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Hyperbolically embedded subgroups and quasi-isometries of pairs

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Abstract. We give technical conditions for a quasi-isometry of pairs to preserve a subgroup being hyperbolically embedded. We consider applications to the quasi-isometry and commensurability invariance of acylindrical hyperbolicity of finitely generated groups.

1 Introduction

A group G is *acylindrically hyperbolic* if it admits a nonelementary, acylindrical action on a hyperbolic space. An alternative characterization is that G is acylindrically hyperbolic if and only if G contains a *hyperbolically embedded subgroup* H, denoted $H \hookrightarrow_h G$, and we will give a characterization from [13] in Proposition 3.1.

The class of acylindrically hyperbolic groups generalizes the classes of nonelementary hyperbolic and relatively hyperbolic groups while sharing many similar properties [17]. In spite of this, there are still foundational questions that remain open, for instance, it is known that a group being hyperbolic or relatively hyperbolic is invariant under quasi-isometry [8, 9], but the corresponding question for acylindrical hyperbolicity is still open.

Question 1.1 [17, Question 2.20(a)] Is the class of finitely generated acylindrically hyperbolic groups closed under quasi-isometry?

Some partial results are known, for instance, acylindrical hyperbolicity passes to finite-index subgroups and is preserved by quotienting out a finite normal subgroup [15]. If the group is $\mathcal{AH}$-accessible, then acylindrical hyperbolicity can be passed to finite extensions [16]. The property of being $\mathcal{AH}$-accessible also passes to finite-index overgroups [3]. However, not every finitely presented acylindrically hyperbolic group

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is $\mathcal{AH}$-accessible [1, Theorem 2.18]. Some experts in the field do not expect a complete positive answer to Question 1.1.

This article relies on the notion of quasi-isometry of pairs, and our results provide technical conditions to ensure that a quasi-isometry of pairs carries the property of being a hyperbolically embedded subgroup.

Definition 1.1 (Quasi-isometry of pairs) Let X and Y be metric spaces, and let $\mathcal{A}$ and $\mathcal{B}$ be collections of subspaces of X and Y, respectively. A quasi-isometry $q: X \to Y$ is a *quasi-isometry of pairs* $q: (X, \mathcal{A}) \to (Y, \mathcal{B})$ if there is $M > 0$:

(1) For any $A \in \mathcal{A}$, the set $\{B \in \mathcal{B}: \mathsf{hdist}_Y(q(A), B) < M\}$ is nonempty.
(2) For any $B \in \mathcal{B}$, the set $\{A \in \mathcal{A}: \mathsf{hdist}_Y(q(A), B) < M\}$ is nonempty.

In this case, if $q: X \to Y$ is an (L, C)-quasi-isometry, then $q: (X, \mathcal{A}) \to (Y, \mathcal{B})$ is called an (L, C, M)-*quasi-isometry*. If there is a quasi-isometry of pairs $(X, \mathcal{A}) \to (Y, \mathcal{B})$, we say that $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are *quasi-isometric pairs*.

We specialize the previous definition to the case of finitely generated groups with finite collections of subgroups as follows.

Definition 1.2 (Quasi-isometry of group pairs) Consider two pairs $(G, \mathcal{P})$ and $(H, \mathcal{Q})$ where G and H are finitely generated groups with chosen word metrics dist_G and dist_H. Denote the Hausdorff distance between subsets of H by hdist_H. An (L, C)-quasi-isometry $q: G \to H$ is an (L, C, M)-*quasi-isometry of pairs* $q: (G, \mathcal{P}) \to (H, \mathcal{Q})$ if the relation

$$\dot{q} = \{(A, B) \in G/\mathcal{P} \times H/\mathcal{Q}: \mathsf{hdist}_H(q(A), B) < M\}$$

satisfies that the projections into $G/\mathcal{P}$ and $H/\mathcal{Q}$ are surjective.

Example 1.1 (Quasi-isometry of pairs and finite extensions) Let H be a finite index normal subgroup of finitely generated group G, and let $\mathcal{Q}$ be a finite collection of subgroups of H. Then the inclusion $(H, \mathcal{Q}) \hookrightarrow (G, \mathcal{Q})$ is a quasi-isometry of pairs if the collection $\{hQh^{-1}: h \in H \text{ and } Q \in \mathcal{Q}\}$ is invariant under conjugation by G (see Proposition 4.1).

Recall that the *commensurator* of a subgroup P of a group G is the subgroup

$$Comm_G(P) = \{g \in G: P \cap gPg^{-1} \text{ is a finite index subgroup of } P \text{ and } gPg^{-1}\}.$$

Definition 1.3 (Refinements) Let $\mathcal{P}$ be a collection of subgroups of group G. A *refinement* $\mathcal{P}^*$ of $\mathcal{P}$ is a set of representatives of conjugacy classes of the collection of subgroups

$$\{\mathrm{Comm}_G(gPg^{-1}): P \in \mathcal{P} \text{ and } g \in G\}.$$

Example 1.2 (Refinements and qi of pairs) Let $\mathcal{Q}$ be a finite collection of subgroups of a finitely generated group H, and let $\mathcal{Q}^*$ be a refinement. If each $Q \in \mathcal{Q}$ is finite index in $\mathrm{Comm}_H(Q)$, then the identity map on G is a quasi-isometry of pairs $(H, \mathcal{Q}) \to (H, \mathcal{Q}^*)$.

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Example 1.3 (Refinements and finite extensions) Let A be a group, let $\mathcal{H}$ be an almost malnormal collection of infinite subgroups, and let $F \leq \mathrm{Aut}(A)$ be a finite subgroup. If F acts freely on $\mathcal{H}$ and $\mathcal{H}_F$ is a collection of representatives of F-orbits in $\mathcal{H}$, then a refinement of $\mathcal{H}$ in $A \rtimes F$ is $\mathcal{H}_F$.

Definition 1.4 (Reduced collections) A collection of subgroups $\mathcal{P}$ of a group G is *reduced* if for any $P, Q \in \mathcal{P}$ and $g \in G$, if P and gQg^{-1} are commensurable, then $P = Q$ and $g \in P$.

Our first result, Theorem A, describes a strategy to obtain positive results to Question 1.1. For a group G with a generating set S, let $\Gamma(G, S)$ denote the corresponding *Cayley graph* (see Definition 2.3).

Theorem A (Theorem 3.1) *Let $q\colon G \to H$ be a quasi-isometry of finitely generated groups, let $\mathcal{P}$ and $\mathcal{Q}$ be finite collections of subgroups of G and H, respectively, and let S and T be (not necessarily finite) generating sets of G and H, respectively. Suppose that:*

(1) *$q\colon (G, \mathcal{P}) \to (H, \mathcal{Q})$ is a quasi-isometry of pairs and*
(2) *$q\colon \Gamma(G, S) \to \Gamma(H, T)$ is a quasi-isometry.*

The following statements hold:

(1) *If $\mathcal{P}$ and $\mathcal{Q}$ are reduced collections in G and H, respectively, then $\mathcal{P} \hookrightarrow_h (G, S)$ if and only if $\mathcal{Q} \hookrightarrow_h (G, T)$.*
(2) *If $\mathcal{Q}$ contains only infinite subgroups and $\mathcal{Q} \hookrightarrow_h (H, T)$, then $\mathcal{P}^* \hookrightarrow_h (G, S)$.*

1.1 Qi-characteristic collections

The first numbered hypothesis of Theorem A raises the following general problem: Given a finite collection of subgroups $\mathcal{Q}$ of a group H and a quasi-isometry $q\colon G \to H$ of finitely generated groups, is there a collection $\mathcal{P}$ of subgroups of G such that $q\colon (G, \mathcal{P}) \to (H, \mathcal{Q})$ is a quasi-isometry of pairs?

