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REAL FLEXIBLE DIVISION ALGEBRAS

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1. Introduction. In this paper we classify finite-dimensional flexible division algebras over the real numbers. We show that every such algebra is either (i) commutative and of dimension one or two, (ii) a slight variant of a noncommutative Jordan algebra of degree two, or (iii) an algebra defined by putting a certain product on the 3×3 complex skew-Hermitian matrices of trace zero. A precise statement of this result is given at the end of this section after we have developed the necessary background and terminology. In Section 3 we show that, if one also assumes that the algebra is Lie-admissible, then the structure follows rapidly from results in [**2**] and [**3**].

All algebras in this paper will be assumed to be finite-dimensional. A nonassociative algebra A is called *flexible* if $(xy)x = x(yx)$ for all $x, y \in A$. Most frequently, we will be using the linearized form

$$(1.1) \qquad (xy)z + (zy)x = x(yz) + z(yx)$$

of the flexible identity. A is *noncommutative Jordan* if it is flexible and also satisfies the identity $(x^2y)x = x^2(yx)$ for all $x, y \in A$. If $xy = 0$ for $x, y \in A$ implies that either $x = 0$ or $y = 0$, A is said to be a *division* algebra. Every real division algebra is known to have dimension 1, 2, 4, or 8, and hence so does every subalgebra of a real division algebra ([**7**] and [**8**]).

Defining the products $[x, y] = xy - yx$ and $x \circ y = xy + yx$ in A, we denote by A^- and A^+ the algebras obtained from A by replacing the product xy in A by $[x, y]$ and $x \circ y$ respectively. It is known [**1**] that the operator ad_x on A defined by $\mathrm{ad}_x(y) = [x, y]$ is a derivation of A^+ for all $x \in A$ if and only if A is flexible. A is called *Lie-admissible* if A^- is a Lie algebra. A helpful reformulation of this definition is that A is Lie-admissible if and only if ad_x is a derivation of A^- for each $x \in A$. From this it is clear that A is both flexible and Lie-admissible if and only if ad_x is a derivation of A for all $x \in A$.

The algebra A is called *quadratic* if A has an identity element e, and for each $x \in A$ there exist real numbers $T(x)$ and $N(x)$ such that

$$x^2 - T(x)x + N(x)e = 0.$$

We need to use a few of the basic properties of such algebras developed

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in [**12**]. In a real quadratic algebra A, the set

$$V = \{x \in A \mid x \notin \mathbf{R}e, x^2 \in \mathbf{R}e\}$$

where $\mathbf{R}$ denotes the real numbers is a vector space, and $A = \mathbf{R}e \oplus V$. For $x, y \in V$, we write $xy = -(x, y)e + x \times y$, where $-(x, y)e$ and $x \times y$ are the components of xy in $\mathbf{R}e$ and V respectively. Then (x, y) is a bilinear form on V, and $x \times y = -y \times x$. It is convenient to extend the bilinear form to all of A by defining $(e, e) = 1$ and $(e, x) = 0$ for $x \in V$. For future reference, we state the most relevant elementary facts about quadratic algebras pertaining to flexibility as

LEMMA 1.2. *For a quadratic algebra A, the following properties are equivalent:*

(i) *The bilinear form (x, y) on A is symmetric,*

(ii) $xy + yx = -2(x, y)e$ *for* $x, y \in V$

(iii) $[x, y] \in V$ *for* $x, y \in V$.

If these properties hold, then the following conditions are equivalent:

(iv) $(x, xy) = 0$, *for all* $x, y \in V$,

(v) $(xy, z) = (x, yz)$, *for all* $x, y, z \in V$.

Furthermore, A is flexible if and only if all these properties hold.

A real quadratic algebra A is a division algebra if and only if the bilinear form is positive definite on A and if for any linearly independent $x, y \in V$, the elements x, y, $x \times y$ are linearly independent. If A is a flexible quadratic algebra, then A is also noncommutative Jordan, since

$$\begin{aligned}(x^2y)x &= ((T(x)x - N(x)e)y)x = T(x)(xy)x - N(x)(ey)x \\ &= T(x)x(yx) - N(x)e(yx) = (T(x)x - N(x)e)(yx) = x^2(yx).\end{aligned}$$

Let P and Q be two nonsingular linear transformations on an algebra A, and let "$*$" be the operation on A defined by $x * y = P(x)Q(y)$. The algebra $(A, *)$ is called a *principal isotope* of A. If A is a real flexible quadratic algebra, and if P and Q are defined by $P(e) = e = Q(e)$ and by $P(x) = \alpha x = Q(x)$ for all $x \in V$ and for some fixed nonzero $\alpha \in \mathbf{R}$, then we call $(A, *)$ the *scalar isotope of A determined by α*, and we denote it by ${}_\alpha A$. We want to show that ${}_\alpha A$ is flexible if A is flexible. Choosing $\beta, \gamma \in \mathbf{R}$ and $x, y \in V$, and using $(x, y) = (y, x)$, $(x, x \times y) = 0$, and the fact that e commutes with everything in ${}_\alpha A$, we obtain

$$\begin{aligned}&[(\beta e + x) * (\gamma e + y)] * (\beta e + x) - (\beta e + x) * [(\gamma e + y) * (\beta e + x)] \\ &\quad = [\beta\gamma e - \alpha^2(x, y)e + \alpha\gamma x + \alpha\beta y + \alpha^2 x \times y] * (\beta e + x) \\ &\qquad - (\beta e + x) * [\gamma\beta e - \alpha^2(y, x)e + \alpha\gamma x + \alpha\beta y + \alpha^2 y \times x] \\ &\quad = [\alpha\gamma x + \alpha\beta y + \alpha^2 x \times y] * (\beta e + x) \\ &\qquad - (\beta e + x) * [\alpha\gamma x + \alpha\beta y + \alpha^2 y \times x] \\ &\quad = [\alpha\gamma x + \alpha\beta y + \alpha^2 x \times y] * x - x * [\alpha\gamma x + \alpha\beta y + \alpha^2 y \times x] \\ &\qquad + \alpha^2\beta(x \times y) * e - \alpha^2\beta e * (y \times x) \\ &\quad = \alpha^3\beta y \times x + \alpha^4(x \times y) \times x - \alpha^3\beta x \times y - \alpha^4 x \times (y \times x) \\ &\qquad + \alpha^3\beta x \times y - \alpha^3\beta y \times x = 0.\end{aligned}$$

Thus ${}_\alpha A$ is flexible when A is. It is also clear from the definition of ${}_\alpha A$ that it is a division algebra whenever A is.

Next, we recall that the set of $n \times n$ complex skew-Hermitian matrices of trace zero forms a Lie algebra under the commutator $[x, y]$, and this is a simple Lie algebra over the reals which is denoted by $su(n)$. When $n = 3$, this set of matrices is also closed under the commutative operation $i\{x \circ y - \frac{2}{3} \operatorname{tr}(xy)I\}$, where tr is the trace, $i = \sqrt{-1}$, and I is the 3×3 identity matrix. It is shown in Section 3 of [**3**] that the set $S(\alpha, \beta)$ of 3×3 complex skew-Hermitian matrices of trace zero under the operation

$$x * y = \alpha[x, y] + \beta i\{x \circ y - \tfrac{2}{3} \operatorname{tr}(xy)I\}$$

is a flexible real division algebra for any nonzero $\alpha, \beta \in \mathbf{R}$. Now $S(\alpha, \beta)$ is isomorphic to $S(\alpha\gamma, \beta\gamma)$ for any nonzero $\gamma \in \mathbf{R}$ using the map $x \rightarrow \gamma x$ for all $x \in S(\alpha, \beta)$. Thus, up to isomorphism, we can restrict our attention to the algebra $S(\delta, \frac{1}{2})$, which has the product

$$x * y = \delta[x, y] + \tfrac{1}{2} i\{x \circ y - \tfrac{2}{3} \operatorname{tr}(xy)I\} \tag{1.3}$$

for some nonzero $\delta \in \mathbf{R}$. When $\delta = \pm\sqrt{3/2}$, this algebra was first studied by Okubo [**9**], and was called by him the pseudo-octonions. For this reason, we call the algebras $S(\delta, \frac{1}{2})$ defined using (1.3) *generalized pseudo-octonion* algebras, or GP-algebras for short.

We are now able to state our main result precisely.

THEOREM 1.4. *If A is a finite-dimensional real algebra, then A is a flexible division algebra if and only if A has one of the following forms:*

(a) *A is a commutative division algebra of dimension* 1 *or* 2,

(b) *A is isomorphic to a scalar isotope ${}_\alpha B$ of some quadratic real division algebra B which is flexible* (*and hence noncommutative Jordan*), *or*

(c) *A is a generalized pseudo-octonion algebra.*

It is clear from the preceding discussion that algebras of these three types are real flexible division algebras, and most of the remainder of the paper is devoted to showing that these are the only types which occur. Real commutative division algebras are considered in Section 2, and we classify them in Theorem 2.3. Scalar isotopes of flexible quadratic division algebras are discussed in Section 5, where we show (Theorem 5.13) that A has to be of this type if A is a real flexible division algebra of dimension 4 or 8 containing an idempotent commuting with every element of A. In Section 6, the real flexible division algebras of dimension 8 without such an idempotent are shown to be GP-algebras. The reduction to the cases considered in Sections 5 and 6 and the basic results needed for these sections (and hence for Theorem 1.4) are found in Section 4.

The structure of real flexible Lie-admissible division algebras can be found by combining the results of Theorem 1.4 with Proposition 5.15. However, this important special case can be derived much more quickly using results from [**2**] and [**3**], and without using any of the results in Sections 4–6. Such a proof of the Lie-admissible case is given in Section 3.

2. Commutative division algebras. The task of investigating real commutative division algebras is greatly simplified by the following result of Hopf.

THEOREM 2.1. [**6**]. *Every real commutative division algebra has dimension* 1 *or* 2.

We begin our study with

LEMMA 2.2. *Every real commutative division algebra A contains an idempotent.*

Proof. Let $x \in A$, $x \neq 0$. If the subalgebra generated by x is 1-dimensional, then $x^2 = \delta x$ for some $\delta \in \mathbf{R}$, and $e = (1/\delta)x$ is an idempotent. By Theorem 2.1, we may assume that x generates a 2-dimensional subalgebra with basis x, x^2 and multiplication given by

$$xx^2 = x^2x = ax + bx^2, \quad (x^2)^2 = cx + dx^2$$

for some $a, b, c, d \in \mathbf{R}$. Then for any $\lambda \in \mathbf{R}$, the element $y = \lambda x + x^2$ satisfies

$$y^2 = \lambda^2 x^2 + 2\lambda(ax + bx^2) + cx + dx^2.$$

Thus, $y^2 = \mu y$ for some $\mu \in \mathbf{R}$ if and only if

$$2\lambda a + c = \lambda\mu, \quad \lambda^2 + 2\lambda b + d = \mu.$$

This system has a solution if and only if

$$\lambda^3 + 2\lambda^2 b + (d - 2a)\lambda - c = 0$$

has a solution. Since every cubic polynomial possesses a root in $\mathbf{R}$, there exists some $\lambda \in \mathbf{R}$ which satisfies this equation. For this λ, $y^2 = \mu y$ where $\mu = \lambda^2 + 2\lambda b + d$, and $e = (1/\mu)y$ is an idempotent of A.

THEOREM 2.3. *If A is a real commutative division algebra of dimension* 2, *then A has a basis e, x with the property that multiplication is given by one of the tables*

(2.4)

	e	x
e	e	αx
x	αx	$-e + \gamma x$

	e	x
e	e	$e + x$
x	$e + x$	βe

from some $\alpha, \beta, \gamma \in \mathbf{R}$. The algebras over $\mathbf{R}$ defined by these tables are division algebras if and only if the constants in the multiplication tables satisfy the relations $4\alpha^2 > \gamma^2$ and $\beta < 0$ respectively.

Proof. By Lemma 2.2, A contains an idempotent e. Let L_e be the linear transformation on A defined by $L_e y = ey$ for $y \in A$, and suppose first that L_e is diagonalizable. Then there exists $\alpha \in \mathbf{R}$ and $x \in A$ such that $ex = \alpha x$. Letting $x^2 = \beta e + \gamma x$, we calculate that the matrix of left multiplication by $\lambda_1 e + \lambda_2 x$ relative to the basis e, x has determinant given by

$$\begin{vmatrix} \lambda_1 & \beta\lambda_2 \\ \alpha\lambda_2 & \alpha\lambda_1 + \gamma\lambda_2 \end{vmatrix} = \alpha\lambda_1^2 + \gamma\lambda_1\lambda_2 - \alpha\beta\lambda_2^2.$$

Thus, A is a division algebra if and only if the quadratic form $\alpha\lambda_1^2 + \gamma\lambda_1\lambda_2 - \alpha\beta\lambda_2^2$ does not represent zero over $\mathbf{R}$, which is equivalent to the discriminant of the form being negative:

$$0 > \gamma^2 + 4\alpha^2\beta.$$

If A is a division algebra, clearly $\beta < 0$. Replacing x by $(-\beta)^{-1/2}x$, we can assume that $\beta = -1$, to give the first table in (2.4). When $\beta = -1$, the condition for A to be a division algebra reduces to $4\alpha^2 > \gamma^2$.

If L_e is not diagonalizable, then there exists $y \in A$ with $ey = e + y$. If $y^2 = \delta e + \gamma y$, the element $x = -\frac{1}{2}\gamma e + y$ satisfies

$$ex = -\tfrac{1}{2}\gamma e + e + y = e + x, \; x^2 = \tfrac{1}{4}\gamma^2 e - \gamma(e + y) + \delta e + \gamma y = \beta e,$$

where $\beta = \frac{1}{4}\gamma^2 - \gamma + \delta$. This gives the second table in (2.4). The determinant of the matrix of left multiplication by $\lambda_1 e + \lambda_2 x$ relative to the basis e, x is

$$\begin{vmatrix} \lambda_1 + \lambda_2 & \lambda_1 + \beta\lambda_2 \\ \lambda_2 & \lambda_1 \end{vmatrix} = \lambda_1^2 - \beta\lambda_2^2,$$

which does not represent zero if and only if $\beta < 0$.

3. Real flexible Lie-admissible division algebras. In this section we classify real flexible division algebras under the added assumption of Lie-admissibility. Since the hypothesis that A is flexible and Lie-admissible is equivalent to assuming that ad_a is a derivation of A for each a in A, our technique is to use the derivation algebra Der A of A to severely limit the possibilities for A. Our proof relies heavily on the following theorem which determines those Lie algebras which can arise as the derivation algebra of a real division algebra.

THEOREM 3.1. [**2**]. *Let S be a subalgebra of* Der A *for A a real division algebra.*

(i) *If* dim $A = 1, 2$, *then* $S = 0$.

(ii) *If* dim $A = 4$, *then S is isomorphic to* $su(2)$ *or* dim S $= 0$ *or* 1.

(iii) *If* dim $A = 8$, *then S is isomorphic to one of the following Lie algebras*:

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1. *compact* G_2
2. $su(3)$
3. $su(2) \oplus su(2)$
4. $su(2) \oplus N$ *where* N *is an abelian ideal and* $\dim N = 0$ *or* 1
5. N *where* N *is abelian and* $\dim N = 0, 1,$ *or* 2.

In particular, if $S = \operatorname{Der} A$, *then* $\operatorname{Der} A$ *is one of the Lie algebras listed above.*

One may view A as a module for a subalgebra S of $\operatorname{Der} A$, and if S is semisimple, then A decomposes into the direct sum of irreducible S-submodules. In a subsequent investigation [**3**], Theorem 3.1 together with the representation theory of semisimple Lie agebras was used to analyze the structure of A.

