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The solvability for a nonlinear degenerate hyperbolic–parabolic coupled system arising from nematic liquid crystals

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Abstract. This paper focuses on the Cauchy problem for a one-dimensional quasilinear hyperbolic–parabolic coupled system with initial data given on a line of parabolicity. The coupled system is derived from the Poiseuille flow of full Ericksen–Leslie model in the theory of nematic liquid crystals, which incorporates the crystal and liquid properties of the materials. The main difficulty comes from the degeneracy of the hyperbolic equation, which makes that the system is not continuously differentiable and then the classical methods for the strictly hyperbolic–parabolic coupled systems are invalid. With a choice of a suitable space for the unknown variable of the parabolic equation, we first solve the degenerate hyperbolic problem in a partial hodograph plane and express the smooth solution in terms of the original variables. Based on the smooth solution of the hyperbolic equation, we then construct an iterative sequence for the unknown variable of the parabolic equation by the fundamental solution of the heat equation. Finally, we verify the uniform convergence of the iterative sequence in the selected function space and establish the local existence and uniqueness of classical solutions to the degenerate coupled problem.

1 Introduction

Liquid crystals are a phase of matter intermediate between isotropic fluids and crystalline solids, which combine typical properties of liquids (flow properties) with those of solids (anisotropy). These multi-faceted properties make liquid crystal materials widely used in industry. Microscopically, to generate liquid crystal phase, the molecules have to be anisotropic, for example, rod-like or disk-like, and therefore they tend to organize themselves [11, 40]. According to the structure of the constituent molecules or groups of molecules, liquid crystals have many forms, such as nematic, smectic, and cholesteric. A liquid crystal is nematic if only long-range orientational order is present (rod-like molecules tend to align in a common direction), whereas is smectic if the order is partially positional (molecules are usually in layers). For the cholesteric liquid crystals, the molecules are aligned parallel to a certain direction, but this direction changes in space and makes a helix. Nematic liquid crystals are one of the simplest and most important members of liquid crystals and have been

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extensively studied theoretically, numerically, and experimentally [8, 11, 13, 30, 40]. There are various continuum theories for nematic liquid crystals [35]; we adopt in this paper the Ericksen–Leslie theory, which is adequate for the treatment of both static and dynamic phenomena [34]. Macroscopically, the motion of nematic liquid crystals can be characterized by the coupling of velocity field $\mathbf{u}$ for the flow and director field $\mathbf{n}$ for the alignment of the rod-like feature. The governing system is the so-called Ericksen–Leslie equations [13, 29, 30, 34], which reads that

$$\begin{cases} \nu\ddot{\mathbf{n}} = \lambda\mathbf{n} - \dfrac{\partial W}{\partial \mathbf{n}} - \mathbf{g} + \nabla\cdot\left(\dfrac{\partial W}{\partial\nabla\mathbf{n}}\right), \\ \rho\dot{\mathbf{u}} + \nabla P = \nabla\cdot\sigma - \nabla\cdot\left(\dfrac{\partial W}{\partial\nabla\mathbf{n}}\otimes\nabla\mathbf{n}\right), \\ \nabla\cdot\mathbf{u} = 0, \quad |\mathbf{n}| = 1, \end{cases} \tag{1.1}$$

where the notation $\dot{f}$ denotes the material derivative, that is, $\dot{f} = f_t + \mathbf{u}\cdot\nabla f$. In system (1.1), ρ is the density, P is the pressure, ν is the inertial coefficient of $\mathbf{n}$, and λ is the Lagrangian multiplier of the constraint $|\mathbf{n}| = 1$. Furthermore, the function $W = W(\mathbf{n}, \nabla\mathbf{n})$ is the Oseen–Franck potential energy density

$$W(\mathbf{n},\nabla\mathbf{n}) = \frac{1}{2}k_1(\nabla\cdot\mathbf{n})^2 + \frac{1}{2}k_2(\mathbf{n}\cdot\nabla\times\mathbf{n})^2 + \frac{1}{2}k_3|\mathbf{n}\times(\nabla\times\mathbf{n})|^2,$$

with elastic constants k_i $(i = 1,2,3)$, and $\mathbf{g}$ and σ are, respectively, the kinematic transport tensor and the viscous stress tensor given by

$$\begin{aligned} \mathbf{g} &= \gamma_1 N + \gamma_2 D\mathbf{n}, \\ \sigma &= \alpha_1(\mathbf{n}^T D\mathbf{n})\mathbf{n}\otimes\mathbf{n} + \alpha_2 N\otimes\mathbf{n} + \alpha_3\mathbf{n}\otimes N + \alpha_4 D \\ &\quad + \alpha_5(D\mathbf{n})\otimes\mathbf{n} + \alpha_6\mathbf{n}\otimes(D\mathbf{n}), \end{aligned}$$

where N represents the rigid rotation part of director changing rate by fluid vorticity and D represents the rate of strain tensor expressed as

$$N = \dot{\mathbf{n}} - \frac{1}{2}(\nabla\mathbf{u} - \nabla^T\mathbf{u})\mathbf{n}, \quad D = \frac{1}{2}(\nabla\mathbf{u} + \nabla^T\mathbf{u}).$$

The material coefficients γ_1, γ_2 in the expression of $\mathbf{g}$ and the Leslie coefficients α_i $(i = 1,\dots,6)$ in the expression of σ satisfy some specific relations and inequalities. For the one-dimensional Poiseuille flows with the following special form [7]

$$\mathbf{n}(x,t) = (\sin\theta(x,t), 0, \cos\theta(x,t))^T, \quad \mathbf{u}(x,t) = (0,0,u(x,t))^T, \tag{1.2}$$

Chen, Huang, and Liu [9] investigated the full Ericksen–Leslie equations (1.1) and derived the following one-dimensional hyperbolic–parabolic coupled system

$$\begin{cases} \nu\theta_{tt} + \gamma_1\theta_t = c(\theta)(c(\theta)\theta_x)_x - h(\theta)u_x, \\ \rho u_t = a + (g(\theta)u_x + h(\theta)\theta_t)_x, \end{cases} \tag{1.3}$$

where a is a constant, and

$$\begin{aligned} c^2(\theta) &= k_1\cos^2\theta + k_3\sin^2\theta, \qquad h(\theta) = \alpha_3\cos^2\theta - \alpha_2\sin^2\theta, \\ g(\theta) &= \alpha_1\sin^2\theta\cos^2\theta + \frac{\alpha_5-\alpha_2}{2}\sin^2\theta + \frac{\alpha_3+\alpha_6}{2}\cos^2\theta + \frac{\alpha_4}{2}. \end{aligned}$$

For the derivation of (1.3), we refer the reader to the work of Chen, Huang, and Liu [9]. By choosing appropriate parameters such that $h(\theta) = g(\theta) \equiv 1$ and $\nu = \rho = 1$, $a = 0$, $\gamma_1 = 2$, system (1.3) reduces to

$$\begin{cases} \theta_{tt} + 2\theta_t - c(\theta)(c(\theta)\theta_x)_x + u_x = 0, \\ u_t - (u_x + \theta_t)_x = 0. \end{cases} \tag{1.4}$$

We here provide a detailed introduction to the variables in (1.4). $t \geq 0$ is the time-independent variable, $x \in \mathbb{R}$ is the space-independent variable, the unknown function $\theta(t,x) \in \mathbb{R}$ represents the angle between the director field $\mathbf{n}$ and the z-axis, and the unknown function $u(t,x) \in \mathbb{R}$ represents the component of flow velocity $\mathbf{u}$ along the z-axis. See (1.2) for the details. The special physical significance of these unknown variables comes from the plane shear and Poiseuille flow geometries (assuming director stays in the x–z plane). Moreover, the first hyperbolic equation in (1.4) (and also (1.3)) describes the "crystal" property in nematic liquid crystals since it characterizes the propagation of the orientation waves in the director field, whereas the second parabolic equation in (1.4) (and also (1.3)) describes the "liquid" property of the liquid crystals.

When neglecting the fluid effect in system (1.3), it is obtained the well-known variational wave equation

$$\nu\theta_{tt} + \gamma_1\theta_t - c(\theta)(c(\theta)\theta_x)_x = 0, \tag{1.5}$$

which was introduced by Hunter and Saxton [22] and had been widely studied under the positiveness condition of the wave speed c. See works on the singularity formations of smooth solutions [16], on the existence of dissipative weak solutions [42, 43], and on the well-posedness of conservative weak solutions [2–5]. One may also consult the works of the related systems of variational wave equations in [6, 18, 44, 45]. On the other hand, the elastic coefficients $k_i (i = 1, 2, 3)$ may be negative in some cases physically, for example, the bend elastic constant k_3 is negative for the twist-bend nematic liquid crystals (see, e.g., [1, 12, 36]), so that the wave speed $c(\cdot)$ may degenerate at some time. Moreover, the motion of long waves on a neutral dipole chain in the continuum limit can also be described by a mixed-type equation similar to (1.5) [52]. In [38], Saxton studied the boundary blowup properties of smooth solutions to the degenerate equation (1.5) with $\gamma_1 = 0$ by allowing either k_1 or k_3 to be zero. In addition, equation (1.5) with $c(\theta) = \theta$ and $\gamma_1 = 0$ is corresponded to the one-dimensional second sound equation which describes the wave of temperature in superfluids (see [26, 27]). Under the initial assumption $\theta(0,x) \geq \delta > 0$, they investigated the local well-posedness and blowup of solutions to the one-dimensional second sound equation. In [21], Hu and Wang considered the degenerate variational wave equation (1.5) and discussed the local existence of smooth solutions near the degenerate line.

For the hyperbolic–parabolic coupled system (1.4) with a strictly positive wave function c, the local existence and uniqueness of smooth solutions to its Cauchy problem can be obtained from the classical results of the general hyperbolic–parabolic coupled systems by Li, Yu, and Shen [31]. Moreover, the local classical solutions to the initial-boundary value problems for the general hyperbolic–parabolic coupled systems were presented in [32, 33, 47]. The global existence of smooth solutions for the initial value problems and initial-boundary value problems to a class of hyperbolic–parabolic coupled systems were established by Zheng and Shen [48, 49, 51]. For more relevant results about the hyperbolic–parabolic coupled systems, we refer the reader to, for example, [17, 28, 39, 41, 50] and the references therein. Furthermore, for system (1.4), Chen, Huang, and Liu [9] shown the cusp-type singularity formations of smooth solutions in finite time and established the global existence of Hölder continuous weak solutions for its Cauchy problem. Recently, Chen, Liu, and Sofiani [10] generalized the existence results to the general system (1.3). Some relevant studies for the full Ericksen–Leslie equations (1.1) were presented among others in [23–25]. To the best of our knowledge, studies on the hyperbolic–parabolic coupled systems with a degenerate wave speed are still very limited and there is no general theory for these kinds of problems. Based on an estimate of the solution for the heat equation, we [19] shown that the smooth solution of (1.4) may break down in finite time even for an arbitrarily small initial energy under the initial condition $c(\theta(0,x)) \geq \delta > 0$.

In this paper, we are concerned with the Cauchy problem for the nonlinear degenerate hyperbolic–parabolic coupled system (1.4) with initial data given on a line of parabolicity. More precisely, we consider the local solvability of classical solutions to (1.4) with the degenerate initial data $c(\theta(0,x)) = 0$. This type of degenerate problem is very meaningful and interesting in both mathematical theory and physical applications. This situation corresponds physically to the case that the orientation wave in the director field has no potential energy initially. At later times, the orientation wave converts kinetic energy into potential energy due to the coupling effect of director field and flow field. Exploring this issue may help us better understand the process of energy conversion. Mathematically, the solvability of degenerate hyperbolic–parabolic coupled systems is a fundamental problem for coupled systems. Although there have been many results on the solvability for the strictly hyperbolic–parabolic coupled systems and the single degenerate hyperbolic equations, they cannot be applied to the current degenerate hyperbolic–parabolic coupled problem. In the present paper, we need to overcome the coupling difficulty of degeneracy on variables at different scales in terms of technology. In addition, it is also very valuable to investigate how the unknown function of parabolic equation will be affected by the degeneracy of the hyperbolic equation.

For the convenience of calculation, we discuss the special wave speed $c(\theta) = \theta$ here, and a similar result for the general wave speed $c(\cdot)$ with $|c'(\cdot)| \geq c_0 > 0$ can be established by the mean value theorem in the derivation process. By introducing a new variable $v(t,x)$

$$v(t,x) = \int_{-\infty}^{x} u(t,z)\,\mathrm{d}z, \tag{1.6}$$

and hence $v_t = u_x + \theta_t$, system (1.4) can be rewritten as in terms of (θ, v)

$$\begin{cases} \theta_{tt} + \theta_t - \theta(\theta\theta_x)_x = -v_t, \\ v_t - v_{xx} = \theta_t. \end{cases} \tag{1.7}$$

We study the Cauchy problem to (1.7) with the following initial data:

$$\theta(0,x) = 0, \quad \theta_t(0,x) = \theta_0(x), \quad v(0,x) = v_0(x). \tag{1.8}$$

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From (1.8), the wave speed is equal to zero on the initial line, then the hyperbolic equation in (1.7) is degenerate at $t = 0$. We emphasize that this degenerate Cauchy problem is not trivial or easy, and it cannot be solved by applying the classical method for the strictly hyperbolic–parabolic coupled systems (e.g., [31]). The main reason is that the corresponding system is not a continuously differentiable system by the degeneracy. Moreover, due to the coupling of hyperbolicity and parabolicity, this degenerate Cauchy problem can also not be solved directly from the previous strategy of studying degenerate hyperbolic equations (e.g., [20, 21, 46]) since the scales of the hyperbolic and parabolic equations are different. In the present paper, we solve the degenerate Cauchy problem (1.7), (1.8) by the fixed point iteration pattern in two steps. First, we introduce a suitable function space for the variable v, and establish the existence of classical solutions for the degenerate hyperbolic system in a partial hodograph coordinate plane. Here, due to the regularity of variable v, the approach of solving the current degenerate hyperbolic problem, different from the method in previous papers [20, 21, 46], is inspired by the work done by Protter [37] for studying the well-posedness of the Cauchy problem to the second-order linear degenerate hyperbolic equation. We solve the quasilinear degenerate hyperbolic problem and derive the uniform estimates of solutions near the parabolic degenerating line in the partial hodograph plane. By transforming the smooth solution back to the original coordinate variables, we obtain the information of variable θ and its corresponding estimates. Second, we use the fundamental solution of one-dimensional heat equation to express the variable v and then establish its series of estimates based on the information of θ. The uniform convergence of the iterative sequence generated by this pattern is verified in the selected space for a sufficiently small time.

