MEASURABLE COVER FUNCTIONS

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1. Introduction. Let μ^* be an outer measure on (X, S) with σ- algebra S and let μ_* be the inner measure induced by μ^*. A set M is a measurable cover of a set $A \subseteq X$ if $A \subseteq M$, M is measurable, and $\mu_*(M-A) = 0$. We assume that every subset of X has a measurable cover; this holds, for example, if μ^* is the outer measure induced by a measure which is σ- finite on X [2, theorem C, p. 50]. For each $x \varepsilon X$ and each $A \subseteq X$, $D(x, A)$ is a non-negative real number with the properties:

(i) if $A \subseteq B \subseteq X$ and $x \varepsilon X$, then $D(x, A) \leq D(x, B)$;

(ii) if A is a measurable subset of X, then $D(x, A) > 0$ for almost all $x \varepsilon A$ and $D(x, A) = 0$ for almost all $x \notin A$;

(iii) if M is a measurable cover of $A \subseteq X$ and $x \epsilon X$, then $D(x, A) = D(x, M)$.

It is easily seen [1, theorem 11] that, for each $A \subseteq X$, the set $A \cup \{x \mid D(x, A) > 0\}$ is a measurable cover of A.

2. Measurable Cover Functions. Let f be a real-valued function with domain X and, for each real number a, let $M(a)$ be the measurable cover of $\{x \mid f(x) > a\}$ as above. For each $x \varepsilon X$, let $\bar{f}(x)$ be the supremum of $\{a \mid x \varepsilon M(a)\}$. The function $\bar{f}$ will be called the cover function of f.

THEOREM 1. If f is a bounded real-valued function, then $\bar{f}$ is a measurable function and $\bar{f}(x) \geq f(x)$ for all $x \varepsilon X$.

Proof. The set $\{a \mid x \varepsilon M(a)\}$ is not empty, since $X \varepsilon M(a)$ if $a < f(x)$ and is bounded above by any upper bound for f. Thus, $\bar{f}(x)$ is defined for all x, and because $x \varepsilon M(a)$ if $a < f(x)$, $f(x) \leq \bar{f}(x)$ for all x. For each real number y,

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$$\{x \mid \bar{f}(x) > y\} = \bigcup_{n=1}^{\infty} \bigcap M(a),$$

where the intersection is taken over all rational numbers a such that $a < y + 1/n$. Thus, $\bar{f}$ is measurable.

THEOREM 2. Let f be a bounded real-valued function. If h is a measurable function such that $h(x) \geq f(x)$ for all x, then $h(x) \geq \bar{f}(x)$ for almost all x. Thus, if f is measurable, then $f(x) = \bar{f}(x)$ for almost all x.

Proof. For each real number r,

$$\{x \mid h(x) < r < \bar{f}(x)\} \subseteq M(r) - \{x \mid f(x) > r\} .$$

so the result follows from the definition of 'measurable cover'.

THEOREM 3. Let f be a bounded real-valued function and let $\varepsilon > 0$ be a real number. Then

$$\mu_* (\{x \mid f(x) + \varepsilon < \bar{f}(x)\}) = 0 .$$

Proof. Suppose that there is a measurable set E such that $\mu^* (E) > 0$ and $f(x) + \varepsilon < \bar{f}(x)$ for all $x \varepsilon E$. Then a contradiction to Theorem 2 can be obtained by considering the function h defined by:

$$h(x) = \begin{cases} \bar{f}(x) - \varepsilon & \text{if } x \in E; \\ \bar{f}(x) & \text{if } x \notin E. \end{cases}$$

3. Examples. In the following three examples, $X = [0, 1]$, μ^* is Lebesgue outer measure, and A is a non-measurable subset of $[0, 1]$ constructed by partitioning $[0, 1]$ with the relation 'a is related to b if $a - b$ is rational' and then choosing one and only one member from each of the resulting equivalence classes. For each positive integer n, the set A_n is $A + r_n$, the addition being modulo 1, where (r_n) is some enumeration of the rational numbers in $[0, 1]$ with $r_1 = 0$. The sets $\{A_n\}$ are disjoint, each of them has inner measure 0, their union is $[0, 1]$, and, for each n, $\mu^*(A_n) = \mu^*(A_1) > 0$.

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Example 1: We give an example of a function f such that $\mu_*(\{x \mid f(x) < \bar{f}(x)\}) > 0$. For each $x \in [0, 1]$ let $f(x) = 1 - 1/n$ if $x \in A_n$. Since $\{x \mid f(x) > 1 - 1/n\} = \bigcup_{r=n+1}^{\infty} A_r$, we have $\mu^*(M(1 - 1/n) \geq \mu^*(A_1)$ so that $\mu^*(\bigcap_{n=1}^{\infty} M(1 - 1/n) \geq \mu^*(A_1)$.