Suppose now that A is a real flexible Lie-admissible division algebra. Then A^- is a Lie algebra, and the transformation $A^- \to \operatorname{Der} A$ given by ad is a Lie homomorphism with kernel Z where

$$Z = \{z \in A \mid [z, x] = 0 \quad \text{for all } x \in A\}.$$

LEMMA 3.2. *Z is a subalgebra of A.*

Proof. For $y, z \in Z$ and any $x \in A$ we have

$$[yz, x] = [y, x]z + y[z, x] = 0.$$

The space Z is also a Lie ideal of A^- and A^-/Z is isomorphic to a Lie subalgebra S of $\operatorname{Der} A$, where S then is one of the Lie algebras listed in Theorem 3.1. For any Lie subalgebra L of $\operatorname{Der} A$, the space Z also has one further property: Z is an L-submodule of A. For if $\partial \in L$, $z \in Z$ and $x \in A$, then

$$[\partial(z), x] = \partial[z, x] - [z, \partial(x)] = 0.$$

We are now ready to state and prove our classification result.

THEOREM 3.3. *Assume A is a real flexible Lie-admissible division algebra.*

(i) *If* $\dim A = 1$ *or* 2, *then* A *is commutative.*

(ii) *If* $\dim A = 4$, *then there is a basis* e, x_1, x_2, x_3 *of* A *such that the multiplication is given by Table* 3.4 *below for* $\alpha, \beta \in \mathbf{R}$ *with* $\alpha \neq 0$ *and* $\beta > 0$.

(3.4)

	e	x_1	x_2	x_3
e	e	αx_1	αx_2	αx_3
x_1	αx_1	$-\beta e$	x_3	$-x_2$
x_2	αx_2	$-x_3$	$-\beta e$	x_1
x_3	αx_3	x_2	$-x_1$	$-\beta e$

(iii) *If* $\dim A = 8$, *then* A *is a generalized pseudo-octonion algebra.*

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Proof. As we noted previously, if A is a flexible Lie-admissible real division algebra, then A^-/Z is isomorphic to a subalgebra S of Der A.

Assume first that dim $A = 1$ or 2. Then by Theorem 3.1, $S = 0$, so that $A = Z$ and A is commutative.

We suppose next that dim $A = 4$. If $S = 0$, then A is commutative, which is impossible by Hopf's result. The case dim $S = 1$ cannot occur, because if it did the subalgebra Z would be 3-dimensional. Therefore S must be isomorphic to $su(2)$, and dim $Z = 1$. In [**3**], 4-dimensional real division algebras with $su(2)$ as derivations were shown to have a basis e, x_1, x_2, x_3 with multiplication as prescribed in Table 3.4 except that the constant appearing in the first column was not necessarily equal to α. Thus, we may suppose that $x_ie = \gamma x_i$ and that the remainder of the products are given by (3.4). Further, we assume that $0 \neq z = a_0e + \sum_{i=1}^{3} a_ix_i$ lies in Z. Then

$$0 = [x_1, z] = a_0(\gamma - \alpha)x_1 + 2a_2x_3 - 2a_3x_2,$$

and hence $a_2 = a_3 = 0$. Similarly from the product $[x_2, z] = 0$ we obtain $a_1 = 0$. Since $z \neq 0$, it must be that $a_0 \neq 0$, $\gamma = \alpha$, and $e \in Z$. It is shown in [**3**] that such algebras are division algebras if and only if $\alpha\beta\gamma > 0$. In the present situation $\alpha = \gamma$, so that A is a division algebra exactly when $\beta > 0$. It will follow from Theorem 5.13 and Proposition 5.15 in Section 5, that each algebra with multiplication given by Table 3.4 is flexible and Lie-admissible. The algebra A is also a special case of a more general construction of flexible Lie-admissible algebras studied in [**4**], and these properties could be deduced from results in that paper as well.

Assume now that dim $A = 8$. Using the fact that Z is a subalgebra of A, we obtain dim $S = 8, 7, 6, 4$ or 0. We can eliminate the cases dim $S = 0$ and dim $S = 4$ immediately by the Hopf result since Z is a commutative subalgebra. Also the case dim $S = 7$ can be ruled out by a dimension comparison with the list in Theorem 3.1.

Let us suppose then dim $S = 6$. Under this assumption, S is isomorphic to $su(2) \oplus su(2)$, and by ([**3**], Theorem 5.1) the algebra A decomposes into three irreducible S-summands which have dimensions 1, 3 and 4. Since Z is an S-submodule, and since the dimensions of the summands are distinct, Z must be a sum of some of the irreducibles. But this is impossible since dim $Z = 2$.

The only possibility that remains is that dim $S = 8$, and S is isomorphic to $su(3)$. In this situation A^- is isomorphic to $su(3)$, and thus is a simple Lie algebra. The algebra A cannot decompose into more than one irreducible $su(3)$-summand because such summands would be proper Lie ideals of A^-. By ([**3**], Theorem 3.2) any real division algebra A on which $su(3)$ acts irreducibly as derivations can be constructed from 3×3 complex skew-Hermitian matrices of trace zero by defining the product to be that given in (1.3), and thus, A is a generalized pseudo-octonion algebra.

As we mentioned in Section 1, it is verified in [**3**] by a straightforward calculation that such algebras are flexible, and are division algebras if and only if $\delta \neq 0$. Lie-admissibility follows easily from the fact that $x * y - y * x = 2\delta[x, y]$. Thus, the only real flexible Lie-admissible division algebras are the ones listed in Theorem 3.3, and each of those algebras is flexible, Lie-admissible, and division.

4. Idempotent decompositions. We initiate our study of arbitrary real flexible division algebras A by showing that A possesses an idempotent e, and then use the operators L_e, R_e (left and right multiplication by e) to investigate the structure of A. Ultimately, we prove that either (i) A possesses an idempotent which commutes with all of A, or (ii) $\dim A = 8$ and A has an idempotent commuting with a 4-dimensional subalgebra. The two possibilities are then explored individually in subsequent sections.

PROPOSITION 4.1. [**13**]. *If $\{x_i\}$ is a set of commuting elements in a flexible algebra A over a field of characteristic not* 2, *then the subalgebra generated by the x_i's is commutative.*

Proof. Although this result appears in [**13**] we include the brief proof for the convenience of the reader.

If y and z are two monomials in the x_i's, we must prove that $[y, z] = 0$. Since the x_i's commute, we may suppose that one of the elements, y, z, say z, is not an x_i. Then $z = uv$ where u, v are monomials in the x_i's. Inducting on $\deg y + \deg z$, we may assume that $[y, u] = 0 = [y, v] = [u, v]$. But then

$$[y, z] = [y, uv] = \tfrac{1}{2}[y, u \circ v] = \tfrac{1}{2}[y, u] \circ v + \tfrac{1}{2}u \circ [y, v] = 0$$

as was to be shown.

COROLLARY 4.2. *Every element of A generates a commutative subalgebra.*

Using the result of Hopf cited in Section 2, we have

COROLLARY 4.3. *In a real flexible division algebra if two elements commute, they generate a subalgebra of dimension* 1 *or* 2.

An immediate consequence of Corollary 4.3 and Lemma 2.2 is

COROLLARY 4.4. *Every real flexible division algebra contains an idempotent.*

We assume henceforth that A is a real flexible division algebra of dimension $\geqq 2$, and e is an idempotent of A. The following subspaces, defined using right and left multiplication by e, play a crucial role in our investigations:

$$A_\alpha = \{x \in A \mid (L_e + R_e - 2\alpha I)x = 0\}, \quad \text{where } \alpha \in \mathbf{R},$$
$$E_\alpha = \{x \in A \mid (L_e + R_e - 2\alpha I)^n x = 0, \quad \text{for some } n\}, \quad \alpha \in \mathbf{R},$$
$$B_1 = \{x \in A \mid (L_e + R_e - 2I)^2 x = 0\},$$
$$C(e) = \{x \in A \mid [e, x] = 0\}.$$

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Clearly, $A_\alpha \subseteq E_\alpha$ for every $\alpha \in \mathbf{R}$, and $E_\alpha \cap E_\beta = 0$ for $\alpha \neq \beta$. Often, a helpful way of restating $x \in A_\alpha$ is that $e \circ x = 2\alpha x$.

THEOREM 4.5. *Either*
(i) $A = \mathbf{R}e \oplus E_{1/2}$, *or*
(ii) $C(e)$ *is a subalgebra*, $A = C(e) \oplus E_{1/2}$, *and either*

$$C(e) = B_1 \quad or \quad C(e) = \mathbf{R}e \oplus A_\alpha \text{ for some } \alpha \neq 1, \tfrac{1}{2}.$$

The proof of this theorem will be the end-product of a sequence of results which begins with

LEMMA 4.6. (i) $A_\alpha \subseteq C(e)$ *for all* $\alpha \neq \frac{1}{2}$.
(ii) *The roots of the minimum polynomial of* $L_e + R_e$ *lie in* $\mathbf{R}$.

Proof. For $x \in A_\alpha$, the linearized flexible identity gives

$$(4.7) \qquad (xe)e + (ee)x = x(ee) + e(ex),$$

or

$$(2\alpha x - ex)e + ex = xe + e(2\alpha x - xe).$$

Since $(ex)e = e(xe)$, the last equation reduces to $(2\alpha - 1)[e, x] = 0$, which implies part (i): $x \in C(e)$ when $\alpha \neq \frac{1}{2}$. This argument depends only on the flexibility of A, and so must work for the complexification $A_\mathbf{C}$ of A also. Thus, if $\frac{1}{2} \neq \gamma \in \mathbf{C}$ and

$$(A_\mathbf{C})_\gamma = \{x \in A_\mathbf{C} | (L_e + R_e - 2\gamma I)x = 0\},$$

then $(A_\mathbf{C})_\gamma \subseteq C(e)$. Suppose $\gamma \in \mathbf{C} - \mathbf{R}$ and

$$0 \neq y \in \{x \in A | (L_e + R_e - 2\gamma I)(L_e + R_e - 2\bar{\gamma} I)x = 0\}.$$

Then in $A_\mathbf{C}$, $y = y_1 + y_2$ where $y_1 \in (A_\mathbf{C})_\gamma$ and $y_2 \in (A_\mathbf{C})_{\bar{\gamma}}$ so that $y \in C(e)$. Since e and y commute, the subalgebra they generate is at most 2-dimensional, and so $ey = \beta_1 e + \beta_2 y$ for some $\beta_1, \beta_2 \in \mathbf{R}$. But then on $A_\mathbf{C}$

$$\beta_1 e - \beta_2 y_1 + \beta_2 y_2 = ey = ey_1 + ey_2 = \gamma y_1 + \bar{\gamma} y_2.$$

This gives $\gamma = \beta_2 = \bar{\gamma}$, which is a contradiction to $\gamma \in \mathbf{C} - \mathbf{R}$. Since such an element y would exist if $L_e + R_e$ had roots in $\mathbf{C} - \mathbf{R}$, we conclude that assertion (ii) must hold.

LEMMA 4.8. (i) $A_\alpha = E_\alpha$ *for all* $\alpha \neq \frac{1}{2}, 1$.
(ii) $E_1 = B_1 \subseteq C(e)$.
(iii) *There is a basis* $e, x, y_1, \ldots, y_m$ *of* B_1 *with* $ex = x + \epsilon e$ *where* $\epsilon = 0$ *or* 1, *and* $ey_i = y_i$ *for all* $i = 1, \ldots, m$.

Remark. It is to be understood that if $\dim B_1 = 2$, then there are no y's, and if $\dim B_1 = 1$, then the basis is just e.

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Proof. Suppose that

$$(L_e + R_e - 2\alpha I)^n x = 0,\ (L_e + R_e - 2\alpha I)^{n-1} x \neq 0, \quad \text{and} \quad (L_e + R_e - 2\alpha I)x = w \neq 0$$

where $w \in C(e)$. Then $ex + xe - 2\alpha x = w$, and (4.7) becomes

$$(w + 2\alpha x - ex)e + ex = xe + e(w + 2\alpha x - xe).$$

Since $w \in C(e)$, this equation reduces to

$$(4.9) \qquad (1 - 2\alpha)[e, x] = 0.$$

Thus, if $\alpha \neq \frac{1}{2}$, then $x \in C(e)$ also. Now $x \notin \mathbf{R}e$, since $w \neq 0$, and consequently x, e generate a 2-dimensional commutative subalgebra. But then $w = ex + xe - 2\alpha x$ is a linear combination of e, x, say $w = \rho_1 e + \rho_2 x$. Because $(L_e + R_e - 2\alpha I)^{n-1}$ annihilates w but not x, the scalar ρ_1 is nonzero in this situation.

We now apply these results to three different choices of x to obtain Lemma 4.8. First, let us suppose that $E_\alpha \neq A_\alpha$ for some $\alpha \neq 1, \frac{1}{2}$. Then there is an $x \in E_\alpha - A_\alpha$ such that

$$(L_e + R_e - 2\alpha I)^2 x = 0 \quad \text{and} \quad 0 \neq w = (L_e + R_e - 2\alpha I)x \in A_\alpha.$$

Since $A_\alpha \subseteq C(e)$ by Lemma 4.6, the above argument implies that $w = \rho_1 e + \rho_2 x$ where $\rho_1 \neq 0$. But solving for e, we obtain $e \in E_\alpha \cap E_1 = 0$, a contradiction. Therefore, no such x can be found, and $E_\alpha = A_\alpha$.

We now consider the case that $x \in B_1 - A_1$. Here,

$$0 \neq w = (L_e + R_e - 2I)x \in A_1,$$

which lies in $C(e)$ by Lemma 4.6. We conclude first from (4.9) that $B_1 \subseteq C(e)$, and secondly that $w = \rho_1 e + \rho_2 x$ where $\rho_1 \neq 0$. If $\rho_2 \neq 0$, then we would obtain $x \in A_1$, contrary to our assumption. Thus, we have shown that for every $x \in B_1 - A_1$,

$$(L_e + R_e - 2I)x = w \in \mathbf{R}e.$$

We argue next that $B_1 = E_1$. If this is not the case, then there is an $x \in E_1 - B_1$ such that

$$0 \neq w = (L_e + R_e - 2I)x \in B_1 \subseteq C(e) \quad \text{and} \quad w \notin A_1.$$

But $w = \rho_1 e + \rho_2 x$, and if $\rho_2 \neq 0$ then $x \in B_1$, while if $\rho_2 = 0$ then $w \in A_1$. These contradictions show that no such x exists, and hence $E_1 = B_1 \subseteq C(e)$ as claimed. Since $B_1 \subseteq C(e)$, we have

$$(L_e + R_e - 2I) = 2(L_e - I) \quad \text{on } B_1,$$

so that for every $x \in B_1$, $ex - x \in \mathbf{R}e$. If $ex - x \neq 0$ for some $x \in B_1$ then by suitably modifying x by a scalar if necessary, we may assume $ex = x + e$. Let $w_1, \ldots, w_m$ together with e, x form a basis for B_1, where

$ew_i = w_i + \eta_i e$. Then for $y_i = w_i - \eta_i x$, $ey_i = y_i$, and the elements $e, x, y_1, \ldots, y_m$ form the desired basis. If no such x exists, then $B_1 = A_1$ and any choice of elements $x, y_1, \ldots, y_m$, which together with e form a basis of A_1 will satisfy (iii).

LEMMA 4.10. (i) *If* $g \in C(e)$, *then* $[g, A_\alpha] \subseteq A_\alpha$ *for all* α.
(ii) $g^2 \in \mathbf{R}e + \mathbf{R}g$ *for all* $g \in C(e)$.
(iii) A_1, B_1 *and* $\mathbf{R}e \oplus A_\alpha$ *for* $\alpha \neq \frac{1}{2}, 1$ *are subalgebras of* A.

Proof. Assume $h \in A_\alpha$ and $g \in C(e)$. Then, it follows from

$$2\alpha[g, h] = [g, e \circ h] = [g, e] \circ h + e \circ [g, h] = e \circ [g, h]$$

that $[g, h] \in A_\alpha$, and (i) is verified. Part (ii) is just an easy consequence of the fact that e and g generate a commutative subalgebra.