The main conclusion of this paper can be stated as follows.

Theorem 1.1 *Suppose that the functions $\theta_0(x)$ and $v_0(x)$ satisfy*

$$\begin{gathered} \theta_0(x) \in C^3, \quad v_0(x) \in C^4, \\ |\theta_0(x)| \geq \underline{\theta} > 0,\ |\theta_0^{(j)}| < \infty\ (j = 0, \dots, 3),\ |v_0^{(j)}| < \infty\ (j = 0, \dots, 4), \end{gathered} \tag{1.9}$$

for all $x \in \mathbb{R}$ and some positive constant $\underline{\theta}$. Then there exists a constant $\delta > 0$ such that the degenerate Cauchy problem (1.7), (1.8) admits a unique classical solution on $[0, \delta] \times \mathbb{R}$.

We comment that the technique developed in the paper can be applied to study the more general hyperbolic–parabolic coupled systems, e.g., system (1.3). With a choice of a suitable space for the unknown variable u, one can obtain the function θ and its properties near the degenerate line by solving the degenerate hyperbolic equation based on almost the same process. Subsequently, we may utilize the parametrix method [15] to construct the iterative sequence for the variable u and then show that it is uniformly convergent in the selected function space. The detailed process is relatively complicated and will be considered in the future.

The rest of the paper is organized as follows. In Section 2, we reformulate the problem in terms of new dependent variables and then restate the main result. Section 3 is devoted to solving the degenerate hyperbolic problem for a given variable v in a suitable function space. In Section 3.1, we introduce a partial hodograph coordinate system to transform the hyperbolic equation into a new system with a transparent singularity-regularity structure. In Section 3.2, we construct an iterative sequence generated by the new problem in the partial hodograph plane. Sections 3.3 and 3.4 are devoted to, respectively, establish a series of lemmas for the iterative sequence and show the existence and uniqueness of smooth solutions for the new problem. The smooth solution is converted in the partial hodograph plane to that in the original physical plane in Section 3.5. In Section 4, we explore the local existence and uniqueness of classical solutions for the hyperbolic–parabolic coupled problem. In Section 4.1, we present the preliminary results for the one-dimensional heat equation. Section 4.2 is devoted to constructing an iterative sequence for the parabolic equation and verifying that it belongs to the selected function space. In Sections 4.3–4.5, we establish a series of properties for the iterative sequence and its derivatives in the selected function space, including the regularity and convergence of the iterative sequence. Finally, in Section 4.6, we complete the proof of Theorem 1.1.

2 Reformulation of the problem and the main result

In this section, we reformulate the problem by introducing a series of new dependent variables and then restate the main result in terms of these variables. We discuss the case $\theta_0(x) \geq \underline{\theta} > 0$; the other case $\theta_0(x) \leq -\underline{\theta} < 0$ is analogous.

To handle the term v_t in the first equation of (1.7), we introduce

$$R = \theta_t + \theta\theta_x + v, \quad S = \theta_t - \theta\theta_x + v. \tag{2.1}$$

Thus,

$$\theta_t = \frac{R+S}{2} - v, \quad \theta_x = \frac{R-S}{2\theta}. \tag{2.2}$$

In terms of variables (v, R, S, θ), system (1.7) can be written as

$$\begin{cases} v_t - v_{xx} = \dfrac{R+S}{2} - v, \\ R_t - \theta R_x = \dfrac{R+S-2v}{4\theta}(R-S) - \dfrac{R+S}{2} + v - \theta v_x, \\ S_t + \theta S_x = \dfrac{R+S-2v}{4\theta}(S-R) - \dfrac{R+S}{2} + v + \theta v_x, \\ \theta_t = \dfrac{R+S}{2} - v. \end{cases} \tag{2.3}$$

Corresponding to (1.8), the initial conditions of (2.3) are

$$v(0,x) = v_0(x), \quad R(0,x) = S(0,x) = \theta_0(x) + v_0(x), \quad \theta(0,x) = 0. \tag{2.4}$$

Furthermore, we introduce the following variables $(\widetilde{v}, \widetilde{R}, \widetilde{S})$ to homogenize the initial data:

$$\begin{cases} \widetilde{v}(t,x) = v(t,x) - v_0(x), \\ \widetilde{R}(t,x) = R(t,x) - [\theta_0(x) + v_0(x)] + \theta(t,x), \\ \widetilde{S}(t,x) = S(t,x) - [\theta_0(x) + v_0(x)] + \theta(t,x). \end{cases} \tag{2.5}$$

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Then one has by (2.3) and (2.4)

$$\begin{aligned} \widetilde{v}(0,x) = \widetilde{R}(0,x) = \widetilde{S}(0,x) = \theta(0,x) = 0, \\ \widetilde{R}_t(0,x) = \widetilde{S}_t(0,x) = 0. \end{aligned} \tag{2.6}$$

Here, the homogeneous initial values of $(\widetilde{R}_t, \widetilde{S}_t)$ come from the equations of (R,S) in (2.3) and the fact $\frac{R-S}{2\theta}\Big|_{t=0} = \theta_x|_{t=0} = 0$. By a direct calculation, we can obtain a new system in terms of the variables $(\widetilde{v}, \widetilde{R}, \widetilde{S}, \theta)$ as follows:

$$\begin{cases} \widetilde{v}_t - \widetilde{v}_{xx} = \dfrac{\widetilde{R}+\widetilde{S}}{2} - \theta - \widetilde{v} + \theta_0 + v_0'', \\ \widetilde{R}_t - \theta\widetilde{R}_x = \dfrac{\widetilde{R}+\widetilde{S}-2\widetilde{v}+2\theta_0}{4\theta}(\widetilde{R}-\widetilde{S}) - (\widetilde{R}-\widetilde{S}) + \theta\theta_0' - \theta\widetilde{v}_x, \\ \widetilde{S}_t + \theta\widetilde{S}_x = \dfrac{\widetilde{R}+\widetilde{S}-2\widetilde{v}+2\theta_0}{4\theta}(\widetilde{S}-\widetilde{R}) + (\widetilde{R}-\widetilde{S}) - \theta\theta_0' + \theta\widetilde{v}_x, \\ \theta_t = \dfrac{\widetilde{R}+\widetilde{S}}{2} - \theta - \widetilde{v} + \theta_0. \end{cases} \tag{2.7}$$

For the singular Cauchy problem (2.7), (2.6), we have the following.

Theorem 2.1 *Assume that the functions $v_0(x)$ and $\theta_0(x)$ satisfy (1.9) and*

$$0 < \underline{\theta} \le \theta_0(x) \le \overline{\theta}, \tag{2.8}$$

for all $x \in \mathbb{R}$ and two positive constants $\underline{\theta}, \overline{\theta}$. Then there exists a constant $\delta > 0$ such that the singular Cauchy problem (2.7), (2.6) has a unique classical solution on $[0,\delta] \times \mathbb{R}$. Moreover, the solution $(\widetilde{v}, \widetilde{R}, \widetilde{S}, \theta)(t,x)$ satisfies

$$\begin{aligned} |\widetilde{v}(t,x)|, |\widetilde{v}_x(t,x)| \le \widehat{M}\theta(t,x),\ |\widetilde{R}(t,x)|, |\widetilde{S}(t,x)| \le \widetilde{M}\theta^2(t,x), \\ \frac{1}{2}\underline{\theta}t \le \theta(t,x) \le 2\overline{\theta}t, \end{aligned} \tag{2.9}$$

for any $(t,x) \in [0,\delta] \times \mathbb{R}$, where $\widehat{M}$ and $\widetilde{M}$ are two positive constants.

We shall use the iteration method to show Theorem 2.1, from which and (2.5) one establishes the existence of classical solutions for the Cauchy problem (2.3), (2.4). Then Theorem 1.1 can be achieved by the relations in (2.1).

Here, we present the results of our choices of the constants $\widetilde{M}, \widehat{M}$, and δ for the reader first, which come from the construction process in Sections 3 and 4. Denote

$$\begin{aligned} \overline{\theta}_0 &= \max\left\{\max_{x\in\mathbb{R}}|\theta_0'(x)|, \max_{x\in\mathbb{R}}|\theta_0''(x)|, \max_{x\in\mathbb{R}}|\theta_0'''(x)|\right\}, \\ \overline{K} &= \max\left\{1, \frac{2}{\underline{\theta}}, \frac{4}{\underline{\theta}^2}, \frac{8}{\underline{\theta}^3}, \overline{\theta}_0, \frac{2\overline{\theta}_0}{\underline{\theta}}, \frac{4\overline{\theta}_0}{\underline{\theta}^2}, \frac{8\overline{\theta}_0^2}{\underline{\theta}^2}\right\}, \\ \widehat{K} &= \max\left\{1, \max_{x\in\mathbb{R}}|\theta_0(x) + v_0''(x)|,\ \max_{x\in\mathbb{R}}|\theta_0'(x) + v_0'''(x)|,\right. \\ &\qquad \left. \max_{x\in\mathbb{R}}|\theta_0''(x) + v_0''''(x)|\right\}. \end{aligned} \tag{2.10}$$

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It is noted that $\overline{K}$ and $\widehat{K}$ are constants depending only on $\underline{\theta}$, $\overline{\theta}$, the C^3 norms of $\theta_0(x)$, and the C^4 norms of $v_0(x)$. Moreover, we choose

$$\begin{aligned} &\overline{M} = 16\overline{K} \geq 16, \\ &\widehat{M}_0 = \max\left\{32\widehat{K},\ 12\overline{M}^2(1+\overline{\theta})^2, 32\overline{\theta}\left(C_{2,1} + 2C_{3,1/2} + C_{3,0}\right)\right\}, \\ &\widetilde{M} = 3\overline{M}, \quad \widehat{M} = \frac{2}{\underline{\theta}}\widehat{M}_0, \end{aligned} \tag{2.11}$$

and

$$\widetilde{\delta} = \min\left\{\frac{1}{16\overline{M}}, \frac{1}{16\overline{K}\widehat{M}_0}, \frac{1}{\widehat{M}_0^2}\right\}, \quad \delta = \min\left\{\frac{\widetilde{\delta}}{2\overline{\theta}}, \frac{1}{\widehat{M}_0}, \frac{\underline{\theta}^2}{16\widehat{M}_0\overline{\theta}}\right\}. \tag{2.12}$$

In (2.11), $C_{j,\beta}(j \geq 1, \beta \geq 0)$ are positive constants that make the following inequality valid:

$$\left| |x|^\beta \frac{\partial^j}{\partial x^j} \exp\left(-\frac{x^2}{4t}\right)\right| \leq C_{j,\beta} t^{-\frac{j-\beta}{2}} \exp\left(-\frac{x^2}{16t}\right), \tag{2.13}$$

which come from the estimates for the fundamental solution of the heat equation.

3 The degenerate hyperbolic problem

In this section, we choose a suitable function space for the variable $\widetilde{v}$ and then solve the degenerate hyperbolic problem for the variables $(\widetilde{R}, \widetilde{S}, \theta)(t,x)$.

Set

$$\Sigma(\delta_0) = \left\{ \widetilde{v}(t,x) \;\middle|\; \begin{array}{l} \widetilde{v}(t,x) \in C^1,\ \widetilde{v}_x(t,x) \in C^1,\ \forall\ (t,x) \in [0,\delta_0] \times \mathbb{R} \\ |\widetilde{v}| \leq \widehat{M}_0 t,\ |\widetilde{v}_x| \leq \widehat{M}_0 t \\ |\widetilde{v}_t| \leq \widehat{M}_0,\ |\widetilde{v}_{xt}| \leq \widehat{M}_0,\ |\widetilde{v}_{xx}| \leq \widehat{M}_0\sqrt{t} \end{array} \right\}, \tag{3.1}$$

where $\widehat{M}_0$ is a positive constant which will be determined in Section 4, and δ_0 is an arbitrary positive number which may be assumed to be 1. Let $\widetilde{v}$ be any element in $\Sigma(1)$ and denote $w(t,x) = \widetilde{v}_x(t,x) \in C^1$. We consider the degenerate hyperbolic problem

$$\begin{cases} \widetilde{R}_t - \theta\widetilde{R}_x = \dfrac{\widetilde{R} + \widetilde{S} - 2\widetilde{v} + 2\theta_0}{4\theta}(\widetilde{R} - \widetilde{S}) - (\widetilde{R} - \widetilde{S}) + \theta\theta_0' - \theta w, \\ \widetilde{S}_t + \theta\widetilde{S}_x = \dfrac{\widetilde{R} + \widetilde{S} - 2\widetilde{v} + 2\theta_0}{4\theta}(\widetilde{S} - \widetilde{R}) + (\widetilde{R} - \widetilde{S}) - \theta\theta_0' + \theta w, \\ \theta_t = \dfrac{\widetilde{R} + \widetilde{S}}{2} - \theta - \widetilde{v} + \theta_0, \end{cases} \tag{3.2}$$

with the homogeneous initial conditions

$$\widetilde{R}(0,x) = \widetilde{S}(0,x) = \theta(0,x) = \widetilde{R}_t(0,x) = \widetilde{S}_t(0,x) = 0. \tag{3.3}$$

For the Cauchy problem (3.2), (3.3), we have the following theorem.

Theorem 3.1 *Let the conditions in Theorem 2.1 hold. Then there exists a constant $\overline{\delta} > 0$ such that the Cauchy problem (3.2), (3.3) has a unique classical solution on $[0, \overline{\delta}] \times \mathbb{R}$.*

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Moreover, the solution $(\widetilde{R},\widetilde{S},\theta)(t,x)$ *satisfies*

$$|\widetilde{R}(t,x)|,|\widetilde{S}(t,x)| \le \widetilde{M}\theta^2(t,x), \quad \frac{1}{2}\underline{\theta}t \le \theta(t,x) \le 2\overline{\theta}t, \tag{3.4}$$

for any $(t,x) \in [0,\overline{\delta}] \times \mathbb{R}$, *where* $\widetilde{M}$ *is a positive constant.*

We show Theorem 3.1 by transforming the problem into a partial hodograph plane and then returning the solution to the original physical plane.