This problem was studied in [14] where the notion of qi-characteristic collection is introduced and it is proved that if the collection $\mathcal{Q}$ is qi-characteristic in H, then any quasi-isometry of finitely generated groups induces a collection $\mathcal{P}$.

Definition 1.5 (Qi-characteristic [14]) A collection of subgroups $\mathcal{P}$ of a finitely generated group G is *quasi-isometrically characteristic* (or shorter *qi-characteristic*) if $\mathcal{P}$ is finite; each $P \in \mathcal{P}$ has finite index in its commensurator; and for every $L \geq 1$ and $C \geq 0$, there is $M = M(G, \mathcal{P}, L, C) \geq 0$ such that every (L, C)-quasi-isometry $q\colon G \to G$ is an (L, C, M)-quasi-isometry of pairs $q\colon (G, \mathcal{P}) \to (G, \mathcal{P})$.

Example 1.4. The argument by Behrstock, Druțu, and Mosher proving quasi-isometric rigidity of relative hyperbolicity with respect to nonrelatively hyperbolic groups (NRH groups) shows that if H is hyperbolic group relative to a collection $\mathcal{Q}$ of NRH subgroups, then $\mathcal{Q}$ is qi-characteristic [4, Theorems 4.1 and 4.8]. Another example is provided by mapping class groups. Ruling out a few surfaces of low complexity, any self-quasi-isometry of the mapping class group is at uniform distance

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from left multiplication by an element of the group (see the work of Behrstock, Kleiner, Minsky, and Mosher [6, Theorem 1.1]). As a consequence, the hyperbolically embedded (virtually cyclic) subgroup generated by a pseudo-Anosov is qi-characteristic. More generally, any finite collection of subgroups of such mapping class groups are qi-characteristic.

Corollary B *Let G and H be finitely generated groups, let T be a generating set of H, let $\mathcal{Q}$ be a finite collection of subgroups of H such that $\mathcal{Q} \hookrightarrow_h (H, T)$, and let $q: G \to H$ be a quasi-isometry. If:*

(1) *$\mathcal{Q}$ is a qi-characteristic collection of subgroups of H and*
(2) *there is a generating set $S \subset G$ such that $q: \Gamma(G, S) \to \Gamma(H, T)$ is a quasi-isometry,*

then there is a finite collection $\mathcal{P}$ of subgroups of G such that $\mathcal{P} \hookrightarrow_h (G, S)$ and $q: (G, \mathcal{P}) \to (H, \mathcal{Q})$ is a quasi-isometry of pairs.

Proof Without loss of generality, assume that all subgroups in $\mathcal{Q}$ are proper infinite subgroups. Note that removing finite subgroups from $\mathcal{Q}$ preserves being qi-characteristic and that $\mathcal{Q} \hookrightarrow_h (H, T)$. On the other hand, if $\mathcal{Q}$ contains H, then the theorem is trivial by taking $\mathcal{P}$ the collection that contains only G and S any finite generating set of G. Since $\mathcal{Q}$ is qi-characteristic, the quasi-isometry $q: G \to H$ induces a finite collection $\mathcal{P}$ such that $q: (G, \mathcal{P}) \to (H, \mathcal{Q})$ is a quasi-isometry of pairs, and this is precisely [14, Theorem 1.1]. Then the second statement of Theorem A and $\mathcal{Q} \hookrightarrow_h (H, T)$ imply that $\mathcal{P}^* \hookrightarrow_h (G, S)$. ∎

1.2 Uniform quasi-actions

The second numbered hypothesis of Theorem A raises the problem: Given a group H with a generating set T and a quasi-isometry $q: G \to H$ of finitely generated groups, is there a generating set $S \subset G$ such that $q: \Gamma(G, S) \to \Gamma(H, T)$ is a quasi-isometry of Cayley graphs?

We show that a positive answer to this question is equivalent to asking that the quasi-action of G on H induced by q is T-uniform in the following sense (see Proposition C).

Definition 1.6 (Uniform induced quasi-action) Let G and H be finitely generated groups, and let $q: G \to H$ be a quasi-isometry with quasi-inverse $\bar{q}$. Let $T \subset H$ be a generating set (possibly infinite). We say that the quasi-action of G on H induced by q is *uniform with respect to* T if there are constants $L \geq 1$, $C \geq 0$ such that, for each $g \in G$, the function $q_g: H \to H$ given by $q_g(h) = q(g \cdot \bar{q}(h))$ is an (L, C)-quasi-isometry $q_g: \Gamma(H, T) \to \Gamma(H, T)$.

Example 1.5 (Uniform quasi-action and finite extensions) Let H be a finite index normal subgroup of finitely generated group G, and let T be a generating set of H invariant under conjugation by G. The G-action by conjugation on H preserves the word metric induced by T. On the other hand, any transversal R of H in G induces

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a quasi-isometry $q: G \to H$ given by $q(hg) = h$ for $h \in H$ and $g \in R$. In this case, the quasi-action of G on H induced by q is uniform with respect to T (see Lemma 2.1).

Proposition C (Proposition 2.1) *Let G and H be groups with finite generating sets S_0 and T_0, and let $q: \Gamma(G, S_0) \to \Gamma(H, T_0)$ be a quasi-isometry. Let $T \subset H$ containing T_0. The following statements are equivalent:*

(1) *The quasi-action of G on H induced by q is uniform with respect to T.*
(2) *There is $S \subset G$ containing S_0 such that $q: \Gamma(G, S) \to \Gamma(H, T)$ is a quasi-isometry.*

Corollary D *Let G and H be finitely generated groups with finite collections of infinite subgroups $\mathcal{P}$ and $\mathcal{Q}$, respectively. Suppose that $q: (G, \mathcal{P}) \to (H, \mathcal{Q})$ is a quasi-isometry of pairs inducing a T-uniform quasi-action of G on H. If $\mathcal{Q} \hookrightarrow_h (H, T)$, then $\mathcal{P}^* \hookrightarrow_h G$.*

Proof Since the quasi-action of G on H induced by q is T-uniform, Proposition C implies that there is a generating set S of G such that $q: \Gamma(G, S) \to \Gamma(H, T)$ is a quasi-isometry. Then the second statement of Theorem A and $\mathcal{Q} \hookrightarrow_h (H, T)$ imply that $\mathcal{P}^* \hookrightarrow_h (G, S)$. ■

Let us remark that for this last corollary, in the case that T is finite, then there is a finite $S \subset G$ such that $\mathcal{P} \hookrightarrow_h (G, S)$; this case is implied by the results on quasi-isometric rigidity of relative hyperbolicity in [4].