For convenience, let S denote $\mathbf{R}e \oplus A_\alpha$ or A_1 and observe that $S \subseteq C(e)$ by Lemma 4.6. For $s, t \in S$ we deduce from (i) that $[s, t] \in S$. Now, by (ii)

$$(s + t)^2 \in \mathbf{R}e + \mathbf{R}(s + t) \subseteq \mathbf{R}e + \mathbf{R}s + \mathbf{R}t,$$
$$s^2 \in \mathbf{R}e + \mathbf{R}s, \quad \text{and} \quad t^2 \in \mathbf{R}e + \mathbf{R}t,$$

and combining these statements we have

$$s \circ t \in \mathbf{R}e + \mathbf{R}s + \mathbf{R}t \subseteq S.$$

Since $[s, t] \in S$ also, we obtain $st \in S$ and S is a subalgebra.

What remains to be shown is that B_1 is a subalgebra. We assume that B_1 has a basis $e, x, y_1, \ldots, y_m$ of the type described in Lemma 4.8. Then since $A_1 + \mathbf{R}x = B_1 \subseteq C(e)$, and A_1 was just shown to be a subalgebra, all that is left is to show xB_1 and B_1x are contained in B_1. The argument above can be copied verbatim to give $x \circ B_1 \subseteq B_1$, and in fact $B_1 \circ B_1 \subseteq B_1$. Since $x \in C(e)$, part (i) implies that

$$[x, B_1] = [x, \mathbf{R}x + A_1] \subseteq A_1 \subseteq B_1.$$

Therefore, $xB_1 \subseteq B_1$ and $B_1x \subseteq B_1$, and B_1 is a subalgebra as claimed.

An immediate consequence of Lemma 4.8 and Lemma 4.10 (iii) is

COROLLARY 4.11. *If* $B_1 \neq A_1$, *then* $A_1 = \mathbf{R}e$ *and* $B_1 = \mathbf{R}e + \mathbf{R}x$ *where* $ex = x + e = xe$.

The next lemma combined with Lemma 4.6 drastically reduces the number of possible α with $A_\alpha \neq 0$.

LEMMA 4.12. *There cannot exist elements* x, y *in* $C(e) - \mathbf{R}e$ *such that* $x \in A_\alpha$, $y \in A_\beta$ *and* $\alpha \neq \beta$, *or such that* $x \in A_\alpha$ *for* $\alpha \neq 1$ *and* $y \in B_1$.

Proof. Assume the contrary, that such x, y do exist. Then by Lemmas 4.6 and 4.8 the subalgebra S generated by e and $x + y$ is commutative. If $x \in A_\alpha$, $y \in A_\beta$, then $\alpha x + \beta y = e(x + y) \in S$ and since $\alpha \neq \beta$ we obtain $x, y \in S$. If $x \in A_\alpha$ and $y \in B_1$, then

$$\alpha x + y + \rho e = e(x + y) \in S \quad \text{for some } \rho \in \mathbf{R},$$

and since $e \in S$ and $\alpha \neq 1$ we obtain $x, y \in S$ as before. Therefore, in both instances we have produced a commutative subalgebra S with $\dim S \geqq 3$. This gives the desired contradiction.

All the necessary ingredients for proving the theorem are in place.

Proof of Theorem 4.5. Assume $A \neq \mathbf{R}e \oplus E_{1/2}$. Then because $L_e + R_e$ has no roots in $\mathbf{C} - \mathbf{R}$, either $A_\alpha \neq 0$ for some $\alpha \neq \frac{1}{2}, 1$ in $\mathbf{R}$ or $B_1 \neq \mathbf{R}e$. Both of these spaces lie in $C(e)$ by Lemmas 4.6 and 4.8, and by Lemma 4.12 either $C(e) = \mathbf{R}e \oplus A_\alpha$ or $C(e) = B_1$. It follows from Lemma 4.10 (iii) that $C(e)$ is a subalgebra. Also, Lemma 4.12 with Lemmas 4.6, 4.8 eliminates the possibility of any other nonzero space occurring except for $E_{1/2}$. Finally, since $E_\alpha \cap E_\beta = 0$ for $\alpha \neq \beta$ we have $A = C(e) \oplus E_{1/2}$, and the proof is complete.

We now turn our attention to gathering information about the structure of $E_{1/2}$.

LEMMA 4.13. (i) *For all* $x \in A_{1/2}$, $x^2 \in C(e)$.
(ii) $E_{1/2} = U \oplus V$ *where*

$$U = \{x \in E_{1/2} | (L_e - \tfrac{1}{2}I)x = 0\} \subseteq C(e)$$

and V *is an* L_e*-invariant subspace on which* L_e *has eigenvalues in* $\mathbf{C} - \mathbf{R}$.

Proof. For each $x \in A_{1/2}$,

$$(e + x)^2 = e + x + x^2 \quad \text{and} \quad 0 = [e + x, (e + x)^2] = [e, x^2]$$

giving (i).

Suppose now that γ is a real eigenvalue of L_e and let

$$U = \{x \in E_{1/2} | (L_e - \gamma I)x = 0\}.$$

Since $L_e + R_e - I$ commutes with $L_e - \gamma I$, it leaves U invariant, so there is an $x \in U$, $x \neq 0$, with

$$(L_e + R_e - I)x = 0 = (L_e - \gamma I)x.$$

But then

$$(4.14) \quad (xe)x = x(ex) = \gamma x^2$$

and because A is a division algebra we have $xe = \gamma x$. Therefore $(L_e + R_e - I)x = 0$ implies $2\gamma - 1 = 0$ and $\gamma = \frac{1}{2}$. Thus, $\frac{1}{2}$ is the only real eigenvalue of L_e on $E_{1/2}$, and

$$U = \{x \in E_{1/2} | (L_e - \tfrac{1}{2}I)x = 0\}.$$

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Our argument using (4.14) shows also that U is contained in $C(e)$, and $xe = \frac{1}{2}x$ for all $x \in U$.

Consider now the generalized eigenspace

$$W = \{x \in E_{1/2} | (L_e - \tfrac{1}{2}I)^n x = 0 \quad \text{for some } n\}.$$

Since W and U are invariant under the transformations L_e and $L_e + R_e$, if $W/U \neq 0$ there is a $w \neq 0$ in W with the following properties: $e \circ w = w + y$, $ew = \frac{1}{2}w + x$, where $x \neq 0$ and $x, y \in U$. From the linearized flexible identity

$$x(ew) + w(ex) = (xe)w + (we)x$$

it follows that

$$\tfrac{1}{2}xw + x^2 + \tfrac{1}{2}wx = \tfrac{1}{2}xw + \tfrac{1}{2}wx + yx - x^2.$$

Hence, $2x^2 = yx$, $2x = y$, and $we = \frac{1}{2}w + x = ew$. Because $w \in C(e)$, the subalgebra generated by e and w is 2-dimensional, so that

$$x = ew - \tfrac{1}{2}w \in \mathbf{R}e + \mathbf{R}w.$$

If $x = \mu e + \nu w$ where $\mu \neq 0$, then $e \in E_{1/2}$ which cannot be, but if $x = \nu w$ where $\nu \neq 0$, then $w \in U$, contrary to our choice of w. These contradictions show that $W = U$. Since L_e has no real eigenvalues besides $\frac{1}{2}$, the space $E_{1/2}$ has the structure claimed in (ii).

For $\zeta \in \mathbf{C} - \mathbf{R}$ define

$$V_\zeta = \{v \in E_{1/2} | (L_e - \zeta I)(L_e - \bar{\zeta} I)v = 0\}.$$

LEMMA 4.15. *There is a* $z \in A_{1/2}$ *with* $z^2 \notin \mathbf{R}e + \mathbf{R}z$ *if and only if the minimum polynomial of* L_e *on* $E_{1/2}$ *has roots in* $\mathbf{C} - \mathbf{R}$.

Proof. If there is a $z \in A_{1/2}$ with $z^2 \notin \mathbf{R}e + \mathbf{R}z$, then e fails to commute with this z. Thus, in the notation of the previous lemma, $E_{1/2} \neq U$ and so L_e must have complex roots. Conversely, suppose $\zeta \in \mathbf{C} - \mathbf{R}$ is a root of L_e. Then $V_\zeta \neq 0$ and since V_ζ is $L_e + R_e$-invariant, $V_\zeta \cap A_{1/2} \neq 0$. Let $z \in V_\zeta \cap A_{1/2}$, $z \neq 0$, and assume further that $z^2 \in \mathbf{R}e + \mathbf{R}z$. If $z^2 \in \mathbf{R}z$, then since $[e, z^2] = 0$, we obtain $[e, z] = 0$ and $ez = ze = \frac{1}{2}z$. However, this contradicts the fact that L_e on V_ζ has eigenvalues in $\mathbf{C} - \mathbf{R}$. Therefore, $z^2 \notin \mathbf{R}z$, and since we are assuming $z^2 \in \mathbf{R}e + \mathbf{R}z$, we have $\mathbf{R}z^2 + \mathbf{R}z = \mathbf{R}e + \mathbf{R}z$. Since the space on the left is commutative, we again obtain $[e, z] = 0$ which gives a contradiction. Thus, $z^2 \notin \mathbf{R}e + \mathbf{R}z$ and we are done.

COROLLARY 4.16. *If* $z \in V_\zeta \cap A_{1/2}$ *and* $z \neq 0$, *then* $z^2 \notin \mathbf{R}e + \mathbf{R}z$.

COROLLARY 4.17. (i) $A = C(e)$ *if and only if* $z^2 \in \mathbf{R}e + \mathbf{R}z$ *for all* $z \in A_{1/2}$.

(ii) *If* $A = C(e)$ *and* $\dim A \geqq 4$, *then* $A = \mathbf{R}e \oplus A_\alpha$ *for some* $\alpha \in \mathbf{R}$.

Proof. If e and z commute, then $z^2 \in \mathbf{R}e + \mathbf{R}z$, so the "only if" portion of (i) is clear. Conversely, assume $z^2 \in \mathbf{R}e + \mathbf{R}z$ for all $z \in A_{1/2}$. Then by Lemma 4.15, L_e must have real roots on $E_{1/2}$, and by Lemma 4.13

$$E_{1/2} = U = \{x \in E_{1/2} | (L_e - \tfrac{1}{2}I)x = 0\} \subseteq C(e).$$

Since $xe = ex = \frac{1}{2}x$ for all $x \in E_{1/2}$, we have $E_{1/2} = A_{1/2}$, and if $E_{1/2} \neq 0$, then by Lemma 4.12 $A = \mathbf{R}e \oplus A_{1/2} = C(e)$. If $E_{1/2} = 0$, then the hypothesis $z^2 \in \mathbf{R}e + \mathbf{R}z$ is vacuously satisfied. However, Theorem 4.5 implies that $A = C(e)$ in this case where $C(e) = B_1$ or $\mathbf{R}e \oplus A_\alpha$. In the course of proving (i) we showed that when A satisfies the hypotheses in (i), then $A = \mathbf{R}e + A_\alpha$ or $A = B_1$. But if $A = B_1$ and $\dim A \geqq 4$, then $B_1 = A_1$ by Corollary 4.11, and so assertion (ii) must hold.

COROLLARY 4.18. *Every flexible real division algebra of dimension 2 is commutative.*

Proof. The condition $z^2 \in \mathbf{R}e + \mathbf{R}z$ is trivially satisfied by 2-dimensional algebras.

We next consider the case when L_e has complex roots on $E_{1/2}$.

LEMMA 4.19. *If* $z \in V_\zeta \cap A_{1/2}$, $z \neq 0$, *then for some* $\delta \in \mathbf{R}$, $f = (1/\delta)z^2 \in C(e)$ *is an idempotent of* A *not equal to* e, *and* $\dim C(f) \geqq 3$.

Proof. By Lemma 4.13, z^2 commutes with e and therefore $(z^2)^2$ lies in the subalgebra $\mathbf{R}e + \mathbf{R}z^2$. If $(z^2)^2 \notin \mathbf{R}z^2$, then $\mathbf{R}z^2 + \mathbf{R}(z^2)^2$ is 2-dimensional and is contained in the subalgebra $\mathbf{R}z + \mathbf{R}z^2$, so in fact the two are equal. However, e commutes with everything in $\mathbf{R}z^2 + \mathbf{R}(z^2)^2$, and thus would commute with z. But then $ez = ze = \frac{1}{2}z$, which cannot be true for $z \in V_\zeta$. Thus, it must be that $(z^2)^2 \in \mathbf{R}z^2$, and $f = (1/\delta)z^2$ is an idempotent if $(z^2)^2 = \delta z^2$. The elements e, f are linearly independent since z commutes with f but not e, and $\dim C(f) \geqq 3$ because $e, z, f \in C(f)$.

With these preliminaries in hand, we have the information needed to prove

THEOREM 4.20. *Let* A *be a real flexible division algebra. Then there is an idempotent* e *in* A *such that*

(i) $A = C(e)$, *or else*

(ii) $\dim A = 8$, *and* $A = C(e) \oplus E_{1/2}$ *where* $C(e)$ *is a 4-dimensional subalgebra with* $C(e) = \mathbf{R}e + A_\alpha$ *for some* $\alpha \neq \frac{1}{2}$, *and* L_e *on* $E_{1/2}$ *has eigenvalues in* $\mathbf{C} - \mathbf{R}$.

Proof. To begin, let e be any idempotent of A and define the spaces $C(e)$, A_α, etc. with respect to e. If $z^2 \in \mathbf{R}e + \mathbf{R}z$ for all $z \in A_{1/2}$, then by Corollary 4.17, $A = C(e)$, and we are in case (i). We may assume then that $z^2 \notin \mathbf{R}e + \mathbf{R}z$ for some $z \in A_{1/2}$, which by Lemma 4.15, is equivalent to saying L_e has complex roots on $E_{1/2}$. Let $\zeta \in \mathbf{C} - \mathbf{R}$ be such a root, and suppose $0 \neq z \in V_\zeta \cap A_{1/2}$. Then by Lemma 4.19, $f = (1/\delta)z^2 \in C(e)$ is an idempotent for some $\delta \in \mathbf{R}$ and $\dim C(f) \geqq 3$. If $C(f) = A$ we are in case (i), so we may assume that $A \neq C(f)$ and $\dim C(f) \geqq 3$. We now apply Theorem 4.5 to the idempotent f, to conclude that there are two possibilities: either $A = C(f) \oplus E_{1/2}(f)$ where $C(f)$ is a subalgebra with $C(f) = B_1(f)$ or $C(f) = \mathbf{R}f \oplus A_\alpha(f)$ for some $\alpha \neq 1, \frac{1}{2}$, or else $A = \mathbf{R}f \oplus E_{1/2}(f)$. (We have modified the notation to emphasize that the decompositions are relative to f.) In the first case, it follows from Lemmas 4.12 and 4.13 that L_f has eigenvalues in $\mathbf{C} - \mathbf{R}$ on $E_{1/2}(f)$. Also in this case, $\dim C(f) = 4$, since $C(f)$ is a subalgebra of dimension at least 3, and whenever $C(f) = B_1(f)$, then $C(f) = A_1(f)$ by Corollary 4.11. Thus, in the first possibility, we are in case (ii) relative to the idempotent f.

