J. Austral. Math. Soc. Ser. B 26 (1985), 503–516

AN UNCOUPLING PROCEDURE FOR A CLASS OF COUPLED LINEAR PARTIAL DIFFERENTIAL EQUATIONS

A. MCNABB[1]

(Received 14 June 1983; revised 21 December 1983)

Abstract

A Fredholm operator exists which maps the solutions of a system of linear partial differential equations of the form $\partial u/\partial t = DLu + Au$ coupled by a matrix A onto those solutions of a similar system coupled by a matrix B which have the same initial values. The kernels of this operator satisfy a hyperbolic system of equations. Since these equations are independent of the linear partial differential operator L, the same operator serves as a mapping for a large class of equations. If B is chosen diagonal, the solutions of a coupled system with matrix A may be obtained from the uncoupled system with matrix B.

1. Introduction

Hill [3] considered the coupled system

$$\frac{\partial u_1}{\partial t} = d_1 L u_1 - a_1 u_1 + b_1 u_2, \quad u_1(x,0) = f_1(x),$$

$$\frac{\partial u_2}{\partial t} = d_2 L u_2 + b_2 u_1 - a_2 u_2, \quad u_2(x,0) = f_2(x),$$

where L denotes a linear constant coefficient differential operator involving spatial derivatives only. He showed that if h_1, h_2 are solutions of the uncoupled system,

$$\frac{\partial h_i}{\partial t} = L h_i, \quad h_i(x,0) = f_i(x),$$

[1] Applied Mathematics Divsion, Department of Scientific and Industrial Research, Wellington, New Zealand.

https://doi.org/10.1017/S0334270000004689 Published online by Cambridge University Press

then a solution of the coupled system is given by

$$u_1(x,t) = e^{-a_1 t}h_1(x, d_1 t) + \frac{e^{\lambda t}}{(d_1 - d_2)}\int_{d_2 t}^{d_1 t} e^{-\mu\xi} \times \left[\left\{\frac{b_1 b_2(\xi - d_2 t)}{(d_1 t - \xi)}\right\}^{1/2} I_1(\eta)h_1(x,\xi) + b_1 I_0(\eta)h_2(x,\xi)\right] d\xi,$$

$$u_2(x,t) = e^{-a_2 t}h_2(x, d_2 t) + \frac{e^{\lambda t}}{(d_1 - d_2)}\int_{d_2 t}^{d_1 t} e^{-\mu\xi} \times \left[\left\{\frac{b_1 b_2(d_1 t - \xi)}{(\xi - d_2 t)}\right\}^{1/2} I_1(\eta)h_2(x,\xi) + b_2 I_0(\eta)h_1(x,\xi)\right] d\xi,$$

where

$$\lambda = (a_1 d_2 - a_2 d_1)/(d_1 - d_2), \quad \mu = (a_1 - a_2)/(d_1 - d_2),$$
$$\eta = 2[b_1 b_2(d_1 t - \xi)(\xi - d_2 t)]^{1/2}/(d_1 - d_2),$$

and I_0 and I_1 are modified Bessel functions.

A matrix formulation and generalisation of this two variable case is studied in this paper. Suppose h is a mapping from R^{m+1} to R^N satisfying

$$\begin{aligned} \frac{\partial h}{\partial t} &= DLh \quad \text{in } \{(x,t): x \in \Omega \text{ in } R^m, 0 < t < T\}, \\ h(x,0) &= f(x), \quad x \in \Omega, \end{aligned} \tag{1.1}$$

where D is a constant nonsingular diagonal matrix and the operator L commutes with any $B \in S_N(0, T_0)$, the set of regulated mappings from $(0, T)$ into $N \times N$ matrices with real ijth element B_{ij} [1]. We show that, given $A \in S_N(0, T_0)$, there is a $J \in S_N(0, T_0)$, and a pair of kernels $k^+(t,s)$, $k^-(t,s)$ such that

$$\begin{aligned} u(x,t) &= Jh(x,t) + \int_0^t k^+(t,s)h(x,s)\,ds + \int_t^{T_0} k^-(t,s)h(x,s)\,ds \\ &\equiv (J + K)h(x,t). \end{aligned} \tag{1.2}$$

is a solution of the equations

$$\frac{\partial u}{\partial t} = DLu + Au, \quad u(x,0) = f(x). \tag{1.3}$$

If it is assumed such an operator $J + K$ exists, then equations (1.1), (1.2) and (1.3) imply J must satisfy an ordinary differential equation and commute with D, while k^+ and k^- satisfy a hyperbolic system of equations.

In Section 2 we describe these equations and show they have a unique solution dependent only on D and A, and that the operator $J + K$ so constructed does indeed map solutions of (1.1) into solutions of (1.3). This operator $J + K$ is shown to be invertible on the space of regulated functions on $(0, T_0)$ with an inverse $J^{-1} - H$ of similar form with kernels h^+, h^- satisfying a hyperbolic system of equations in a sense adjoint to those for k^+, k^-.

