THE KO-COHOMOLOGY RING OF $SU(2n)/SO(2n)$

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Abstract. The KO-cohomology ring of the symmetric space $SU(2n)/SO(2n)$ is computed by using the Bott exact sequence and some facts on the real and quaternionic representation rings of $SU(2n)$ and $SO(2n)$.

1. Introduction and statement of result. For each integer $n \geq 1$, through the natural inclusion $\mathbf{R} \subset \mathbf{C}$, the rotation group $SO(n)$ may be viewed as a closed subgroup of the special unitary group $SU(n)$, and we have a homogeneous space $SU(n)/SO(n)$. It is a symmetric space, because the complex conjugation $\sigma = {}^{-}: SU(n) \to SU(n)$ is an involutive automorphism of $SU(n)$, and its fixed point subgroup is $SO(n)$. The cohomology and K-theory of $SU(n)/SO(n)$ are known (see [**4**] and [**5**]), and so is the KO-theory of $SU(2n+1)/SO(2n+1)$ (see [**7**]). The purpose of this paper is to compute the KO-theory of $SU(2n)/SO(2n)$. For this we need the following result of H. Minami [**5**, Proposition 8.2] on the K-theory of $SU(2n)/SO(2n)$.

We begin with some notation and terminology. There is a fibre sequence

$$SO(2n) \xrightarrow{i} SU(2n) \xrightarrow{\pi} SU(2n)/SO(2n) \xrightarrow{j} BSO(2n) \xrightarrow{Bi} BSU(2n).$$

In general, let G be a topological group and $\mathbf{K} = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$. The set of G-$\mathbf{K}$-isomorphism classes $[V]$ of G-$\mathbf{K}$-modules V corresponds bijectively to the set of equivalence classes of homomorphisms $\varphi_V: G \to GL(\dim V, \mathbf{K})$ of topological groups (see [**8**]). For a while we confine our attention to the case $\mathbf{K} = \mathbf{C}$. Let $R(G)$ be the complex representation ring of G. $\mathbf{C}^{2n}$ becomes a $SU(2n)$-$\mathbf{C}$-module in the natural manner. We put $\lambda^k = [\Lambda^k(\mathbf{C}^{2n})]$ for each integer $k \geq 0$, where Λ^k denotes the k-th exterior power functor. Then, as in [**3**, Theorem 13(3.1)],

$$R(SU(2n)) = \mathbb{Z}[\lambda_1, \lambda_2, \ldots, \lambda_{2n-2}, \lambda_{2n-1}]$$

and, as in [**5**, (6.2)], the induced homomorphism $\sigma^*: R(SU(2n)) \to R(SU(2n))$ satisfies

$$\sigma^*(\lambda_k) = \lambda_{2n-k} \quad \text{for} \quad k = 1, 2, \ldots, 2n-1. \tag{1}$$

We put $\mu_k = [\Lambda^k((\mathbf{R}^{2n})^{\mathbf{C}})]$, where $(\mathbf{R}^{2n})^{\mathbf{C}}$ is the complexification of the $SO(2n)$-$\mathbf{R}$-module $\mathbf{R}^{2n}$. Then, as in [**3**, Theorem 13(10.3)],

$$R(SO(2n)) = \mathbf{Z}[\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n^+, \mu_n^-]/(r_n),$$

where

$$r_n = (\mu_n^+ + \mu_{n-2} + \ldots)(\mu_n^- + \mu_{n-2} + \ldots) - (\mu_{n-1} + \mu_{n-3} + \ldots)^2$$

and elements μ_n^+, μ_n^- are given as follows. By the definitions of λ_k and μ_k, the induced homomorphism $i^*: R(SU(2n)) \to R(SO(2n))$ satisfies

$$i^*(\lambda_k) = \mu_k \quad (k = 1, 2, \ldots, 2n-1). \tag{2}$$

Since $\sigma \circ i = i$, it follows from (1) and (2) that $\mu_k = \mu_{2n-k}$ for each $k = 1, 2, \ldots, 2n-1$.

There is an $SO(2n)$-$\mathbf{C}$-isomorphism $f:\Lambda^k((\mathbf{R}^{2n})^{\mathbf{C}})\to\Lambda^{2n-k}((\mathbf{R}^{2n})^{\mathbf{C}})$ that gives rise to the equation and satisfies $f\circ f=(-1)^{k(2n-k)}$. In particular, for $k=n$, we have an isomorphism $f:\Lambda^n((\mathbf{R}^{2n})^{\mathbf{C}})\to\Lambda^n((\mathbf{R}^{2n})^{\mathbf{C}})$ with $f\circ f=(-1)^n$. If n is odd, f has two eigenvalues $\pm\sqrt{-1}$; if n is even, f has two eigenvalues ± 1. Let μ_n^+ denote the $SO(2n)$-$\mathbf{C}$-isomorphism class of the eigenspace belonging to $\sqrt{-1}$ if n is odd or to 1 if n is even, and μ_n^- that of the eigenspace belonging to $-\sqrt{-1}$ if n is odd or to -1 if n is even. Then

$$\mu_n=\mu_n^++\mu_n^- \tag{3}$$

and $\dim\mu_n^+=\dim\mu_n^-=\binom{2n}{n}/2$.