3.1 The problem in a partial hodograph plane

We introduce the new independent variables

$$\tau = \theta(t,x), \quad y = x. \tag{3.5}$$

It is easy to calculate the Jacobian of this transformation

$$J := \frac{\partial(\tau,y)}{\partial(t,x)} = \tau_t y_x - \tau_x y_t = \theta_t = \frac{\widetilde{R}+\widetilde{S}}{2} - \tau - \widetilde{v} + \theta_0,$$

which, together with (3.1) and (3.3), gives $J|_{t=0} = \theta_0 \ge \underline{\theta} > 0$. Furthermore, it acquires that

$$\partial_t = \left(\frac{\widetilde{R}+\widetilde{S}}{2} - \theta - \widetilde{v} + \theta_0\right)\partial_\tau, \quad \partial_x = \frac{\widetilde{R}-\widetilde{S}}{2\tau}\partial_\tau + \partial_y. \tag{3.6}$$

From system (3.2) and the relations in (3.6), we can obtain a closed quasilinear hyperbolic system in terms of $(\widetilde{R},\widetilde{S},t)$

$$\begin{cases} \widetilde{R}_\tau - \frac{\tau}{f+\widetilde{S}}\widetilde{R}_y = \frac{\widetilde{R}-\widetilde{S}}{2\tau} + \frac{1}{f+\widetilde{S}}\cdot\frac{(\widetilde{R}-\widetilde{S})^2}{4\tau} - \frac{\widetilde{R}-\widetilde{S}}{f+\widetilde{S}} + \frac{\theta_0'}{f+\widetilde{S}}\tau - \frac{w(t,y)}{f+\widetilde{S}}\tau, \\ \widetilde{S}_\tau + \frac{\tau}{f+\widetilde{R}}\widetilde{S}_y = \frac{\widetilde{S}-\widetilde{R}}{2\tau} + \frac{1}{f+\widetilde{R}}\cdot\frac{(\widetilde{R}-\widetilde{S})^2}{4\tau} + \frac{\widetilde{R}-\widetilde{S}}{f+\widetilde{R}} - \frac{\theta_0'}{f+\widetilde{R}}\tau + \frac{w(t,y)}{f+\widetilde{R}}\tau, \\ t_\tau = \frac{1}{\frac{\widetilde{R}+\widetilde{S}}{2}-\tau-\widetilde{v}(t,y)+\theta_0(y)}, \end{cases} \tag{3.7}$$

where $f = \theta_0(y) - \widetilde{v}(t,y) - \tau$. Obviously, system (3.7) has a clear regularity-singularity structure. The initial values of $(\widetilde{R},\widetilde{S},t)$ are

$$\widetilde{R}(0,y) = \widetilde{S}(0,y) = t(0,y) = 0. \tag{3.8}$$

It is easily seen that the three eigenvalues of system (3.7) are

$$\lambda_- = -\frac{\tau}{f+\widetilde{S}}, \quad \lambda_+ = \frac{\tau}{f+\widetilde{R}}, \quad \lambda_0 = 0. \tag{3.9}$$

Moreover, the three characteristics passing through a point (ξ,η) are defined by

$$\begin{cases} \dfrac{\mathrm{d}y_\pm(\tau;\xi,\eta)}{\mathrm{d}\tau} = \lambda_\pm(\tau,y,t,\widetilde{R},\widetilde{S})(\tau,y_\pm(\tau;\xi,\eta)), \\ y_\pm(\xi;\xi,\eta) = \eta, \end{cases} \quad y_0(\tau;\xi,\eta) = \eta. \tag{3.10}$$

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Suppose that $\widetilde{M}$ is a sufficiently large positive constant determined later. We first choose a sufficiently small positive constant $\widetilde{\delta}_1 \le 1$ such that

$$\widetilde{\delta}_1 \le \frac{1}{2}\underline{\theta}, \quad \widetilde{M}\cdot\widetilde{\delta}_1 \le 1, \quad \left(2 + \frac{2\widehat{M}_0}{\underline{\theta}}\right)\widetilde{\delta}_1 \le \frac{1}{2}\underline{\theta}. \tag{3.11}$$

We use the notation $\widetilde{\Sigma}(\widetilde{\delta}_1)$ to denote the function class which incorporates all continuously vector functions $\mathbf{F} = (\widetilde{R}, \widetilde{S}, t)^T : [0, \widetilde{\delta}_1] \times \mathbb{R} \to \mathbb{R}^3$ satisfying the following properties:

$$\begin{aligned} &(\mathrm{P}_1) : (\widetilde{R}, \widetilde{S}, t)(\tau, y) \text{ are continuous on } [0, \widetilde{\delta}_1] \times \mathbb{R};\\ &(\mathrm{P}_2) : (\widetilde{R}, \widetilde{S}, t)(0, y) = \mathbf{0}, \ \forall\ y \in \mathbb{R};\\ &(\mathrm{P}_3) : |\widetilde{R}(t, y)|, |\widetilde{S}(t, y)| \le \widetilde{M}\tau^2, \ \frac{\tau}{2\overline{\theta}} \le t(\tau, y) \le \frac{2\tau}{\underline{\theta}} \le 1,\\ &\qquad \text{for all } (\tau, y) \in [0, \widetilde{\delta}_1] \times \mathbb{R}. \end{aligned} \tag{3.12}$$

If $(\widetilde{R}, \widetilde{S}, t)$ is an arbitrary element in $\widetilde{\Sigma}(\widetilde{\delta}_1)$, we see by (3.11) and (3.12) that

$$\begin{aligned} f + \widetilde{R} =& \theta_0(y) - \widetilde{v}(t, y) - \tau + \widetilde{R} \ge \underline{\theta} - \widehat{M}_0 t - \tau - |\widetilde{R}|\\ \ge& \underline{\theta} - \widehat{M}_0 \cdot \frac{2\tau}{\underline{\theta}} - \tau - \widetilde{M}\tau^2 \ge \underline{\theta} - \left(\frac{2\widehat{M}_0}{\underline{\theta}} + 1 + \widetilde{M}\widetilde{\delta}_1\right)\tau \ge \frac{1}{2}\underline{\theta} \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} f + \widetilde{R} =& \theta_0(y) - \widetilde{v}(t, y) - \tau + \widetilde{R} \le \overline{\theta} + \widehat{M}_0 t + \tau + |\widetilde{R}|\\ &\le \overline{\theta} + \widehat{M}_0 \cdot \frac{2\tau}{\underline{\theta}} + \tau + \widetilde{M}\tau^2\\ &\le \overline{\theta} + \left(\frac{2\widehat{M}_0}{\underline{\theta}} + 1 + \widetilde{M}\widetilde{\delta}_1\right)\tau \le \overline{\theta} + \frac{1}{2}\underline{\theta} \le 2\overline{\theta}. \end{aligned} \tag{3.14}$$

Similarly, one has

$$\frac{1}{2}\underline{\theta} \le f + \widetilde{S} \le 2\overline{\theta} \tag{3.15}$$

and

$$\frac{1}{2}\underline{\theta} \le \frac{\widetilde{R} + \widetilde{S}}{2} - \tau - \widetilde{v}(t, y) + \theta_0(y) \le \overline{\theta} + \frac{1}{2}\underline{\theta} \le 2\overline{\theta}. \tag{3.16}$$

Furthermore, we have by (3.9) and (3.13)–(3.15)

$$\frac{\tau}{2\overline{\theta}} \le -\lambda_-, \lambda_+ \le \frac{2\tau}{\underline{\theta}}, \tag{3.17}$$

from which and (3.10) one achieves

$$|y_+(\tau; \xi, \eta) - y_-(\tau; \xi, \eta)| \le 2\int_0^{\xi} \frac{2\tau}{\underline{\theta}}\, \mathrm{d}\tau = \frac{2}{\underline{\theta}}\xi^2. \tag{3.18}$$

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3.2 The iterative sequence

For any $(\xi,\eta)\in[0.\widetilde{\delta}_1]\times\mathbb{R}$, integrating the differential system (3.7) along the characteristic curves $y=y_i(\tau;\xi,\eta)$ $(i=\pm,0)$ defined in (3.10), we utilize the boundary conditions (3.8) to gain a system of integral equations

$$
\left\{
\begin{aligned}
\widetilde{R}(\xi,\eta) &= \int_0^{\xi}\bigg\{\frac{\widetilde{R}-\widetilde{S}}{2\tau}+A\cdot\frac{(\widetilde{R}-\widetilde{S})^2}{4\tau}\\
&\qquad -A(\widetilde{R}-\widetilde{S})+A\theta_0'\tau-Aw(t,y)\tau\bigg\}(\tau,y_-(\tau;\xi,\eta))\,\mathrm{d}\tau,\\
\widetilde{S}(\xi,\eta) &= \int_0^{\xi}\bigg\{\frac{\widetilde{S}-\widetilde{R}}{2\tau}+B\cdot\frac{(\widetilde{R}-\widetilde{S})^2}{4\tau}\\
&\qquad +B(\widetilde{R}-\widetilde{S})-B\theta_0'\tau+Bw(t,y)\tau\bigg\}(\tau,y_+(\tau;\xi,\eta))\,\mathrm{d}\tau,\\
t(\xi,\eta) &= \int_0^{\xi}C(\tau,y_0(\tau;\xi,\eta))\,\mathrm{d}\tau,
\end{aligned}
\right. \tag{3.19}
$$

where

$$
A=\frac{1}{f+\widetilde{S}},\quad B=\frac{1}{f+\widetilde{R}},\quad C=\frac{1}{\frac{\widetilde{R}+\widetilde{S}}{2}-\tau-\widetilde{v}(t,y)+\theta_0(y)}. \tag{3.20}
$$

One can apply the integral system (3.19) to construct the iterative sequences. For any $(\tau,y)\in[0,\widetilde{\delta}_1]\times\mathbb{R}$, set

$$
\widetilde{R}^{(0)}(\tau,y)=\widetilde{S}^{(0)}(\tau,y)=0,\quad t^{(0)}(\tau,y)=\frac{\tau}{2\overline{\theta}}, \tag{3.21}
$$

from which we define the characteristic curves $y=y_i^{(0)}(\tau)=:y_i^{(0)}(\tau;\xi,\eta)$ $(i=\pm,0)$ for $\tau\in[0,\xi]$ as

$$
\left\{
\begin{aligned}
&\frac{\mathrm{d}y_\pm^{(0)}(\tau)}{\mathrm{d}\tau}=\lambda_\pm(\tau,y,t^{(0)},\widetilde{R}^{(0)},\widetilde{S}^{(0)})(\tau,y_\pm^{(0)}(\tau)),\\
&y_\pm^{(0)}(\xi)=\eta,
\end{aligned}
\right.
\qquad y_0^{(0)}(\tau)=\eta. \tag{3.22}
$$

Thus, the functions $(\widetilde{R}^{(1)},\widetilde{S}^{(1)},t^{(1)})(\tau,y)$ can be defined as

$$
\left\{
\begin{aligned}
\widetilde{R}^{(1)}(\xi,\eta) &= \int_0^{\xi}\bigg\{\frac{\widetilde{R}^{(0)}-\widetilde{S}^{(0)}}{2\tau}+A^{(0)}\cdot\frac{(\widetilde{R}^{(0)}-\widetilde{S}^{(0)})^2}{4\tau}\\
&\quad -A^{(0)}(\widetilde{R}^{(0)}-\widetilde{S}^{(0)})+A^{(0)}\theta_0'\tau-A^{(0)}w(t^{(0)},y)\tau\bigg\}(\tau,y_-^{(0)}(\tau))\,\mathrm{d}\tau,\\
\widetilde{S}^{(1)}(\xi,\eta) &= \int_0^{\xi}\bigg\{\frac{\widetilde{S}^{(0)}-\widetilde{R}^{(0)}}{2\tau}+B^{(0)}\cdot\frac{(\widetilde{R}^{(0)}-\widetilde{S}^{(0)})^2}{4\tau}\\
&\quad +B^{(0)}(\widetilde{R}^{(0)}-\widetilde{S}^{(0)})-B^{(0)}\theta_0'\tau+B^{(0)}w(t^{(0)},y)\tau\bigg\}(\tau,y_+^{(0)}(\tau))\,\mathrm{d}\tau,\\
t^{(1)}(\xi,\eta) &= \int_0^{\xi}C^{(0)}(\tau,y_0^{(0)}(\tau))\,\mathrm{d}\tau,
\end{aligned}
\right. \tag{3.23}
$$

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where

$$A^{(0)} = \frac{1}{f^{(0)} + \widetilde{S}^{(0)}}, \quad B^{(0)} = \frac{1}{f^{(0)} + \widetilde{R}^{(0)}},$$
$$C^{(0)} = \frac{1}{\frac{\widetilde{R}^{(0)}+\widetilde{S}^{(0)}}{2} - \tau - \widetilde{v}(t^{(0)}, y) + \theta_0(y)},$$

and $f^{(0)} = \theta_0(y) - \widetilde{v}(t^{(0)}, y) - \tau$. After obtaining the functions $(\widetilde{R}^{(k)}, \widetilde{S}^{(k)}, t^{(k)})(\tau, y)$, one gets the characteristic curves $y = y_i^{(k)}(\tau) =: y_i^{(k)}(\tau; \xi, \eta)$ $(i = \pm, 0)$ by solving the following ODE equations:

$$(3.24)\qquad \begin{cases} \dfrac{dy_\pm^{(k)}(\tau)}{d\tau} = \lambda_\pm(\tau, y, t^{(k)}, \widetilde{R}^{(k)}, \widetilde{S}^{(k)})(\tau, y_\pm^{(k)}(\tau)), \\ y_\pm^{(k)}(\xi) = \eta, \end{cases} \qquad y_0^{(k)}(\tau) = \eta.$$

Hence, we determine the functions $(\widetilde{R}^{(k+1)}, \widetilde{S}^{(k+1)}, t^{(k+1)})(\tau, y)$ by the following relations:

$$(3.25)\qquad \begin{cases} \widetilde{R}^{(k+1)}(\xi, \eta) = \displaystyle\int_0^\xi \Big\{ \frac{\widetilde{R}^{(k)} - \widetilde{S}^{(k)}}{2\tau} + A^{(k)} \cdot \frac{(\widetilde{R}^{(k)} - \widetilde{S}^{(k)})^2}{4\tau} \\ \quad -A^{(k)}(\widetilde{R}^{(k)} - \widetilde{S}^{(k)}) + A^{(k)}\theta_0'\tau - A^{(k)} w(t^{(k)}, y)\tau \Big\}(\tau, y_-^{(k)}(\tau))\, d\tau, \\ \widetilde{S}^{(k+1)}(\xi, \eta) = \displaystyle\int_0^\xi \Big\{ \frac{\widetilde{S}^{(k)} - \widetilde{R}^{(k)}}{2\tau} + B^{(k)} \cdot \frac{(\widetilde{R}^{(k)} - \widetilde{S}^{(k)})^2}{4\tau} \\ \quad +B^{(k)}(\widetilde{R}^{(k)} - \widetilde{S}^{(k)}) - B^{(k)}\theta_0'\tau + B^{(k)} w(t^{(k)}, y)\tau \Big\}(\tau, y_+^{(k)}(\tau))\, d\tau, \\ t^{(k+1)}(\xi, \eta) = \displaystyle\int_0^\xi C^{(k)}(\tau, y_0^{(k)}(\tau))\, d\tau, \end{cases}$$

where

$$A^{(k)} = \frac{1}{f^{(k)} + \widetilde{S}^{(k)}}, \quad B^{(k)} = \frac{1}{f^{(k)} + \widetilde{R}^{(k)}},$$
$$C^{(k)} = \frac{1}{\frac{\widetilde{R}^{(k)}+\widetilde{S}^{(k)}}{2} - \tau - \widetilde{v}(t^{(k)}, y) + \theta_0(y)},$$

and $f^{(k)} = \theta_0(y) - \widetilde{v}(t^{(k)}, y) - \tau$.