In Section 3 the hyperbolic systems are solved for the two-variable case and we obtain the results given by Hill [3]. Results for the nonhomogeneous problem are given in Section 4.

2. Fredholm operator and its inverse for homogeneous problem

Let $\Omega \subset R^m$ denote a bounded domain and G the region $\{(x, t)\colon x \in \Omega, 0 < t < T(x) \leqslant T_0\}$. Suppose $v(x, t)(\,:\, G \to R^N)$ is a solution of the linear partial differential equation,

$$\frac{\partial v}{\partial t} = DLv \quad \text{in } G, \qquad v(x,0) = f(x) \quad \text{in } \Omega, \tag{2.1}$$

where D is a constant, nonsingular, diagonal matrix with diagonal elements $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_N$, and the operator L satisfies the commutative relation

$$LB = BL$$

for all $B \in S_N(0, T_0)$, the set of regulated mappings from $(0, T_0)$ into $N \times N$ matrices.

THEOREM 1. *Given a bounded matrix $A \subset S_N(0, T_0)$ with ijth element denoted by A_{ij}, let*

(1) *$A^0 \in S_N$ be such that $A^0_{ij} = A_{ij}$ when $d_i = d_j$ and $A^0_{ij} = 0$ otherwise;*

(2) *$J \in S_N$ be the solution of the differential equation*

$$\frac{dJ}{dt} = A^0 J \quad \text{in } (0, T_0), \quad J(0) = I, \tag{2.2}$$

where I is the unit matrix in S_N.

Then there is a unique pair of functions

$$k^+(t, s)\colon \{(t, s)\colon 0 \leqslant t \leqslant T_0, 0 \leqslant s \leqslant t\} \to N \times N \text{ matrices},$$

$$k^-(t, s)\colon \{(t, s)\colon 0 \leqslant s \leqslant T_0, 0 \leqslant t \leqslant s\} \to N \times N \text{ matrices}$$

which satisfy the hyperbolic equations

$$\begin{aligned} \frac{\partial k^+}{\partial t} + D\frac{\partial k^+}{\partial s}D^{-1} &= Ak^+ \quad \text{in } 0 \leqslant t \leqslant T_0, 0 \leqslant s \leqslant t, \\ \frac{\partial k^-}{\partial t} + D\frac{\partial k^-}{\partial s}D^{-1} &= Ak^- \quad \text{in } 0 \leqslant s \leqslant T_0, 0 \leqslant t \leqslant s, \end{aligned} \tag{2.3}$$

and boundary conditions

$$k^+(t,0) = k^-(0, s) = 0,$$

$$[k^+(t, t) - k^-(t, t)] - D[k^+(t, t) - k^-(t, t)]D^{-1} = (A - A^0)J(t) \tag{2.4}$$

and the function $u(x, t)\colon G \to R^N$ *defined by*

$$u(x,t) = J(t)v(x,t) + \int_0^t k^+(t,s)v(x,s)\,ds + \int_t^{T_0} k^-(t,s)v(x,s)\,ds$$
$$\equiv (J+K)v(x,t) \tag{2.5}$$

is a solution of the differential equation

$$\frac{\partial u}{\partial t} = DLu + Au \text{ in } G, \quad u(x,0) = f(x) \tag{2.6}$$

for $0 < t < \min_{i,j}(d_i/d_j)T_0$.

PROOF. The existence proof for $k^{\pm}(t, s)$ follows classical lines [2] and is merely sketched here. Functions $k^{\pm}(t,s)$ satisfying equations (2.3), (2.4) can be constructed iteratively as follows. Let

$$k_{ij}(t,s) = \begin{cases} k_{ij}^+(t,s) & \text{in } 0 \leqslant t \leqslant T_0, 0 \leqslant s < t, \\ k_{ij}^-(t,s) & \text{in } 0 \leqslant s \leqslant T_0, 0 \leqslant t < s, \end{cases}$$

and $t = \tau$, $s = \phi + (d_i/d_j)\tau$, so that

$$\frac{\partial k_{ij}}{\partial \tau}(\tau,\phi) = \sum_{\alpha=1}^{N} A_{i\alpha}k_{\alpha j}(\tau,\phi)$$

except on $\tau = \phi + (d_i/d_j)\tau$.