Let (G,σ) be a symmetric pair (see [**4**]). That is, roughly speaking, G is a Lie group and σ is an involutive automorphism of G. Let G^σ denote the fixed point subgroup of σ. Then we have a map $\xi:G/G^\sigma\to G$ defined by

$$\xi(xG^\sigma)=x\sigma(x)^{-1} \tag{4}$$

for $xG^\sigma\in G/G^\sigma$. For $(G,\sigma)=(SU(2n),{}^-)$ we have $\xi:SU(2n)/SO(2n)\to SU(2n)$.

The element λ_k may be regarded as a homomorphism $SU(2n)\to U\left(\binom{2n}{k}\right)$ of topological groups. Let U be the infinite unitary group and $\iota_U:U\left(\binom{2n}{k}\right)\to U$ the canonical injection. Then we have an element

$$\beta(\lambda_k-\lambda_{2n-k}):=[\iota_U\circ\lambda_k\circ\xi]\in[SU(2n)/SO(2n),U]=\tilde{K}^{-1}(SU(2n)/SO(2n)).$$

On the other hand, let $\alpha:R(G)\to K^0(BG)$ be the homomorphism of Atiyah–Hirzebruch [**2**]. More precisely, the restriction of α to the augmentation ideal $I(G)$ is given by

$$\alpha([V]-\dim V)=[B\iota_U\circ B\varphi_V]\in[BG,BU]=\tilde{K}^0(BG), \tag{5}$$

where $B\iota_U:BU(\dim V)\to BU$ is the canonical injection. We denote by $\alpha(\widetilde{\mu_n^+})$ the image of $\mu_n^+-\binom{2n}{n}/2\in I(SO(2n))$ under the composite

$$I(SO(2n))\xrightarrow{\alpha}\tilde{K}^0(BSO(2n))\xrightarrow{j^*}\tilde{K}^0(SU(2n)/SO(2n)).$$

That is,

$$\alpha(\widetilde{\mu_n^+}):=[B\iota_U\circ B\mu_n^+\circ j]\in\tilde{K}^0(SU(2n)/SO(2n)).$$

With the above notation, Minami [**5**] showed that

$$\begin{aligned}&K^*(SU(2n)/SO(2n))\\&\quad=K^*(pt)\otimes\Lambda_{\mathbf{Z}}(\beta(\lambda_1-\lambda_{2n-1}),\beta(\lambda_2-\lambda_{2n-2}),\ldots,\beta(\lambda_{n-1}-\lambda_{n+1}),\alpha(\widetilde{\mu_n^+})).\end{aligned} \tag{6}$$

Let $g\in K^{-2}(pt)$ be the Bott generator and $c:KO^*(X)\to K^*(X)$ the complexification. Our main result is as follows.

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THEOREM 1. *There exist elements*

$$\lambda_{k,2n-k} \in \widetilde{KO}^1(SU(2n)/SO(2n)), \text{ for } k = 1, 2, \ldots, n-1,$$
$$\mu_{n,+} \in \widetilde{KO}^2(SU(2n)/SO(2n)), \text{ if } n \text{ is odd, or}$$
$$\mu_{n,+} \in \widetilde{KO}^0(SU(2n)/SO(2n)), \text{ if } n \text{ is even,}$$

such that

$$c(\lambda_{k,2n-k}) = g^{-1}\beta(\lambda_k - \lambda_{2n-k}),$$
$$c(\mu_{n,+}) = \begin{cases} g^{-1}\alpha(\widetilde{\mu_n^+}), & \text{if } n \text{ is odd,} \\ \alpha(\widetilde{\mu_n^+}), & \text{if } n \text{ is even,} \end{cases}$$

and

$$KO^*(SU(2n)/SO(2n)) = KO^*(pt) \otimes \Lambda_{\mathbb{Z}}(\lambda_{1,2n-1}, \lambda_{2,2n-2}, \ldots, \lambda_{n-1,n+1}, \mu_{n,+}).$$

