take $x \in S_i$ and consider $(xV_C + V_C)/V_C$. For $s \in S$ and $v \in V_C$, we have $s(xv) = x(sv) + [s, x]v$. But $[s, x] \in S_{i-1}$, so $[s, x]v \in V_C$. Thus the

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map $v \mapsto xv + V_C \in (xV_C + V_C)/V_C$ is an S-module homomorphism. Thus the character of every composition factor of $\bar{F} \otimes_F ((xV_C + V_C)/V_C)$ is in C, which implies that $xV_C \subseteq V_C$. □

Remark 4.3. The decomposition given by Theorem 4.2 depends on the p-operation, not merely on the algebra S. Changing the p-operation may change the decomposition, as is shown by the following example. This opens the possibility that, where the minimal p-envelope of S has nontrivial centre, judicious variation of the p-operation may give useful different direct decompositions.

Example 4.4. Let $L = \langle a_1, a_2 \mid [a_1, a_2] = 0\rangle$ and let $V = \langle v_1, v_2\rangle$ with $a_i v_i = v_i$ and $a_i v_j = 0$ for $i \neq j$. With $a_1^{[p]} = 0$ and $a_2^{[p]} = -a_1$, V has the character c with $c(a_1) = 0$ and $c(a_2) = 1$. The cluster decomposition with respect to $(L, [p])$ is simply $V = V_c$. However, with the p-operation $[p]'$ with $a_i^{[p]'} = 0$, the submodule $\langle v_1\rangle$ has character c_1 with $c_1(a_1) = 1$ and $c_1(a_2) = 0$, while $\langle v_2\rangle$ has character c_2 with $c_2(a_1) = 0$ and $c_2(a_2) = 1$. This gives the cluster decomposition $V = V_{c_1} \oplus V_{c_2}$.

5. $\mathfrak{F}$-central and $\mathfrak{F}$-eccentric modules

Let $\mathfrak{F}$ be a saturated formation of soluble Lie algebras over F. Comparing Theorem 4.2 with [2, Lemma 1.1] suggests a further relationship between clusters and saturated formations beyond that of [2, Theorem 6.4].

Theorem 5.1. *Let $\mathfrak{F}$ be a saturated formation and suppose $S \in \mathfrak{F}$. Let $(L, [p])$ be a p-envelope of S and suppose that $z^{[p]} = 0$ for all z in the centre of L. Let V, W be irreducible S-modules. Suppose that $\mathrm{Cl}(V) = \mathrm{Cl}(W)$. Then V, W are either both $\mathfrak{F}$-central or both $\mathfrak{F}$-eccentric.*

Proof. Suppose to the contrary, that V is $\mathfrak{F}$-central and that W is $\mathfrak{F}$-eccentric. By [1, Theorem 2.3], $\mathrm{Hom}(V, W)$ is $\mathfrak{F}$-hypereccentric. But from [7, Theorem 5.2.7] it follows that the characters of the composition factors of $\mathrm{Hom}(\bar{V}, \bar{W})$ are all of the functions $c_2 - c_1$ where $c_1 \in \mathrm{Cl}(V)$ and $c_2 \in \mathrm{Cl}(W)$. Since $\mathrm{Cl}(V) = \mathrm{Cl}(W)$, we have that $0 \in \mathrm{Cl}(\mathrm{Hom}(V, W))$.

By assumption, we have that $z^{[p]} = 0$ for all z in the centre of L. As $(L, [p])$ is a p-envelope of S, we have $S \trianglelefteq L$. By [2, Theorem 6.4], a composition factor X of $\mathrm{Hom}(V, W)$ with $\mathrm{Cl}(X) = \{0\}$ is $\mathfrak{F}$-central, contrary to $\mathrm{Hom}(V, W)$ being $\mathfrak{F}$-hypereccentric. □

6. C-induced modules

Let $(L, [p])$ be a restricted Lie algebra over the perfect field F and let S be a $[p]$-subalgebra of L. Let W be an S-module and let C be a cluster of characters of L whose restriction to S is $\mathrm{Cl}(W)$. We require that distinct members of C have distinct restrictions to S, in which case we say that C restricts simply to S.