By integrating this equation along characteristics ϕ = constant and using the boundary conditions (2.4), we find: when $i < j$ and $d_i > d_j$,

$$s > \frac{d_i}{d_j}t,\ k_{ij}(t,s) = \sum_{\alpha=1}^{N} \int_0^t A_{i\alpha}k_{\alpha j}\left(\theta, s - \frac{d_i}{d_j}(t-\theta)\right) d\theta;$$

$$t < s < \frac{d_i}{d_j}t,\ k_{ij}(t,s) = \frac{d_j}{d_j - d_i}\sum_{\alpha=1}^{N}\left(A_{i\alpha} - A_{i\alpha}^0\right)J_{\alpha j}\left(\frac{d_i t - d_j s}{d_i - d_j}\right)$$
$$+ \sum_{\alpha=1}^{N}\int_{t-(d_j/d_i)s}^{t} A_{i\alpha}k_{\alpha j}\left(\theta, s - \frac{d_i}{d_j}(t-\theta)\right) d\theta;$$

$$s < t,\ k_{ij}(t,s) = \sum_{\alpha=1}^{N}\int_{t-(d_j/d_i)s}^{t} A_{i\alpha}k_{\alpha j}\left(\theta, s - \frac{d_i}{d_j}(t-\theta)\right) d\theta;$$

when $i > j$ and $d_i < d_j$,

$$s > t,\ k_{ij}(t,s) = \sum_{\alpha=1}^{N} \int_0^t A_{i\alpha} k_{\alpha j}\left(\theta, s - \frac{d_i}{d_j}(t-\theta)\right) d\theta;$$

$$t > s > \frac{d_i}{d_j} t,\ k_{ij}(t,s) = \frac{-d_j}{d_j - d_i} \sum_{\alpha=1}^{N} \left(A_{i\alpha} - A_{i\alpha}^0\right) J_{\alpha j}\left(\frac{d_i t - d_j s}{d_i - d_j}\right)$$

$$+ \sum_{\alpha=1}^{N} \int_0^t A_{i\alpha} k_{\alpha j}\left(\theta, s - \frac{d_i}{d_j}(t-\theta)\right) d\theta;$$

$$\frac{d_i}{d_j} t > s,\ k_{ij}(t,s) = \sum_{\alpha=1}^{N} \int_{t-(d_j/d_i)s}^{t} A_{i\alpha} k_{\alpha j}\left(\theta, s - \frac{d_i}{d_j}(t-\theta)\right) d\theta;$$

when $d_i = d_j$,

$$s > t,\ k_{ij}(t,s) = \sum_{\alpha=1}^{N} \int_0^t A_{i\alpha} k_{\alpha j}(\theta, s - t + \theta)\, d\theta;$$

$$s < t,\ k_{ij}(t,s) = \sum_{\alpha=1}^{N} \int_{t-s}^{t} A_{i\alpha} k_{\alpha j}(\theta, s - t + \theta)\, d\theta.$$

Replace k_{ij} by k_{ij}^{n+1} on the left and by k_{ij}^{n} on the right side of all these equations, set $k_{ij}^0 \equiv 0$ and solve the system iteratively.

If

$$a = \sup \sum_{\alpha=1}^{N} |A_{i\alpha}|, \quad b = \sup \frac{d_j}{d_j - d_i} |J_{\alpha\beta}| \quad \text{for } d_i \neq d_j,$$

then, by induction, $|k_{ij}^{n+1}(t,s) - k_{ij}^{n}(t,s)| \leqslant b a^{n+1} t^n / n!$, so that

$$\sum_{n=1}^{\infty} \left(k_{ij}^{n}(t,s) - k_{ij}^{n-1}(t,s)\right)$$

converges uniformly to a solution $k_{ij}(t,s)$ of the integral equations.

This solution is unique, for if $u_{ij}(t,s) = k_{ij}^1 - k_{ij}^2$, the difference of any two solutions k_{ij}^1 and k_{ij}^2 of the integral equations above, then it satisfies the homogeneous equations derived from these by setting $J_{\alpha j}$ to zero.

Let $U(T) = \sup|u_{ij}(t,s)|$ for all i, j and $t, s \in (0, T)$. Suppose $U(T_1) = 0$, where $0 \leqslant T_1 < T_0$ and $U(T^*) > 0$ for any $T^* > T_1$. Since $U(T^*) = |u_{ij}(t,s)|$ for some i, j and t, s in $(T_1, T^*]$ and

$$u_{ij}(t,s) = \sum_{\alpha=1}^{N} \int_c^t A_{i\alpha} u_{\alpha j}\left(\theta, s - \frac{d_i}{d_j}(t-\theta)\right) d\theta,$$

where $c =$ greater of 0 and $t - (d_j/d_i)s$ then it follows that

$$U(T^*) \leqslant \int_{T_1}^{T^*} \alpha U(T^*)\, d\theta = \alpha U(T^*)(T^* - T_1).$$

https://doi.org/10.1017/S0334270000004689 Published online by Cambridge University Press

But T^* can be chosen so that $a(T^* - T_1) = \frac{1}{2}$ in which case the inequality leads to a contradiction. We must conclude $U(T_0) = 0$, and only one solution exists.