1.3 Finite extensions

The following application is a particular instance of Theorem 4.1 in the main body of the article.

Theorem E (Theorem 4.1) *Let H be a finite index normal subgroup of a finitely generated group G, and let $\mathcal{Q}$ be a finite collection of infinite subgroups of H such that $\mathcal{Q} \hookrightarrow_h (H, T)$. Suppose that:*

(1) *The set T is invariant under conjugation by G.*
(2) *The collection $\{hQh^{-1}: h \in H \text{ and } Q \in \mathcal{Q}\}$ is invariant under conjugation by G.*

If $\mathcal{Q}^*$ is a refinement of $\mathcal{Q}$ in G, then $\mathcal{Q}^* \hookrightarrow_h G$.

Example 1.6. Let $G = \langle a, b, t: tat^{-1} = b,\ t^2 = 1\rangle \cong F_2 \rtimes \mathbb{Z}_2$, let $H = \langle a, b\rangle$, and let $\mathcal{Q} = \{\langle a\rangle, \langle b\rangle\}$. Note that $\mathcal{Q} \hookrightarrow_h H$, and, for instance, one can take $\mathcal{Q}^* = \{\langle a\rangle\}$ and observe that $\mathcal{Q}^* \hookrightarrow_h G$. In contrast, for $\mathcal{Q}_0 = \{\langle a\rangle\} \hookrightarrow_h H$, the theorem does not apply since the conjugates of $\langle a\rangle$ in H are not invariant under conjugation by elements of G.

The next result illustrates concrete examples where Theorem E applies.

Theorem F (Theorem 5.1) *Let A be a finitely generated group with a (not necessarily finite) generating set T, and let $\mathcal{H}$ be a finite collection of infinite subgroups such that $\mathcal{H} \hookrightarrow_h (A, T)$. If $F \leq \mathrm{Aut}(A)$ is finite, T and $\mathcal{H}$ are F-invariant, and the F-action on $\mathcal{H}$ is free, then $\mathcal{H}_F \hookrightarrow_h (A \rtimes F, T \cup F)$ where $\mathcal{H}_F$ is collection of representatives of F-orbits in $\mathcal{H}$.*

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Example 1.7. Let $A = *_{i=1}^{n} B_i$ with each B_i isomorphic to a fixed finitely generated group B. Let $F = \mathbb{Z}_n$ act on A by cyclically permuting the copies of B. Consider the generating set of A given by $T = \bigcup_{i=1}^{n} B_i \setminus \{1\}$, then T is F-invariant. Now, the collection $\mathcal{H} = \{B_1, \ldots, B_n\}$ is hyperbolically embedded into (A, T) and F acts freely by conjugation on $\mathcal{H}$. All of the hypotheses of the previous theorem have been verified, so we conclude that $B_1 \hookrightarrow_h (A \rtimes F, T \cup F)$.

1.4 Organization

The rest of the article is divided into five sections. Section 2 is on quasi-actions; it contains the proof of Proposition C as well as some corollaries. The proof of Theorem A is the content of Section 3. Then Sections 4 and 5 contain the proofs of Theorems E and F, respectively. Finally, Section 6 contains some questions and discussion about related to the results in this article and the definition of a quasi-isometry of pairs.

2 Uniform quasi-actions

Definition 2.1 (Uniform quasi-action) Let G be a group, and let X be a metric space. Let $\mathsf{QI}(X)$ denote the set of quasi-isometries $X \to X$. A function $G \to \mathsf{QI}(X)$, $g \mapsto f_g$, is a *quasi-action* if there is $K \geq 0$ such that for any $g_1, g_2 \in G$:

(1) the map $f_{g_1 g_2}$ is at distance at most K from the map $f_{g_1} \circ f_{g_2}$ in the L_∞-distance and

(2) the map $f_{g_1} \circ f_{g_1^{-1}}$ is at distance at most K from the identity.

The quasi-action $G \to \mathsf{QI}(X)$ is *uniform* if there are constants $L \geq 1$ and $C \geq 0$ such that, for any $g \in G$, the map f_g is an (L, C)-quasi-isometry.

It is well known that a quasi-isometry $q\colon G \to H$ of finitely generated groups induces a uniform quasi-action of G on H:

Definition 2.2 (Uniform quasi-action induced by a quasi-isometry) Let G be a group with a word metric induced by a finite generating set, let X be a metric space, and let $q\colon G \to X$ and $\bar{q}\colon X \to G$ be (L_0, C_0)-quasi-isometries such that $q \circ \bar{q}$ and $\bar{q} \circ q$ are at distance less than C_0 from the identity maps on X and G, respectively. For $g \in G$, let

$$L_g\colon G \to G, \qquad x \mapsto gx,$$

and let

$$q_g\colon X \to X, \qquad q_g = q \circ \mathbf{g} \circ \bar{q}.$$

It is an exercise to verify that there are constants $L \geq 1$ and $C \geq 0$ such that:

- For $g \in G$, $q_g\colon X \to X$ is an (L, C)-quasi-isometry.
- (G quasi-acts on X) For $g_1, g_2 \in G$, the map $q_{g_1 g_2}$ is at distance at most C from the map $q_{g_1} \circ q_{g_2}$, and the map $q_{g_1} \circ q_{g_1^{-1}}$ is at distance at most C from the identity.

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- (G acts C_0-transitively on X) For every $x, y \in X$, there is $g \in G$ such that $\mathsf{dist}_G(x, q_g(y)) \leq C$.

The map $G \to \mathsf{QI}(X)$ given by $g \mapsto q_g$ is called the *uniform quasi-action of G on X induced by q and $\bar{q}$*.

Remark 2.1 (Equivalence of Definitions 1.6 and 2.2) In the context of Definition 1.6, if the induced quasi-action of G on H is uniform with respect to T, then $G \to \mathsf{QI}(\Gamma(H, T))$ given by $g \mapsto q_q$ is a uniform quasi-action in the sense of Definition 2.2. Indeed, since T contains a finite generating set of H, there is $M > 0$ such that $\mathsf{dist}_{(H,T)} \leq M\,\mathsf{dist}_{(H,T_0)}$. Hence, if two functions $H \to H$ are at finite L_∞-distance with respect to $\mathsf{dist}_{(H,T_0)}$, then the same holds for $\mathsf{dist}_{(H,T)}$.

Definition 2.3 (Cayley graph) Let G be a group with a generating set S. The *Cayley graph $\Gamma(G, S)$ of G with respect to S* is the G-graph with vertex set G and edge set $\{\{g, gs\} : g \in G,\ s \in S\}$.

Proposition 2.1 (Proposition C) *Let G and H be groups with finite generating sets S_0 and T_0, and let $q : \Gamma(G, S_0) \to \Gamma(H, T_0)$ be a quasi-isometry. Let $T \subset H$ containing T_0. The following statements are equivalent:*

(1) *The quasi-action of G on H induced by q is uniform with respect to T.*
(2) *There is $S \subset G$ containing S_0 such that $q : \Gamma(G, S) \to \Gamma(H, T)$ is a quasi-isometry.*

Proof That the second statement implies the first one is immediate. Conversely, suppose that q and $\bar{q}$ are (L_0, C_0)-quasi-isometries $\Gamma(G, S_0) \to \Gamma(H, T_0)$ and $\Gamma(H, T_0) \to \Gamma(G, S_0)$, respectively. Without loss of generality, assume that $q(e) = e$ and $\bar{q}(e) = e$ where e denotes the identity in each corresponding group.