If $x \in \bigcap_{n=1}^{\infty} M(1 - 1/n)$ then $x \in M(a)$ for all $a < 1$, so that $\{x \mid \bar{f}(x) \geq 1\} \supseteq \bigcap_{n=1}^{\infty} M(1 - 1/n)$. From this $\mu_*(\{x \mid \bar{f}(x) > f(x)\}) \geq \mu^*(A_1) > 0$.

Example 2: This example shows the difficulty in generalizing the concept of cover function to non-bounded functions: we construct a finite-valued function f such that $\mu^*(\{x \mid x \in M(a)$ for every $a\}) > 0$. Define f by $f(x) = n$ if $x \in A_n$. Then $\{x \mid f(x) > n\} = \bigcup_{r=n+1}^{\infty} A_r$ so that $\mu^*(M(n)) \geq \mu^*(A_1)$ for all n, and the stated condition follows immediately from this.

Example 3: It is easily seen that if $\{f_n\}$ is an increasing sequence of bounded functions which converges to a bounded function f, then $\{\bar{f}_n\}$ converges almost everywhere to $\bar{f}$. We now show that 'increasing' cannot be replaced by 'decreasing' in this statement. For each positive integer n and $x \in [0, 1]$ we let

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \bigcup_{m=n}^{\infty} A_m \\ 0 & \text{otherwise.} \end{cases}$$

The sequence $\{f_n\}$ is obviously decreasing and converges to 0 for all x. We show that $\bar{f}_n(x) = 1$ for all n and x. First, it is easy to see that the difference set of $\bigcup_{m=1}^{n-1} A_m$ contains just a finite number of rational numbers, so that

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[2, p. 68] the inner measure of $\bigcup_{m=1}^{n-1} A_m$ is 0. Therefore [0, 1] is a measurable cover for $\bigcup_{m=n}^{\infty} A_m$ and so $D(x, \bigcup_{m=n}^{\infty} A_m) = 1$ for all x. From this $\bar{f}_n(x) = 1$ for all n and x.

4. Applications to Local Measurability: In this section we consider the case in which X is the set of real numbers, μ^* is Lebesgue outer measure, and D(x, A) is the strong upper density of A at x. Thus, in addition to the conditions of § 1, D satisfies:

(iv) if A, B are sets of real numbers and x is a real number, then $D(x, A \cup B) \leq D(x, A) + D(x, B)$.

Let f, g be real-valued functions of a real variable and let x be a real number. We define d(f, g) to be $d(f, g) = D(x, \{y \mid f(y) \neq g(y)\})$ and let C_x be the class of those functions such that there is a measurable function g with the property d(f, g) = 0. The class C_x is discussed extensively in [3], where it is referred to as the class of locally measurable functions.

THEOREM 4. *A bounded real-valued function f is in C_x if and only if $d(f, \bar{f}) = 0$.*

Proof. Let $f \in C_x$ and let g be a measurable function such that d(f, g) = 0. Let M be a measurable cover of $\{y \mid f(y) \neq g(y)\}$. Then D(x, M) = 0. Let h be the function:

$$h(y) = \begin{cases} \bar{f}(y) & \text{if } y \in M; \\ g(y) & \text{if } y \notin M. \end{cases}$$

Then h is measurable and $h(y) \geq f(y)$ for all y, so that

$$\begin{aligned} d(f, \bar{f}) &= D(x, \{y \mid f(y) < \bar{f}(y)\}) \\ &\leq D(x, \{y \mid f(y) < h(y)\}) \\ &\leq D(x, \{y \mid f(y) < h(y)\} \cap M) + \\ &\quad D(x, \{y \mid f(y) < h(y)\} \cap \tilde{M}) \\ &= 0. \end{aligned}$$

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If $d(f, \bar{f}) = 0$, then $f \in C_x$ because $\bar{f}$ is measurable.

THEOREM 5. Let f be a bounded real-valued function. Then $\{x \mid f \in C_x\}$ is a measurable set.

Proof. Let $F = \{y \mid f(y) \neq \bar{f}(y)\}$, let $A = \{x \mid f \in C_x\}$ and let M be a measurable cover of F By the preceding theorem, $A = \{x \mid D(x, M) = 0\}$, so that, by (ii) of § 1, $A = M$, modulo a null set.

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