Consider now the function $u(x, t)$ defined by equation (2.5). Since $J(0) = I$ and $k^-(0, s) = 0$, we have

$$u(x,0) = v(x,0) = f(x).$$

Moreover,

$$\begin{aligned}\frac{\partial u}{\partial t} - DLu - Au = {} & (A^0 - A)Jv + (JD - DJ)Lv + Dk^+(t,0)D^{-1}f(x) \\ & + \{[k^+(t,t) - k^-(t,t)] - D[k^+(t,t) - k^-(t,t)]D^{-1}\}v \\ & - Dk^-(t,T_0)D^{-1}v(x,T_0) \\ & + \int_0^t \left(\frac{\partial k^+}{\partial t} + D\frac{\partial k^+}{\partial s}D^{-1} - Ak^+\right)(t,\theta)v(x,\theta)\,d\theta \\ & + \int_t^{T_0} \left(\frac{\partial k^-}{\partial t} + D\frac{\partial k^-}{\partial s}D^{-1} - Ak^-\right)(t,\theta)v(x,\theta)\,d\theta.\end{aligned} \tag{2.7}$$

From the definition of A^0 it follows that

$$DA^0 - A^0D = 0$$

and hence if $\phi \equiv DJ - JD$, then $\phi = 0$ at $t = 0$ and

$$\frac{d\phi}{dt} = DA^0J - A^0JD = A^0(DJ - JD) = A^0\phi.$$

Thus

$$\phi = 0.$$

Now $k^-(0, s) = 0$ and the characteristics of the equations for $k^-(t, s)$ are lines of the form

$$s = s_0 + (d_i/d_j),$$

so that $k^-(t, s) = 0$ in $s \geqslant$ max over i, j of $(d_i/d_j)t$ and hence $k^-(t, T_0) = 0$ if $t \leqslant$ min over i, j of $(d_j/d_i)T_0$.

These results, together with equations (2.3), (2.4) make the right side of equation 2.7 vanish. *Q.E.D.*

If A commutes with D, then $A = A^0$ and $k(t, s) = k^-(t, s) = 0$. In this case

$$u = Jv, \quad \text{where } \frac{dJ}{dt} = AJ, \quad J(0) = I.$$

The operator $J + K$ maps regulated functions $w(t)$ on $(0, T_0)$ into regulated functions on $(0, T_0)$, and at least for T_0 small enough will have an inverse $J^{-1} - H$ such that $(J + K)(J^{-1} - H) = I$ or

$$KJ^{-1}w = JHw + KHw. \tag{2.8}$$

If we assume Hw is of the form

$$Hw = \int_0^t h^+(t,s)w(s)\,ds + \int_t^{T_0} h^-(t,s)w(s)\,ds$$

it will be sufficient to find kernels $h^{\pm}(t,s)$ satisfying the equations

$$\begin{aligned}\phi^+(t,s) \equiv k^+(t,s)J^{-1}(s) &= J(t)h^+(t,s) + \int_0^s k^+(t,\theta)h^-(\theta,s)\,d\theta \\ &\quad + \int_s^t k^+(t,\theta)h^+(\theta,s)\,d\theta + \int_t^{T_0} k^-(t,\theta)h^+(\theta,s)\,d\theta, \qquad (2.9)\\ \phi^-(t,s) \equiv k^-(t,s)J^{-1}(s) &= J(t)h^-(t,s) + \int_0^t k^+(t,\theta)h^-(\theta,s)\,d\theta \\ &\quad + \int_t^s k^-(t,\theta)h^-(\theta,s)\,d\theta + \int_s^{T_0} k^-(t,\theta)h^+(\theta,s)\,d\theta,\end{aligned}$$

derived from the operator equation (2.8).

THEOREM 2. *There exists a unique pair of functions $h^+(t,s)$, $h^-(t,s)$ which satisfy the hyperbolic equations*

$$\begin{aligned}&\frac{\partial h^+}{\partial t} + D\frac{\partial h^+}{\partial s}D^{-1} + Dh^+D^{-1}A = 0 \quad \text{in } 0 \leqslant t \leqslant T_0,\ 0 \leqslant s \leqslant t,\\ &\frac{\partial h^-}{\partial t} + D\frac{\partial h^-}{\partial s}D^{-1} + Dh^-D^{-1}A = 0 \quad \text{in } 0 \leqslant s \leqslant T_0,\ 0 \leqslant t \leqslant s,\end{aligned} \qquad (2.10)$$

and boundary conditions

$$\begin{gathered}h^+(t,0) = h^-(0,s) = 0,\\ [h^+(t,t) - h^-(t,t)] - D[h^+(t,t) - h^-(t,t)]D^{-1} = J^{-1}(A - A^0).\end{gathered} \qquad (2.11)$$

These functions also satisfy equations (2.9), *and if u satisfies equations* (2.6) *in* $(0, T_0)$, *the function $v = (J^{-1} - H)u$ satisfies equation* (2.1) *for* $0 < t < \min_{i,j}(d_i/d_j)T_0$.