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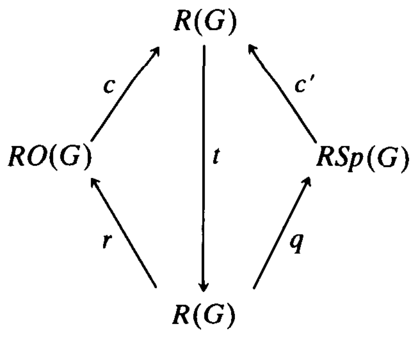
2. Proof of main result. This section is devoted to proving Theorem 1. Let $RO(G)$ and $RSp(G)$ be the real and quaternionic representation rings of G, respectively. As in [**1**, p. 26], there are five homomorphisms between representation rings in the following diagram, which is not commutative.



Their properties which we shall use are

$$q \circ c' = 2, \tag{7}$$

$$q \circ t = q. \tag{8}$$

Corresponding to these homomorphisms, there are five maps between (infinite) classical Lie groups. We shall use the same letters to denote them. For example, $t: U \to U$ is defined to be the limit of the complex conjugations $t = \bar{\ }: U(n) \to U(n)$, so that the diagram

$$\begin{array}{ccc} U(n) & \xrightarrow{\iota_U} & U \\ {\scriptstyle t}\Big\downarrow & & \Big\downarrow{\scriptstyle t} \\ U(n) & \xrightarrow{\iota_U} & U \end{array}$$

is commutative. In this way we have maps $c: O \to U$ and $q: U \to Sp$, which yield fibre sequences

$$O \xrightarrow{c} U \xrightarrow{\pi_c} U/O \xrightarrow{j_c} BO \xrightarrow{Bc} BU$$

and

$$U \xrightarrow{q} Sp \xrightarrow{\pi_q} Sp/U \xrightarrow{j_q} BU,$$

respectively. We also have a map $\xi_c: U/O \to U$ defined analogously to (4). Recall that an Ω-spectrum $\mathbf{KO} = \{\mathbf{KO}_k\}_{k \in \mathbf{Z}}$ representing KO-theory is given by

$$\mathbf{KO}_k = BO \times \mathbf{Z},\ U/O,\ Sp/U,\ Sp,\ BSp \times \mathbf{Z},\ U/Sp,\ O/U,\ O$$

according as $k \equiv 0, 1, 2, 3, 4, 5, 6, 7 \pmod 8$. It would seem that the following result is implicitly shown in a topological proof of the Bott periodicity.

Lemma 2. *A map* $\mathbf{c} = \{\mathbf{c}_k\}_{k \in \mathbf{Z}}$ *of* Ω*-spectra that represents the complexification* c: $KO^*(X) \to K^*(X)$ *is given by the following*:
(0) $\mathbf{c}_0 = Bc \times 1: BO \times \mathbf{Z} \to BU \times \mathbf{Z}$;
(1) $\mathbf{c}_1 = \xi_c: U/O \to U$;
(2) $\mathbf{c}_2 = (j_q, 0): Sp/U \to BU \times \mathbf{Z}$;
and so on.

The following lemma, which is due to Seymour [6], is an exercise on Bott's exact sequence

$$\ldots \to KO^{*+1}(X) \to KO^*(X) \xrightarrow{c} K^*(X) \xrightarrow{\delta} KO^{*+2}(X) \to \ldots,$$

where $\delta(gx) = r(x)$ for $x \in K^{*+2}(X)$.

Lemma 3. *Let* X *be a space such that*
(1) $K^*(X)$ *is free as a* $K^*(pt)$*-module, and so, as a* $K^*(pt)$*-algebra, we may write*

$$K^*(X) = K^*(pt) \otimes A(b_1, b_2, \ldots, b_m)$$

for some algebra $A(b_1, b_2, \ldots, b_m)$ *over* $\mathbf{Z}$ *with generators* $b_1, b_2, \ldots, b_m \in \tilde{K}^*(X)$;

(2) *There exist uniquely determined* $a_1, a_2, \ldots, a_m \in \widetilde{KO}^*(X)$ *such that* $c(a_i) = b_i$ *for each* i.

Then $KO^*(X)$ *is a free* $KO^*(pt)$*-module. Moreover*

$$KO^*(X) = KO^*(pt) \otimes A(a_1, a_2, \ldots, a_m)$$

as a $KO^*(pt)$*-algebra.*

By (6) the first assumption of Lemma 3 is satisfied for $X = SU(2n)/SO(2n)$. We will show that the second assumption of Lemma 3 is also satisfied for $X = SU(2n)/SO(2n)$.