Let $K_0 = L_0 + C_0 + 1$ and define

$$S = \{f^{-1}g \in G : \text{there are } h \in H \text{ and } t \in T \text{ such that } \mathsf{dist}_{(H,T_0)}(q(f), h) \leq K_0 \text{ and } \mathsf{dist}_{(H,T_0)}(q(g), ht) \leq K_0\}.$$

Note that $S_0 \subset S$ since $q : \Gamma(G, S_0) \to \Gamma(H, T_0)$ is an (L_0, C_0)-quasi-isometry. In particular, S is a generating set of G.

Let $L_1 \geq 1$ and $C_1 \geq 0$ be such that the G-action on H induced by q is (L_1, C_1)-uniform with respect to T. In particular, for every $g \in G$, the function $q_g : H \to H$ is an (L_1, C_1)-quasi-isometry $\Gamma(H, T) \to \Gamma(H, T)$.

Now, we prove that if the induced quasi-action of G on H is uniform with respect to T, then $q : \Gamma(G, S) \to \Gamma(H, T)$ is a quasi-isometry. Observe that every vertex of $\Gamma(H, T)$ is at distance at most C_0 from $q(G)$ with respect to $\mathsf{dist}_{(H,T_0)}$ and hence with respect to $\mathsf{dist}_{(H,T)}$. Below, we prove inequalities (2.1) and (2.2), which will conclude proof.

Claim *There is constant $\bar{L}$ such that*

$$\mathsf{dist}_{(H,T)}(q(a), q(b)) \leq \bar{L}\,\mathsf{dist}_{(G,S)}(a, b), \tag{2.1}$$

for any $a, b \in G$.

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Proof of claim Let $s \in S$. Then there are $f, g \in G$, $h \in H$, and $t \in T$ such that $s = f^{-1}g$ and

$$\mathsf{dist}_{(H,T_0)}(q(f),h) \le K_0, \quad \mathsf{dist}_{(H,T_0)}(q(g),ht) \le K_0.$$

It follows that

$$\mathsf{dist}_{(H,T)}(q_f(e),q_g(e)) = \mathsf{dist}_{(H,T)}(q(f),q(g)) \le 2K_0 + 1.$$

Since the quasi-action of G on $\Gamma(H,T)$ is (L_1,C_1)-uniform, the previous inequality implies that

$$\begin{aligned}\mathsf{dist}_{(H,T)}(e,q(s)) &= \mathsf{dist}_{(H,T)}(q_e(e),q_{f^{-1}g}(e))\\ &\le L_1\,\mathsf{dist}_{(H,T)}(q_f \circ q_e(e), q_f \circ q_{f^{-1}g}(e)) + C_1\\ &\le L_1\,\mathsf{dist}_{(H,T)}(q_f(e),q_g(e)) + 3C_1\\ &\le L_1(2K_0+1) + 3C_1 =: \bar{L}_0.\end{aligned}$$

For any $g \in G$ and $s \in S$, we have that

$$\begin{aligned}\mathsf{dist}_{(H,T)}(q(g),q(gs)) &= \mathsf{dist}_{(H,T)}(q_g(e),q_{gs}(e))\\ &\le L_1\,\mathsf{dist}_{(H,T)}(q_{g^{-1}} \circ q_g(e), q_{g^{-1}} \circ q_{gs}(e)) + C_1\\ &\le L_1\,\mathsf{dist}_{(H,T)}(e, q_{g^{-1}gs}(e)) + 3C_1\\ &\le L_1\,\mathsf{dist}_{(H,T)}(q(e),q(s)) + 3C_1,\end{aligned}$$

and hence

$$\mathsf{dist}_{(H,T)}(q(g),q(gs)) \le \mathsf{dist}_{(H,T_0)}(q(g),q(gs)) \le \bar{L},$$

where $\bar{L} = L_1(\bar{L}_0) + 3C_1$. If $a, b \in G$ and $[u_0,\dots,u_\ell]$ is a geodesic in $\Gamma(G,S)$ from a to b, then the triangle inequality implies inequality (2.1). ■

Claim *For any $a, b \in G$, we have*

$$\mathsf{dist}_{(G,S)}(a,b) \le \mathsf{dist}_{(H,T)}(q(a),q(b)). \tag{2.2}$$

Proof of claim Suppose that $[h_0,\dots,h_\ell]$ is a geodesic in $\Gamma(H,T)$ from $q(a)$ to $q(b)$. Since $q\colon \Gamma(G,S_0) \to \Gamma(H,T_0)$ is an (L_0,C_0)-quasi-isometry, for each i, there is $g_i \in G$ such that $\mathsf{dist}_{(H,T_0)}(q(g_i),h_i) \le C_0$. Let $g_0 = a$ and $g_\ell = b$. Observe that $g_i^{-1}g_{i+1} \in S$ for $0 \le i < \ell$, and hence $\mathsf{dist}_{(G,S)}(g_i,g_{i+1}) \le 1$. Now, $[g_0,\dots,g_\ell]$ is a path in $\Gamma(G,S)$ from a to b and therefore $\mathsf{dist}_{(G,S)}(a,b) \le \mathsf{dist}_{(H,T)}(q(a),q(b))$ proving inequality (2.2). ■

■

Corollary 2.1 *Let G and H be groups with finite generating sets S_0 and T_0. Let $q\colon G \to H$ be a group homomorphism which is also an (L_0,C_0)-quasi-isometry $q\colon \Gamma(G,S_0) \to \Gamma(H,T_0)$. If $T \subset H$ contains T_0, then there is $S \subset G$ containing S_0 such that $q\colon \Gamma(G,S) \to \Gamma(H,T)$ is a quasi-isometry.*

Proof Let $\bar{q}\colon H \to G$ be a quasi-inverse of q and, by increasing L_0 and C_0 if necessary, assume that $\bar{q}\colon \Gamma(H,T_0) \to \Gamma(G,S_0)$ is an (L_0,C_0)-quasi-isometry. Moreover,

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suppose $q \circ \bar{q}$ and $\bar{q} \circ q$ are at distance at most C_0 from the corresponding identity maps with respect to $\mathsf{dist}_{(H,T_0)}$ and $\mathsf{dist}_{(G,S_0)}$. Note that for any $g \in G$,

$$q_g(h) = q(g \cdot \bar{q}(h)) = q(g) \cdot q(\bar{q}(h)).$$

Hence, q_g is a $(1, C_0)$-quasi-isometry since it is the composition of $q \circ \bar{q}$ followed by the isometry given by multiplication on the left by $q(g)$. Then the proof concludes by invoking Proposition 2.1. ∎

The following result is the particular case of Corollary 2.1 in which H is a finite index subgroup of G. In this case, one can give a more algebraic description of the generating set S. The proof follows the same lines as the previous argument modulo Lemma 2.1.