Suppose now $A = \mathbf{R}f \oplus E_{1/2}(f)$ and $\dim C(f) \geqq 3$. Since e commutes with f, they generate a 2-dimensional subalgebra, on which L_f has eigenvalues 1, $\frac{1}{2}$, Therefore, $fe = \frac{1}{2}e + \xi f$, and relative to the basis f, e the transformation $L_{\lambda_1 f+\lambda_2 e}$ has the matrix

$$\begin{pmatrix} \lambda_1 + \lambda_2\xi & \lambda_1\xi \\ \frac{1}{2}\lambda_2 & \frac{1}{2}\lambda_1 + \lambda_2 \end{pmatrix}.$$

Thus, $L_{\lambda_1 f+\lambda_2 e}$ is nonsingular if and only if for λ_1, λ_2 not both zero

$$\tfrac{1}{2}\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2\xi \neq 0.$$

The discriminant $1 - 2\xi$ must be negative for this to happen, and hence, $\xi > \frac{1}{2}$. But relative to the basis f, e, L_e has matrix

$$\begin{pmatrix} \xi & 0 \\ \frac{1}{2} & 1 \end{pmatrix}$$

and hence ξ is an eigenvalue for L_e on $C(e)$. Since $\xi > \frac{1}{2}$, $C(e)$ is a subalgebra by Theorem 4.5, and $C(e) = \mathbf{R}e \oplus A_\xi(e)$ if $\xi \neq 1$ or else $C(e) = B_1(e)$. If $\dim C(e) = 4$, then A satisfies (ii) with respect to the idempotent e, and we are done. Consequently, we have argued that except when $\dim C(e) = 2$ for our initial choice of an idempotent e, cases (i) and (ii) hold. Therefore, beginning with the idempotent f, we can proceed as we did with e, and cases (i) and (ii) must hold since $\dim C(f) \geqq 3$. This concludes the proof.

5. The case $A = C(e)$. Since we have already covered the case when $\dim A = 2$ in Corollary 4.18 and Theorem 2.3, we can assume here that $\dim A = 4$ or 8. By Corollary 4.17 it follows that $A = \mathbf{R}e + A_\alpha$ for some nonzero $\alpha \in \mathbf{R}$. Then A has a basis $e, y_1, \ldots, y_m$ where $ey_i = \alpha y_i = y_ie$ for $1 \leqq i \leqq m = \dim A - 1$. Since e and y_i commute,

$$y_i^2 = \beta_ie + \delta_iy_i \quad \text{for some } \beta_i, \delta_i \in \mathbf{R}$$

by Corollary 4.3, and hence the element $x_i = y_i - \frac{1}{2}\delta_i\alpha^{-1}e$ satisfies

$$x_i^2 = \left(y_i - \frac{1}{2}\frac{\delta_i}{\alpha}e\right)^2 = y_i^2 - \delta_i y_i + \frac{1}{4}\frac{\delta_i^2}{\alpha^2}e = \beta_i e - \frac{1}{4}\frac{\delta_i^2}{\alpha^2}e \in \mathbf{R}e.$$

Changing to the new basis $e, x_1, \ldots, x_m$, we obtain

$$ex_i = \alpha y_i - \frac{1}{2}\delta_i e = \alpha x_i + \left(\frac{1}{2}\frac{\delta_i}{\alpha} - \frac{1}{2}\delta_i\right)e.$$

We have achieved a basis $e, x_1, \ldots, x_m$ with the properties that

(5.1) $\quad x_i^2 \in \mathbf{R}e, \quad ex_i = \alpha x_i + \eta_i e$

for some $\eta_i \in \mathbf{R}$ and for $1 \leqq i \leqq m$.

LEMMA 5.2. (i) *The set* $V = \{y \in A | y \notin \mathbf{R}e, y^2 \in \mathbf{R}e\}$ *is a subspace of* A *of dimension* $m = \dim A - 1$, *and* $x_1, \ldots, x_m$ *are a basis of* V.
(ii) *If* $y, z \in V$, *then* $y \circ z \in \mathbf{R}e$.

Proof. Let $y, z \in V$, say $y^2 = \rho e$ and $z^2 = \sigma e$. For each $\lambda \in \mathbf{R}$, $\lambda y + z$ commutes with e, so there exist $\mu, \nu \in \mathbf{R}$ depending on λ such that

$$(\lambda y + z)^2 = \mu e + \nu(\lambda y + z).$$

Thus,

(5.3) $\quad \lambda^2\rho e + \lambda(y \circ z) + \sigma e = \mu e + \nu\lambda y + \nu z,$

which shows that $y \circ z$ is a linear combination of e, y, z, say

$$y \circ z = \tau_1 e + \tau_2 y + \tau_3 z.$$

Substituting this into (5.3) and assuming that e, y, z are linearly independent, we may equate the coefficients of y and z on the two sides of (5.3) to get $\lambda\tau_2 = \nu\lambda$ and $\lambda\tau_3 = \nu$. Thus, $\tau_2 = \nu = \lambda\tau_3$ for all nonzero values of $\lambda \in \mathbf{R}$, which implies that $\tau_2 = 0 = \tau_3$ since τ_2 and τ_3 do not depend on λ. This shows that $y \circ z \in \mathbf{R}e$, and hence $(\lambda y + z)^2 \in \mathbf{R}e$ for e, y, z linearly independent. If e, y, z are linearly dependent, then y and z are linearly dependent, and again $y \circ z \in \mathbf{R}e$ and $(\lambda y + z)^2 \in \mathbf{R}e$. Thus, $\lambda y + z \in V$. Since $x_1, \ldots, x_m \in V$ and $e \notin V$, clearly $\dim V = m = \dim A - 1$ and $x_1, \ldots, x_m$ are a basis of V.

Part (ii) of Lemma 5.2 allows us to define a bilinear form (y, z) on V by $y \circ z = -2(y, z)e$, and it is immediate that $(y, z) = (z, y)$ for all $y, z \in V$. This bilinear form can be extended to a symmetric bilinear form on all of A by defining $(e, e) = 1$ and $(e, x) = 0 = (x, e)$ for $x \in V$.

LEMMA 5.4. (i) *If* $y \in V$ *and* $a \in A$, *then* $y \circ a \in \mathbf{R}e$ *if and only if* $a \in V$.
(ii) *If* $y, z \in V$ *and if* $(y, z) = 0$, *then* $yz \in V$ *and* $(y, yz) = 0$.
(iii) *If* $y, z \in V$, *then* $[y, z] \in V$.

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Proof. If $y \in V$ and $a \in A$, then $a = \beta e + z$ for some $\beta \in \mathbf{R}$ and $z \in V$, and $y \circ a = 2\beta ey + y \circ z$. Since $y \circ z \in \mathbf{R}e$, clearly $y \circ a \in \mathbf{R}e$ if and only if $a \in V$. For part (ii), suppose that $y, z \in V$ with $(y, z) = 0$. Then $yz = -zy$, and so

$$y \circ (yz) = y(yz) + (yz)y = -y(zy) + (yz)y = 0.$$

By part (i), $yz \in V$, and hence $(y, yz) = 0$. Finally, for part (iii), given $y, z \in V$, we may find $\delta \in \mathbf{R}$ and $z_1 \in V$ satisfying $(y, z_1) = 0$ such that $z = z_1 + \delta y$. Then

$$[y, z] = [y, z_1 + \delta y] = [y, z_1] = yz_1 - z_1 y.$$

Since $yz_1, z_1 y \in V$ by part (ii), we have $[y, z] \in V$.

LEMMA 5.5. *The bilinear form (y, z) is associative and positive definite on V.*

Proof. For $y, z \in V$, we again write $z = z_1 + \delta y$ for $\delta \in \mathbf{R}$, and $z_1 \in V$ and $(y, z_1) = 0$. Then

$$(y, yz) = (y, yz_1 + \delta y^2) = (y, yz_1) + \delta(y, y^2) = 0$$

using part (ii) of Lemma 5.4 and $y^2 \in \mathbf{R}e$. Linearizing y in this relation gives $(x, yz) + (y, xz) = 0$, which yields

$$(xz, y) = (y, xz) = -(x, yz) = -(x, y \circ z) + (x, zy) = (x, zy).$$

To prove that the form is positive definite, we must show that $(y, y) > 0$ for any nonzero $y \in V$, We have

$$y^2 = \tfrac{1}{2} y \circ y = -(y, y)e,$$

and the relation $ey = \alpha y + \eta e$ for some $\eta \in \mathbf{R}$ follows from (5.1). The determinant of left multiplication by the element $\lambda_1 e + \lambda_2 y$ in the subalgebra spanned by e and y is

$$\begin{vmatrix} \lambda_1 + \eta\lambda_2 & \eta\lambda_1 - (y, y)\lambda_2 \\ \alpha\lambda_2 & \alpha\lambda_1 \end{vmatrix} = \alpha(\lambda_1^2 + (y, y)\lambda_2^2).$$

Since A is a division algebra, this determinant cannot be zero for any $\lambda_1, \lambda_2 \in \mathbf{R}$ not both zero, and hence $(y, y) > 0$.

Now that we have a symmetric positive definite form on V, we want to make our basis $x_1, \ldots, x_m$ of V behave nicely with respect to this form. By the Gram-Schmidt orthonormalization process, we can modify the x_i's so that they form an orthonormal basis of V. Thus, we may assume that $x_i^2 = -e$ and $x_i x_j = -x_j x_i$ for $1 \leqq i, j \leqq m$ and $i \neq j$.

LEMMA 5.6. *If $x_1, \ldots, x_m$ is an orthonormal basis of V, then*

$$(5.7) \qquad x_i x_j = \sum_{k=1}^{m} \gamma_{ijk} x_k - \delta_{ij} e$$

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where δ_{ij} is the Kronecker delta, and where $\gamma_{ijk} \in \mathbf{R}$ and γ_{ijk} is skew-symmetric in its subscripts.

Proof. If $i \neq j$, then $x_i x_j \in V$ by Lemma 5.4 (ii), showing that the e-component of (5.7) is correct. Also, $\gamma_{ijk} = -\gamma_{ijk}$, since $x_i x_j = -x_j x_i$. Finally,

$$-\gamma_{ijk}e = (\gamma_{ijk}x_k, x_k) = \left(\sum_l \gamma_{ijl}x_l - \delta_{ij}e, x_k\right) = (x_i x_j, x_k)$$

$$= (x_i, x_j x_k) = \left(x_i, \sum_l \gamma_{jkl}x_l - \delta_{jk}e\right) = (x_i, \gamma_{jki}x_i) = -\gamma_{jki}e,$$

or $\gamma_{ijk} = \gamma_{jki}$. It follows that γ_{ijk} is skew-symmetric on its subscripts.

LEMMA 5.8. *For any $y \in V$, $ey = \alpha y$.*

Proof. It is sufficient to show that, for an orthonormal basis $x_1, \dots, x_m$ of V, the constants η_i of (5.1) are all zero. From (1.1) we have for $i \neq j$

$$\begin{aligned} 0 &= (x_i x_j)e + (ex_j)x_i - x_i(x_j e) - e(x_j x_i) \\ &= 2(x_i x_j)e + (\alpha x_j + \eta_i e)x_i - x_i(\alpha x_j + \eta_i e) \\ &= 2(x_i x_j)e - 2\alpha x_i x_j = 2\sum_k \gamma_{ijk}x_k e - 2\alpha \sum_k \gamma_{ijk}x_k \\ &= 2\sum_k \gamma_{ijk}\eta_k e, \end{aligned}$$

or

$$(5.9) \qquad 0 = \sum_{k=1}^{m} \gamma_{ijk}\eta_k$$

for any i, j not equal. If dim $A = 4$, then $m = 3$ and (5.9) gives

$$0 = \gamma_{121}\eta_1 + \gamma_{122}\eta_2 + \gamma_{123}\eta_3 = \gamma_{123}\eta_3,$$

since $\gamma_{121} = 0 = \gamma_{122}$ by the skew-symmetry of the subscripts on γ_{ijk}. Now $\gamma_{123} \neq 0$ since $x_1 x_2 \neq 0$, and so $\eta_3 = 0$. Similarly, $\eta_1 = 0 = \eta_2$.

If dim $A = 8$, we consider the six equations

$$(5.10) \qquad \sum_{k=1}^{6} \gamma_{7jk}\eta_k = 0$$

for $1 \leqq j \leqq 6$. To show that $\eta_1, \dots, \eta_6$ are zero, it is sufficient to show that the determinant $|\gamma_{7jk}|$ for $1 \leqq j, k \leqq 6$ is nonzero. For this, we consider left multiplication by x_7 on the space V' spanned by $x_1, \dots, x_6$. Since V' is the orthogonal complement on $\mathbf{R}x_7$ in V, we have $x_7 V' \subseteq V$, and the relation $0 = (x_7, x_7 V')$ implies that $x_7 V' \subseteq V'$. In fact, $x_7 V' = V'$ because A is a division algebra. But the determinant of left multiplication by x_7 acting on V' is just $|\gamma_{7jk}|$ for $1 \leqq j, k \leqq 6$, so this determinant is

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nonzero. Thus, (5.10) implies that $\eta_1, \ldots, \eta_6$ are zero, and $\eta_7 = 0$ by symmetry.

LEMMA 5.11. *For each $a \in A$, there exist $\beta_1, \beta_2 \in \mathbf{R}$ and $x \in V$ such that $a = \beta_1 e + \beta_2 x$ and $x^2 = -e$. Every subalgebra B of A of dimension 2 has as a basis e, x for some $x \in V$ with $x^2 = -e$.*

Proof. Given $a \in A$, we can find $\beta_1 \in \mathbf{R}$ and $y \in V$ such that $a = \beta_1 e + y$. If $y = \sum_i \zeta_i x_i \neq 0$ for $\zeta_i \in \mathbf{R}$, then $y^2 = -(\sum_i \zeta_i^2)e$ by (5.7). Setting $\beta_2 = (\sum_i \zeta_i^2)^{1/2}$ and $x = \beta_2^{-1} y$, we have $x^2 = -e$ and $a = \beta_1 e + \beta_2 x$ as desired. If B is any subalgebra of A of dimension 2, then $B \cap V \neq 0$ since $\dim V = \dim A - 1$, and so B contains a nonzero element $y \in V$. Just as above, $y = \beta_2 x$ for $x^2 = -e$, and $x \in B$. Clearly, $e \in B$, and e, x form a basis for B.

At this point, the only additional property that would be needed to make A into a flexible quadratic algebra would be to have $\alpha = 1$. Since α is not necessarily 1 in general, we can only achieve this by taking an appropriate isotope.

LEMMA 5.12. *Let Q be the linear transformation on A defined by $Q(e) = e$ and $Q(x) = \alpha^{-1}x$ for $x \in V$, and let $(A, *)$ be the principal isotope of A defined by $a * b = Q(a)Q(b)$ for all $a, b \in A$. Then $(A, *)$ is a flexible quadratic algebra, and A is a scalar isotope of $(A, *)$.*

Proof. The element e is the identity element in $(A, *)$, since

$$e * e = e^2 = e, \; e * x = eQ(x) = e(\alpha^{-1}x) = x = (\alpha^{-1}x)e = x * e$$

for $x \in V$. For $\beta e + x$ where $\beta \in \mathbf{R}$ and $x \in V$, we define

$$T(\beta e + x) = 2\beta \quad \text{and} \quad N(\beta e + x) = \beta^2 + \alpha^{-2}(x, x),$$

and we calculate that

$$\begin{aligned}(\beta e + x) * (\beta e + x) &- T(\beta e + x)(\beta e + x) + N(\beta e + x)e \\ &= \beta^2 e + 2\beta x + x * x - 2\beta(\beta e + x) + \beta^2 e + \alpha^{-2}(x, x)e \\ &= -\alpha^{-2}(x, x)e + \alpha^{-2}(x, x)e = 0.\end{aligned}$$

Thus $(A, *)$ is a quadratic algebra.