We next choose the constants $\widetilde{M}$ and $\widetilde{\delta} \le \widetilde{\delta}_1$ to verify the uniform convergence of the sequences $(\widetilde{R}^{(k)}, \widetilde{S}^{(k)}, t^{(k)})(\tau, y)$.

3.3 Several lemmas

It first follows by the detailed expressions of A, B, C in (3.20) that

$$(3.26)\qquad \begin{aligned} &A_{\widetilde{S}} = -A^2, \quad B_{\widetilde{R}} = -B^2, \quad C_{\widetilde{R}} = C_{\widetilde{S}} = -\frac{1}{2}C^2, \\ &A_t = A^2\widetilde{v}_t(t, y), \quad A_y = -A^2[\theta_0' - w(t, y)], \quad B_t = B^2\widetilde{v}_t(t, y), \\ &B_y = -B^2[\theta_0' - w(t, y)], \quad C_t = C^2\widetilde{v}_t(t, y), \quad C_y = C^2[w(t, y) - \theta_0']. \end{aligned}$$

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If $(\widetilde{R}, \widetilde{S}, t)(\tau, y) \in \widetilde{\Sigma}(\widetilde{\delta}_1)$, we have by (3.13)–(3.16) and (3.26)

$$|A|, |B| \le \frac{2}{\underline{\theta}}, \quad \frac{1}{2\overline{\theta}} \le C \le \frac{2}{\underline{\theta}}, \quad |A_{\widetilde{S}}|, |B_{\widetilde{R}}|, |C_{\widetilde{R}}|, |C_{\widetilde{S}}| \le \left(\frac{2}{\underline{\theta}}\right)^2, \tag{3.27}$$
$$|A_t|, |B_t|, |C_t| \le \left(\frac{2}{\underline{\theta}}\right)^2 \widehat{M}_0, \ |A_y|, |B_y|, |C_y| \le \left(\frac{2}{\underline{\theta}}\right)^2 \left(|\theta_0'| + \frac{2}{\underline{\theta}}\widehat{M}_0 \tau\right).$$

Denote $\overline{\theta}_0 = \max\{\max_{z\in\mathbb{R}} |\theta_0'(z)|, \max_{z\in\mathbb{R}} |\theta_0''(z)|, \max_{z\in\mathbb{R}} |\theta_0'''(z)|\}$ and

$$\overline{K} = \max\left\{1, \frac{2}{\underline{\theta}}, \frac{4}{\underline{\theta}^2}, \frac{8}{\underline{\theta}^3}, \overline{\theta}_1, \frac{2\overline{\theta}_0}{\underline{\theta}}, \frac{4\overline{\theta}_0}{\underline{\theta}^2}, \frac{8\overline{\theta}_0^2}{\underline{\theta}^2}\right\}, \tag{3.28}$$

which along with (3.27) give

$$\begin{gathered} |A|, |B|, |A\theta_0'|, |B\theta_0'|, |A_{\widetilde{S}}|, |B_{\widetilde{R}}|, |C_{\widetilde{R}}|, |C_{\widetilde{S}}| \le \overline{K}, \\ |A_t|, |B_t|, |C_t| \le \overline{K}\widehat{M}_0, \quad |Aw|, |Bw| \le \overline{K}\widehat{M}_0 \tau, \\ |A_y|, |B_y|, |C_y|, |(A\theta_0')_y|, |(B\theta_0')_y| \le \overline{K}(1 + \widehat{M}_0 \tau). \end{gathered} \tag{3.29}$$

We now choose $\widetilde{M}$, $\overline{M}$, and $\widetilde{\delta}$ satisfying

$$\widetilde{M} = 3\overline{M}, \quad \overline{M} = 16\overline{K}, \quad \widetilde{\delta} = \min\left\{\frac{1}{16\overline{M}}, \frac{1}{16\overline{K}\widehat{M}_0}, \frac{1}{\widehat{M}_0^2}\right\} \le \widetilde{\delta}_1, \tag{3.30}$$

such that there hold

$$\overline{M}\widetilde{\delta} \le \frac{1}{16}, \quad \overline{K}\widehat{M}_0\widetilde{\delta} \le \frac{1}{16}, \quad \widehat{M}_0\sqrt{\widetilde{\delta}} \le 1, \quad e^{\overline{M}\widetilde{\delta}^2} \le \exp\left\{\frac{1}{64}\right\} \le \frac{16}{15}. \tag{3.31}$$

Hence, if $(\widetilde{R}, \widetilde{S}, t)(\tau, y) \in \widetilde{\Sigma}(\widetilde{\delta})$, one obtains

$$\begin{gathered} \frac{1}{2\overline{\theta}} \le C \le \frac{2}{\underline{\theta}}, \quad \widetilde{\delta}|A_t|, \widetilde{\delta}|B_t|, \widetilde{\delta}|C_t| \le \frac{1}{16}, \quad |Aw|, |Bw| \le \overline{K} \le \frac{\overline{M}}{16}, \\ |A|, |B|, |A\theta_0'|, |B\theta_0'|, |A_{\widetilde{S}}|, |B_{\widetilde{R}}|, |C_{\widetilde{R}}|, |C_{\widetilde{S}}| \le \overline{K} \le \frac{\overline{M}}{16}, \\ |A_y|, |B_y|, |C_y|, |(A\theta_0')_y|, |(B\theta_0')_y| \le 2\overline{K} \le \frac{\overline{M}}{8}. \end{gathered} \tag{3.32}$$

Thanks to (3.31) and (3.32), we have the following lemma.

Lemma 3.1 *Let the sequences $(\widetilde{R}^{(k)}, \widetilde{S}^{(k)}, t^{(k)})$ be defined in (3.25). For all $k \ge 1$, the following inequalities*

$$\left|\widetilde{R}^{(k)}(\xi,\eta)\right|, \left|\widetilde{S}^{(k)}(\xi,\eta)\right| \le \overline{M}\xi^2 \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j, \quad \frac{\xi}{2\overline{\theta}} \le t^{(k)}(\xi,\eta) \le \frac{2\xi}{\underline{\theta}}, \tag{3.33}$$
$$\left|\widetilde{R}^{(k)}(\xi,\eta) - \widetilde{S}^{(k)}(\xi,\eta)\right| \le \overline{M}\xi^2 \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j$$

hold in $[0, \widetilde{\delta}] \times \mathbb{R}$.

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Proof We show this lemma by the standard argument of induction. That is, we first check that each inequality in (3.33) is true for $n = 1$, then assume that they all hold for $n = k$ and establish (3.33) for $n = k + 1$.

Obviously, the functions $(\widetilde{R}^{(0)}, \widetilde{S}^{(0)}, t^{(0)})(\xi, \eta)$ defined in (3.21) are in $\widetilde{\Sigma}(\widetilde{\delta})$, then we find by (3.32) that

$$\frac{1}{2\overline{\theta}} \le C^{(0)} \le \frac{2}{\underline{\theta}}, \quad |I^{(0)}|, |I^{(0)}\theta_0'|, |I^{(0)}w(t^{(0)}, y)| \le \frac{\overline{M}}{16}, \tag{3.34}$$

for $I = A, B$. It concludes by (3.23) and (3.34) that

$$\begin{aligned} \left|\widetilde{R}^{(1)}(\xi,\eta)\right| &\le \int_0^{\xi} \left\{ |A^{(0)}\theta_0'|\tau + |A^{(0)}w(t^{(0)}, y)|\tau \right\} \mathrm{d}\tau \\ &\le \int_0^{\xi} \frac{\overline{M}}{8}\tau \,\mathrm{d}\tau = \frac{\overline{M}}{16}\xi^2 \le \overline{M}\xi^2 \sum_{j=0}^{1} \left(\frac{2}{3}\right)^j. \end{aligned} \tag{3.35}$$

The estimate (3.35) is also true for $\widetilde{S}^{(1)}(\xi, \eta)$. For the variable $t^{(1)}(\xi, \eta)$, one arrives at

$$\frac{1}{2\overline{\theta}}\xi \le t^{(1)}(\xi,\eta) = \int_0^{\xi} C^{(0)}(\tau, y_0^{(0)}(\tau))\,\mathrm{d}\tau \le \frac{2}{\underline{\theta}}\xi. \tag{3.36}$$

Furthermore, applying (3.23) and (3.34) again acquires

$$\begin{aligned} &\left|\widetilde{R}^{(1)}(\xi,\eta) - \widetilde{S}^{(1)}(\xi,\eta)\right| \\ &\le \int_0^{\xi} \left\{ |A^{(0)}\theta_0'|\tau + |A^{(0)}w(t^{(0)}, y)|\tau + |B^{(0)}\theta_0'|\tau + |B^{(0)}w(t^{(0)}, y)|\tau \right\} \mathrm{d}\tau \\ &\le \int_0^{\xi} \frac{\overline{M}}{4}\tau \,\mathrm{d}\tau = \frac{\overline{M}}{8}\xi^2 \le \overline{M}\xi^2 \sum_{j=0}^{1} \left(\frac{2}{3}\right)^j. \end{aligned} \tag{3.37}$$

We combine (3.35)–(3.37) to achieve (3.33) for $n = 1$. In addition, the functions $(\widetilde{R}^{(1)}, \widetilde{S}^{(1)}, t^{(1)})(\xi, \eta)$ satisfy

$$\left|\widetilde{R}^{(1)}(\xi,\eta)\right|, \left|\widetilde{S}^{(1)}(\xi,\eta)\right| \le 3\overline{M}\xi^2 = \widetilde{M}\xi^2, \quad \frac{1}{2\overline{\theta}}\xi \le t^{(1)}(\xi,\eta) \le \frac{2}{\underline{\theta}}\xi, \tag{3.38}$$

which means that $(\widetilde{R}^{(1)}, \widetilde{S}^{(1)}, t^{(1)})(\xi, \eta) \in \widetilde{\Sigma}(\widetilde{\delta})$.

Suppose that all inequalities in (3.33) hold for $n = k$. Thus,

$$\left|\widetilde{R}^{(k)}(\xi,\eta)\right|, \left|\widetilde{S}^{(k)}(\xi,\eta)\right| \le \overline{M}\xi^2 \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j \le 3\overline{M}\xi^2 = \widetilde{M}\xi^2, \tag{3.39}$$

from which we see that $(\widetilde{R}^{(k)}, \widetilde{S}^{(k)}, t^{(k)})(\xi, \eta) \in \widetilde{\Sigma}(\widetilde{\delta})$. Therefore, one has by (3.32)

$$\frac{1}{2\overline{\theta}} \le C^{(k)} \le \frac{2}{\underline{\theta}}, \quad |I^{(k)}|, |I^{(k)}\theta_0'|, |I^{(k)}w(t^{(k)}, y)| \le \frac{\overline{M}}{16}, \tag{3.40}$$

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for $I = A, B$. In view of (3.25) and (3.40), we employ the induction assumptions to obtain

$$\begin{aligned}\left|\widetilde{R}^{(k+1)}(\xi,\eta)\right| &\le \int_0^\xi \left\{\frac{1}{2}\overline{M}\tau\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j + \frac{\overline{M}}{16}\cdot\frac{1}{4}\left(\overline{M}\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j\right)^2\tau^3 \right.\\ &\qquad \left. + \frac{\overline{M}}{16}\cdot\overline{M}\tau^2\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j + \frac{\overline{M}}{16}\tau + \frac{\overline{M}}{16}\tau\right\} d\tau \\ &\le \overline{M}\xi^2\left\{\frac{1}{16} + \left(\frac{1}{4} + \frac{3(\overline{M}\widetilde{\delta})^2}{256} + \frac{\overline{M}\widetilde{\delta}}{48}\right)\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j\right\} \\ &\le \overline{M}\xi^2\left\{1 + \frac{2}{3}\cdot\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j\right\} = \overline{M}\xi^2\sum_{j=0}^{k+1}\left(\frac{2}{3}\right)^j. \end{aligned} \tag{3.41}$$

The above estimate (3.41) also holds for $\widetilde{S}^{(k+1)}(\xi,\eta)$. Moreover, it is easy to find by (3.40) that

$$\frac{1}{2\overline{\theta}}\xi \le t^{(k+1)}(\xi,\eta) = \int_0^\xi C^{(k)}(\tau, y_0^{(k)}(\tau))\, d\tau \le \frac{2}{\underline{\theta}}\xi. \tag{3.42}$$

For the term $|\widetilde{R}^{(k+1)}(\xi,\eta) - \widetilde{S}^{(k+1)}(\xi,\eta)|$, we proceed by using (3.25) and (3.40) again

$$\begin{aligned}&\left|\widetilde{R}^{(k+1)}(\xi,\eta) - \widetilde{S}^{(k+1)}(\xi,\eta)\right| \\ &\le \int_0^\xi \left\{\overline{M}\tau\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j + \frac{\overline{M}}{16}\cdot\frac{1}{2}\left(\overline{M}\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j\right)^2\tau^3 \right.\\ &\qquad \left. + \frac{\overline{M}}{16}\cdot\overline{M}\tau^2\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j + \frac{\overline{M}}{4}\tau\right\} d\tau \\ &\le \overline{M}\xi^2\left\{\frac{1}{8} + \left(\frac{1}{2} + \frac{3}{128} + \frac{1}{48}\right)\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j\right\} \le \overline{M}\xi^2\sum_{j=0}^{k+1}\left(\frac{2}{3}\right)^j. \end{aligned} \tag{3.43}$$