PROOF. Integral equations analogous to those for $k^{\pm}(t,s)$ may be constructed and solved interatively for $h^{\pm}(t,s)$ for all T_0. It is readily shown that both sides of equations (2.9) satisfy the hyperbolic system

$$\frac{\partial \phi^{\pm}}{\partial t} + D\frac{\partial \phi^{\pm}}{\partial s}D^{-1} = A\phi^{\pm} - D\phi^{\pm}D^{-1}A^0$$

and boundary conditions

$$\begin{gathered}\phi^+(t,0) = \phi^-(0,s) = 0,\\ [\phi^+(t,t) - \phi^-(t,t)] - D[\phi^+(t,t) - \phi^-(t,t)]D^{-1} = A - A^0.\end{gathered}$$

This system has a unique solution.

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In similar fashion we find $v = (J^{-1} - H)u$ satisfies equation (2.1) if $h^{-}(t, T^0) = 0$ and this holds if $0 < t < \min_{i,j}(d_i/d_j)T_0$. *Q.E.D.*

There is an interesting relationship between equations (2.3), (2.4) for $k^{\pm}(t, s)$ and equations (2.10), (2.11) for the resolvent kernels $h^{\pm}(t, s)$. If A' denotes the transpose of a matrix $A \in S_N$ and $h'(t, s)$ denotes the transpose matrix of $h(t, s)$ and if $B = -A'$, $J^* = J'^{-1}$, then

$$z^{+}(t, s) = -D'^{-1}h'^{-1}(s, t)D' \quad \text{on } t > s,$$
$$z^{-}(t, s) = -D'^{-1}h'^{+}(s, t)D' \quad \text{on } t < s$$

satisfies the equations

$$\frac{\partial z^-}{\partial t} + D'\frac{\partial z^-}{\partial s}D'^{-1} = -A'z^- = Bz^-, \quad \text{on } 0 < s < T_0, 0 < t < s,$$
$$\frac{\partial z^+}{\partial t} + D'\frac{\partial z^+}{\partial s}D'^{-1} = -A'z^+ = Bz^+, \quad \text{on } 0 < t < T_0, 0 < s < t,$$
$$z^-(0, s) = z^+(t, 0) = 0,$$
$$(z^+(t,t) - z^-(t,t)) - D'(z^+(t,t) - z^-(t,t))D'^{-1}$$
$$= -(A' - A^{0\prime})J'^{-1} = (B - B_0)J^*.$$

Since

$$\frac{dJ}{dt} = A^0 J, \quad J(0) = I,$$

we have

$$\frac{d}{dt}J'^{-1} = (-A'^0)J'^{-1}, \quad J'^{-1}(0) = I,$$

or

$$\frac{dJ^*}{dt} = B^0 J^*, \quad J^*(0) = I.$$

Thus for a pair of kernels generated by a matrix A, we derive the kernels of its inverse from the system generated by a matrix $B = -A'$. Specifically,

$$h^+(t, s) = -D^{-1}z'^{-1}(s, t)D, \quad h^-(t, s) = -D^{-1}z'^{+}(s, t)D. \tag{2.12}$$

If A is skew symmetric, then $B = A$ and $z^{\pm}(t, s) = k^{\pm}(t, s)$ while $J = I$.

For each bounded matrix A we have operators $J + K$ and $J^{-1} - H$ and to identify the operator with A we can write it as $J_A + K_A$ *etc.* Evidently, if

$$w = (J_B + K_B)(J_A^{-1} - H_A)u = (J^* + K^*)u$$

and if u satisfies equations (2.6), then w satisfies the same equation with A replaced by B. We find that $J^* = J_B J_A^{-1}$ satisfies the system

$$\frac{dJ^*}{dt} = B_0 J^* - J^* A_0, \quad J^*(0) = I, \tag{2.13}$$

while $k^{*\pm}(t, s)$ satisfy the hyperbolic system

$$\frac{\partial k^{*\pm}}{\partial t} + D\frac{\partial k^{*\pm}}{\partial s}D^{-1} = Bk^{*\pm} - Dk^{*\pm}D^{-1}A, \tag{2.14}$$

and boundary conditions

$$\begin{aligned} k^{*+}(t,0) = k^{*-}(0, s) = 0, \\ [k^{*+}(t,t) - k^{*-}(t,t)] - D[k^{*+}(t,t) - k^{*-}(t,t)]D^{-1} \\ = (B - B^0)J^* - J^*(A - A^0). \end{aligned} \tag{2.15}$$