In order to show that $(A, *)$ is flexible, one of the properties that we have to verify (see Lemma (1.2)) is that any commutator lies in V (which is the same in $(A, *)$ as in A). This follows from

$$\begin{aligned}(\beta e + x) * (\delta e + y) - (\delta e + y) * (\beta e + x) &= x * y - y * x \\ &= \alpha^{-2}(xy - yx) \in V\end{aligned}$$

for $\beta, \delta \in \mathbf{R}$ and $x, y \in V$. As we saw in Section 1, this property is equivalent to the property that the natural bilinear form $(x, y)^*$ on $(A, *)$ is symmetric. Using this symmetry, we have for $x, y \in V$

$$(x, y)^*e = -\tfrac{1}{2}(x * y + y * x) = -\tfrac{1}{2}\alpha^{-2}(xy + yx) = \alpha^{-2}(x, y)e,$$

so that $(x, y)^* = \alpha^{-2}(x, y)$. Then the second property needed to establish flexibility in $(A, *)$ follows from

$$(x, x * y)^* = \alpha^{-2}(x, \alpha^{-2}xy) = \alpha^{-4}(x, xy) = 0$$

for $x, y \in V$.

Now that $(A, *)$ is known to be a flexible quadratic algebra, we want to show that A is a scalar isotope of $(A, *)$. Defining the linear transformation P on A by $P(e) = e$ and $P(x) = \alpha x$ for $x \in V$, we see that $P = Q^{-1}$ and that

$$xy = QP(x)QP(y) = P(x) * P(y).$$

We have proved all but the last sentence of

THEOREM 5.13. *Let A be a real flexible division algebra of dimension* 4 *or* 8 *containing an idempotent commuting with all the elements of A. Then A is isomorphic to a scalar isotope ${}_\alpha B$ of some real flexible quadratic division algebra B. If* dim $A = 4$, *then A has a basis which multiplies as in the Table* 3.4. *Conversely an algebra with multiplication as in* (3.4) *where $\alpha \neq 0$ and $\beta > 0$ is a flexible division algebra.*

Proof. If dim $A = 4$, let e, x_1, x_2, x_3 be an orthonormal basis. Then by Lemma 5.6, $\gamma_{123} = \gamma_{231} = \gamma_{312} = -\gamma_{213} = -\gamma_{132} = -\gamma_{321}$, and the other γ_{ijk}'s are zero. Writing γ for γ_{123} and using (5.7) and Lemma 5.8, the multiplication table for A is given by

(5.14)

	e	x_1	x_2	x_3
e	e	αx_1	αx_2	αx_3
x_1	αx_1	$-e$	γx_3	$-\gamma x_2$
x_2	αx_2	$-\gamma x_3$	$-e$	γx_1
x_3	αx_3	γx_2	$-\gamma x_1$	$-e$

If, in this table, we replace each x_i by γ^{-1} times itself and write $\beta = \gamma^{-2}$, we obtain Table 3.4.

To see that this algebra is flexible, we note that all commutators lie in V, and that $(x_i x_j, x_k) = (x_i, x_j x_k)$ for all $i, j, k \in \{1, 2, 3\}$. Thus, flexibility follows from Lemma 1.2.

We asserted in Theorem 3.3 that the algebra defined by (3.4) is Lie-admissible. This is proved in

PROPOSITION 5.15. *Let A be a scalar isotope of a flexible quadratic division algebra. If* dim $A = 4$, *then A is Lie-admissible. If* dim $A = 8$, *then A is not Lie-admissible.*

Proof. Since e commutes with all elements of A, e is in the center of A^-. If $\dim A = 4$, the products between x_1, x_2, x_3 in A^- are

$$[x_1, x_2] = 2x_3, \quad [x_2, x_3] = 2x_1, \quad \text{and} \quad [x_3, x_1] = 2x_2.$$

Thus, V^- is isomorphic to $su(2)$, and A is Lie-admissible. If $\dim A = 8$, we observe that the multiplication in A^- differs only by a scalar multiple from the multiplication in B^- where B is the flexible quadratic algebra of which A is a scalar isotope. Thus, we only need to consider the case where A itself is a flexible quadratic division algebra. As we saw in Section 1, the fact that A is a division algebra implies that, for any linearly independent $x, y \in V$, the elements x, y, and $[x, y] = 2x \times y$ are linearly independent. It is clear from this that V^- is a simple algebra of dimension 7. But, since there are no simple Lie algebras of dimension 7, V^- cannot be a Lie algebra, and hence A is not Lie-admissible.

One can readily see from Theorem 5.13 that every flexible quadratic division algebra of dimension 4 is a scalar isotope of the quaternions, but it is not the case that every 8-dimensional one is a scalar isotope of the octonions. There are many more flexible quadratic division algebras of dimension 8 than of dimension 4, and we find no method of classifying them which further illuminates their structure. Basically, the reason that there are so many more algebras of dimension 8 is that the condition that xy is orthogonal to both x and y for $x, y \in V$ does not come close to determining xy, as it does in the case of dimension 4. To illustrate this diversity for dimension 8, we end this section by constructing a class of real flexible division algebras which are not scalar isotopes of the octonions.

PROPOSITION 5.16. *Let A be the real algebra with basis $e, x_1, \ldots, x_7$ and with multiplication determined by Table 5.17 where $\alpha, \beta, \delta \in \mathbf{R}$ and $\alpha \neq 0$. Then A is flexible, and A is a division algebra if $\beta > 0$ and $|\delta| < 2$. If, in addition, $\delta \neq 0$, then A is not isomorphic to any of the algebras defined in Section 2 of* [**3**], *and hence A is not isomorphic to a scalar isotope of the octonions.*

(5.17)

	e	x_1	x_2	x_3	x_4	x_5	x_6	x_7
e	e	αx_1	αx_2	αx_3	αx_4	αx_5	αx_6	αx_7
x_1	αx_1	$-\beta e$	x_4	x_7	$-x_2$	x_6	$-x_5 + \delta x_7$	$-x_3 - \delta x_6$
x_2	αx_2	$-x_4$	$-\beta e$	x_5	x_1	$-x_3$	x_7	$-x_6$
x_3	αx_3	$-x_7$	$-x_5$	$-\beta e$	x_6	x_2	$-x_4$	x_1
x_4	αx_4	x_2	$-x_1$	$-x_6$	$-\beta e$	x_7	x_3	$-x_5$
x_5	αx_5	$-x_6$	x_3	$-x_2$	$-x_7$	$-\beta e$	x_1	x_4
x_6	αx_6	$x_5 - \delta x_7$	$-x_7$	x_4	$-x_3$	$-x_1$	$-\beta e$	$x_2 + \delta x_1$
x_7	αx_7	$x_3 + \delta x_6$	x_6	$-x_1$	x_5	$-x_4$	$-x_2 - \delta x_1$	$-\beta e$

Proof. Letting V denote the subspace of A spanned by $x_1, \ldots, x_7$, and Q the linear transformation of A defined by $Q(e) = e$ and $Q(x) = \alpha^{-1}x$ for $x \in V$, we see that e is the identity element for $(A, *)$. By our remarks in Section 1, it is sufficient to show that $(A, *)$ is a flexible division algebra in order to show that A is (see also Lemma 5.12). If $x, y \in V$, then $x * y = \alpha^{-2}xy$, implying that multiplication in $(A, *)$ between elements of V will be exactly as in (5.17) if we replace x_i by αx_i for $1 \leqq i \leqq 7$. Thus it is sufficient to establish the proposition when $\alpha = 1$. We shall assume $\alpha = 1$ in the remainder of the proof.

If $a \in A$, then there exists $\eta \in \mathbf{R}$ and $y \in V$ such that $a = \eta e + y$, and $ey = y = ye$ and $y^2 = -\zeta e$ for some $\zeta \in \mathbf{R}$. Thus,

$$a^2 - 2\eta a + (\eta^2 + \zeta)e = \eta^2 e + 2\eta y - \zeta e - 2\eta^2 e - 2\eta y + (\eta^2 + \zeta)e = 0,$$

showing that A is quadratic. The condition that $[x, y] \in V$ for $x, y \in V$ follows immediately from the form of (5.17). Then by Lemma 1.2, to show that A is flexible, we need only show that the bilinear form is associative on V. The form is defined here by

$$-2(x_i, x_i)e = 2x_i^2 = -2\beta e \quad \text{and}$$
$$-2(x_i, x_j)e = x_i x_j + x_j x_i = 0 \quad \text{for } i \neq j,$$

giving $(x_i, x_i) = \beta$ and $(x_i, x_j) = 0$ for $i \neq j$. We have

$$x_i x_j = \sum_k \gamma_{ijk} x_k - \eta_{ij} e \quad \text{for some } \gamma_{ijk}, \eta_{ij} \in \mathbf{R},$$

and from (5.17) we obtain $\eta_{ii} = \beta$, $\eta_{ij} = 0$ for $i \neq j$, and $\gamma_{ijk} = -\gamma_{jik}$. The condition that the form is associative is that γ_{ijk} is skew-symmetric in its subscripts, since

$$(x_i x_j, x_k) = \left(\sum_l \gamma_{ijl} x_l, x_k\right) = (\gamma_{ijk} x_k, x_k) = \gamma_{ijk}\beta,$$
$$(x_i, x_j x_k) = \left(x_i, \sum_l \gamma_{jkl} x_l\right) = (x_i, \gamma_{jki} x_i) = \gamma_{jki}\beta.$$

The skew-symmetry of γ_{ijk} can be verified directly from (5.16), for example,

$$\delta = \gamma_{167} = \gamma_{671} = \gamma_{716} = -\gamma_{617} = -\gamma_{176} = -\gamma_{761}.$$

Except when $\{i, j, k\} = \{1, 6, 7\}$, the numbers γ_{ijk} are the same here as in the octonions. Thus, we have shown that A is flexible.

To show that A is a division algebra when $\beta > 0$ and $|\delta| < 2$, we shall verify that the linear transformation defined by left multiplication by the element $\lambda_0 e + \sum_{i=1}^7 \lambda_i x_i$ is nonsingular for any $\lambda_0, \ldots, \lambda_7 \in \mathbf{R}$ not all zero.

The matrix of this linear transformation with respect to the basis e, $x_1, \ldots, x_7$ is

$$\begin{pmatrix} \lambda_0 & -\beta\lambda_1 & -\beta\lambda_2 & -\beta\lambda_3 & -\beta\lambda_4 & -\beta\lambda_5 & -\beta\lambda_6 & -\beta\lambda_7 \\ \lambda_1 & \lambda_0 & -\lambda_4 & -\lambda_7 & \lambda_2 & -\lambda_6 & \lambda_5 - \delta\lambda_7 & \lambda_3 + \delta\lambda_6 \\ \lambda_2 & \lambda_4 & \lambda_0 & -\lambda_5 & -\lambda_1 & \lambda_3 & -\lambda_7 & \lambda_6 \\ \lambda_3 & \lambda_7 & \lambda_5 & \lambda_0 & -\lambda_6 & -\lambda_2 & \lambda_4 & -\lambda_1 \\ \lambda_4 & -\lambda_2 & \lambda_1 & \lambda_6 & \lambda_0 & -\lambda_7 & -\lambda_3 & \lambda_5 \\ \lambda_5 & \lambda_6 & -\lambda_3 & \lambda_2 & \lambda_7 & \lambda_0 & -\lambda_1 & -\lambda_4 \\ \lambda_6 & -\lambda_5 + \delta\lambda_7 & \lambda_7 & -\lambda_4 & \lambda_3 & \lambda_1 & \lambda_0 & -\lambda_2 - \delta\lambda_1 \\ \lambda_7 & -\lambda_3 - \delta\lambda_6 & -\lambda_6 & \lambda_1 & -\lambda_5 & \lambda_4 & \lambda_2 + \delta\lambda_1 & \lambda_0 \end{pmatrix}$$

Multiplying the first row of this matrix by β^{-1}, we obtain a matrix which we denote by M. We shall show that $|M| \neq 0$ for all $\lambda_0, \ldots, \lambda_7 \in \mathbf{R}$ not all zero, when $\beta > 0$ and $|\delta| < 2$. If N denotes the matrix obtained from M by setting $\beta = 1$ and $\delta = 0$, then N is the left multiplication matrix that comes from the octonions, and so $|N| \neq 0$ for any choice of $\lambda_0, \ldots, \lambda_7$ not all zero. Thus, in order to show that $|M| \neq 0$ for $\lambda_0, \ldots, \lambda_7$ not all zero, it is sufficient to verify that $|MN^t| \neq 0$ for $\lambda_0, \ldots, \lambda_7 \in \mathbf{R}$ not all zero, where N^t is the transpose of N. We calculate that MN^t is as given below, where the asterisks denote entries which need not be computed.

$$|MN^t| = \begin{vmatrix} \beta^{-1}\lambda_0^2 + \sum_{i=1}^{7} \lambda_i^2 & * & * & * & * & * & * & * \\ 0 & \sum \lambda_i^2 - \delta\lambda_5\lambda_7 + \delta\lambda_3\lambda_5 & * & * & * & * & -\delta\lambda_0\lambda_7 - \beta\lambda_2\lambda_6 & -\delta\lambda_2\lambda_7 + \beta\lambda_0\lambda_6 \\ 0 & 0 & \sum\lambda_i^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sum\lambda_i^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sum\lambda_i^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sum\lambda_i^2 & 0 & 0 \\ 0 & \delta\lambda_0\lambda_7 - \delta\lambda_1\lambda_3 & * & * & * & * & \sum \lambda_i^2 - \delta\lambda_5\lambda_7 + \delta\lambda_1\lambda_2 & -\delta\lambda_3\lambda_7 - \delta\lambda_0\lambda_1 \\ 0 & -\delta\lambda_0\lambda_6 + \delta\lambda_1\lambda_5 & * & * & * & * & \delta\lambda_5\lambda_6 + \delta\lambda_0\lambda_1 & \sum \lambda_i^2 + \delta\lambda_3\lambda_6 + \delta\lambda_1\lambda_2 \end{vmatrix}$$

Expanding $|MN^t|$ on the first column and then on the rows with only one nonzero entry, we obtain

$$|MN^t| = \left(\beta^{-1}\lambda_0^2 + \sum_{i=1}^{7} \lambda_i^2\right)\left(\sum \lambda_i^2\right)^4 D$$

where

$$D = \begin{vmatrix} \sum \lambda_i^2 - \delta\lambda_5\lambda_7 + \delta\lambda_3\lambda_6 & -\delta\lambda_0\lambda_7 - \delta\lambda_2\lambda_6 & -\delta\lambda_2\lambda_7 + \delta\lambda_0\lambda_6 \\ \delta\lambda_0\lambda_7 - \delta\lambda_1\lambda_3 & \sum \lambda_i^2 - \delta\lambda_5\lambda_7 + \delta\lambda_1\lambda_2 & -\delta\lambda_3\lambda_7 - \delta\lambda_0\lambda_1 \\ -\delta\lambda_0\lambda_6 + \delta\lambda_1\lambda_5 & \delta\lambda_5\lambda_6 + \delta\lambda_0\lambda_1 & \sum \lambda_i^2 + \delta\lambda_3\lambda_6 + \delta\lambda_1\lambda_2 \end{vmatrix}$$

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Since $\beta > 0$, $\beta^{-1}\lambda_0^2 + \sum_{i=1}^{7} \lambda_i^2 \neq 0$ if $\lambda_0, \ldots, \lambda_7 \in \mathbf{R}$ are not all zero. Hence, A is a division algebra if $D \neq 0$ for $\lambda_0, \ldots, \lambda_7$ not all zero.