Proposition 2.2 *Let H be a finite index normal subgroup of a finitely generated group G. Let T be a generating set of H, let R be a right transversal of H in G, and let $S = T \cup R$. If the G-action by conjugation on H is a uniform quasi-action on $\Gamma(H, T)$, then the inclusion $\Gamma(H, T) \hookrightarrow \Gamma(G, S)$ is a quasi-isometry.*

We divert the proof of the proposition after the following lemma.

Lemma 2.1 *Let H be a finite index normal subgroup of a finitely generated group G. Let T be a generating set of H containing a finite generating set T_0, let R be transversal of H in G, let S_0 be a finite generating set of G, and let $q \colon \Gamma(G, S_0) \to \Gamma(H, T_0)$ be the quasi-isometry defined by $q(hg) = h$ for $h \in H$ and $g \in R$. The following statements are equivalent:*

(1) *The G-action by conjugation on H is a uniform quasi-action on $\Gamma(H, T)$.*
(2) *The quasi-action of G on H induced by q is uniform with respect to T.*

Proof Take as the quasi-inverse of q the inclusion $H \hookrightarrow G$. For $h \in H$, let $L_h \colon H \to H$ be given by $L_h(x) = hx$, i.e., multiplication on the left. Note that $L_h \colon \Gamma(H, T) \to \Gamma(H, T)$ is an isometry for every $h \in H$.

Let $g \in G$ and suppose that $g = h_* g_*$ where $h_* \in H$ and $g_* \in R$. Then

$$q_g(h) = q(gh) = q(ghg^{-1}h_* g_*) = ghg^{-1}h_* = h_* g_* h g_*^{-1} h_*^{-1} h_* = h_* g_* h g_*^{-1}$$

and hence

$$q_g = L_{h_*} \circ \mathrm{Ad}(g_*),$$

where $\mathrm{Ad}(g_*)$ is conjugation by g_*. It follows $q_g \colon \Gamma(H, T) \to \Gamma(H, T)$ is an (L, C)-quasi-isometry for all $g \in G$ if and only if $\mathrm{Ad}(g_*) \colon \Gamma(H, T) \to \Gamma(H, T)$ is an (L, C)-quasi-isometry for all $g_* \in R$. In particular, the first statement implies the second by Remark 2.1, and the second statement implies the first since the constants L and C hold for all conjugations. ∎

Proof [**Proof of Proposition 2.2.**] Let $T_0 \subset T$ be a finite generating set of H, and let $S_0 = T_0 \cup R$. Note that S_0 is a finite generating set of G. Then $q \colon \Gamma(G, S_0) \to \Gamma(H, T_0)$

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is an (L_0, C_0) quasi-isometry for some $L_0 \geq 1$ and $C_0 \geq 0$, and the quasi-inverse $\bar{q}$ can be taken as the inclusion $\Gamma(H, T_0) \hookrightarrow \Gamma(G, S_0)$.

Observe that, in $\Gamma(G, S)$, the vertices $g = hr$ and $q(g) = h$ are adjacent since $r \in S$. Therefore, if $[v_0, \ldots, v_\ell]$ is a geodesic path in $\Gamma(H, T)$ from $q(a)$ to $q(b)$, then $[a, v_0, \ldots, v_\ell, b]$ is a path in $\Gamma(G, S)$ from a to b, and hence

$$\mathsf{dist}_{(G,S)}(a, b) \leq \mathsf{dist}_{(H,T)}(q(a), q(b)) + 2.$$

We now prove the other inequality. Since the G-action on H by conjugation is a uniform quasi-action on $\Gamma(H, T)$, Lemma 2.1 implies that the quasi-action of G on H induced by q is (L_1, C_1)-uniform with respect to T, for some $L_1 \geq 1$ and $C_1 \geq 0$.

Let $K_0 = L_0 + C_0 + 1$. Observe that

$$S \subseteq \{f^{-1}g \in G\colon \text{there are } h \in H \text{ and } t \in T \text{ such that } \mathsf{dist}_{(H,T_0)}(q(f), h) \leq K_0 \text{ and } \mathsf{dist}_{(H,T_0)}(q(g), ht) \leq K_0\}.$$

Indeed, let $s \in S = T \cup R$, and there are two cases. First, if $s \in T$, let $f = h = e$ and $g = t = s$, and second, if $s \in R$, let $f = h = e$, $g = s$, and t be any element of T_0. Then, exactly as in the first claim in the proof of Proposition 2.1, one defines a constant $\bar{L} = \bar{L}(L_1, C_1, K_0)$ and deduces the inequality

$$\mathsf{dist}_{(H,T)}(q(a), q(b)) \leq \bar{L}\, \mathsf{dist}_{(G,S)}(a, b). \tag{2.3}$$

It remains to show that

$$\mathsf{dist}_{(G,S)}(a, b) \leq \mathsf{dist}_{(H,T)}(q(a), q(b)) + 2, \tag{2.4}$$

for any $a, b \in G$, concluding the proof. This is clear since $\Gamma(H, T)$ is a subgraph of $\Gamma(G, T)$ and $\mathsf{dist}_{G,S}(g, q(g)) \leq 1$ for any $g \in G$. ∎

The following example by Minasyan and Osin illustrates the need for the hypothesis relating to the conjugation action in Corollary 2.2.

Example 2.1 [16] Let $H = \langle a, b \rangle$ be the free group of rank 2, let $G = \langle a, b, t\colon tat^{-1} = b, t^2 = e \rangle$, let $T = \{b, a, a^{-1}, a^2, a^{-2}, \ldots\}$, and let $S = T \cup \{t\}$. The inclusion $\Gamma(H, T) \to \Gamma(G, S)$ is not a quasi-isometry. Indeed, in G, we have $ta^n t^{-1} = b^n$, and hence $\mathsf{dist}_{(G,S)}(e, b^n) = 3$, but $\mathsf{dist}_{(H,T)}(e, b^n) = n$ for every n. In particular, the map $\Gamma(H, T) \to \Gamma(H, T)$ given by $h \mapsto tht^{-1}$ is not a quasi-isometry, and hence the G-action on H by conjugation is not an action by quasi-isometries.

3 Quasi-isometries and hyperbolically embedded subgroups

In this section, we will prove Theorem A. The theorem is obtained by putting together a simple characterization of hyperbolically embedded subgroups in terms of coned-off Cayley graphs which appeared in the work of Rashid and the second author (see [13, Propositions 1.5 and 5.8]), some results about quasi-isometries of pairs from [11], and some basic facts about hyperbolically embedded subgroups from [7]. Below, we state these results and then we discuss the proof of Theorem 3.1.

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Definition 3.1 (Reduced collections) A collection of subgroups $\mathcal{Q}$ of a group H is *reduced* if for any $P, Q \in \mathcal{Q}$ and $g \in H$, if P and gQg^{-1} are commensurable subgroups, then $P = Q$ and $g \in P$.

Remark 3.1. An almost malnormal collection is reduced.

Definition 3.2 (Fine) Let Γ be a graph, and let v be a vertex of Γ. Let

$$T_v\Gamma = \{w \in V(\Gamma) \mid \{v, w\} \in E(\Gamma)\}$$

denote the set of the vertices adjacent to v. For $x, y \in T_v\Gamma$, the *angle metric* $\angle_v(x, y)$ is the length of the shortest path in the graph $\Gamma \backslash \{v\}$ between x and y, with $\angle_v(x, y) = \infty$ if there is no such path. The graph Γ is *fine at* v if $(T_v\Gamma, \angle_v)$ is a locally finite metric space. The graph Γ is *fine at* $C \subseteq V(\Gamma)$ if Γ is fine at v for all $v \in C$.