Combining (3.41)–(3.43) ends the proof of the lemma. ∎

By virtue of Lemma 3.1, the functions $(\widetilde{R}^{(k)}, \widetilde{S}^{(k)}, t^{(k)})(\xi,\eta)$ are in the space $\widetilde{\Sigma}(\widetilde{\delta})$ for each $k \ge 0$. Thus, the estimates in (3.32) are true for the functions $(\widetilde{R}^{(k)}, \widetilde{S}^{(k)}, t^{(k)})$ $(k \ge 0)$. To derive the uniform convergence of the iterative sequences $(\widetilde{R}^{(k)}, \widetilde{S}^{(k)}, t^{(k)})$, one needs the properties of the sequences $(\widetilde{R}_\eta^{(k)}, \widetilde{S}_\eta^{(k)}, t_\eta^{(k)})(\xi,\eta)$. We differentiate system (3.25) with respect to η to calculate

$$\begin{aligned}\widetilde{R}_\eta^{(k+1)}(\xi,\eta) &= \int_0^\xi \left\{\frac{\widetilde{R}_y^{(k)} - \widetilde{S}_y^{(k)}}{2\tau} + A^{(k)}\cdot\frac{(\widetilde{R}^{(k)} - \widetilde{S}^{(k)})(\widetilde{R}_y^{(k)} - \widetilde{S}_y^{(k)})}{2\tau}\right.\\ &\quad - A^{(k)}(\widetilde{R}_y^{(k)} - \widetilde{S}_y^{(k)}) + A_{11}^{(k)}A_{\widetilde{S}}^{(k)}\widetilde{S}_y^{(k)} + [A_{11}^{(k)}A_t^{(k)} - A^{(k)}w_t\tau]t_y^{(k)} \\ &\quad \left. + A_{11}^{(k)}A_y^{(k)} + A^{(k)}(\theta_0'' - w_y)\tau\right\}\frac{\partial y_-^{(k)}(\tau)}{\partial\eta}(\tau, y_-^{(k)}(\tau))\, d\tau, \end{aligned} \tag{EQ.a}$$

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where

$$
(3.44)\qquad \begin{aligned}
A_{11}^{(k)} &= \frac{(\widetilde{R}^{(k)}-\widetilde{S}^{(k)})^2}{4\tau} - (\widetilde{R}^{(k)}-\widetilde{S}^{(k)}) + \theta_0'\tau - w(t^{(k)},y)\tau,\\
A_{\widetilde{S}}^{(k)} &= -(A^{(k)})^2, \quad A_t^{(k)} = (A^{(k)})^2\widetilde{v}_t(t^{(k)},y),\\
A_y^{(k)} &= -(A^{(k)})^2[\theta_0' - w(t^{(k)},y)],
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{S}_\eta^{(k+1)}(\xi,\eta) = \int_0^\xi \Bigg\{ &\frac{\widetilde{S}_y^{(k)} - \widetilde{R}_y^{(k)}}{2\tau} + B^{(k)}\cdot\frac{(\widetilde{R}^{(k)}-\widetilde{S}^{(k)})(\widetilde{R}_y^{(k)}-\widetilde{S}_y^{(k)})}{2\tau}\\
&+ B^{(k)}(\widetilde{R}_y^{(k)}-\widetilde{S}_y^{(k)}) + B_{11}^{(k)}B_{\widetilde{R}}^{(k)}\widetilde{R}_y^{(k)} + [B_{11}^{(k)}A_t^{(k)} + B^{(k)}w_t\tau]t_y^{(k)}\\
&+ B_{11}^{(k)}B_y^{(k)} - B^{(k)}(\theta_0'' - w_y)\tau\Bigg\}\frac{\partial y_+^{(k)}(\tau)}{\partial\eta}(\tau, y_+^{(k)}(\tau))\,d\tau,
\end{aligned}\qquad \text{(EQ.b)}
$$

where

$$
(3.45)\qquad \begin{aligned}
B_{11}^{(k)} &= \frac{(\widetilde{R}^{(k)}-\widetilde{S}^{(k)})^2}{4\tau} + (\widetilde{R}^{(k)}-\widetilde{S}^{(k)}) - \theta_0'\tau + w(t^{(k)},y)\tau,\\
B_{\widetilde{R}}^{(k)} &= -(B^{(k)})^2, \quad B_t^{(k)} = (B^{(k)})^2\widetilde{v}_t(t^{(k)},y),\\
B_y^{(k)} &= -(B^{(k)})^2[\theta_0' - w(t^{(k)},y)],
\end{aligned}
$$

and

$$
\text{(EQ.c)}\qquad t_\eta^{(k+1)}(\xi,\eta) = \int_0^\xi \left\{ C_{\widetilde{R}}^{(k)}\widetilde{R}_y^{(k)} + C_{\widetilde{S}}^{(k)}\widetilde{S}_y^{(k)} + C_t^{(k)}t_y^{(k)} + C_y^{(k)} \right\}(\tau,\eta)\,d\tau,
$$

where

$$
(3.46)\qquad \begin{aligned}
&C_{\widetilde{R}}^{(k)} = C_{\widetilde{S}}^{(k)} = -\frac{1}{2}(C^{(k)})^2,\\
&C_t^{(k)} = (C^{(k)})^2\widetilde{v}_t(t^{(k)},y), \quad C_y^{(k)} = (C^{(k)})^2[w(t^{(k)},y) - \theta_0'].
\end{aligned}
$$

The functions $\frac{\partial y_\pm^{(k)}(\tau)}{\partial\eta}$ in (EQ.a) and (EQ.b) are

$$
(3.47)\qquad \frac{\partial y_\pm^{(k)}}{\partial\eta}(\tau) = \exp\left\{ \int_\xi^\tau \frac{\partial\lambda_\pm^{(k)}}{\partial y}(s, y_\pm^{(k)}(s))\,ds \right\},
$$

where

$$
(3.48)\qquad \begin{aligned}
\frac{\partial\lambda_-^{(k)}}{\partial y} &= -s\big[A_{\widetilde{S}}^{(k)}\widetilde{S}_y^{(k)} + A_t^{(k)}t_y^{(k)} + A_y^{(k)}\big],\\
\frac{\partial\lambda_+^{(k)}}{\partial y} &= s\big[B_{\widetilde{R}}^{(k)}\widetilde{R}_y^{(k)} + B_t^{(k)}t_y^{(k)} + B_y^{(k)}\big].
\end{aligned}
$$

Due to $(\widetilde{R}^{(k)}, \widetilde{S}^{(k)}, t^{(k)}) \in \widetilde{\Sigma}(\widetilde{\delta})$, for $k \geq 1$, it suggests by Lemma 3.1 and (3.27), (3.30), and (3.32) that

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$$
\begin{aligned}
|A_{11}^{(k)}|, |B_{11}^{(k)}| &\le \frac{9}{4}\overline{M}^2\tau^3 + \frac{3}{2}\overline{M}\tau^2 + (|\theta_0'| + |w|)\tau \\
&\le 4\overline{M}\tau^2 + \left(|\theta_0'| + \frac{2}{\underline{\theta}}\widehat{M}_0\widetilde{\delta}\right)\tau,
\end{aligned} \tag{3.49}
$$

and then

$$
\begin{aligned}
&|A_{11}^{(k)}A_{\widetilde{S}}^{(k)}| \le 4\overline{M}\tau^2 \cdot \frac{\overline{M}}{16} + \left(\frac{2}{\underline{\theta}}\right)^2 \cdot \left(|\theta_0'| + \frac{2}{\underline{\theta}}\widehat{M}_0\widetilde{\delta}\right)\tau \le \frac{\overline{M}^2}{4}\tau^2 + \frac{\overline{M}}{8}\tau, \\
&|A_{11}^{(k)}A_t^{(k)} - A^{(k)}w_t\tau| \le \left\{4\overline{M}\tau^2 + \left(|\theta_0'| + \frac{2}{\underline{\theta}}\widehat{M}_0\widetilde{\delta}\right)\tau\right\} \cdot \left(\frac{2}{\underline{\theta}}\right)^2 \widehat{M}_0 + \frac{2}{\underline{\theta}}\widehat{M}_0\tau \\
&\le \widehat{M}_0\tau\left\{\frac{2}{\underline{\theta}} + \frac{16\overline{M}\tau}{\underline{\theta}^2} + \left(\frac{2}{\underline{\theta}}\right)^2\left(|\theta_0'| + \frac{2}{\underline{\theta}}\widehat{M}_0\widetilde{\delta}\right)\right\} \le 4\overline{K}\widehat{M}_0\widetilde{\delta} \le \frac{1}{4}, \\
&|A_{11}^{(k)}A_y^{(k)}| \le \left\{4\overline{M}\tau^2 + \left(|\theta_0'| + \frac{2}{\underline{\theta}}\widehat{M}_0\widetilde{\delta}\right)\tau\right\}\left(\frac{2}{\underline{\theta}}\right)^2\left(|\theta_0'| + \frac{2}{\underline{\theta}}\widehat{M}_0\widetilde{\delta}\right) \\
&\le 4\overline{M}\tau^2 \cdot \frac{\overline{M}}{8} + \frac{\overline{M}}{8}\tau \le \frac{\overline{M}}{4}\tau, \\
&|A^{(k)}(\theta_0'' - w_y)| \le \frac{2}{\underline{\theta}}\left(|\theta_0''| + \widehat{M}_0\sqrt{\widetilde{\delta}}\right) \le \frac{\overline{M}}{8}.
\end{aligned} \tag{3.50}
$$

We obtain analogously

$$
\begin{aligned}
&|B_{11}^{(k)}B_{\widetilde{R}}^{(k)}| \le \frac{\overline{M}^2}{4}\tau^2 + \frac{\overline{M}}{8}\tau, \quad |B_{11}^{(k)}B_t^{(k)} + B^{(k)}w_t\tau| \le \frac{1}{4}, \\
&|B_{11}^{(k)}B_y^{(k)}| \le \frac{\overline{M}}{4}\tau, \qquad\qquad |B^{(k)}(\theta_0'' - w_y)| \le \frac{\overline{M}}{8}.
\end{aligned} \tag{3.51}
$$

Furthermore, there also hold

$$
|C_{\widetilde{R}}^{(k)}|, |C_{\widetilde{S}}^{(k)}| \le \frac{\overline{M}}{16}, \quad |C_t^{(k)}| \le \overline{K}\widehat{M}_0, \quad |C_y^{(k)}| \le \frac{\overline{M}}{8}, \tag{3.52}
$$

and

$$
\left|\frac{\partial\lambda_{\pm}^{(k)}}{\partial y}\right| \le s\left\{\frac{\overline{M}}{16}(|\widetilde{R}_y^{(k)}| + |\widetilde{S}_y^{(k)}|) + \overline{K}\widehat{M}_0|t_y^{(k)}| + \frac{\overline{M}}{8}\right\}. \tag{3.53}
$$

For the sequences $(\widetilde{R}_\eta^{(k)}, \widetilde{S}_\eta^{(k)}, t_\eta^{(k)})(\xi,\eta)$, we have the following lemma.

Lemma 3.2 *For all $k \ge 1$, the following inequalities*

$$
\begin{aligned}
&\left|\widetilde{R}_\eta^{(k)}(\xi,\eta)\right|, \left|\widetilde{S}_\eta^{(k)}(\xi,\eta)\right| \le \overline{M}\xi^2\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j, \ \left|t_\eta^{(k)}(\xi,\eta)\right| \le \overline{M}\xi\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j, \\
&\left|\widetilde{R}_\eta^{(k)}(\xi,\eta) - \widetilde{S}_\eta^{(k)}(\xi,\eta)\right| \le \overline{M}\xi^2\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j
\end{aligned} \tag{3.54}
$$

hold in $[0,\widetilde{\delta}] \times \mathbb{R}$.

Proof The proof is also based on the standard argument of induction. According to

$$(\widetilde{R}^{(0)}, \widetilde{S}^{(0)}, t^{(0)})(\xi,\eta) \in \widetilde{\Sigma}(\widetilde{\delta}),$$

we get by (3.53) and (3.47)

$$\left|\frac{\partial \lambda_{\pm}^{(0)}}{\partial y}\right| \le s\cdot\frac{\overline{M}}{8}, \quad \left|\frac{\partial y_{\pm}^{(0)}}{\partial \eta}(\tau)\right| \le \exp\left\{\int_0^{\xi}\frac{\overline{M}}{8}s\,\mathrm{d}s\right\} \le \exp\left(\frac{\overline{M}\xi^2}{16}\right). \tag{3.55}$$

One combines (EQ.a), (3.50), and (3.55) to acquire

$$\begin{aligned} |\widetilde{R}_{\eta}^{(1)}(\xi,\eta)| &\le \int_0^{\xi}\left\{|A_{11}^{(0)}A_{y}^{(0)}| + |A^{(0)}(\theta_0''-w_y)|\tau\right\}\left|\frac{\partial y_{-}^{(0)}(\tau)}{\partial\eta}\right|\mathrm{d}\tau \\ &\le \int_0^{\xi}\left\{\frac{\overline{M}}{4}\tau+\frac{\overline{M}}{8}\tau\right\}\exp\left(\frac{\overline{M}\xi^2}{16}\right)\mathrm{d}\tau \le \overline{M}\xi^2\cdot\frac{3}{16}\exp\left(\frac{\overline{M}\widetilde{\delta}^2}{16}\right) \\ &\le \overline{M}\xi^2\sum_{j=0}^{1}\left(\frac{2}{3}\right)^j. \end{aligned} \tag{3.56}$$

The estimate in (3.56) is also valid for the function $\widetilde{S}_{\eta}^{(1)}(\xi,\eta)$. For the function $t_{\eta}^{(1)}(\xi,\eta)$, it concludes by (EQ.c) and (3.52) that

$$\left|t_{\eta}^{(1)}(\xi,\eta)\right| \le \int_0^{\xi}|C_y^{(0)}|(\tau,\eta)\,\mathrm{d}\tau \le \int_0^{\xi}\frac{\overline{M}}{8}\,\mathrm{d}\tau \le \overline{M}\xi\sum_{j=0}^{1}\left(\frac{2}{3}\right)^j. \tag{3.57}$$

Moreover, for the term $|\widetilde{R}_{\eta}^{(1)}(\xi,\eta)-\widetilde{S}_{\eta}^{(1)}(\xi,\eta)|$, it is easy to check that

$$\left|\widetilde{R}_{\eta}^{(1)}(\xi,\eta)-\widetilde{S}_{\eta}^{(1)}(\xi,\eta)\right| \le \overline{M}\xi^2\cdot\frac{3}{8}\exp\left(\frac{\overline{M}\widetilde{\delta}^2}{16}\right) \le \overline{M}\xi^2\sum_{j=0}^{1}\left(\frac{2}{3}\right)^j, \tag{3.58}$$

which along with (3.56) and (3.57) give (3.54) for $n=1$.