Multiplying the second row of D by λ_6, and then adding λ_1 times the first row and λ_7 times the third row to the second row, yields

$$\lambda_6 D = \begin{vmatrix} \sum \lambda_i^2 - \delta\lambda_5\lambda_7 + \delta\lambda_3\lambda_6 & -\delta\lambda_0\lambda_7 - \delta\lambda_2\lambda_6 & -\delta\lambda_2\lambda_7 + \delta\lambda_0\lambda_6 \\ \lambda_1 \sum \lambda_i^2 & \lambda_6 \sum \lambda_i^2 & \lambda_7 \sum \lambda_i^2 \\ -\delta\lambda_0\lambda_6 + \delta\lambda_1\lambda_5 & \delta\lambda_5\lambda_6 + \delta\lambda_0\lambda_1 & \sum \lambda_i^2 + \delta\lambda_3\lambda_6 + \delta\lambda_1\lambda_2 \end{vmatrix}$$

Dividing the second row by $\sum \lambda_i^2$, then adding $\delta\lambda_2$ times the second row to the first row and subtracting $\delta\lambda_5$ times the second row from the third row, we obtain

$$\frac{\lambda_6}{\sum \lambda_i^2} D = \begin{vmatrix} \sum \lambda_i^2 - \delta\lambda_5\lambda_7 + \delta\lambda_3\lambda_6 + \delta\lambda_1\lambda_2 & -\delta\lambda_0\lambda_7 & \delta\lambda_0\lambda_6 \\ \lambda_1 & \lambda_6 & \lambda_7 \\ -\delta\lambda_0\lambda_6 & \delta\lambda_0\lambda_1 & \sum \lambda_i^2 + \delta\lambda_3\lambda_6 + \delta\lambda_1\lambda_2 - \delta\lambda_5\lambda_7 \end{vmatrix}$$

If we expand out the right side and divide each term by λ_6, we arrive at

$$\frac{1}{\sum \lambda_i^2} D = \left(\sum \lambda_i^2 - \delta\lambda_5\lambda_7 + \delta\lambda_3\lambda_6 + \delta\lambda_1\lambda_2\right)^2 + \delta^2\lambda_0^2\lambda_1^2 + \delta^2\lambda_0^2\lambda_7^2 - \delta^2\lambda_0^2\lambda_6^2.$$

The right side of this equation can only be zero if

$$\begin{aligned} 0 &= \sum \lambda_i^2 - \delta\lambda_5\lambda_7 + \delta\lambda_3\lambda_6 + \delta\lambda_1\lambda_2 \\ &= \lambda_0^2 + (\lambda_1^2 + \delta\lambda_1\lambda_2 + \lambda_2^2) + (\lambda_3^2 + \delta\lambda_3\lambda_6 + \lambda_6^2) \\ &\qquad + (\lambda_5^2 - \delta\lambda_5\lambda_7 + \lambda_7^2). \end{aligned}$$

Since $|\delta| < 2$, this can only be zero if all the λ's are zero. Thus, we have proved that A is a division algebra when $\beta > 0$ and $|\delta| < 2$.

To see that, when $\delta \neq 0$, the algebra A is not isomorphic to any of the algebras defined in Section 2 of [**3**], we need only note that the latter algebras cannot be generated by 2 elements, whereas A is generated by x_1 and x_5 when $\delta \neq 0$.

6. The generalized pseudo-octonion algebras. Throughout this section we assume that A is an 8-dimensional flexible division algebra over $\mathbf{R}$ which contains no idempotent commuting with all of A. By Theorem 4.20, we know that there exists an idempotent $e \in A$ which decomposes A into $A = \mathbf{R}e + A_\alpha \oplus E_{1/2}$ from some $\alpha \in \mathbf{R}$, $\alpha \neq \frac{1}{2}$. Moreover, $C(e) = \mathbf{R}e + A_\alpha$ is a 4-dimensional subalgebra and the eigenvalues of L_e restricted to $E_{1/2}$ are in $\mathbf{C} - \mathbf{R}$. The main purpose of this section is to prove that A is a GP-algebra. To aid our discussion, we present the following example.

Example 6.1. Recall that the GP-algebra $S = S(\delta, \frac{1}{2})$ is described in Section 1 as the algebra of 3×3 complex skew-Hermitian matrices of trace zero with "$*$" given by:

$$(6.2) \qquad x * y = \delta[x, y] + \tfrac{1}{2}i\{x \circ y - \tfrac{2}{3} \operatorname{tr}(xy)I\}.$$

Since S is an 8-dimensional flexible Lie-admissible real division algebra with minus algebra isomorphic to the simple Lie algebra $su(3)$, it contains no idempotent which commutes with every element of S. Therefore, S is an algebra of the type that we are studying in this section.

Let $\{e_{jk}\}$ be a set of 3×3 matrix units. Then a basis for S is given by:

$$(6.3) \qquad \begin{aligned} e &= ie_{11} + ie_{22} - 2ie_{33}, \\ f &= -2ie_{11} + ie_{22} + ie_{33}, \\ x &= \sqrt{3}\, ie_{12} + \sqrt{3}\, ie_{21}, \\ fx &= (3\sqrt{3}\,\delta + \tfrac{1}{2}\sqrt{3}\, i)e_{12} + (-3\sqrt{3}\,\delta + \tfrac{1}{2}\sqrt{3}\, i)e_{21}, \\ z &= \sqrt{3}\, ie_{23} + \sqrt{3}\, ie_{32}, \\ ez &= (-3\sqrt{3}\,\delta + \tfrac{1}{2}\sqrt{3}\, i)e_{23} + (3\sqrt{3}\,\delta + \tfrac{1}{2}\sqrt{3}\, i)e_{32}, \\ w &= (-3\delta - \tfrac{3}{2}i)e_{13} + (3\delta - \tfrac{3}{2}i)e_{31}, \\ ew &= (3\delta - (\tfrac{3}{4} + 9\delta^2)i)e_{13} + (-3\delta - (\tfrac{3}{4} + 9\delta^2)i)e_{31}. \end{aligned}$$

Table 6.5 is a multiplication table for S. The coefficients in this table are all rational polynomials in δ which have been simplified by using:

$$(6.4) \qquad \begin{aligned} \tau &= \tfrac{1}{4} + 9\delta^2, & \psi &= \tfrac{1}{3}(2\tau + 1), \\ \sigma &= \tfrac{1}{3}(4 - \tau), & \varphi &= \tfrac{1}{3}(4\tau - 1), \\ \rho &= \tfrac{1}{3}(\tau - 1), & \gamma &= \tfrac{1}{3}(\tau + 2). \end{aligned}$$

It is apparent from Table 6.5 that e, f and $g = -e - f$ are idempotents in S. One can verify that $C(e)$ is the linear span of $\{e, f, x, fx\}$ and

$$S_{1/2} = \{y \in S \mid (L_e + R_e - I)y = 0\}$$

is the linear span of $\{z, ez, w, ew\}$. For $\zeta = \frac{1}{2} + 3\delta i$,

$$V_\zeta = \{y \in S \mid (L_e^2 - L_e + \zeta\bar{\zeta}I)y = 0\}$$

is such that $V_\zeta = S_{1/2}$. Furthermore, decomposing $C(e)$ as in Theorem 4.20 gives $C(e) = \mathbf{R}e \oplus S_{-1}$.

Guided by this example, we begin our study of the algebra A using the idempotent e given to us by Theorem 4.20. Our first aim is to show that the decomposition of A relative to this idempotent into the spaces defined in Section 4 has the property that $E_{1/2} = A_{1/2} = V_\zeta$. This involves

TABLE 6.5.

The constants in this table are defined by (6.4).

	e	f	x	fx	z	ez	w	ew
e	e	$-e-f$	$-x$	$-fx$	ez	$ez - \tau z$	ew	$ew - \tau w$
f	$-e-f$	f	fx	$fx - \tau x$	$-z$	$-ez$	$w - ew$	τw
x	$-x$	$x - fx$	$-e$	$2\rho e + \varphi f$	w	ew	$\sigma z + ez$	$\tau z - \rho ez$
fx	$-fx$	τx	$-\psi e - \varphi f$	$-\tau e$	$w - ew$	τw	$\tau z - \rho ez$	$-\tau\rho z$
z	$z - ez$	$-z$	$-\rho\gamma^{-1}w + \gamma^{-1}ew$	$\psi\gamma^{-1}w$	$-f$	$\varphi e + 2\rho f$	γx	$\gamma x - \gamma fx$
ez	τz	$-ez$	$-\tau\gamma^{-1}w + \sigma\gamma^{-1}ew$	$-\tau\rho\gamma^{-1}w + \tau\gamma^{-1}ew$	$-\varphi e - \psi e$	$-\tau f$	γfx	$\tau\gamma x$
w	$w - ew$	ew	γz	γez	$-\rho x + fx$	$-\tau x + \sigma fx$	$-\gamma g$	$\varphi\gamma e + 2\rho\gamma g$
ew	τw	$ew - \tau w$	γez	$-\tau\gamma z - \gamma ez$	$\psi x + \rho fx$	$-\tau\rho x + \tau fx$	$-\varphi\gamma e - \psi\gamma g$	$-\gamma\tau g$

considering the action of L_e on $A_{\mathbf{C}} = \mathbf{C} \otimes A$. For $\zeta \in \mathbf{C} - \mathbf{R}$, let

$$U_\zeta = \{x \in E_{1/2} | (L_e^2 - (\zeta + \bar{\zeta})L_e + \zeta\bar{\zeta}I)^2 x = 0\},$$
$$(A_{\mathbf{C}})_{1/2} = \{x \in A_{\mathbf{C}} | ex + xe = x\},$$
$$W_\zeta = \{x \in A_{\mathbf{C}} | (L_e - \zeta I)x = 0\}.$$

By Theorem 4.20, we know that the eigenvalues of L_e restricted to $E_{1/2}$ are complex. Since $E_{1/2}$ is 4-dimensional, the characteristic polynomial of L_e on $E_{1/2}$ is

$$(\lambda^2 - (\zeta + \bar{\zeta})\lambda + \zeta\bar{\zeta})(\lambda^2 - (\mu + \bar{\mu})\lambda + \mu\bar{\mu})$$

for some $\zeta, \mu \in \mathbf{C} - \mathbf{R}$. Therefore, when $\zeta = \mu$ or $\bar{\mu}$, $E_{1/2} = U_\zeta$ and when $\zeta \neq \mu$ or $\bar{\mu}$, $E_{1/2} = V_\zeta \oplus V_\mu$.

LEMMA 6.6. *For* $\zeta, \mu \in \mathbf{C} - \mathbf{R}$,
(i) $V_\zeta \cap A_{1/2} \subseteq [W_\zeta \cap (A_{\mathbf{C}})_{1/2}] \oplus [W_{\bar{\zeta}} \cap (A_{\mathbf{C}})_{1/2}]$,
(ii) *if* $V_\zeta \neq 0$ *then* $\zeta + \bar{\zeta} = 1$,
(iii) *if* $V_\zeta \neq V_\mu$ *are nonzero, then* $A_{1/2} = V_\zeta \oplus V_\mu = E_{1/2}$,
(iv) *if* $x \in V_\zeta$, $y \in V_\mu$, $V_\zeta \neq V_\mu$, *then* $xy = -yx$.

Proof. There are polynomials $h_1(\lambda), h_2(\lambda) \in \mathbf{C}[\lambda]$ such that

$$x = h_1(L_e)(L_e - \zeta I)x + h_2(L_e)(L_e - \bar{\zeta}I)x.$$

Clearly, this representation of x satisfies part (i) for $x \in V_\zeta \cap A_{1/2}$.
Let $x \in (A_{\mathbf{C}})_{1/2} \cap W_\zeta$, $y \in (A_{\mathbf{C}})_{1/2} \cap W_\mu$. Then

$$xe = x - ex = (1 - \zeta)x \quad \text{and} \quad ye = y - ey = (1 - \mu)y$$

and using the flexible identity, (1.1), we obtain

$$(6.7) \qquad (1 - \zeta - \mu)xy = (xe)y - x(ey) = y(ex) - (ye)x = (\zeta + \mu - 1)yx.$$

Therefore, if $\zeta + \mu \neq 1$, then $xy = -yx$.
Now, let $w \in A_{1/2} \cap V_\zeta$ with $w = w_1 + w_2$ where

$$w_1 \in (A_{\mathbf{C}})_{1/2} \cap W_\zeta, \quad w_2 \in (A_{\mathbf{C}})_{1/2} \cap W_{\bar{\zeta}},$$

and similarly let $z \in A_{1/2} \cap V_\mu$ with $z = z_1 + z_2$ where

$$z_1 \in (A_{\mathbf{C}})_{1/2} \cap W_\mu, \quad z_2 \in (A_{\mathbf{C}})_{1/2} \cap W_{\bar{\mu}}.$$

Then if $\zeta + \mu \neq 1 \neq \zeta + \bar{\mu}$, we can use equation (6.7) to obtain

$$(6.8) \qquad wz = (w_1 + w_2)(z_1 + z_2) = w_1z_1 + w_1z_2 + w_2z_1 + w_2z_2 = -(z_1 + z_2)(w_1 + w_2) = -zw,$$

so that $wz = -zw$. From this, we see that if $\zeta + \bar{\zeta} \neq 1$ then $w^2 = -w^2$ for all $w \in A_{1/2} \cap V_\zeta$, which cannot occur with a nonzero w in a division algebra. Therefore, if $V_\zeta \neq 0$, then $\zeta + \bar{\zeta} = 1$ so that (ii) is proved.

If $V_\zeta \neq V_\mu$ are nonzero, then $\dim V_\zeta = 2 = \dim V_\mu$ and if $0 \neq w \in V_\zeta \cap A_{1/2}$, $0 \neq z \in V_\mu \cap A_{1/2}$ then w, ew is a basis for V_ζ contained in $A_{1/2}$ and z, ez is a basis for V_μ contained in $A_{1/2}$. That is, if $V_\zeta \neq V_\mu$ are nonzero then $V_\zeta, V_\mu \subseteq A_{1/2}$, and we see that $A_{1/2} = V_\zeta \oplus V_\mu$.

Part (iv) follows from (iii) and (6.8).

LEMMA 6.9. (i) $E_{1/2} = A_{1/2}$.

(ii) $U_\zeta = V_\zeta$ *for all* $\zeta \in \mathbf{C} - \mathbf{R}$.

Proof. Since $\zeta + \bar{\zeta} = 1$ for $V_\zeta \neq 0$, upon setting $\tau = \zeta\bar{\zeta}$ we have

$$U_\zeta = \{z \in A: (L_e^2 - L_e + \tau I)^2 z = 0\}.$$

Suppose $E_{1/2} \neq A_{1/2}$. Since $E_{1/2} = U_\zeta$ or $E_{1/2} = V_\zeta \oplus V_\mu$ for ζ, $\mu \in \mathbf{C} - \mathbf{R}$ with $V_\zeta \neq V_\mu$, we see by Lemma 6.6 (iii) that this supposition implies $E_{1/2} = U_\zeta$. Hence, there is some $z \in U_\zeta$ such that $ez + ze = z + x$ for some $x \in A_{1/2}$, $x \neq 0$. Let

$$z' = (L_e^2 - L_e + \tau I)z \in V_\zeta.$$

Then

$$\begin{aligned} z' &= e(ez) - ez + \tau z = e(z + x - ze) - ez + \tau z \\ &= ex - eze + \tau z = ex - (z + x - ze)e + \tau z \\ &= ex - xe - ze + (ze)e + \tau z \end{aligned}$$

and since $x \in A_{1/2}$ this implies

$$(6.10) \quad (ze)e - ze = z' + x - 2ex + \tau z.$$

From the definition of z', the flexible identity, and (6.10) we have

$$z' - \tau z = e(ez) - ez = (ze)e - ze = z' + x - 2ex - \tau z$$

which upon simplifying gives

$$(6.11) \quad x = 2ex.$$

But (6.11) implies that $\frac{1}{2}$ is an eigenvalue of L_e restricted to $E_{1/2}$, contrary to Theorem 4.20. Therefore, $E_{1/2} = A_{1/2}$ and $U_\zeta \subseteq A_{1/2}$.