Note that, given a simple cluster C_S of S, we can easily construct a cluster C of L which restricts simply to C_S. We take a cobasis $\{e_1, \ldots, e_n\}$ of S in L, that is,

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a basis of some subspace complementary to S. A character $c : S \to \bar{F}$ can be extended to L by assigning arbitrarily the values $c(e_i) \in \bar{F}$. If these are chosen in $F[c]$, then any automorphism which fixes the given c also fixes its extension.

We want to apply the construction of c-induced modules (see Strade and Farnsteiner [7, Section 5.6]) to the c-components $\bar{W}_c$ of $\bar{W}$. This construction only works for modules with character c. Every compostion factor of $\bar{W}_c$ has character c, but $\bar{W}_c$ itself need not. This leads to the following definition.

Definition 6.1. We say that the S-module W is amenable (for induction) if, for all $c \in \mathrm{Cl}(W)$, $\bar{W}_c$ has character c.

Note that if $\bar{W}_c$ has character c, then for each conjugate c^α of c, $\bar{W}_{c^\alpha}$ has character c^α. It would be nice to have a way of determining if a module W is amenable which does not require analysis of $\bar{W}$. The following lemmas achieve that.

Lemma 6.2. *Let $\{s_1, \ldots, s_n\}$ be a basis of S and let W be an S-module. Let $m_i(t)$ be the minimum polynomial of ϕ_{s_i}. Then W is amenable if and only if for all i, $\gcd(m_i(t), m_i'(t)) = 1$.*

Proof. The module W is amenable if and only if, for all $c \in \mathrm{Cl}(W)$ and all i, we have $(\phi_{s_i} - c(s_i)^p 1)\bar{W}_c = 0$. So W is amenable if and only if for all i, in $\bar{F}[t]$, $m_i(t)$ has no repeated factors, that is, if and only if $\gcd(m_i(t), m_i'(t)) = 1$. As the calculation of $\gcd(m_i(t), m_i'(t))$ in $F[t]$ is the same as in $\bar{F}[t]$, the result follows. □

Lemma 6.3. *Let W be an irreducible S-module. Then W is amenable.*

Proof. For any $s \in S$, $s^p - s^{[p]}$ is in the centre of the universal enveloping algebra of S and so, for any representation ρ of S, we have $[\rho(s_1)^p - \rho(s_1^{[p]}), \rho(s_2)] = 0$ for all $s_1, s_2 \in S$. For $c \in \mathrm{Cl}(W)$, put $f_s(t) = \Pi(t - c^\alpha(s)^p)$ where the product is taken over the distinct conjugates of $c(s)$. Then $f_s(t)$ is a polynomial over F, and $\rho(s_2)$ commutes with $f_{s_1}(\phi_{s_1})$ for all $s_1, s_2 \in S$. Thus $W_0 = \{w \in W \mid f_s(\phi_s)w = 0 \text{ for all } s \in S\}$ is a submodule of W. The conditions $f_s(\phi_s)w = 0$ are linear equations over F with nonzero solutions over $\bar{F}$ and so have nonzero solutions over F. Thus $W_0 \neq 0$, which implies $W_0 = W$. □

As the construction being developed can be applied separately to each direct summand of W, we suppose that C is simple. Take a basis $\{b^1, \ldots, b^k\}$ of W. Corresponding to each $c \in C$, we have a component $\bar{W}_c$ of $\bar{W} = \bigoplus_\alpha \bar{W}_{c^\alpha}$. For each $w \in \bar{W}$, we have $w = \sum_c w_c$ with $w_c \in \bar{W}_c$.