Let (G, σ) and (G', σ') be symmetric pairs, and let $\lambda : G \to G'$ be a homomorphism of topological groups such that $\lambda \circ \sigma = \sigma' \circ \lambda$. Then we have a map $\underline{\lambda} : G/G^{\sigma} \to G'/G'^{\sigma'}$ defined by $\underline{\lambda}(xG^{\sigma}) = \lambda(x)G'^{\sigma'}$ for $xG^{\sigma} \in G/G^{\sigma}$, which makes the following square commute.

$$\begin{array}{ccc} G/G^{\sigma} & \xrightarrow{\ \xi\ } & G \\ {\scriptstyle \underline{\lambda}}\big\downarrow & & \big\downarrow{\scriptstyle \lambda} \\ G'/G'^{\sigma'} & \xrightarrow{\ \xi'\ } & G' \end{array}$$

Does there exist an element in $KO^{-2i-1}(SU(2n)/SO(2n))$ such that its image under c is $g^i\beta(\lambda_k - \lambda_{2n-k})$ for some $i \in \mathbf{Z}$ and $k = 1, 2, \ldots, n-1$? Let us consider the element $\lambda_k \in R(SU(2n))$. By (2), $i^*(\lambda_k) = \mu_k$ in $R(SO(2n))$. By definition, μ_k is real. That is, there exists an element $\widehat{\mu_k} \in RO(SO(2n))$ such that $c(\widehat{\mu_k}) = \mu_k$. (Since $c : RO(SO(2n)) \to R(SO(2n))$ is injective [**1**, Proposition 3.27], such a $\widehat{\mu_k}$ is unique.) Hence $i^*(\lambda_k) = c(\widehat{\mu_k})$. This implies that the left square in the following diagram is commutative.

$$\begin{array}{ccccccc} SO(2n) & \xrightarrow{\ i\ } & SU(2n) & \xrightarrow{\ \pi\ } & SU(2n)/SO(2n) & \xrightarrow{\ \xi\ } & SO(2n) \\ {\scriptstyle \widehat{\mu_k}}\big\downarrow & & \big\downarrow{\scriptstyle \lambda_k} & & \big\downarrow{\scriptstyle \underline{\lambda_k}} & & \big\downarrow{\scriptstyle \lambda_k} \\ O\left(\binom{2n}{k}\right) & \xrightarrow{\ c\ } & U\left(\binom{2n}{k}\right) & \xrightarrow{\ \pi_c\ } & U\left(\binom{2n}{k}\right)\Big/O\left(\binom{2n}{k}\right) & \xrightarrow{\ \xi_c\ } & U\left(\binom{2n}{k}\right) \end{array}$$

Since $\sigma^*(\lambda_k) = \lambda_{2n-k} = t(\lambda_k)$ by (1) and [**1**, Theorem 7.4], we see that not only the middle square but also the right square is commutative. Therefore, if $\iota_{U/O} : U\left(\binom{2n}{k}\right)/O\left(\binom{2n}{k}\right) \to U/O$ is the canonical injection, we have an element

$$\lambda_{k,2n-k} := [\iota_{U/O} \circ \underline{\lambda_k}] \in [SU(2n)/SO(2n), U/O] = \widetilde{KO}^1(SU(2n)/SO(2n))$$

such that $\xi_{c*}(\lambda_{k,2n-k}) = [\iota_U \circ \lambda_k \circ \xi]$. Since $\xi_{c*} : [X, U/O] \to [X, O]$ is just $c : \widetilde{KO}^1(X) \to \tilde{K}^1(X)$ by Lemma 2(1), this implies that $c(\lambda_{k,n-k}) = g^{-1}\beta(\lambda_k - \lambda_{2n-k})$.

Does there exist an element in $\widetilde{KO}^{-2i}(SU(2n)/SO(2n))$ such that its image under c is $g^i\alpha(\mu_n^+)$ for some $i \in \mathbf{Z}$? Let us consider the element $\mu_n^+ \in R(SO(2n))$. Our argument is divided into two cases.

Consider first the case that n is even. By [**1**, Theorem 7.9], $\mu_n^+ \in R(SO(2n))$ is real. That is, there exists an element $\widehat{\mu_n^+} \in RO(SO(2n))$ such that $c(\widehat{\mu_n^+}) = \mu_n^+$. Let