Let all the inequalities in (3.54) hold for $n=k$. Then

$$|\widetilde{R}_y^{(k)}|, |\widetilde{S}_y^{(k)}| \le 3\overline{M}\xi^2, \quad |t_y^{(k)}| \le 3\overline{M}\xi. \tag{3.59}$$

Combining (3.53), (3.47), and (3.59) yields

$$\left|\frac{\partial\lambda_{\pm}^{(k)}}{\partial y}\right| \le s\left\{\frac{\overline{M}}{16}\cdot 6\overline{M}s^2 + \overline{K}\widehat{M}_0\cdot 3\overline{M}s + \frac{\overline{M}}{8}\right\} \tag{3.60}$$

and then

$$\begin{aligned} \left|\frac{\partial y_{\pm}^{(k)}}{\partial\eta}(\tau)\right| &\le \exp\left\{\int_0^{\xi}s\left(\frac{\overline{M}}{16}\cdot 6\overline{M}s^2+\overline{K}\widehat{M}_0\cdot 3\overline{M}s+\frac{\overline{M}}{8}\right)\mathrm{d}s\right\} \\ &\le \exp\left\{\overline{M}\widetilde{\delta}^2\left(\frac{\overline{M}\widetilde{\delta}^2}{8}+\overline{K}\widehat{M}_0\widetilde{\delta}+\frac{1}{16}\right)\right\} \le e^{\overline{M}\widetilde{\delta}^2}. \end{aligned} \tag{3.61}$$

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Utilizing the induction assumptions, we have by (EQ.a), (3.50), and (3.61)

$$
\begin{aligned}
\left|\widetilde{R}_\eta^{(k+1)}(\xi,\eta)\right| \le & \int_0^\xi \left\{\frac{1}{2}\overline{M}\tau\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j + \frac{\overline{M}}{16}\cdot\frac{3\overline{M}\tau}{2}\cdot\overline{M}\tau^2\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j\right.\\
&+\frac{\overline{M}}{16}\cdot\overline{M}\tau^2\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j + \left(\frac{\overline{M}^2\tau^2}{4}+\frac{\overline{M}\tau}{8}\right)\cdot\overline{M}\tau^2\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j\\
&\left.+\frac{1}{4}\cdot\overline{M}\tau\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j+\frac{\overline{M}}{4}\tau+\frac{\overline{M}}{8}\tau\right\}e^{\overline{M}\widetilde{\delta}^2}\,\mathrm{d}\tau\\
\le\; & \overline{M}\xi^2\left\{\frac{3}{16}+\left(\frac{1}{4}+\frac{(\overline{M}\widetilde{\delta})^2}{32}+\frac{\overline{M}\widetilde{\delta}}{32}+\left(\frac{(\overline{M}\widetilde{\delta})^2}{4}+\frac{\overline{M}\widetilde{\delta}}{8}\right)\cdot\frac{\widetilde{\delta}}{3}+\frac{1}{8}\right)\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j\right\}e^{\overline{M}\widetilde{\delta}^2}\\
\le\; & \overline{M}\xi^2\left\{\frac{15}{32}+\left[\frac{1}{4}+\frac{(\overline{M}\widetilde{\delta})^2}{32}+\frac{\overline{M}\widetilde{\delta}}{32}+\left(\frac{(\overline{M}\widetilde{\delta})^2}{4}+\frac{\overline{M}\widetilde{\delta}}{8}\right)\cdot\frac{\widetilde{\delta}}{3}+\frac{1}{32}\right]\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j\right\}e^{\overline{M}\widetilde{\delta}^2}\\
\le\; & \overline{M}\xi^2\left\{\frac{15}{32}+\left[\frac{9}{32}+\frac{1}{32\times16}+\frac{1}{32\times4}+\left(\frac{1}{4\times16}+\frac{1}{8\times4}\right)\cdot\frac{1}{3}\right]\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j\right\}e^{\overline{M}\widetilde{\delta}^2}
\end{aligned}
$$

(3.62)

$$
\le \overline{M}\xi^2\left\{\frac{15}{32}e^{\overline{M}\widetilde{\delta}^2}+\left(\frac{5}{16}e^{\overline{M}\widetilde{\delta}^2}\right)\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j\right\}\le\overline{M}\xi^2\sum_{j=0}^{k+1}\left(\frac{2}{3}\right)^j.
$$

The above estimate also holds for the function $\widetilde{S}_\eta^{(k+1)}(\xi,\eta)$. For the term $|\widetilde{R}_\eta^{(k+1)}(\xi,\eta)-\widetilde{S}_\eta^{(k+1)}(\xi,\eta)|$, we can perform the process as in (3.62) to obtain

$$
\left|\widetilde{R}_\eta^{(k+1)}(\xi,\eta)-\widetilde{S}_\eta^{(k+1)}(\xi,\eta)\right|\le\overline{M}\xi^2\left\{\frac{15}{16}e^{\overline{M}\widetilde{\delta}^2}+\frac{5}{8}e^{\overline{M}\widetilde{\delta}^2}\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j\right\},\tag{3.63}
$$

which, together with the fact $15e^{\overline{M}\widetilde{\delta}^2}\le 16$ by (3.31), leads to

$$
\left|\widetilde{R}_\eta^{(k+1)}(\xi,\eta)-\widetilde{S}_\eta^{(k+1)}(\xi,\eta)\right|\le\overline{M}\xi^2\sum_{j=0}^{k+1}\left(\frac{2}{3}\right)^j.\tag{3.64}
$$

For the function $t_\eta^{(k+1)}(\xi,\eta)$, it follows by (EQ.c), (3.52), and the induction assumptions that

$$
\begin{aligned}
\left|t_\eta^{(k+1)}(\xi,\eta)\right|\le&\int_0^\xi\left\{\frac{\overline{M}}{16}\cdot\overline{M}\tau^2\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j\times2+\overline{K}\widehat{M}_0\cdot\overline{M}\tau\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j+\frac{\overline{M}}{8}\right\}\mathrm{d}\tau\\
&=\overline{M}\xi\left\{\frac{1}{8}+\left(\frac{\overline{M}\widetilde{\delta}^2}{24}+\frac{\overline{K}\widehat{M}_0\widetilde{\delta}}{2}\right)\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j\right\}
\end{aligned}
$$

$$
\le\overline{M}\xi\left\{1+\left(\frac{1}{24}+\frac{1}{2}\right)\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j\right\}\le\overline{M}\xi\sum_{j=0}^{k+1}\left(\frac{2}{3}\right)^j.\tag{3.65}
$$

We combine (3.62), (3.64), and (3.65) to complete the proof of the lemma. ■

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By means of Lemmas 3.1 and 3.2, one achieves the following lemma.

Lemma 3.3 *For all $k \geq 0$, the following inequalities*

$$\begin{aligned}
\left|\widetilde{R}^{(k+1)}(\xi,\eta) - \widetilde{R}^{(k)}(\xi,\eta)\right| &\leq \overline{M}\xi^2\left(\frac{2}{3}\right)^k,\\
\left|\widetilde{S}^{(k+1)}(\xi,\eta) - \widetilde{S}^{(k)}(\xi,\eta)\right| &\leq \overline{M}\xi^2\left(\frac{2}{3}\right)^k,\\
\left|t^{(k+1)}(\xi,\eta) - t^{(k)}(\xi,\eta)\right| &\leq \overline{M}\xi\left(\frac{2}{3}\right)^k
\end{aligned} \tag{3.66}$$

hold in $[0,\widetilde{\delta}] \times \mathbb{R}$.

Proof We use again the argument of induction to verify the lemma. It is obvious to see by the functions $(\widetilde{R}^{(0)}, \widetilde{S}^{(0)}, t^{(0)})(\xi,\eta)$ in (3.21) and (3.33) that the inequalities in (3.66) are valid for $n = 1$. Suppose that each inequality in (3.66) holds for $n \leq k-1$. We shall show that they are true for $n = k$.

We first estimate the term $|t^{(k+1)}(\xi,\eta) - t^{(k)}(\xi,\eta)|$. In view of (3.25) and (3.32), we apply the induction assumptions and the fact $y_0^k(\tau) \equiv \eta$ to find that

$$\begin{aligned}
&\left|t^{(k+1)}(\xi,\eta) - t^{(k)}(\xi,\eta)\right| \leq \int_0^\xi \left|C^{(k)}(\tau,\eta) - C^{(k-1)}(\tau,\eta)\right| \mathrm{d}\tau\\
&\leq \int_0^\xi \Big\{|C_{\widetilde{R}}| \cdot \left|\widetilde{R}^{(k)}(\tau,\eta) - \widetilde{R}^{(k-1)}(\tau,\eta)\right| + |C_{\widetilde{S}}| \cdot \left|\widetilde{S}^{(k)}(\tau,\eta) - \widetilde{S}^{(k-1)}(\tau,\eta)\right|\\
&\qquad + |C_t| \cdot \left|t^{(k)}(\tau,\eta) - t^{(k-1)}(\tau,\eta)\right|\Big\} \mathrm{d}\tau\\
&\leq \int_0^\xi \left\{\frac{\overline{M}}{16} \cdot \overline{M}\tau^2\left(\frac{2}{3}\right)^{k-1} \times 2 + \frac{1}{16\widetilde{\delta}} \cdot \overline{M}\tau\left(\frac{2}{3}\right)^{k-1}\right\} \mathrm{d}\tau\\
&\leq \left(\frac{\overline{M}\widetilde{\delta}^2}{24} + \frac{1}{32}\right) \cdot \overline{M}\xi\left(\frac{2}{3}\right)^{k-1} \leq \overline{M}\xi\left(\frac{2}{3}\right)^k.
\end{aligned} \tag{3.67}$$

To derive the difference between $\widetilde{R}^{(k+1)}(\xi,\eta)$ and $\widetilde{R}^{(k)}(\xi,\eta)$, we need to first obtain the estimate of $|y_-^{(k)}(\tau) - y_-^{(k-1)}(\tau)|$. By virtue of (3.24), it suggests that for $\tau \in [0,\xi]$,

$$\begin{aligned}
&y_-^{(k)}(\tau) - \int_\tau^\xi \frac{s}{f^{(k)} + \widetilde{S}^{(k)}}(s, y_-^{(k)}(s))\, \mathrm{d}s\\
&= \eta = y_-^{(k-1)}(\tau) - \int_\tau^\xi \frac{s}{f^{(k-1)} + \widetilde{S}^{(k-1)}}(s, y_-^{(k-1)}(s))\, \mathrm{d}s,
\end{aligned} \tag{3.68}$$

from which we see by Lemma 3.1 and (3.15) that

$$\begin{aligned}
\left|y_-^{(k)}(\tau) - y_-^{(k-1)}(\tau)\right| &\leq \int_\tau^\xi \frac{T_1^{(k)} + T_2^{(k)} + T_3^{(k)}}{|f^{(k)} + \widetilde{S}^{(k)}| \cdot |f^{(k-1)} + \widetilde{S}^{(k-1)}|} \cdot s\, \mathrm{d}s\\
&\leq \frac{4}{\underline{\theta}^2} \int_0^\xi (T_1^{(k)} + T_2^{(k)} + T_3^{(k)}) s\, \mathrm{d}s,
\end{aligned} \tag{3.69}$$

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where

$$
\begin{aligned}
T_1^{(k)} &= \left|\theta_0(y_-^{(k)}(s)) - \theta_0(y_-^{(k-1)}(s))\right|,\\
T_2^{(k)} &= \left|\widetilde{S}^{(k)}(s, y_-^{(k)}(s)) - \widetilde{S}^{(k-1)}(s, y_-^{(k-1)}(s))\right|,\\
T_3^{(k)} &= \left|\widetilde{v}\big(t^{(k)}(s, y_-^{(k)}(s)), y_-^{(k)}(s)\big) - \widetilde{v}\big(t^{(k-1)}(s, y_-^{(k-1)}(s)), y_-^{(k-1)}(s)\big)\right|.
\end{aligned}
\tag{3.70}
$$

For the term $T_1^{(k)}$, one easily finds by the mean value theorem that

$$
T_1^{(k)} \le \max|\theta_0'| \cdot \left|y_-^{(k)}(s) - y_-^{(k-1)}(s)\right| \le \overline{\theta}_0 \left|y_-^{(k)}(s) - y_-^{(k-1)}(s)\right|. \tag{3.71}
$$

For the term $T_2^{(k)}$, we have by Lemma 3.2 and the induction assumptions

$$
\begin{aligned}
T_2^{(k)} &\le \left|\widetilde{S}^{(k)}(s, y_-^{(k)}(s)) - \widetilde{S}^{(k)}(s, y_-^{(k-1)}(s))\right|\\
&\quad + \left|\widetilde{S}^{(k)}(s, y_-^{(k-1)}(s)) - \widetilde{S}^{(k-1)}(s, y_-^{(k-1)}(s))\right|\\
&\le 3\overline{M}s^2\left|y_-^{(k)}(s) - y_-^{(k-1)}(s)\right| + \overline{M}s^2\left(\frac{2}{3}\right)^{k-1}.
\end{aligned}
\tag{3.72}
$$