Lemma 6.4. *Let $w = \sum \lambda_i b^i \in \bar{W}$. Then w is invariant under the automorphisms of $\bar{F}/F$ if and only if the $\lambda_i \in F$, in which case, $(w_c)^\alpha = w_{c^\alpha}$. Further, sw is also invariant for all $s \in S$.*

Proof. If $w = \sum \lambda_i b^i$ is invariant, then λ_i is invariant. Since F is perfect, this implies $\lambda_i \in F$. If $\lambda_i \in F$ for all i, then clearly w is invariant. As W is an S-module, also sw is invariant. The action of α permutes the $\bar{W}_c$ and does not change the direct decomposition. It follows that $(w_c)^\alpha = w_{c^\alpha}$. □

Suppose that C is a simple cluster of characters of L which restricts simply to S and that W is an amenable S-module with $\mathrm{Cl}(W) = C|S$. For each $c \in C$, we form the c-reduced enveloping algebras $u(L, c)$ and $u(S, c)$. (See Strade and Farnsteiner [7, page 226].) Since W is amenable, we can construct the c-induced $\bar{L}$-modules

$$\bar{V}_c = \mathrm{Ind}_{\bar{S}}^{\bar{L}}(\bar{W}_c, c) = u(\bar{L}, c) \otimes_{u(\bar{S},c)} \bar{W}_c$$

and put $\bar{V} = \bigoplus_c \bar{V}_c$. From $\bar{V}$, we shall select an F-subspace V with $\bar{F} \otimes_F V = \bar{V}$, which we shall show to be an L-module with $\mathrm{Cl}(V) = C$.

For $x, y \in L$, in the following, we need to distinguish their product in the associative algebra $u(L, c)$ from their product in the Lie algebra. We denote the Lie algebra product by $[x, y]$. Take a cobasis $\{e_1, \dots, e_n\}$ for S in L. Then the elements $e_1^{r_1} e_2^{r_2} \cdots e_n^{r_n} \otimes w_c$ with $r_i \le p - 1$ and $w_c \in \bar{W}_c$ span $\bar{V}_c$. To simplify the notation, we write $e(r)$ for $e_1^{r_1} e_2^{r_2} \cdots e_n^{r_n}$. For an element $w = \sum_c w_c \in \bar{W}$, it is convenient to abuse notation and write $e(r) \otimes w$ for the element $\sum_c e(r) \otimes w_c$. It should be remembered that in this sum, the $e(r)$ come from different algebras $u(\bar{L}, c)$ with different multiplication, and that the tensor products are over different algebras $u(\bar{S}, c)$.

Any element $w_c \in \bar{W}_c$ is an $\bar{F}$-linear combination of the b^i, so an element of $\bar{V}_c$ is expressible as an $\bar{F}$-linear combination of the $e(r) \otimes b^i$. It follows that the $e(r) \otimes b^i$ form a basis of $\bar{V}$. An automorphism α maps $e(r) \otimes w$ to $e(r) \otimes w^\alpha$. Thus the invariant elements of $\bar{V}$ are the F-linear combinations of the basis.

Lemma 6.5. *Let $v \in \bar{V}$ be invariant. Then xv is invariant for all $x \in L$.*

Proof. We use induction over k to show that $x_1 \cdots x_k \otimes b^i$ is invariant for all $x_1, \dots, x_k \in L$. The result then follows trivially.

For $s \in S$, we have $s(1 \otimes b^i) = 1 \otimes sb^i$, which is invariant by Lemma 6.4. For e_j, we have $e_j(1 \otimes b^i) = e_j \otimes b^i$, which is invariant. Note that in this case, the multiplication is the same in all the $u(\bar{L}, c)$. Thus the result holds for $k = 1$.

Suppose that $k > 1$. We express each of the x_t as a linear combination of the e_j and an element of S. We then use the commutation rules $xy - yx = [x, y]$ to move each factor to its correct position, giving a sum of terms of the form $e_1^{r_1} e_2^{r_2} \cdots e_n^{r_n} s_1 \cdots s_m \otimes b^i$, but with the r_j not restricted to be less than p. The terms coming from a commutator $[x, y]$ all have fewer than k factors and so are invariant. Any elements of S at the end move past the tensor product, giving $e_1^{r_1} e_2^{r_2} \dots e_n^{r_n} \otimes s_1 \cdots s_m b^i$. By Lemma 6.4, $1 \otimes s_1 \cdots s_m b^i$ is invariant and since, in this case, $e_1^{r_1} \cdots e_n^{r_n}$ has fewer than k factors, the term is invariant. Thus we are left to consider terms of the form $e_1^{r_1} \cdots e_n^{r_n} \otimes b^i$. If $r_j < p$ for all j, then the term is one of our basis elements and so is invariant.