3. Kernels for two variable case

Since the differential operators $\partial/\partial t$ and $\partial/\partial s$ commute with constants, the hyperbolic equation for $k^+(t, s)$ and $k^-(t, s)$ may, in the case where A and D are constant, be written in the matrix form

$$\left[A - I\frac{\partial}{\partial t} - \frac{D}{d_j}\frac{\partial}{\partial s}\right]k_j^{\pm} = 0, \tag{3.1}$$

where $k_j^{\pm}$ is the jth column of $k^{\pm}$. If this equation is multiplied by the matrix adjugate operator we see that each element k_{ij} of the vector k_j satisfies the differential equation

$$\left|A - I\frac{\partial}{\partial t} - \frac{D}{d_j}\frac{\partial}{\partial s}\right|\phi = 0, \tag{3.2}$$

where $|B|$ denotes the formal determinant of the matrix B.

In the case where A is triangular this differential equation has a simple form:

$$\begin{aligned} \left|A - I\frac{\partial}{\partial t} - \frac{D}{d_j}\frac{\partial}{\partial s}\right|\phi = \left(a_{11} - \frac{\partial}{\partial t} - \frac{d_1}{d_j}\frac{\partial}{\partial s}\right)\left(a_{22} - \frac{\partial}{\partial t} - \frac{d_2}{d_j}\frac{\partial}{\partial s}\right) \\ \cdots\left(e_{NN} - \frac{\partial}{\partial t} - \frac{d_N}{d_j}\frac{\partial}{\partial s}\right)\phi \\ = 0. \end{aligned}$$

J and the kernels $k^{\pm}$ are also triangular.

In the case $N = 2, j = 1$, this equation is

$$\begin{vmatrix} a_{11} - \frac{\partial}{\partial t} - \frac{\partial}{\partial s} & a_{12} \\ a_{21} & a_{22} - \frac{\partial}{\partial t} - \frac{d_2}{d_1}\frac{\partial}{\partial s} \end{vmatrix}\phi = 0. \tag{3.3}$$

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Introduce new variables

$$\tau_1 = t - s, \quad \phi = \psi \exp\left(\frac{a_{22}\tau_1 + a_{11}x_1}{c_1}\right),$$

$$x_1 = s - \frac{d_2 t}{d_1}, \quad c_1 = 1 - \frac{d_2}{d_1},$$

and assume $d_1 > d_2 > 0$. Equation (3.3) for ϕ gives rise to the equation

$$\begin{vmatrix} -c_1\dfrac{\partial}{\partial x_1} & a_{12} \\ a_{21} & -c_1\dfrac{\partial}{\partial \tau_1} \end{vmatrix} \psi = 0 \tag{3.4}$$

for ψ. Since $d_2/d_1 < 1$, $k_1^- = 0$ and hence on $t = s$ or $\tau_1 = 0$,

$$c_1 k_{21}^+(t,t) = ((A - A_0)J_1(t))_{21} = a_{21}e^{a_{11}t}, \quad \text{and} \quad \psi_{21}^+(x,0) = a_{21}/c_1.$$

Thus,

$$\frac{\partial^2}{\partial\tau_1\partial x_1}\psi_{21}^+ = \frac{a_{12}a_{21}}{c_1^2}\psi_{21}^+; \quad \text{on } \tau_1 > 0, \quad x_1 > \frac{-d_2\tau_1}{d_1},$$

$$\psi_{21} = 0 \quad \text{on } x_1 = \frac{-d_2\tau_1}{d_1}, \quad \psi_{21} = \frac{a_{21}}{c_1} \quad \text{on } \tau_1 = 0.$$

This implies $\psi_{21}^+ = 0$ in the sector $x_1 < 0$, so that

$$\psi_{21} = 0 \quad \text{on } x_1 = 0, \quad \psi_{21} = \frac{a_{21}}{c_1} \quad \text{on } \tau_1 = 0,$$

and this problem has a similarity solution

$$\psi_{21}^+ = \frac{a_{21}}{c_1} I_0\left(2\sqrt{\frac{a_{12}a_{21}}{c_1^2}x_1\tau_1}\right).$$

The first element k_{11}^+ of the vector k_1^+ may be found from the relationships

$$k_{11}^+ = \psi_{11}\exp\left(\frac{a_{22}\tau_1}{c_1} + \frac{a_{11}x_1}{c_1}\right),$$

where

$$a_{21}\psi_{11} - c_1\frac{\partial\psi_{21}}{\partial\tau_1} = 0.$$

The case $j = 2$ in analogous fashion shows

$$k_2^+ = 0, \quad k_{12}^- = \frac{a_{12}}{c_2} I_0\left(2\sqrt{\frac{a_{12}a_{21}}{c_2^2}x_2\tau_2}\right)\exp\left(\frac{a_{11}\tau_2}{c_2} + \frac{a_{22}x_2}{c_2}\right),$$

where $c_2 = (d_1/d_2) - 1$, $x_2 = (d_1/d_2)t - s$, $\tau_2 = s - t$, and

$$a_{12}k_{22}^- + \left(a_{11} - c_2\frac{\partial}{\partial\tau_2}\right)k_{12}^- = 0.$$