Once again let $z \in U_\zeta$ with $z' = (L_e^2 - L_e + \tau I)z$ and let $w \in V_\zeta$. Then substituting w, e, ez into the flexible identity we see that

$$w(e(ez)) + (ez)(ew) = (we)(ez) + (eze)w.$$

After recalling that $e(ez) - ez + \tau z = z'$ and $w, z \in A_{1/2}$, we have

$$w(ez - \tau z + z') + (ez)(ew) = (w - ew)(ez) + (ez - (e(ez))w$$

or

$$(6.12) \quad (ez) \circ (ew) = \tau(w \circ z) - w \circ z'.$$

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Now substituting z, e, ew into the flexible identity, we obtain

$$z(ew - \tau w) + (ew)(ez) = z(e(ew)) + (ew)(ez)$$
$$= (ze)(ew) + (ewe)z = (z - ez)(ew) + (e(w - ew))z$$
$$= z(ew) - (ez)(ew) + (ew)z - (e(ew))z,$$

and therefore

(6.13) $(ez) \circ (ew) = \tau(w \circ z)$.

From (6.12) and (6.13), we have $w \circ z' = 0$. Setting $w = z'$ gives $(z')^2 = 0$ which implies $z' = 0$. This means that

$$(L_e^2 - L_e + \tau I)z = 0 \quad \text{for all } z \in U_\zeta,$$

and hence $V_\zeta = U_\zeta$.

LEMMA 6.14. (i) *$C(e)A_{1/2} \subseteq A_{1/2}$ and $A_{1/2}C(e) \subseteq A_{1/2}$,*
(ii) *$A = C(e) \oplus V_\zeta$, i.e., $A_{1/2} = V_\zeta$.*

Proof. Since $C(e) = \mathbf{R}e + A_\alpha$ for $\alpha \in \mathbf{R}$, $\alpha \neq \frac{1}{2}$, and L_e, R_e map $A_{1/2}$ onto $A_{1/2}$, the proof of (i) is complete once we have shown $A_\alpha A_{1/2}$, $A_{1/2}A_\alpha \subseteq A_{1/2}$. By Lemmas 6.6 and 6.9, $A_{1/2} = V_\zeta$ or $V_\zeta \oplus V_\mu$ and hence it suffices to show $A_\alpha V_\zeta$, $V_\zeta A_\alpha \subseteq A_{1/2}$. Also, by Lemmas 6.6 and 6.9,

$$V_\zeta \subseteq W_\zeta \oplus W_{\bar{\zeta}} \subseteq (A_{\mathbf{C}})_{1/2}$$

and therefore we need only show that $A_\alpha W_\zeta$, $W_\zeta A_\alpha \subseteq (A_{\mathbf{C}})_{1/2}$. We prove only $A_\alpha W_\zeta \subseteq (A_{\mathbf{C}})_{1/2}$, since the other proof is similar.

Let $x \in A_\alpha$ and $z \in W_\zeta$. Then

$$e \circ [x, z] = [x, e \circ z] = [x, z]$$

and hence $[x, z] \in (A_{\mathbf{C}})_{1/2}$. Also, the flexible identity gives

$$(ze)x + (xe)z = z(ex) + x(ez)$$

or

$$\bar{\zeta}zx + \alpha xz = \alpha zx + \zeta xz$$

so that

$$(\alpha - \zeta)xz = (\alpha - \bar{\zeta})zx.$$

Hence

$$(\alpha - \zeta)xz = (\alpha - \zeta)zx + (\zeta - \bar{\zeta})zx,$$

and therefore

$$(\zeta - \bar{\zeta})zx = (\alpha - \zeta)[x, z] \in (A_{\mathbf{C}})_{1/2}.$$

The proof of (i) is complete.

Suppose $V_\zeta \neq V_\mu$ and $0 \neq z \in V_\zeta$ and $0 \neq w \in V_\mu$. Then, by Lemma 6.6, we have

$$wz = -zw = \tfrac{1}{2}[w, z].$$

Also, $w \circ [e, z] = 0$ since $[e, z] \in V_\mu$. Thus

$$(wz) \circ e = \tfrac{1}{2}[w, z] \circ e = \tfrac{1}{2}[w \circ e, z] - \tfrac{1}{2}w \circ [e, z] = \tfrac{1}{2}[w, z] = wz$$

so that $wz \in A_{1/2}$. But this implies R_z maps the 6-dimensional space $C(e) + V_\mu$ into the 4-dimensional space $A_{1/2}$, and hence R_z is singular, contrary to A being a division algebra.

Having completed our first task of proving $E_{1/2} = A_{1/2} = V_\zeta$, we turn our attention to showing that $\alpha = -1$. Our proof involves idempotents other than e, so that from this point on we denote $A_{1/2}$ by $A_{1/2}(e)$, A_α by $A_\alpha(e)$ etc.

LEMMA 6.15. *Let $\alpha \neq \frac{1}{2}$ be that real number such that $C(e) = \mathbf{R}e + A_\alpha(e)$. Then*

(i) $\alpha < \frac{1}{2}$

(ii) *each 2-dimensional subalgebra B of $C(e)$ has a basis e, x where $x \in A_\alpha(e)$, $x^2 = -e$, and B contains exactly three idempotents, namely, e and the two elements given by*

$$(6.16) \quad u = \frac{1}{2\alpha} e \pm \frac{\sqrt{1 - 2\alpha}}{2\alpha} x.$$

Proof. By Lemma 4.19, there exist idempotents in $C(e)$ distinct from e. Let u be any such idempotent. Then, by Lemma 5.11, $u = \beta_1 e + \beta_2 x$ for some $\beta_1, \beta_2 \in \mathbf{R}$, $x \in A_\alpha(e)$ with $x^2 = -e$. Now,

$$\beta_1 e + \beta_2 x = u = u^2 = \beta_1{}^2 e + 2\beta_1\beta_2\alpha x - \beta_1{}^2 e$$

which is equivalent to

$$\beta_2 = 2\beta_1\beta_2\alpha \quad \text{and} \quad \beta_2{}^2 = \beta_1{}^2 - \beta_1.$$

Since $u \neq e$, $\beta_2 \neq 0$ and hence

$$\beta_1 = \frac{1}{2\alpha} \quad \text{and} \quad \beta_2{}^2 = \beta_1{}^2 - \beta_1 = \frac{1}{4\alpha^2} - \frac{1}{2\alpha} = \frac{1}{4\alpha^2}(1 - 2\alpha).$$

In particular, we see that $\alpha < \frac{1}{2}$ since $\beta^2 > 0$, and u is an idempotent whenever

$$u = \frac{1}{2\alpha} e \pm \frac{\sqrt{1 - 2\alpha}}{2\alpha} x.$$

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Since by Lemma 5.11, each 2-dimensional subalgebra B of $C(e)$ has such a basis e, x, the argument above gives part (ii).

LEMMA 6.17. *Let $z \in A_{1/2}(e)$ and $u = (1/\delta)z^2$ be an idempotent. Then*
(i) *$C(u)$ is a 4-dimensional subalgebra,*
(ii) *$C(u) = \mathbf{R}e \oplus A_{\alpha'}(u)$ for some $\alpha' \in \mathbf{R}, \alpha' < \frac{1}{2}$,*
(iii) *$A = C(u) \oplus A_{1/2}(u)$ and the eigenvalues of L_u restricted to $A_{1/2}(u)$ are in $\mathbf{C} - \mathbf{R}$.*

Proof. From Lemma 4.19, e, u form a basis for a 2-dimensional subalgebra of $C(e)$. Hence by Lemma 6.15, u is given by (6.16), and

$$L_u(e) = \frac{1}{2\alpha}e \pm \frac{\sqrt{1-2\alpha}}{2}x,$$

$$L_u(x) = \frac{\mp\sqrt{1-2\alpha}}{2\alpha} + \frac{1}{2}x.$$

The restriction of $L_u + R_u$ to $\mathbf{R}e \oplus \mathbf{R}x$ has characteristic polynomial

$$\lambda^2 - \left(\frac{1}{\alpha} + 1\right)\lambda + \frac{1}{\alpha} + \frac{1}{\alpha}(1 - 2\alpha).$$

This polynomial does not have 1 for a root, since $\alpha < \frac{1}{2}$. Therefore, $A \neq \mathbf{R}u \oplus E_{1/2}(u)$ and hence, by Theorem 4.5, $C(u)$ is a subalgebra. Since $C(u)$ is not 8-dimensional by assumption and u, e, z are linearly independent in $C(u)$, we conclude that $C(u)$ is 4-dimensional.

From Theorem 4.5 and Corollary 4.17, it follows that

$$A = C(u) \oplus E_{1/2}(u) \quad \text{and} \quad C(u) = \mathbf{R}u + A_{\alpha'}(u)$$

for some $\alpha' \in \mathbf{R}$. By Corollary 4.17 and Lemma 4.15, L_u has complex eigenvalues of $E_{1/2}(u)$ and hence, by Lemma 6.9, $E_{1/2}(u) = A_{1/2}(u)$. Finally, we note that Lemma 6.15 gives us $\alpha' < \frac{1}{2}$.

For a fixed $z_0 \in A_{1/2}(e)$, let $x_0 \in A_\alpha(e)$ be such that the subalgebra $\mathbf{R}e \oplus \mathbf{R}z_0{}^2$ has e, x_0 for a basis and $x_0{}^2 = -e$. From this point on, e_2 and e_3 denote the two idempotents given by (6.16) with $x = x_0$. Also, when it is convenient we denote e by e_1, and α by α_1.

LEMMA 6.18. *For $i = 1, 2, 3$, let $\alpha_i \in \mathbf{R}$ be such that*

$$C(e_i) = \mathbf{R}e_i \oplus A_{\alpha_i}(e_i).$$

Then $\alpha_i = -1$.

Proof. Suppose $u = \beta_1 e + \beta_2 x_0$ is e_2 (or e_3). Then by Theorem 5.12, L_u is diagonalizable on $C(u)$ with eigenvalues 1 and $\alpha_0 = \alpha_2$ (or α_3 resp.).

Let $v = \gamma_1 e + \gamma_2 x_0$ be an eigenvector of L_u belonging to α_0. Then

$$(\beta_1 e + \beta_2 x_0)(\gamma_1 e + \gamma_2 x_0) = (\beta_1\gamma_1 - \beta_2\gamma)e + (\beta_2\gamma_1 + \beta_1\gamma_2)\alpha x_0$$
$$= \alpha_0\gamma_1 e + \alpha_0\gamma_2 x_0,$$

which yields

$$(\beta_1 - \alpha_0)\gamma_1 - \beta_2\gamma_2 = 0 \quad \text{and} \quad \alpha\beta_2\gamma_1 + \alpha(\beta_1 - \alpha_0)\gamma_2 = 0.$$

The determinant of the coefficient matrix of this linear system in γ_1, γ_2 is

$$\Delta(\beta_1, \beta_2) = \alpha_0^2 - (\alpha + 1)\beta_1\alpha_0 + \alpha(\beta_1^2 + \beta_2^2)$$

which is zero since $v \neq 0$. Substituting the coordinates of e_2 or e_3 into $\Delta(\beta_1, \beta_2)$, we have

$$\alpha_0^2 + \frac{\alpha + 1}{2\alpha}\alpha_0 + \frac{1 - \alpha}{2\alpha} = 0,$$

or equivalently

$$2\alpha\alpha_0^2 - (1 + \alpha)\alpha_0 + (1 - \alpha) = 0.$$

From the quadratic formula $\alpha_0 = 1$ or $(1 - \alpha)/2\alpha$, and hence $\alpha_0 = (1 - \alpha)/2\alpha$ since $\alpha_0 < \frac{1}{2}$. In particular, $\alpha_2 = \alpha_3$. Similarly, $\alpha = \alpha_2$. Thus

$$\alpha = (1 - \alpha)/2\alpha \quad \text{or} \quad 2\alpha^2 + \alpha - 1 = 0.$$

Factoring this polynomial, we find that the possible values of α are -1 and $\frac{1}{2}$, and hence $\alpha = -1$. Therefore $\alpha = \alpha_2 = \alpha_3 = -1$.

From Lemma 6.18 and the definition of e_2 and e_3, it follows that each of $e = e_1, e_2, e_3$ is the negative of the sum of the other two. Also, Lemmas 6.17 and 6.18 show us some similarities among the decompositions of A relative to these idempotents which are further exhibited by

LEMMA 6.19. *For each $i = 1, 2, 3$, let $\zeta_i \in \mathbf{C} - \mathbf{R}$ be such that $A_{1/2}(e_i) = V_{\zeta_i}$ and assume $\tau_i = \zeta_i\bar{\zeta}_i$. Then, for i, j, k distinct, we have*

(i) dim $C(e_j) \cap A_{1/2}(e_k) = 2$ *and*

$$A_{1/2}(e_k) = C(e_i) \cap A_{1/2}(e_j) \oplus C(e_j) \cap A_{1/2}(e_i),$$

(ii) $\tau_1 = \tau_2 = \tau_3$.

Proof. Using $e_k = -e_i - e_j$ and $x \in C(e_i) \cap A_{1/2}(e_j)$, we have

$$(R_{ek}^2 - L_{ek} + \tau_j I)x = (-e_i - e_j)(x - e_j x) - (x - e_j x) + \tau_j x$$
$$= e_j(e_j x) - e_j x + \tau_j x = 0.$$

The roots of $\lambda^2 - \lambda + \tau_j$ are complex. Therefore, x is in the subspace on which L_{ek} has complex roots. This means

$$x \in V_{\zeta k} = A_{1/2}(e_k).$$

But the minimum polynomial of L_{e_k} on $A_{1/2}(e_k)$ is $\lambda^2 - \lambda + \tau_k$, and hence it must be the case that $\tau_j = \tau_k$.

In order to obtain the direct sum in part (i), first note that the above argument gives us that

$$C(e_i) \cap A_{1/2}(e_j) + C(e_j) \cap A_{1/2}(e_i) \subseteq A_{1/2}(e_k)$$

and note that

$$(6.20) \quad [C(e_i) \cap A_{1/2}(e_j)] \cap [C(e_j) \cap A_{1/2}(e_i)] \subseteq C(e_i) \cap A_{1/2}(e_i) = 0.$$

Let $x \in C(e_i) \cap A_{1/2}(e_j)$ and $z \in C(e_j) \cap A_{1/2}(e_i)$. Then x and e_jx are linearly independent in $C(e_i) \cap A_{1/2}(e_j)$, and z and e_iz are linearly independent in $C(e_j) \cap A_{1/2}(e_i)$. Now, it follows easily that

$$A_{1/2}(e_k) = C(e_i) \cap A_{1/2}(e_j) \oplus C(e_j) \cap A_{1/2}(e_i)$$

and $C(e_j) \cap A_{1/2}(e_k)$ is 2-dimensional.

Lemma 6.6 tells us that $\zeta_j + \bar{\zeta}_j = 1$. Combining this with the second part of Lemma 6.19, we deduce that there is some real number $\delta \neq 0$ such that ζ_j is one of the complex numbers given by $\frac{1}{2} \pm \delta i$. From this, we see that the common value $\tau_1 = \tau_2 = \tau_3$, which we denote by τ, is greater than $\frac{1}{4}$.

For each pair $i \neq j$ in $\{1, 2, 3\}$, it is convenient to introduce the notation: $X_i = C(e_i) \cap A_{1/2}(e_j)$ and $C = C(e_i) \cap C(e_j)$. Since

$$C(e_i) \cap A_{1/2}(e_k) = C(e_i) \cap [C(e_i) \cap A_{1/2}(e_j) \oplus C(e_j) \cap A_{1/2}(e_i)] = C(e_i) \cap A_{1/2}(e_j),$$

the definition of X_i is independent of which $e_j \neq e_i$ we use to define it. Also, C does not depend on the choice of i and j because $e_k = -e_i - e_i \in C$.

For i, j, k distinct, $X_iX_j \subseteq X_k$ since Lemma 6.14 applies to e_2 and e_3 as well as to $e_1 = e$. We have seen, in Section 5, that L_{e_i} and R_{e_i} are diagonalizable on the subalgebra $C(e_i)$. Since X_i is invariant under L_{ek} and R_{e_i}, we can conclude, by employing Lemma 6.18, that X_i is an eigenspace belonging to eigenvalue -1 for both transformations L_{e_i} and R_{e_i}.