Suppose that for some j, we have $r_j \ge p$. Then we must separate the summands. The term can be written in the form $ee_j^p e' \otimes b^i$, where e, e' are strings of cobasis elements. In the algebra $u(\bar{L}, c^\alpha)$, we have $e_j^p = e_j^{[p]} + c^\alpha(e_j)^p 1$. Thus

$$ee_j^p e' \otimes b^i = ee_j^{[p]} e' \otimes b^i + \sum_\alpha ee' \otimes c^\alpha(e_j)^p b^i_{c^\alpha}.$$

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But $ee_j^{[p]}e' \otimes b^i$ has fewer than k factors and so is invariant. As $\sum_\alpha 1 \otimes c^\alpha(e_j)^p b^i_{c^\alpha}$ is invariant and ee' has fewer than k factors, it follows that $ee_j^p e' \otimes b^i$ also is invariant. □

We define the C-induced module $\mathrm{Ind}_S^L(W, C)$ to be the F-subspace of invariant elements of $\bar{V}$. By Lemma 6.5, it is an L-module. Clearly $\mathrm{Cl}(\mathrm{Ind}_S^L(W, C)) = C$.

To illustrate this, we calculate a simple example.

EXAMPLE 6.6. Let F be the field of three elements and let $L = \langle x, y \mid [x, y] = y\rangle$. Putting $x^{[p]} = x$ and $y^{[p]} = 0$ makes this a restricted Lie algebra. We take $S = \langle x\rangle$ and $W = \langle b^1, b^2\rangle$ with $xb^1 = b^2$ and $xb^2 = -b^1$. Over $\bar{F}$, we have $\bar{W} = \bar{W}_1 \oplus \bar{W}_2$ with $\bar{W}_1 = \langle -b^1 - ib^2\rangle$ and $\bar{W}_2 = \langle -b^1 + ib^2\rangle$, where $i \in \bar{F}$, $i^2 = -1$. Denote the action of S on W by ρ. Then $\rho(x)(-b^1 - ib^2) = -i(-b^1 - ib^2)$ and $\rho(x)(-b^1 + ib^2) = i(-b^1 + ib^2)$.

We have $(\rho(x)^p - \rho(x^{[p]}))(-b^1 - ib^2) = ((-i)^3 - (-i))(-b^1 - ib^2) = -i(-b^1 - ib^2)$. Thus the character c_1 of $\bar{W}_1$ must have $c_1(x)^3 = -i$, so $c_1(x) = i$. Similarly, we have $c_2(x) = -i$. As distinct conjugates of a character on L in C must have distinct restrictions to S, $c_1(y) \in F[i]$. Put $c_1(y) = \lambda = \alpha + i\beta$, where $\alpha, \beta \in F$. Then $c_2(y) = \bar{\lambda}$. Note that in $u(\bar{L}, c_1)$, $y^3 = \lambda^3 = \bar{\lambda}$. In both the algebras, $xy = y + yx$ and $xy^2 = (y + yx)y = y^2 + y(xy) = -y^2 + y^2x$.