Definition 3.3 (Coned-off Cayley graph) Let G be a group, let $\mathcal{P}$ be an arbitrary collection of subgroups of G, and let S be a subset of G. Denote by $G/\mathcal{P}$ the set of all cosets gP with $g \in G$ and $P \in \mathcal{P}$. The *coned-off Cayley graph of G with respect to $\mathcal{P}$* is the graph $\hat{\Gamma}(G, \mathcal{P}, S)$ with vertex set $G \cup G/\mathcal{P}$ and edges are of the following type:

- $\{g, gs\}$ for $s \in S$ and
- $\{x, gP\}$ for $g \in G$, $P \in \mathcal{P}$, and $x \in gP$.

We call vertices of the form gP *cone points*.

Proposition 3.1 [13] *Let $\mathcal{P}$ be a collection of infinite subgroups of G, and let S be a subset of G. Then $\mathcal{P} \hookrightarrow_h (G, S)$ if and only if the Coned-off Cayley graph $\hat{\Gamma}(G, \mathcal{P}, S)$ is a connected hyperbolic graph which is fine at every cone vertex.*

Proposition 3.2 [11, Proposition 5.6] *Let G and H be groups, let $S \subset G$ and $T \subset H$, and let $S_0 \subset S$ and $T_0 \subset T$ be finite generating sets of G and H, respectively. Consider collections $\mathcal{P}$ and $\mathcal{Q}$ of subgroups of G and H, respectively. Let $q: G \to H$ be a function.*

Suppose that q is a quasi-isometry $\Gamma(G, S) \to \Gamma(H, T)$, is a quasi-isometry of pairs $(G, \mathcal{P}, S_0) \to (H, \mathcal{Q}, T_0)$, and $\dot{q}$ is a bijection $G/\mathcal{P} \to H/\mathcal{Q}$.

(1) *Let $\hat{q} = q \cup \dot{q}$, then $\hat{q}$ is a quasi-isometry $\hat{\Gamma}(G, \mathcal{P}, S) \to \hat{\Gamma}(H, \mathcal{Q}, T)$.*
(2) *If $\hat{\Gamma}(H, \mathcal{Q}, T)$ is fine at cone vertices, then $\hat{\Gamma}(G, \mathcal{P}, S)$ is fine at cone vertices.*
(3) *If $\mathcal{Q} \hookrightarrow_h (H, T)$, then $\mathcal{P} \hookrightarrow_h (G, S)$.*

Items (1) and (2) of Proposition 3.2 are taken from [11, Proposition 5.6], and the last item is a direct consequence of Proposition 3.1.

Proposition 3.3 [11, Proposition 5.12] *Let $q: (G, \mathcal{P}) \to (H, \mathcal{Q})$ be an (L, C, M)-quasi-isometry of pairs. Then:*

(1) *$\dot{q}$ is a surjective function $G/\mathcal{P} \to H/\mathcal{Q}$ if $\mathcal{Q}$ is reduced.*
(2) *$\dot{q}$ is a bijection $G/\mathcal{P} \to H/\mathcal{Q}$ if $\mathcal{P}$ and $\mathcal{Q}$ are reduced.*

Proposition 3.4 [11, Proposition 6.2] *Let $\mathcal{P}^*$ be a refinement of a finite collection of subgroups $\mathcal{P}$ of a finitely generated group G. If P is a finite index subgroup of* $\mathrm{Comm}_G(P)$

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for every $P \in \mathcal{P}$, then $(G, \mathcal{P})$ and $(G, \mathcal{P}^*)$ are quasi-isometric pairs via the identity map on G.

Proposition 3.5 [11, Proposition 6.7] *Let $q: (G, \mathcal{P}) \to (H, \mathcal{Q})$ be a quasi-isometry of pairs. If $\mathcal{Q}$ is an almost malnormal finite collection of infinite subgroups and $\mathcal{P}$ is a finite collection, then any refinement $\mathcal{P}^*$ of $\mathcal{P}$ is almost malnormal.*

Proposition 3.6 [7, Proposition 4.33] *Let $\mathcal{P}$ be a collection of subgroups of a group G. If $\mathcal{P} \hookrightarrow_h G$, then $\mathcal{P}$ is an almost malnormal collection.*

We are now ready to prove Theorem A.

Theorem 3.1 (Theorem A) *Let $q: G \to H$ be a quasi-isometry of finitely generated groups, let $\mathcal{P}$ and $\mathcal{Q}$ be finite collections of subgroups of G and H, respectively, and let S and T be (not necessarily finite) generating sets of G and H, respectively. Suppose that:*

(1) *$q: (G, \mathcal{P}) \to (H, \mathcal{Q})$ is a quasi-isometry of pairs and*
(2) *$q: \Gamma(G, S) \to \Gamma(H, T)$ is a quasi-isometry.*

The following statements hold:

(1) *If $\mathcal{P}$ and $\mathcal{Q}$ are reduced collections in G and H, respectively, then $\mathcal{P} \hookrightarrow_h (G, S)$ if and only if $\mathcal{Q} \hookrightarrow_h (H, T)$.*
(2) *If $\mathcal{Q}$ contains only infinite subgroups and $\mathcal{Q} \hookrightarrow_h (H, T)$, then $\mathcal{P}^* \hookrightarrow_h (G, S)$.*

Proof For the first statement, since $\mathcal{P}$ and $\mathcal{Q}$ are reduced, Proposition 3.3 implies that $\dot{q}: G/\mathcal{P} \to H/\mathcal{Q}$ is a bijection. Then Proposition 3.2 implies that $\hat{\Gamma}(G, \mathcal{P}, S)$ is hyperbolic and fine at cone vertices if and only if $\hat{\Gamma}(H, \mathcal{Q}, T)$ is hyperbolic and fine at cone vertices. Then Proposition 3.1 concludes the proof of the first statement.

The second statement is a consequence of the first statement as follows. That $\mathcal{Q} \hookrightarrow_h H$ implies that $\mathcal{Q}$ is an almost malnormal collection of subgroups in H (see Proposition 3.6). It follows that $\mathcal{Q}$ is reduced in H. Then, since $\mathcal{Q}$ contains only infinite subgroups, Proposition 3.5 implies that $\mathcal{P}^*$ is reduced. By Proposition 3.4, $q: (G, \mathcal{P}^*) \to (H, \mathcal{Q})$ is a quasi-isometry of pairs. Then $\mathcal{Q} \hookrightarrow_h H$ and the first statement of the proposition imply that $\mathcal{P}^* \hookrightarrow (G, S)$. ∎

4 Hyperbolically embedded subgroups and commensurability

In this section, we prove Theorem E. The argument uses the following proposition, which is a strengthening of [14, Proposition 2.15]. It essentially follows from the proof in the cited article, but we have included the proof for the convenience of the reader.