Lemma 6.21. *Let φ, ψ and ρ be defined by* (6.4) *and let x_i be a zonzero element of X_i. Then $\{e_i, e_j, x_i, e_jx_i\}$ is a basis for $C(e_i)$, and for some $\theta \in \mathbf{R}$ the corresponding multiplication table is given by:*

	e_i	e_j	x_i	e_jx_i
e_i	e_i	$-e_i - e_j$	$-x_i$	$-e_jx_i$
e_j	$-e_i - e_j$	e_j	e_jx_i	$e_jx_i - \tau x_i$
x_i	$-x_i$	$x_i - e_jx_i$	$-\theta e_i$	$2\rho\theta e_i + \varphi\theta e_j$
e_jx_i	$-e_jx_i$	τx_i	$-\psi\theta e_i - \varphi\theta e_j$	$-\theta\tau e_i$

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Proof. Let $x_i \in X_i$. By the definition of the idempotents e_1, e_2, e_3 it is clear that $e_j \in C(e_i)$. Since $x_i \in X_i = C(e_i) \cap A_{1/2}(e_j)$, we know, by Lemma 6.14, that $e_jx_i \in X_i$. Now, it follows that $\{e_i, e_j, x_i, e_jx_i\}$ is a basis of $C(e_i)$ because x_i, e_jx_i are linearly independent in $A_{1/2}(e_j)$ and e_i, e_j are linearly independent in $C(e_j)$.

Since $-e_i - e_j = e_k = e_k^2 = e_i + e_j + 2e_ie_j$ we know that $e_ie_j = -e_i - e_j$. The validity of the remaining entries in the first two rows and the first two columns follows easily from the definitions of C and X_i.

Lemma 4.19 and Table 5.13 imply that there is some positive θ in $\mathbf{R}$ such that $u = -\theta x_i^2$ is an idempotent in C. Since

$$x_i \in A_{1/2}(e_j) \cap A_{1/2}(e_k),$$

x_i does not commute with e_j or e_k, but clearly x_i does commute with u. Since Lemma 6.15 tells us that the only idempotents in C are e_1, e_2 and e_3, we conclude that $u = e_i$.

Next using (6.13) with $z = x_i = w$ and $e = e_j$ we have

$$(6.22) \quad (e_jx_i)^2 = -\theta\tau e_i.$$

With this we see that only $(e_jx_i)x_i$ and $x_ie_jx_i$ remain to be determined.

Since $(x_ie_j)x_i = (x_i - e_jx_i)x_i = x_i^2 - (e_jx_i)x_i$, it is clear that once $x_ie_jx_i$ is known then $(e_jx_i)x_i$ can be easily computed. This equation can be reformulated as

$$(6.23) \quad (e_jx_i) \circ x_i = x_i^2 = -\theta e_i.$$

From $(e_jx_i)(x_i(e_jx_i)) = ((e_jx_i)x_i)(e_jx_i)$ and (6.23), it follows that

$$(6.24) \quad ((e_jx_i)x_i) \circ (e_jx_i) = \theta e_jx_i.$$

Using $x_i((e_jx_i)x_i) = (x_i(e_jx_i))x_i$ we obtain the corresponding equation

$$(6.25) \quad ((e_jx_i)x_i) \circ x_i = \theta x_i.$$

Suppose now $(e_jx_i)x_i = ae_i + be_j + cx_i + de_jx_i$. Then placing this expression into (6.24) and using that portion of the table already determined and equation (6.23), we have

$$(6.26) \quad (-c\theta - 2d\tau\theta)e_i + (b - 2a)e_jx_i = \theta e_jx_i.$$

Making similar use of (6.25) yields

$$(6.27) \quad (-2c\theta - d\theta)e_i + (b - 2a)x_i = \theta x_i.$$

From (6.26) and (6.27), we can conclude $d\theta(4\tau - 1) = 0$. Noting that $\tau > \frac{1}{4}$ as indicated earlier, we have $d = 0 = c$.

To determine a and b, we apply the flexible identity and deduce

$$\begin{aligned}(6.28) \quad -2\theta\tau e_i &= 2(e_jx_i)^2 \\ &= -((e_jx_i)x_i)e_j + e_j(x_i(e_jx_i)) + (e_jx_i)x_i \\ &= (\theta + 3a)e_i + (\theta + 2a - b)e_j.\end{aligned}$$

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Therefore,

$$a = -\tfrac{1}{3}\theta(2\tau + 1) = -\psi\theta, \quad b = -\theta\varphi.$$

The proof is completed by using (6.23) to find $x_i(e_j x_i)$.

Note that the multiplication table of Lemma 6.21 tells us that $C = \mathbf{R}e_i \oplus \mathbf{R}e_j$ for each $e_i \neq e_j$.

In order to describe certain symmetries that occur within the multiplication on A we define the functions F and G on $A \times A \times A$ into A by

$$F(r, s, t) = r(st) - \rho(rt) - \gamma(tr)$$
$$G(r, s, t) = (st)r - \gamma(rt) - \sigma(tr)$$

where γ, ρ, and σ are as given by (6.4).

LEMMA 6.29. *F and G vanish on* $\{(r, s, t) | s = e_i, r \in X_i, t \in X_j$ *for* $j \neq i\}$.

Proof. Let $s = e_i$, $r \in X_i$, and $t \in X_j$, for $i \neq j$. Then, since $t \in A_{1/2}(e_i) = V_\zeta(e_i)$, we can use (1.1) to obtain

$$r(e_i t) - \tau(rt) - (e_i t)r = r(e_i(e_i t)) + (e_i t)(e_i r)$$
$$= (re_i)(e_i t) + ((e_i t)e_i)r = -r(e_i t) + \tau tr$$

and hence we have

(6.30) $\quad 2r(e_i t) - (e_i t)r = \tau(r \circ t)$.

Also,

$$r(e_i t) + t(e_i r) = (re_i)t + (te_i)r = (re_i)t + tr - (e_i t)r$$

simplifies to become

(6.31) $\quad r \circ (e_i t) = 2tr - rt$.

Combining (6.30) and (6.31) gives

(6.32) $\quad 3r(e_i t) = (\tau + 2)tr + (\tau - 1)rt$

(6.33) $\quad 3(e_i t)r = (4 - \tau)tr - (\tau + 2)rt$.

The lemma follows from (6.32) and (6.33).

LEMMA 6.34. *Let π be the 3-cycle* $(1\ 2\ 3)$. *Then for each* $i = 1, 2, 3$, *there exists* $y_i \in X_i$ *such that*

(i) $y_i^2 = -e_i$,
(ii) $y_{\pi(i)} y_{\pi^2(i)} = \sqrt{\gamma}\, y_i$.

Proof. For each i, let x_i be any nonzero element in X_i and let $\theta_i \in \mathbf{R}$ be the value in the table of Lemma 6.21. Then part (i) is satisfied by

$$y_i = \frac{1}{\sqrt{\theta_i}} x_i.$$

For one fixed i we may choose $x_i = y_{\pi(i)}y_{\pi^2(i)}$ since $X_{\pi(i)}X_{\pi^2(i)} \subseteq X_i$. Then

$$y_{\pi(i)}y_{\pi^2(i)} = \sqrt{\theta_i}\, y_i.$$

We show that this particular θ_i is γ and, also for this i, that

$$y_{\pi^2(i)}y_i = \sqrt{\gamma}\, y_{\pi(i)}.$$

This is sufficient to prove part (ii), because once we have obtained this we can begin the process over again to get

$$y_i y_{\pi(i)} = \sqrt{\gamma}\, y_{\pi^2(i)}.$$

To simplify the notation, we take $\theta_3 = \theta$, $x_3 = y_1y_2$ and show $\theta = \gamma$ and $y_2x_3 = \gamma y_1$.

By Lemma 6.21, we know that x_3, e_1x_3 form a basis for X_3 and hence $y_2y_1 = ax_3 + be_1x_3$ for some $a, b \in \mathbf{R}$. Since $x_3 \in A_{1/2}(e_1) = V_\zeta(e_1)$, when we substitute y_2, y_1, e_1 into (1.1) we obtain

$$(6.35)\quad 0 = (y_2y_1)e_1 + (e_1y_1)y_2 - y_2(y_1e_1) - e_1(y_1y_2) = (2a + b\tau - 1)x_3 + (-a + b - 1)e_1x_3.$$

Solving the two coordinate equations for a and b establishes

$$(6.36)\quad y_2y_1 = \left(\frac{1-\tau}{\tau+2}\right)x_3 + \left(\frac{3}{\tau+2}\right)e_1x_3 = -\rho\gamma^{-1}x_3 + \gamma^{-1}e_1x_3.$$

Let $x_3y_2 = cy_1 + de_2y_1$ and $y_2x_3 = c_1y_1 + d_1e_2y_1$. Substituting these two expressions into $0 = (x_3y_2)e_2 + (e_2y_2)x_3 - x_3(y_2e_2) - e_2(y_2x_3)$ and simplifying produces

$$0 = (2c - c_1 + d\tau + d_1\tau)y_1 + (-c - c_1 + d - 2d_1)e_2y_1.$$

By solving the two coordinate equations for c_1 and d_1 we obtain

$$(6.37)\quad c_1 = \frac{3\tau d + (4-\tau)c}{\tau+2}, \quad d_1 = \frac{-3c + (1-\tau)d}{\tau+2}.$$

Having found one relationship between the coordinates of y_2x_3 and x_3y_2, we generate a second so that c and d may be found. Lemma 6.29 tells us

$$0 = F(y_2, e_2, x_3) = y_2(e_2x_3) - \rho y_2x_3 - \gamma x_3y_2.$$

Substituting $e_2 = -e_1 - e_3$ into this equation gives

$$(6.38)\quad y_2(e_1x_3) = \sigma y_2x_3 - \gamma x_3y_2.$$

Now we substitute y_1, y_2, y_2 into (1.1) and use (6.36) and (6.38) to obtain

$$\begin{aligned} x_3y_2 = (y_1y_2)y_2 &= -y_2^2y_1 + y_1y_2^2 + y_2(y_2y_1) \\ &= -y_1 + 2e_2y_1 + \left(\frac{1-\tau}{\tau+2}\right)y_2x_3 \\ &\quad + \frac{3}{\tau+2}\left[\frac{4-\tau}{3}y_2x_3 - \frac{\tau+2}{3}x_3y_2\right]. \end{aligned}$$

Therefore,

$$(6.39)\quad 2x_3y_2 = -y_1 + 2e_2y_1 + \left(\frac{5-2\tau}{\tau+2}\right)y_2x_3.$$

Expressing each term of (6.39) as a linear combination of y_1 and e_2y_1 and using (6.37), we obtain the following two equations corresponding to the two coordinates:

$$-\tfrac{1}{3}(\tau+2)^2 = (7\tau-4)c - (5\tau-2\tau^2)d,$$
$$\tfrac{2}{3}(\tau+2)^2 = (5-2\tau)c + (5\tau+1)d.$$

The determinant of the coefficient matrix of this linear system in c and d is

$$\Delta(\tau) = 4\tau^3 + 15\tau^2 + 12\tau - 4,$$

which is easily seen to be positive whenever $\tau > \frac{1}{4}$. From this we conclude that there is a unique c and d solving this system. One can check that $c = -\rho$, $d = 1$ is this solution so that from (6.37) we have $c_1 = \gamma$ and $d_1 = 0$. Therefore,

$$(6.40)\quad x_3y_2 = -\rho y_1 + e_2y_1$$

$$(6.41)\quad y_2x_3 = \gamma y_1.$$

Finally, we show $\theta = \gamma$. We use (6.36), (6.40), (6.41), and the table of Lemma 6.21 to simplify

$$-\theta e_3 = x_3^2 = (y_1y_2)x_3 = -(x_3y_2)y_2 + y_1(y_2x_3) + x_3(y_2y_1).$$

After doing this and using the values of equation (6.4), we obtain

$$-\theta e_3 = (-\rho + \psi - \varphi - \gamma + \varphi\theta\gamma^{-1})e_1 + (-\varphi + 3\rho\theta\gamma^{-1})e_3$$

from which it follows easily that $\theta = \gamma$. Therefore $x_3^2 = -\gamma e_3$ so that $y_3 = x_3/\sqrt{\gamma}$ has the properties that $y_3^2 = -e_3$ and, by (6.41), $y_2y_3 = \sqrt{\gamma}\,y_1$. This completes the proof.

We can now prove the main result of this section.

THEOREM 6.42. *Let A be an 8-dimensional flexible division algebra over* **R** *containing no idempotent which commutes with every element of A. Then A is a generalized pseudo-octonion algebra.*

Proof. The algebra A has an idempotent e such that $A = C(e) \oplus A_{1/2}(e)$ where $C(e)$ is a 4-dimensional subalgebra and $A_{1/2}(e) = V_\zeta(e)$, for some $\zeta \in \mathbf{C} - \mathbf{R}$. We prove that A is uniquely determined by the real number $\tau = \zeta\bar{\zeta}$ which we have seen to be greater than $\frac{1}{4}$.

Let e_1, e_2, e_3 be the three idempotents in C and let y_1, y_2, y_3 be the corresponding elements chosen to satisfy Lemma 6.34. Then using y_i instead of x_i in the multiplication table of Lemma 6.21, we see that the multiplication constants of $C(e_i)$ are uniquely determined by τ.

Fix i, j distinct elements in $\{1, 2, 3\}$. To see that the products X_iX_j and X_jX_i are uniquely determined by τ, let y_i, e_jy_i be the basis of X_i and y_j, e_iy_j be the basis of X_j. Without loss of generality, we may assume $i = 2, j = 3$ since we have complete symmetry among the three cases.

Lemma 6.29 allows us to express each of $y_2(e_2y_3)$, $y_3(e_3y_2)$, $(e_2y_3)y_2$, $(e_3y_2)y_3$, $(e_3y_2)(e_2y_3)$ and $(e_2y_3)(e_3y_2)$ as a linear combination of y_2y_3 and y_3y_2 where the two coefficients are certain determined expressions in τ. From (6.40) and (6.41) we have that y_2y_3 and y_3y_2 are linear combinations of y_1 and e_2y_1 whose coefficients are uniquely determined by τ. Therefore, the products X_2X_3 and X_3X_2 are uniquely expressable in terms of τ and, by symmetry, X_iX_j and X_jX_i are. Therefore, since $A = C \oplus X_1 \oplus X_2 \oplus X_3$, we know that the multiplication on A is determined by τ. But as indicated in Example 6.1, corresponding to each $\tau \in \mathbf{R}$, $\tau > \frac{1}{4}$, there is a GP-algebra $S(\delta, \frac{1}{2})$ which decomposes A into $C(e) \oplus A_{1/2}(e)$ with $A_{1/2}(e) = V_\zeta(e)$ and $\tau = \zeta\bar{\zeta}$. Therefore, A is this GP-algebra.

Remark. One can use the relations we have generated to construct a multiplication table for A. The result is exactly Table 6.5 where the basis there is replaced by e_1, e_2, y_1, e_2y_1, y_2, e_1y_2, x_3, e_1x_3 respectively.

Theorem 6.42 is the final result required to establish our main theorem.

Proof of Theorem 1.4. Let A be any finite dimensional flexible division algebra over the real numbers. Then A can be seen to be as indicated by (a), (b) or (c) of Theorem 1.4 in the following way. In the case that A is 1 or 2-dimensional, Corollary 4.18 implies that A is commutative and hence A is described by part (a). When the dimension of A is 4 or 8, Theorem 4.20 reduces the classification of A to two cases. One of these cases, namely when $A = C(e)$ for some idempotent e in A, is treated by Theorem 5.13 where it is shown to be described by part (b). The other case is shown to correspond to part (c) by Theorem 6.42. Since the converse has been argued in Section 1, the proof is complete.

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