$\alpha_{\mathbf{R}}: RO(G) \to KO^0(G)$ be the homomorphism defined analogously to (5). Then there is a commutative diagram

$$\begin{array}{ccc} RO(G) & \xrightarrow{\alpha_{\mathbf{R}}} & [BG, BO \times \mathbf{Z}] = KO^0(BG) \\ \downarrow c & (Bc \times 1)_* \downarrow & \downarrow c \\ R(G) & \xrightarrow{\alpha} & [BG, BU \times \mathbf{Z}] = K^0(BG) \end{array}$$

by [3, p. 191]. Therefore, if $B\iota_O: BO\left(\binom{2n}{n}/2\right) \to BO$ is the canonical injection, we have an element

$$\begin{aligned} \mu_{n,+} &:= (j^* \circ \alpha_{\mathbf{R}})\left(\widehat{\mu_n^+} - \binom{2n}{n}/2\right) \\ &= [B\iota_O \circ B\widehat{\mu_n^+} \circ j] \in \widetilde{KO}^0(SU(2n)/SO(2n)), \end{aligned}$$

and by Lemma 2(0), $c(\mu_{n,+}) = Bc_*(\mu_{n,+}) = [B\iota_U \circ B\mu_n^+ \circ j] = \alpha(\widetilde{\mu_n^+})$.

Consider next the case that n is odd. In this case, by [1, p. 179], the relation

$$t(\mu_n^+) = \mu_n^- \tag{9}$$

holds in $R(SO(2n))$. On the other hand, by [1, Theorem 7.4], the element $\lambda_n \in R(SU(2n))$ is quaternionic. That is, there exists an element $\widehat{\lambda_n} \in RSp(SU(2n))$ such that

$$c'(\widehat{\lambda_n}) = \lambda_n. \tag{10}$$

(Since $c': RSp(SU(2n)) \to R(SU(2n))$ is injective [1, Proposition 3.27], such a $\widehat{\lambda_n}$ is unique.) Then

$$\begin{aligned} 2i^*(\widehat{\lambda_n}) &= (q \circ c')(i^*(\widehat{\lambda_n})), && \text{by (7),} \\ &= (q \circ i^*)(c'(\widehat{\lambda_n})) \\ &= q(i^*(\lambda_n)), && \text{by (10),} \\ &= q(\mu_n), && \text{by (2),} \\ &= q(\mu_n^+ + \mu_n^-), && \text{by (3),} \\ &= q(\mu_n^+ + t(\mu_n^+)), && \text{by (9),} \\ &= q(\mu_n^+) + (q \circ t)(\mu_n^+) \\ &= q(\mu_n^+) + q(\mu_n^+), && \text{by (8),} \\ &= 2q(\mu_n^+) \end{aligned}$$

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in $RSp(SO(2n))$, which is a free abelian group by [**1**, Definition 3.26]. Hence $i^*(\widehat{\lambda_n}) = q(\mu_n^+)$. This implies that the left square in the following diagram is commutative.

$$\begin{array}{ccccccc}
SO(2n) & \xrightarrow{\ i\ } & SU(2n) & \xrightarrow{\ \pi\ } & SU(2n)/SO(2n) & \xrightarrow{\ j\ } & BSO(2n) \\
\downarrow{\scriptstyle \mu_n^+} & & \downarrow{\scriptstyle \widehat{\lambda_n}} & & \downarrow{\scriptstyle \widehat{\lambda_n}} & & \downarrow{\scriptstyle B\mu_n^+} \\
U\left(\binom{2n}{n}\Big/2\right) & \xrightarrow{\ q\ } & Sp\left(\binom{2n}{n}\Big/2\right) & \xrightarrow{\ \pi_q\ } & Sp\left(\binom{2n}{n}\Big/2\right)\Big/U\left(\binom{2n}{n}\Big/2\right) & \xrightarrow{\ j_q\ } & BU\left(\binom{2n}{n}\Big/2\right)
\end{array}$$

The middle and right squares are clearly commutative. Therefore, if $\iota_{S_p/U}$: $Sp\left(\binom{2n}{n}/2\right)/U\left(\binom{2n}{n}/2\right) \to Sp/U$ is the canonical injection, we have an element

$$\mu_{n,+} := [\iota_{Sp/U} \circ \widehat{\lambda_n}] \in [SU(2n)/SO(2n), Sp/U] = \widetilde{KO}^2(SU(2n)/SO(2n))$$

such that $j_{q*}(\mu_{n,+}) = [B\iota_U \circ B\mu_n^+ \circ j]$. Since $j_{q*}: [X, Sp/U] \to [X, BU]$ is just $c: \widetilde{KO}^2(X) \to \tilde{K}^2(X)$ by Lemma 2(2), this implies that $c(\mu_{n,+}) = g^{-1}\alpha(\widetilde{\mu_n^+})$.

Thus all the assumptions of Lemma 3 are satisfied for $X = SU(2n)/SO(2n)$. Now we can apply Lemma 3 to $X = SU(2n)/SO(2n)$. Then Theorem 1 follows.

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