In the notation used above, we have $b^1_{c_1} = -b^1 - ib^2$ and $b^1_{c_2} = -b^1 + ib^2$, while for b^2, we have $b^2_{c_1} = ib^1 - b^2$ and $b^2_{c_2} = -ib^1 - b^2$. The induced module $V = \mathrm{Ind}_S^L(W, C)$ has basis the six elements $v_j^r = y^r \otimes b^j$ for $r = 0, 1, 2$ and $j = 1, 2$. We calculate the actions of x, y on these elements:

$$\begin{aligned}
xv_1^0 &= x(1 \otimes b^1) = v_2^0, & xv_2^0 &= x(1 \otimes b^2) = -v_1^0,\\
xv_1^1 &= (y + yx) \otimes b^1 = v_1^1 + v_2^1, & xv_2^1 &= (y + yx) \otimes b^2 = v_2^1 - v_1^1,\\
xv_1^2 &= (-y^2 + y^2x) \otimes b^1 = -v_1^2 + v_2^2, & xv_2^2 &= (-y^2 + y^2x) \otimes b^2 = -v_2^2 - v_1^2,\\
yv_1^0 &= v_1^1, & yv_2^0 &= v_2^1,\\
yv_1^1 &= v_1^2, & yv_2^1 &= v_2^2.
\end{aligned}$$

The calculations of yv_j^2 are more complicated:

$$\begin{aligned}
yv_1^2 &= y^3 \otimes (b^1_{c_1} + b^1_{c_2}) = 1 \otimes (\bar{\lambda}b^1_{c_1} + \lambda b^1_{c_2})\\
&= 1 \otimes ((\alpha - i\beta)(-b^1 - ib^2) + (\alpha + i\beta)(-b^1 + ib^2))\\
&= 1 \otimes (\alpha b^1 + \beta b^2) = \alpha v_1^0 + \beta v_2^0,\\
yv_2^2 &= y^3 \otimes (b^2_{c_1} + b^2_{c_2}) = 1 \otimes (\bar{\lambda}b^1_{c_2} + \lambda b^2_{c_2})\\
&= 1 \otimes ((\alpha - i\beta)(ib^1 - b^2) + (\alpha + i\beta)(-ib^1 - b^2))\\
&= 1 \otimes (-\beta b^1 + \alpha b^2) = -\beta v_1^0 + \alpha v_2^0.
\end{aligned}$$

REMARK 6.7. As noted earlier, in the notation used above, if we are given an amenable S-module W with simple cluster C_S, we can construct a simple cluster C on L which restricts simply to C_S by choosing arbitrarily the $c_1(e_i)$ in $F[c_1]$. If we choose the $c_1(e_i)$ in F, then we have $c_j(e_i) = c_1(e_i)$ for all j. This simplifies the calculation of the action on the induced module as we then have $e_i^p = e_i^{[p]} + c(e_i)^p 1$ in all the algebras $u(\bar{L}, c_i)$ and it follows that $e_i^p b^j = (e_i^{[p]} + c(e_i)^p 1)b^j$ can be calculated without using the character decomposition of $\bar{W}$. That the action of $x \in L$ on a basis element $e(r) \otimes b^i$ can be calculated without using the decomposition follows by an induction as in the proof of Lemma 6.5. It thus becomes possible to calculate the action on $\mathrm{Ind}_S^L(W, C)$ without having to determine the eigenvalues of the ϕ_{s_i}. In the above example, if we take $\beta = 0$, then the calculations of yv_1^2 and yv_2^2 simplify to $yv_i^2 = y^3 b^i = \alpha^3 b^i = \alpha v_i^0$.

Remark 6.8. Denote the category of amenable L-modules with the cluster C by $\mathrm{AmMod}(L, C)$. The restriction functor $\mathrm{Res}_L^S : \mathrm{AmMod}(L, C) \to \mathrm{AmMod}(S, C|S)$ sends an L-module V to V regarded as an S-module. Suppose that C restricts simply to S. Then $\mathrm{Ind}_S^L(\ , C)$ is a functor $\mathrm{AmMod}(S, C|S) \to \mathrm{AmMod}(L, C)$. In the special case where $C = \{c\}$, the functor $\mathrm{Ind}_S^L(\ , c)$ is a left adjoint to Res_L^S by Strade and Farnsteiner [7, Theorem 5.6.3]. Applying this to the $\bar{W}_c$ in the general case gives that $\mathrm{Ind}_S^L(\ , C)$ is a left adjoint to Res_L^S.

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