Proposition 4.1 *Let H be a finite index subgroup of a finitely generated group G, and let $\mathcal{Q}$ be a finite collection of subgroups of H. The following statements are equivalent:*

(1) *The inclusion $H \hookrightarrow G$ is a quasi-isometry of pairs $(H, \mathcal{Q}) \hookrightarrow (G, \mathcal{Q})$.*
(2) *For any $Q \in \mathcal{Q}$ and $g \in G$, there is $Q' \in \mathcal{Q}$ and $h \in H$ such that $\mathsf{hdist}_G(gQ, hQ') < \infty$.*

Proof That (1) implies (2) is trivial. Assume statement (2). Since H is a finite index subgroup of the finitely generated group G, assume that $H \hookrightarrow G$ is an (L, C) quasi-isometry. Since H is finite index in G, and $\mathcal{Q}$ is a finite collection, the H-action on $G/\mathcal{Q}$ has finitely many orbits. For $gQ \in G/\mathcal{Q}$, let

$$\mathsf{hdist}_G(gQ, H/\mathcal{Q}) := \min\{\mathsf{hdist}_G(gQ, hQ')\colon hQ' \in H/\mathcal{Q}\}.$$

Let $\mathcal{R}$ be a finite collection of orbit representatives of the H-action on $G/\mathcal{Q}$. By hypothesis, for $gQ \in \mathcal{R}$, there is $hQ' \in H/\mathcal{Q}$ such that $\mathsf{hdist}(gQ, hQ') < \infty$ and therefore

$$M = \max\{\mathsf{hdist}_G(gQ, H/\mathcal{Q})\colon gQ \in \mathcal{R}\} < \infty$$

is a well-defined integer since $\mathcal{R}$ is a finite set. Since the subset $H/\mathcal{Q}$ of $G/\mathcal{Q}$ is H-invariant,

$$\mathsf{hdist}_G(gQ, H/\mathcal{Q}) = \mathsf{hdist}_G(hgQ, H/\mathcal{Q})$$

for every $gQ \in \mathcal{R}$ and $h \in H$. Since $\mathcal{R}$ is a collection of representatives of orbits of $G/\mathcal{Q}$,

$$\mathsf{hdist}_G(gQ, H/\mathcal{Q}) \leq M$$

for every $gQ \in G/\mathcal{Q}$. Hence, $(H, \mathcal{Q}) \hookrightarrow (G, \mathcal{Q})$ is an (L, C, M) quasi-isometry of pairs. ∎

Remark 4.1. Let G be a group, and let T and S generating sets with finite symmetric difference. Then the identity map on G is a quasi-isometry $\Gamma(G, T) \to \Gamma(G, S)$.

Theorem 4.1 (Theorem E) *Let H be a finite index normal subgroup of a finitely generated group G, and let $\mathcal{Q}$ be a finite collection of infinite subgroups of H such that $\mathcal{Q} \hookrightarrow_h (H, T)$. Suppose that:*

(1) *The G-action by conjugation on H is a uniform quasi-action on $\Gamma(H, T)$.*
(2) *The collection $\{hQh^{-1}\colon h \in H \text{ and } Q \in \mathcal{Q}\}$ is invariant under conjugation by G.*

If $\mathcal{Q}^*$ is a refinement of $\mathcal{Q}$ in G and R is a transversal of H in G, then $\mathcal{Q}^* \hookrightarrow_h (G, T \cup R)$.

Proof Since H is finitely generated, by adding a finitely many elements, we can assume that T generates H. Note that this preserves $\mathcal{Q} \hookrightarrow_h (H, T)$ by [7, Corollary 4.27], and the quasi-isometry type of $\Gamma(H, T)$ by Remark 4.1. Under this assumption, the conclusion will follow from the second statement of Theorem 3.1 applied to the quasi-isometry of finitely generated groups given by the inclusion $H \hookrightarrow G$.

Since $\mathcal{Q} \hookrightarrow_h (H, T)$, $\mathcal{Q}$ is an almost malnormal collection (see Proposition 3.6). The assumption that $\mathcal{Q}$ consists only of infinite subgroups implies that for any $Q \in \mathcal{Q}$,

$$Q = \mathrm{Comm}_H(Q) = \mathrm{Comm}_G(Q) \cap H.$$

Since H is finite index in G, we have that Q is finite index in $\mathrm{Comm}_G(Q)$. Then Proposition 3.4 implies that the identity map on G is a quasi-isometry of pairs $(G, \mathcal{Q}) \to (G, \mathcal{Q}^*)$. On the other hand, since the collection $\{hQh^{-1}\colon h \in H$ and $Q \in \mathcal{Q}\}$ is invariant under conjugation by elements of G, we have for any $g \in G$ and $Q \in \mathcal{Q}$ there is $h \in H$ such that $gQg^{-1} = hQ'h^{-1}$ and hence

$$\mathsf{hdist}_G(gQ, hQ') \leq \mathsf{hdist}_G(gQ, Q^g) + \mathsf{hdist}_G(Q^g, (Q')^h) + \mathsf{hdist}((Q')^h, hQ') < \infty.$$

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Proposition 4.1 implies that $H \hookrightarrow G$ is a quasi-isometry of pairs $(H, \mathcal{Q}) \to (G, \mathcal{Q})$. It follows that $H \hookrightarrow G$ is a quasi-isometry of pairs $(H, \mathcal{Q}) \to (G, \mathcal{Q}^*)$ as it is the composition $(H, \mathcal{Q}) \hookrightarrow (G, \mathcal{Q}) \to (G, \mathcal{Q}^*)$. Let R be a transversal of H in G, and let $S = T \cup R$. Since the G-action by conjugation on H is uniform on $\Gamma(H, T)$, Proposition 2.2 implies that $H \hookrightarrow G$ is a quasi-isometry $\Gamma(H, T) \to (G, S)$. The hypothesis of Theorem 3.1 has been verified, and therefore $\mathcal{Q} \hookrightarrow_h (H, T)$ implies $\mathcal{Q}^* \hookrightarrow_h (G, S)$. ∎

5 Semidirect products and hyperbolically embedded subgroups

In this section, we will prove Theorem F about semidirect products. The hypothesis of the following proposition and theorem reflects the issues posed by the example of Minasyan and Osin (Example 2.1).