Similarly, it concludes that for the term $T_3^{(k)}$,

$$
\begin{aligned}
T_3^{(k)} &\le \left|\widetilde{v}\big(t^{(k)}(s, y_-^{(k)}(s)), y_-^{(k)}(s)\big) - \widetilde{v}\big(t^{(k)}(s, y_-^{(k)}(s)), y_-^{(k-1)}(s)\big)\right|\\
&\quad + \left|\widetilde{v}\big(t^{(k)}(s, y_-^{(k)}(s)), y_-^{(k-1)}(s)\big) - \widetilde{v}\big(t^{(k-1)}(s, y_-^{(k-1)}(s)), y_-^{(k-1)}(s)\big)\right|\\
&\le \widehat{M}_0 t^{(k)}(s, y_-^{(k)}(s)) \cdot \left|y_-^{(k)}(s) - y_-^{(k-1)}(s)\right|\\
&\quad + \widehat{M}_0\{\left|t^{(k)}(s, y_-^{(k)}(s)) - t^{(k)}(s, y_-^{(k-1)}(s))\right|\\
&\quad + \left|t^{(k)}(s, y_-^{(k-1)}(s)) - t^{(k-1)}(s, y_-^{(k-1)}(s))\right|\}\\
&\le \widehat{M}_0 \frac{2s}{\underline{\theta}} \cdot \left|y_-^{(k)}(s) - y_-^{(k-1)}(s)\right|\\
&\quad + \widehat{M}_0\left\{3\overline{M}s \cdot \left|y_-^{(k)}(s) - y_-^{(k-1)}(s)\right| + \overline{M}s\left(\frac{2}{3}\right)^{k-1}\right\}\\
&= \left(\frac{2\widehat{M}_0 s}{\underline{\theta}} + 3\widehat{M}_0\overline{M}s\right)\left|y_-^{(k)}(s) - y_-^{(k-1)}(s)\right| + \widehat{M}_0\overline{M}s\left(\frac{2}{3}\right)^{k-1}.
\end{aligned}
\tag{3.73}
$$

Putting (3.71)–(3.73) into (3.69) yields for $\tau \in [0, \xi]$,

$$
\begin{aligned}
&\left|y_-^{(k)}(\tau) - y_-^{(k-1)}(\tau)\right|\\
&\le \frac{4}{\underline{\theta}^2}\int_0^\xi \left\{\left(\overline{\theta}_0 + 3\overline{M}s^2 + \frac{2\widehat{M}_0 s}{\underline{\theta}} + 3\widehat{M}_0\overline{M}s\right)\left|y_-^{(k)}(s) - y_-^{(k-1)}(s)\right|\right.\\
&\qquad\left. + \left(\overline{M}s^2 + \widehat{M}_0\overline{M}s\right)\left(\frac{2}{3}\right)^{k-1}\right\} \cdot s\,\mathrm{d}s\\
&\le \int_0^\xi \left\{\left(\overline{K}\widetilde{\delta} + 3\overline{KM}\widetilde{\delta}^3 + \overline{K}\widehat{M}_0\widetilde{\delta}^2 + 3\overline{K}\widehat{M}_0\overline{M}\widetilde{\delta}^2\right)\left|y_-^{(k)}(s) - y_-^{(k-1)}(s)\right|\right.
\end{aligned}
$$

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$$+ 2\overline{K}\widehat{M}_0\overline{M}s^2\left(\frac{2}{3}\right)^{k-1}\Big\} \,\mathrm{d}s$$

$$\leq \int_0^{\xi} \frac{1}{4}\left|y_-^{(k)}(s) - y_-^{(k-1)}(s)\right| \mathrm{d}s + +\frac{2}{3}\overline{K}\widehat{M}_0\overline{M}\xi^3\left(\frac{2}{3}\right)^{k-1}. \tag{3.74}$$

Denote

$$d_-^{(k)} = \max_{\tau\in[0,\xi]}\left|y_-^{(k)}(\tau) - y_-^{(k-1)}(\tau)\right|.$$

Then we obtain by (3.74)

$$d_-^{(k)} \leq \frac{1}{4}d_-^{(k)} + \frac{2}{3}\overline{K}\widehat{M}_0\overline{M}\xi^3\left(\frac{2}{3}\right)^{k-1},$$

which indicates that

$$d_-^{(k)} \leq \overline{K}\widehat{M}_0\overline{M}\xi^3\left(\frac{2}{3}\right)^{k-1}. \tag{3.75}$$

Analogously, one gets

$$\max_{\tau\in[0,\xi]}\left|y_+^{(k)}(\tau) - y_+^{(k-1)}(\tau)\right| \leq \overline{K}\widehat{M}_0\overline{M}\xi^3\left(\frac{2}{3}\right)^{k-1}. \tag{3.76}$$

We now estimate the difference between $\widetilde{R}^{(k+1)}(\xi,\eta)$ and $\widetilde{R}^{(k)}(\xi,\eta)$. Applying (3.25) gives

$$\begin{aligned}
&\left|\widetilde{R}^{(k+1)}(\xi,\eta) - \widetilde{R}^{(k)}(\xi,\eta)\right| \\
&\leq \int_0^{\xi}\left\{T_4^{(k)} + T_5^{(k)} + T_6^{(k)} + (T_7^{(k)} + T_8^{(k)})\tau\right\}\mathrm{d}\tau,
\end{aligned} \tag{3.77}$$

where

$$\begin{aligned}
T_4^{(k)} &= \left|\frac{\widetilde{R}^{(k)} - \widetilde{S}^{(k)}}{2\tau}(\tau, y_-^{(k)}(\tau)) - \frac{\widetilde{R}^{(k-1)} - \widetilde{S}^{(k-1)}}{2\tau}(\tau, y_-^{(k-1)}(\tau))\right|, \\
T_5^{(k)} &= \Bigg|A^{(k)}\frac{(\widetilde{R}^{(k)} - \widetilde{S}^{(k)})^2}{4\tau}(\tau, y_-^{(k)}(\tau)) \\
&\qquad -A^{(k-1)}\frac{(\widetilde{R}^{(k-1)} - \widetilde{S}^{(k-1)})^2}{4\tau}(\tau, y_-^{(k-1)}(\tau))\Bigg|, \\
T_6^{(k)} &= \Big|A^{(k)}(\widetilde{R}^{(k)} - \widetilde{S}^{(k)})(\tau, y_-^{(k)}(\tau)) \\
&\qquad -A^{(k-1)}(\widetilde{R}^{(k-1)} - \widetilde{S}^{(k-1)})(\tau, y_-^{(k-1)}(\tau))\Big|, \\
T_7^{(k)} &= \left|A^{(k)}\theta_0'(\tau, y_-^{(k)}(\tau)) - A^{(k-1)}\theta_0'(\tau, y_-^{(k-1)}(\tau))\right|, \\
T_8^{(k)} &= \left|A^{(k)}w(t^{(k)}, y)(\tau, y_-^{(k)}(\tau)) - A^{(k-1)}w(t^{(k-1)}, y)(\tau, y_-^{(k-1)}(\tau))\right|.
\end{aligned} \tag{3.78}$$

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Thanks to Lemma 3.2 and (3.75), one employs the induction assumptions to arrive at

$$
\begin{aligned}
T_4^{(k)} &\leq \frac{|\widetilde{R}^{(k)}(\tau, y_-^{(k)}(\tau)) - \widetilde{R}^{(k-1)}(\tau, y_-^{(k-1)}(\tau))|}{2\tau} \\
&\quad + \frac{|\widetilde{S}^{(k)}(\tau, y_-^{(k)}(\tau)) - \widetilde{S}^{(k-1)}(\tau, y_-^{(k-1)}(\tau))|}{2\tau} \\
&\leq 3\overline{M}\tau d_-^{(k)} + \overline{M}\tau\left(\frac{2}{3}\right)^{k-1} \\
&\leq \left(1 + 3\overline{K}\widehat{M}_0\overline{M}\widetilde{\delta}^3\right)\overline{M}\tau\left(\frac{2}{3}\right)^{k-1} \leq \frac{97}{96}\overline{M}\tau\left(\frac{2}{3}\right)^{k-1}.
\end{aligned}
\tag{3.79}
$$

Recalling the expressions of $A^{(k)}$ in (3.25) and $T_{1,2,3}^{(k)}$ in (3.70) achieves

$$
\begin{aligned}
&\left|A^{(k)}(\tau, y_-^{(k)}(\tau)) - A^{(k-1)}(\tau, y_-^{(k-1)}(\tau))\right| \leq \frac{4}{\underline{\theta}^2}(T_1^{(k)} + T_2^{(k)} + T_3^{(k)}) \\
&\leq \frac{4}{\underline{\theta}^2}\left\{\left(\overline{\theta}_0 + 3\overline{M}\xi^2 + \frac{2\widehat{M}_0\xi}{\underline{\theta}} + 3\widehat{M}_0\overline{M}\xi\right)d_-^{(k)} + \left(\overline{M}\xi^2 + \widehat{M}_0\overline{M}\xi\right)\left(\frac{2}{3}\right)^{k-1}\right\} \\
&\leq \frac{4}{\underline{\theta}^2}\left\{\left(\overline{\theta}_0 + 3\overline{M}\widetilde{\delta}^2 + \frac{2\widehat{M}_0\widetilde{\delta}}{\underline{\theta}} + 3\widehat{M}_0\overline{M}\widetilde{\delta}\right)\cdot\overline{K}\widehat{M}_0\widetilde{\delta}^2 + \widetilde{\delta} + \widehat{M}_0\right\}\overline{M}\xi\left(\frac{2}{3}\right)^{k-1} \\
&\leq \frac{4}{\underline{\theta}^2}(1 + \widehat{M}_0)\overline{M}\xi\left(\frac{2}{3}\right)^{k-1} \leq 2\overline{K}\widehat{M}_0\overline{M}\xi\left(\frac{2}{3}\right)^{k-1},
\end{aligned}
\tag{3.80}
$$

by the choice of $\widetilde{\delta}$ in (3.31). Therefore, we can use Lemmas 3.1 and 3.2, the induction assumptions, (3.75), and (3.80) to estimate the terms $T_{5,6,7,8}^{(k)}$. In detail, for the term $T_5^{(k)}$, we have

$$
\begin{aligned}
T_5^{(k)} &\leq \left|A^{(k)} - A^{(k-1)}\right|\cdot\frac{(\widetilde{R}^{(k)} - \widetilde{S}^{(k)})^2}{4\tau} + |A^{(k-1)}| \\
&\quad \times \frac{\left|(\widetilde{R}^{(k)} - \widetilde{S}^{(k)})^2(\tau, y_-^{(k)}(\tau)) - (\widetilde{R}^{(k-1)} - \widetilde{S}^{(k-1)})^2(\tau, y_-^{(k-1)}(\tau))\right|}{4\tau} \\
&\leq 2\overline{K}\widehat{M}_0\overline{M}\xi\left(\frac{2}{3}\right)^{k-1}\cdot\frac{(3\overline{M}\tau)^2}{4\tau} + \frac{\overline{M}}{16}\cdot\frac{1}{4\tau}\cdot 6\overline{M}\tau^2\cdot 2\tau T_4^{(k)} \\
&\leq 5\overline{K}\widehat{M}_0\overline{M}^3\widetilde{\delta}\tau^3\left(\frac{2}{3}\right)^{k-1} + \frac{3}{16}(1 + 3\overline{K}\widehat{M}_0\overline{M}\widetilde{\delta}^3)\overline{M}^3\tau^3\left(\frac{2}{3}\right)^{k-1} \\
&\leq \left(5\overline{K}\widehat{M}_0\widetilde{\delta} + \frac{3}{16} + \frac{9}{16}\overline{K}\widehat{M}_0\overline{M}\widetilde{\delta}^3\right)\overline{M}^3\tau^3\left(\frac{2}{3}\right)^{k-1} \leq \overline{M}^3\tau^3\left(\frac{2}{3}\right)^{k-1}.
\end{aligned}
\tag{3.81}
$$

For the term $T_6^{(k)}$, one obtains

$$
\begin{aligned}
T_6^{(k)} &\le \left|A^{(k)} - A^{(k-1)}\right| \cdot \left|\widetilde{R}^{(k)} - \widetilde{S}^{(k)}\right| + \left|A^{(k-1)}\right| \cdot 2\tau T_4^{(k)} \\
&\le 2\overline{K}\widehat{M}_0\overline{M}\xi\left(\frac{2}{3}\right)^{k-1} \cdot 3\overline{M}\tau^2 + \frac{\overline{M}^2}{8} \cdot \tau^2(1 + 3\overline{K}\widehat{M}_0\overline{M}\widetilde{\delta}^3)\left(\frac{2}{3}\right)^{k-1} \\
&\le \left(6\overline{K}\widehat{M}_0\widetilde{\delta} + \frac{1}{8} + \frac{3}{16}\overline{K}\widehat{M}_0\overline{M}\widetilde{\delta}^3\right)\overline{M}^2\tau^2\left(\frac{2}{3}\right)^{k-1} \le \overline{M}^2\tau^2\left(\frac{2}{3}\right)^{k-1}.
\end{aligned} \tag{3.82}
$$

For the terms $T_7^{(k)}$ and $T_8^{(k)}$, we also acquire

$$
\begin{aligned}
T_7^{(k)} &\le \left|A^{(k)} - A^{(k-1)}\right| \cdot \left|\theta_0'\right| + \left|A^{(k-1)}\right| \cdot \left|\theta_0'(y_-^{(k)}(\tau)) - \theta_0'(y_-^{(k-1)}(\tau))\right| \\
&\le \frac{4}{\underline{\theta}^2}(1 + \widehat{M}_0)\overline{M}\widetilde{\delta}\left(\frac{2}{3}\right)^{k-1} \cdot \overline{\theta}_0 + \frac{\overline{M}}{16} \cdot \overline{\theta}_0 \cdot d_-^{(k)} \\
&\le 2\overline{K}\widehat{M}_0\overline{M}\widetilde{\delta}\left(\frac{2}{3}\right)^{k-1} + \frac{\overline{M}}{16} \cdot \overline{\theta}_0 \cdot \overline{K}\widehat{M}_0\overline{M}\widetilde{\delta}^3\left(\frac{2}{3}\right)^{k-1} \\
&= \left(2\overline{K}\widehat{M}_0\widetilde{\delta} + \frac{(\overline{M}\widetilde{\delta}) \cdot (\overline{\theta}_0\widetilde{\delta})}{16} \cdot (\overline{K}\widehat{M}_0\widetilde{\delta})\right)\overline{M}\left(\frac{2}{3}\right)^{k-1} \le \frac{13}{96}\overline{M}\left(\frac{2}{3}\right)^{k-1}
\end{aligned} \tag{3.83}
$$

and

$$
\begin{aligned}
T_8^{(k)} &\le \left|A^{(k)} - A^{(k-1)}\right| \cdot \left|w(t^{(k)}, y_-^{(k)}(\tau))\right| \\
&\qquad + \left|A^{(k-1)}\right| \cdot \left|w(t^{(k)}, y_-^{(k)}(\tau)) - w(t^{(k-1)}, y_-^{(k-1)}(\tau))\right| \\
&\le 2\overline{K}\widehat{M}_0\overline{M}\widetilde{\delta}\left(\frac{2}{3}\right)^{k-1} \cdot \widehat{M}_0 t^{(k)}(\tau, y_-^{(k)}(\tau)) \\
&\quad + \overline{K} \cdot \left\{\widehat{M}_0\left|t^{(k)}(\tau, y_-^{(k)}(\tau)) - t^{(k-1)}(\tau, y_-^{(k-1)}(\tau))\right| + \widehat{M}_0 d_-^{(k)}\right\} \\
&\le 2\overline{K}\widehat{M}_0\overline{M}\widetilde{\delta}\left(\frac{2}{3}\right)^{k-1} \cdot \widehat{M}_0 \cdot \frac{2\tau}{\underline{\theta}} + \overline{K}\widehat{M}_0\left\{(|t_y^{(k)}| + 1)d_-^{(k)} + \overline{M}\tau\left(\frac{2}{3}\right)^{k-1}\right\} \\
&\le \left\{2(\overline{K}\widehat{M}_0\widetilde{\delta})^2 + 2(\overline{K}\widehat{M}_0\widetilde{\delta})^2\widetilde{\delta} + \overline{K}\widehat{M}_0\widetilde{\delta}\right\}\overline{M}\left(\frac{2}{3}\right)^{k-1} \le \frac{7}{96}\overline{M}\left(\frac{2}{3}\right)^{k-1}.
\end{aligned} \tag{3.84}
$$

Here, the facts $\widehat{M}_0 \ge 1$, $16\overline{M}\widetilde{\delta} \le 1$, and $16\overline{K}\overline{M}_0\widetilde{\delta} \le 1$ are used in the above estimation processes.