Thus, for the second order case:

$$J(t) = \begin{pmatrix} e^{ta_{11}} & 0 \\ 0 & e^{ta_{22}} \end{pmatrix},$$

$$k^+(t,s) = \frac{e^{\alpha_1 s - \beta_1 t}}{c_1}\begin{pmatrix} \sqrt{\frac{a_{12}a_{21}x_1}{\tau_1}}\, I_1\left(2\sqrt{\frac{a_{12}a_{21}}{c_1^2}x_1\tau_1}\right) & 0 \\ a_{21}I_0\left(2\sqrt{\frac{a_{12}a_{21}}{c_1^2}x_1\tau_1}\right) & 0 \end{pmatrix}$$

where $\alpha_1 = ((a_{11} - a_{22})d_1)/(d_1 - d_2)$, $\beta_1 = (a_{11}d_2 - a_{22}d_1)/(d_1 - d_2)$, (3.5)

$$k^-(t,s) = \frac{e^{\alpha_2 s - \beta_2 t}}{c_2}\begin{pmatrix} 0 & a_{12}I_0\left(2\sqrt{\frac{a_{12}a_{21}}{c_2^2}x_2\tau_2}\right) \\ 0 & \sqrt{\frac{a_{12}a_{21}x_2}{\tau_2}}\, I_1\left(2\sqrt{\frac{a_{12}a_{21}}{c_2^2}x_2\tau_2}\right) \end{pmatrix}$$

where $\alpha_2 = ((a_{11} - a_{22})d_2)/(d_1 - d_2)$, $\beta_2 = \beta_1$.

The kernels $h^{\pm}(t, s)$ for the resolvent kernels of K may be constructed from its adjoint properties derived in Section 2.

For the present case $N = 2$ and constant A this mapping (2.12) gives

$$h^+(t,s) = \frac{-e^{-\alpha_2 t + \beta_2 s}}{c_2}\begin{pmatrix} 0 \\ -\frac{d_1}{d_2}a_{21}I_0\left(2\sqrt{\frac{a_{12}a_{21}}{c_2^2}\bar{x}_2\bar{\tau}_2}\right); \end{pmatrix.}$$

$$\left. \begin{matrix} 0 \\ \times \sqrt{\frac{a_{12}a_{21}}{\bar{\tau}_2}}\, I_1\left(2\sqrt{\frac{a_{12}a_{21}}{c_2^2}\bar{x}_2\bar{\tau}_2}\right) \end{matrix} \right);$$

$$h^-(t,s) = \frac{-e^{-\alpha_1 t + \beta_1 s}}{c_1}\begin{pmatrix} \sqrt{\frac{a_{12}a_{21}\bar{x}_1}{\bar{\tau}_1}}\, I_1\left(2\sqrt{\frac{a_{12}a_{21}}{c_1^2}\bar{x}_1\bar{\tau}_1}\right); \\ 0 \end{pmatrix.}$$

$$\left. \begin{matrix} -\frac{d_2}{d_1}a_{12}I_0\left(2\sqrt{\frac{a_{12}a_{21}}{c_1^2}\bar{x}_1\bar{\tau}_1}\right) \\ 0 \end{matrix} \right);$$

https://doi.org/10.1017/S0334270000004689 Published online by Cambridge University Press

where $\bar{x}_1 = t - (d_2/d_1)s$, $\bar{\tau}_1 = s - t$, $\bar{x}_2 = (d_1/d_2)s - t$, $\bar{\tau}_2 = t - s$. This agrees with the expression derived by Hill [3].

The function

$$k(t, s) = \begin{cases} k^+(t, s) & \text{for } t > s, \\ k^-(t, s) & \text{for } t < s \end{cases}$$

is bounded by bae^{aT} for $t < T$, $s < T$ (where a and b are the sup norms for K and J used in Section 2) and is discontinuous at only a finite set of points on any line $s = s_0 > 0$, $t = t_0 > 0$. It therefore has a double Laplace Transform:

$$\bar{\bar{k}} = \int_0^\infty e^{-pt} \int_0^\infty e^{-qs} k(t, s)\, dt\, ds, \qquad p > a, q > a,$$

and

$$A\bar{\bar{k}} = \int_0^\infty \int_0^\infty e^{-pt-qs}\left(\frac{\partial k}{\partial t} + D\frac{\partial k}{\partial s}D^{-1}\right) dt\, ds;$$

so that

$$\begin{aligned} A\bar{\bar{k}} - p\bar{\bar{k}} - qD\bar{\bar{k}}D^{-1} &= \int_0^\infty \int_0^\infty \frac{\partial}{\partial t}[e^{-pt-qs}k] + \frac{\partial}{\partial s}[De^{-pt-qs}kD^{-1}]\, dt\, ds \\ &= \int_{s=t} e^{-pt-qs}[k^+ - k^-]\, ds - De^{-pt-qs}(k^+ - k^-)D^{-1}\, dt \\ &= \int_0^\infty e^{-pt-qt}(A - A^0)J(t)\, dt \\ &= (A - A^0)\left[(p + q)I - A^0\right]^{-1}. \end{aligned}$$