Proposition 5.1 *Let A be a group with (not necessarily finite) generating set T, let $\mathcal{H}$ be a collection of subgroups, and let $F \le \mathsf{Aut}(A)$ be a finite subgroup. Suppose that T and $\mathcal{H}$ are F-invariant and that the F-action on $\mathcal{H}$ is free. Let $\mathcal{H}_F$ be a collection of representatives of F-orbits in $\mathcal{H}$. Then the inclusion $A \hookrightarrow A \rtimes F$ induces:*

(1) *a quasi-isometry $\Gamma(A, T) \to \Gamma(A \rtimes F, T \cup F)$ and*
(2) *if A is finitely generated, a quasi-isometry of pairs $(A, \mathcal{H}) \to (A \rtimes F, \mathcal{H}_F)$.*

Proof To prove the first statement, let $S = T \cup F$ and let dist_T and dist_S be the word metrics on A and $A \rtimes F$ induced by T and S, respectively. Let $q\colon A \hookrightarrow A \rtimes F$ be the inclusion, and let $\bar{q}\colon A \rtimes F \to A$ be such that for $a \in A$ and $f \in F$, $\bar{q}(af) = a$. Note that $\bar{q}$ is a well-defined A-equivariant map since each element of $A \rtimes F$ can be expressed as a product af in a unique way. Observe that $\bar{q} \circ q$ is the identity on A, and $q \circ \bar{q}$ is at distance 1 from the identity map on $A \rtimes F$ with respect to dist_S. Since the Cayley graph $\Gamma(A, T)$ is a subgraph of $\Gamma(A \rtimes F, T \cup F)$, it is immediate that for any $u, v \in A$, $\mathsf{dist}_S(q(u), q(v)) \le \mathsf{dist}_T(u, v)$. To conclude the proof of the statement, we show that for any $u, v \in A \rtimes F$, $\mathsf{dist}_T(\bar{q}(u), \bar{q}(v)) \le \mathsf{dist}_S(u, v)$. Note that it is enough to consider the case that $\mathsf{dist}_S(u, v) = 1$. Let $w_1, w_2 \in A \rtimes F$ such that $\mathsf{dist}_S(w_1, w_2) = 1$. Then $w_1 = a_1 f_1$ and $w_2 = a_2 f_2$ and $\bar{q}(w_i) = a_i$. It follows that $g = (a_1 f_1)^{-1} a_2 f_2 \in T \cup F$. Observe that

$$g = f_1^{-1} a_1^{-1} a_2 f_2 = (a_1^{-1})^{f_1^{-1}} f_1^{-1} a_2 f_2 = (a_1^{-1})^{f_1^{-1}} a_2^{f_1^{-1}} f_1^{-1} f_2 = (a_1^{-1} a_2)^{f_1^{-1}} f_1^{-1} f_2 \in T \cup F.$$

There are two cases, either $g \in T$ or $g \in F$, since $T \cap F = \varnothing$. We regard $T \cup F$ and F as a subset and a subgroup of $A \rtimes F$, respectively. If $g \in T$, then $f_1 = f_2$ and hence $(a_1^{-1} a_2)^{f_1^{-1}} \in T$; since T is F-invariant, a_1 and a_2 are adjacent in Γ, and hence $\mathsf{dist}_T(\bar{q}(w_1), \bar{q}(w_2)) = 1$. If $g \in F$, then $a_1 = a_2$ and hence $\mathsf{dist}_T(\bar{q}(w_1), \bar{q}(w_2)) = 0$.

For the second statement, suppose that A is finitely generated and let dist denote word metric on $A \rtimes F$ induced by finite generating set, and let $\mathsf{hdist}_{A \rtimes F}$ be the induced Hausdorff distance. Let $M = \max_{f \in F} \mathsf{dist}(1, f)$. Since the inclusion $A \hookrightarrow A \rtimes F$ is a quasi-isometry of finitely generated groups and $\mathcal{H}_F \subset \mathcal{H}$, it is enough to prove that for any $H \in \mathcal{H}$, there is a left coset in $(A \rtimes F)/\mathcal{H}_F$ at Haudorff distance at most M in $A \rtimes F$. Let $H \in \mathcal{H}$. Since the F-action on $\mathcal{H}$ by conjugation is free, there is a unique

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$f \in F$ and a unique $K \in \mathcal{H}_F$ such that $H = fKf^{-1}$. Observe that

$$\mathsf{hdist}(H, fK) = \mathsf{hdist}(fKf^{-1}, fK) \le \mathsf{dist}(1, f^{-1}) \le M,$$

and this completes the proof. ∎

Theorem 5.1 (Theorem F) *Let A be a finitely generated group with (not necessarily finite) generating set T, and let $\mathcal{H}$ be a finite collection of infinite subgroups such that $\mathcal{H} \hookrightarrow_h (A, T)$. If $F \le \mathsf{Aut}(A)$ is finite, T and $\mathcal{H}$ are F-invariant, and the F-action on $\mathcal{H}$ is free, then $\mathcal{H}_F \hookrightarrow_h (A \rtimes F, T \cup F)$, where $\mathcal{H}_F$ is collection of representatives of F-orbits in $\mathcal{H}$.*

Proof By Proposition 5.1, the inclusion $A \hookrightarrow A \rtimes F$ induces a quasi-isometry $\Gamma(A, T) \to \Gamma(A \rtimes F, T \cup F)$, and a quasi-isometry of pairs $(A, \mathcal{H}) \to (A \rtimes F, \mathcal{H}_F)$. Since $\mathcal{H} \hookrightarrow_h A$, the collection $\mathcal{H}$ is almost malnormal in A; then the assumption that F acts freely on $\mathcal{H}$ implies that a refinement of $\mathcal{H}$ in $A \rtimes F$ is $\mathcal{H}_F$, and this was observed in Example 1.3. Since $\mathcal{H}$ contains only infinite subgroups and $\mathcal{H} \hookrightarrow_h A$, Theorem 3.1 implies that $\mathcal{H}_F \hookrightarrow_h (A \rtimes F, T \cup F)$. ∎

6 Concluding remarks

A positive answer to the following question would allow us to drop the first hypothesis of Theorem A for the relevant groups.

Question 6.1. Let G be a finitely generated NRH acylindrically hyperbolic group. Does G contain a qi-characteristic collection of hyperbolically embedded subgroups?

It is possible that $\mathcal{AH}$-accessibility as defined in [1] may be necessary for a positive answer to Question 6.1. Note that this property does not always hold (see [2]).

It is tempting to weaken the definition of a quasi-isometry of pairs $q: (G, \mathcal{P}) \to (H, \mathcal{Q})$ to remove the uniform constant M bounding the Hausdorff distances on the cosets and instead ask the relation

$$\dot{q} = \{(A, B) \in G/\mathcal{P} \times H/\mathcal{Q} : \mathsf{hdist}_H(q(A), B) < \infty\}$$

satisfies that the projections into $G/\mathcal{P}$ and $H/\mathcal{Q}$ are surjective. We shall call the map q in this modified definition an *almost quasi-isometry of pairs* following [10, Section 5].

Indeed, there is work of Margolis [12] where the main theorems do not require this additional hypothesis. However, Margolis shows that the hypotheses assumed in the main results of loc. cit. in fact imply that such a constant M exists (see [12, Theorem 4.1]). Note that our results in this article rely on the existence of a constant M—primarily due to the use of [11, Proposition 5.6]. Thus, we raise the following question.

Question 6.2. Let G and H be finitely generated groups with finite collections of subgroups $\mathcal{P}$ and $\mathcal{Q}$, respectively. When is an (L, C)-almost quasi-isometry of pairs $q: (G, \mathcal{P}) \to (H, \mathcal{Q})$ an (L, C, M)-quasi-isometry of pairs?

Motivated by results of [5], the referee of the article suggested that it might be interesting to investigate other relaxations of the definition of a quasi-isometry of pairs (Defintion 1.1), for example, in the sense that the image of every element of the collection $\mathcal{A}$ lies at uniform Hausdorff distance of the union of finitely many elements in the collection $\mathcal{B}$. Having a more general notion could allow a broader strategy toward tackling Question 1.1 based on the methods in this article.

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