If $\bar{k}_j$ denotes the single Laplace Transform $\int_0^\infty e^{-qs}k_j(t, s)\, ds$, then we find from the expression for $\bar{\bar{k}}_j$ that

$$\bar{k}_j = e^{t(a_j - q)}\left\{e^{t[A - A^0 - q(D_j - I)]} - I\right\}\left\{A - A^0 - q(D_j - I)\right\}^{-1}(A - A^0)_j$$

in the case A^0 diagonal. a_j is the jth diagonal element of A^0 and $D_j = D/d_j$.

The transform $\bar{k}_j$ may be obtained in another way as follows. The operator $L = -qI$ commutes with A and D and the problems

$$\frac{\partial u}{\partial t} = -Dqu + Au, \quad u(0) = I, \quad \frac{dv}{dt} = -Dqv, \quad v(0) = I,$$

have solutions $u = e^{t(A - Dq)}$ and e^{-tDq}.

The relationship

$$u = Jv + kv$$

gives the result

$$e^{t(A - Dq)} = e^{tA_0}e^{-tDq} + \int_0^t k^+(t, s)e^{-sDq}\, ds + \int_t^{T_0} k^-(t, s)e^{-sDq}\, ds,$$

and hence considering the jth column only we have

$$\int_0^\infty k_j(t,s)e^{-sq^*}\,ds = [e^{t[A-D_jq^*]} - e^{t[A_0-q^*D_j]}]_j$$

where $q^* = qd_j$, $D_j = D/d_j$ and $[B]_j$ denotes the jth column of B.

4. Nonhomogeneous problems

When D and A commute, the solutions of the nonhomogeneous problem

$$\frac{\partial u}{\partial t} = DLu + Au + \phi(x,t), \quad u(x,0) = f(x), \tag{4.1}$$

can be expressed in the form

$$u = Jv, \quad \text{where } \frac{dJ}{dt} = AJ, \quad J(0) = I,$$

where v satisfies the inhomogeneous problem

$$\frac{\partial v}{\partial t} = DLv + \psi(x,t), \quad \psi(x,t) = J^{-1}\phi(x,t), \quad v(x,0) = f(x). \tag{4.2}$$

This result extends to the noncommuting case where D, L, and A have the properties required in Theorem 1.

THEOREM 3. *If $v(x,t)$ satisfies the equation*

$$\frac{\partial v}{\partial t}(x,t) = DLv(x,t) + \psi(x,t), \quad v(x,0) = f(x), \tag{4.3}$$

and $J(t)$, $k^{\pm}(t,s)$ satisfy equations (2.2), (2.3), (2.4), *then*

$$u(x,t) = (J+K)v(x,t) \tag{4.4}$$

satisfies the equations

$$\frac{\partial u}{\partial t} = DLu + Au + \phi \tag{4.5}$$

where

$$\phi(x,t) = D(J+K)D^{-1}\psi(x,t). \tag{4.6}$$

PROOF. From equation (4.4) we have

$$\frac{\partial u}{\partial t} - DLu - Au = (-A + A^0)Jv + J\left(\frac{\partial v}{\partial t} - DLv\right)$$
$$+ [k^+(t, t) - k^-(t, t)]v + \int_0^t \frac{\partial k^+}{\partial t} v$$
$$+ \int_t^{T_0} \frac{\partial k^-}{\partial t} v - \int_0^t Ak^+ v - \int_t^{T_0} Ak^- v - \int_0^t Dk^+ D^{-1}\left(\frac{\partial v}{\partial t} - \psi\right)$$
$$- \int_t^{T_0} Dk^- D^{-1}\left(\frac{\partial v}{\partial t} - \psi\right)$$
$$= J\psi + \int_0^t Dk^+ D^{-1}\psi + \int_t^{t_0} Dk^- D^{-1}\psi$$
$$= D(J + K)D^{-1}\psi$$
$$= \phi. \quad Q.E.D.$$

It can be shown in analogous fashion if u satisfies equation (4.5) and v is defined by

$$v + (J^{-1} - H)u \tag{4.7}$$

where $J^- - H$ is the inverse of $J + K$, then v satisfies equation (4.3) where

$$\psi = D(J^{-1} - H)D^{-1}\phi. \quad Q.E.D. \tag{4.8}$$

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