We now insert (3.79) and (3.81)–(3.84) into (3.77) to gain

$$
\begin{aligned}
\left|\widetilde{R}^{(k+1)}(\xi, \eta) - \widetilde{R}^{(k)}(\xi, \eta)\right| &\le \int_0^\xi \left\{\frac{97}{96}\overline{M}\tau\left(\frac{2}{3}\right)^{k-1}\right. \\
&\quad \left. + \overline{M}^3\tau^3\left(\frac{2}{3}\right)^{k-1} + \overline{M}^2\tau^2\left(\frac{2}{3}\right)^{k-1} + \frac{5}{24}\overline{M}\tau\left(\frac{2}{3}\right)^{k-1}\right\} \mathrm{d}\tau \\
&\le \left\{\frac{97}{192} + \frac{1}{4}(\overline{M}\widetilde{\delta})^2 + \frac{1}{3}(\overline{M}\widetilde{\delta}) + \frac{5}{48}\right\}\overline{M}\xi^2\left(\frac{2}{3}\right)^{k-1}
\end{aligned}
$$

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$$\leq \left\{\frac{97}{192}+\frac{1}{4\times 16^2}+\frac{1}{48}+\frac{5}{48}\right\}\overline{M}\xi^2\left(\frac{2}{3}\right)^{k-1}$$

$$<\frac{122}{192}\overline{M}\xi^2\left(\frac{2}{3}\right)^{k-1}<\overline{M}\xi^2\left(\frac{2}{3}\right)^{k}. \tag{3.85}$$

It is easily checked that the estimate (3.85) is also valid for the term $|\widetilde{S}^{(k+1)}(\xi,\eta)-\widetilde{S}^{(k)}(\xi,\eta)|$. The proof of the lemma is finished. ■

3.4 The existence and uniqueness of solutions

According to Lemma 3.3, it is known that the sequences $(\widetilde{R}^{(k)},\widetilde{S}^{(k)},t^{(k)})(\tau,y)$ are uniformly convergent. We denote the limit functions by $(\widetilde{R},\widetilde{S},t)(\tau,y)$ which are obviously continuous. Furthermore, by means of Lemma 3.1, the functions $(\widetilde{R},\widetilde{S},t)(\tau,y)$ satisfy

$$\begin{aligned}&\left|\widetilde{R}(\tau,y)\right|,\left|\widetilde{S}(\tau,y)\right|\leq 3\overline{M}\tau^2=\widetilde{M}\tau^2,\quad \frac{\tau}{2\overline{\theta}}\leq t(\tau,y)\leq\frac{2\tau}{\underline{\theta}},\\&\left|\widetilde{R}(\tau,y)-\widetilde{S}(\tau,y)\right|\leq 3\overline{M}\tau^2=\widetilde{M}\tau^2,\end{aligned} \tag{3.86}$$

for any $(\tau,y)\in[0,\widetilde{\delta}]\times\mathbb{R}$. In addition, it is clear that the functions $(\widetilde{R},\widetilde{S},t)(\tau,y)$ satisfy the integral system (3.19) and the boundary conditions

$$\widetilde{R}(0,y)=\widetilde{S}(0,y)=t(0,y)=0. \tag{3.87}$$

Thus, we have $(\widetilde{R},\widetilde{S},t)\in\widetilde{\Sigma}(\widetilde{\delta})$.

Next, we show that the functions $(\widetilde{R},\widetilde{S},t)(\tau,y)$ possess first-order continuous derivatives with respect to y. To move forward, one differentiates (3.19) with respect to η to deduce the following linear system of integral equations:

$$\begin{aligned}\widetilde{R}_\eta(\xi,\eta)=\int_0^\xi\Bigg\{&\frac{\widetilde{R}_y-\widetilde{S}_y}{2\tau}+A\cdot\frac{(\widetilde{R}-\widetilde{S})(\widetilde{R}_y-\widetilde{S}_y)}{2\tau}\\&-A(\widetilde{R}_y-\widetilde{S}_y)+A_{11}A_{\widetilde{S}}\widetilde{S}_y+[A_{11}A_t-Aw_t\tau]t_y\\&+A_{11}A_y+A(\theta_0''-w_y)\tau\Bigg\}\frac{\partial y_-(\tau)}{\partial\eta}(\tau,y_-(\tau))\,\mathrm{d}\tau,\end{aligned} \tag{EQ.a1}$$

$$\begin{aligned}\widetilde{S}_\eta(\xi,\eta)=\int_0^\xi\Bigg\{&\frac{\widetilde{S}_y-\widetilde{R}_y}{2\tau}+B\cdot\frac{(\widetilde{R}-\widetilde{S})(\widetilde{R}_y-\widetilde{S}_y)}{2\tau}\\&+B(\widetilde{R}_y-\widetilde{S}_y)+B_{11}B_{\widetilde{R}}\widetilde{R}_y+[B_{11}A_t+Bw_t\tau]t_y\\&+B_{11}B_y-B(\theta_0''-w_y)\tau\Bigg\}\frac{\partial y_+(\tau)}{\partial\eta}(\tau,y_+(\tau))\,\mathrm{d}\tau,\end{aligned} \tag{EQ.b1}$$

$$t_\eta(\xi,\eta)=\int_0^\xi\left\{C_{\widetilde{R}}\widetilde{R}_y+C_{\widetilde{S}}\widetilde{S}_y+C_t t_y+C_y\right\}(\tau,\eta)\,\mathrm{d}\tau. \tag{EQ.c1}$$

Here, the coefficient functions in (EQ.a1)–(EQ.c1) are given in (EQ.a)–(EQ.c) but with the limit functions $(\widetilde{R}, \widetilde{S}, t)$ replacing $(\widetilde{R}^{(k)}, \widetilde{S}^{(k)}, t^{(k)})$. Set

$$(\widetilde{R}_\eta^{(0)}, \widetilde{S}_\eta^{(0)}, t_\eta^{(0)})(\xi, \eta) = \mathbf{0}, \tag{3.88}$$

and construct the iterative sequences $(\widetilde{R}_\eta^{(k)}, \widetilde{S}_\eta^{(k)}, t_\eta^{(k)})(\xi, \eta)$ by the integral system (EQ.a1)–(EQ.c1) as follows:

$$\begin{aligned}\widetilde{R}_\eta^{(k+1)}(\xi, \eta) = \int_0^\xi \Bigg\{ & \frac{\widetilde{R}_y^{(k)} - \widetilde{S}_y^{(k)}}{2\tau} + A\frac{(\widetilde{R} - \widetilde{S})(\widetilde{R}_y^{(k)} - \widetilde{S}_y^{(k)})}{2\tau} \\ & - A(\widetilde{R}_y^{(k)} - \widetilde{S}_y^{(k)}) + A_{11}A_{\widetilde{S}}\widetilde{S}_y^{(k)} + [A_{11}A_t - Aw_t\tau]t_y^{(k)} \\ & + A_{11}A_y + A(\theta_0'' - w_y)\tau \Bigg\} \frac{\partial y_-^{(k)}(\tau)}{\partial \eta}(\tau, y_-(\tau))\, \mathrm{d}\tau,\end{aligned} \tag{EQ.a2}$$

$$\begin{aligned}\widetilde{S}_\eta^{(k+1)}(\xi, \eta) = \int_0^\xi \Bigg\{ & \frac{\widetilde{S}_y^{(k)} - \widetilde{R}_y^{(k)}}{2\tau} + B\frac{(\widetilde{R} - \widetilde{S})(\widetilde{R}_y^{(k)} - \widetilde{S}_y^{(k)})}{2\tau} \\ & + B(\widetilde{R}_y^{(k)} - \widetilde{S}_y^{(k)}) + B_{11}B_{\widetilde{R}}\widetilde{R}_y^{(k)} + [B_{11}A_t + Bw_t\tau]t_y^{(k)} \\ & + B_{11}B_y - B(\theta_0'' - w_y)\tau \Bigg\} \frac{\partial y_+^{(k)}(\tau)}{\partial \eta}(\tau, y_+(\tau))\, \mathrm{d}\tau,\end{aligned} \tag{EQ.b2}$$

$$t_\eta^{(k+1)}(\xi, \eta) = \int_0^\xi \Big\{ C_{\widetilde{R}}\widetilde{R}_y^{(k)} + C_{\widetilde{S}}\widetilde{S}_y^{(k)} + C_t t_y^{(k)} + C_y \Big\}(\tau, \eta)\, \mathrm{d}\tau, \tag{EQ.c2}$$

where

$$\frac{\partial y_\pm^{(k)}}{\partial \eta}(\tau) = \exp\left\{ \int_\xi^\tau \frac{\partial \lambda_\pm^{(k)}}{\partial y}(s, y_\pm(s))\, \mathrm{d}s \right\}, \tag{3.89}$$

and

$$\begin{aligned}\frac{\partial \lambda_-^{(k)}}{\partial y} &= -s\big[A_{\widetilde{S}}\widetilde{S}_y^{(k)} + A_t t_y^{(k)} + A_y\big], \\ \frac{\partial \lambda_+^{(k)}}{\partial y} &= s\big[B_{\widetilde{R}}\widetilde{R}_y^{(k)} + B_t t_y^{(k)} + B_y\big].\end{aligned} \tag{3.90}$$

Since the limit functions $(\widetilde{R}, \widetilde{S}, t)(\tau, y)$ are in the space $\widetilde{\Sigma}(\widetilde{\delta})$, we see that the coefficients of system (EQ.a2) and (EQ.c2) still satisfy the estimates in (3.50)–(3.53). Then we have the following.

Lemma 3.4 *The sequences $(\widetilde{R}_\eta^{(k)}, \widetilde{S}_\eta^{(k)}, t_\eta^{(k)})$ defined by system (EQ.a2) and (EQ.c2) satisfy the following inequalities:*

$$\begin{aligned}&\big|\widetilde{R}_\eta^{(k)}(\xi, \eta)\big|, \big|\widetilde{S}_\eta^{(k)}(\xi, \eta)\big| \le \overline{M}\xi^2 \sum_{j=0}^k \left(\frac{2}{3}\right)^j, \ \big|t_\eta^{(k)}(\xi, \eta)\big| \le \overline{M}\xi \sum_{j=0}^k \left(\frac{2}{3}\right)^j, \\ &\big|\widetilde{R}_\eta^{(k)}(\xi, \eta) - \widetilde{S}_\eta^{(k)}(\xi, \eta)\big| \le \overline{M}\xi^2 \sum_{j=0}^k \left(\frac{2}{3}\right)^j,\end{aligned} \tag{3.91}$$

for all $k \ge 1$ and any $(\xi, \eta) \in [0, \widetilde{\delta}] \times \mathbb{R}$.

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Proof We omit the proof here since the process is exactly the same as that of Lemma 3.2. ■

It concludes by (3.91) that

$$|\widetilde{R}_\eta^{(k)}(\xi,\eta)|, |\widetilde{S}_\eta^{(k)}(\xi,\eta)| \le 3\overline{M}\xi^2, \quad |t_\eta^{(k)}(\xi,\eta)| \le 3\overline{M}\xi, \tag{3.92}$$

which together with (3.89) and (3.90) and the estimates in (3.50)–(3.53) give

$$\begin{aligned} &\left|\frac{\partial\lambda_\pm^{(k)}}{\partial y}\right| \le s\left(\frac{\overline{M}}{16}\cdot 3\overline{M}s^2 + \frac{1}{16\widetilde{\delta}}\cdot 3\overline{M}s + \frac{\overline{M}}{8}\right) \le \overline{M}s, \\ &\left|\frac{\partial y_\pm^{(k)}}{\partial\eta}\right| \le \exp\left\{\int_0^\xi \left|\frac{\partial\lambda_\pm^{(k)}}{\partial y}(s, y_\pm(s))\right| \mathrm{d}s\right\} \le e^{\overline{M}\xi^2}, \end{aligned} \tag{3.93}$$

and then

$$\begin{aligned} \left|\frac{\partial y_\pm^{(k)}}{\partial\eta}(\tau) - \frac{\partial y_\pm^{(k-1)}}{\partial\eta}(\tau)\right| &\le e^{\overline{M}\xi^2}\cdot\int_0^\xi \left|\frac{\partial\lambda_\pm^{(k)}}{\partial y} - \frac{\partial\lambda_\pm^{(k-1)}}{\partial y}\right| \mathrm{d}s \\ &\le e^{\overline{M}\widetilde{\delta}^2}\int_0^\xi s\left\{\frac{\overline{M}}{16}T_9^{(k)} + \frac{1}{16\widetilde{\delta}}|t_y^{(k)} - t_y^{(k-1)}|\right\}\mathrm{d}s, \end{aligned} \tag{3.94}$$

where

$$T_9^{(k)} = |\widetilde{R}_y^{(k)} - \widetilde{R}_y^{(k-1)}| + |\widetilde{S}_y^{(k)} - \widetilde{S}_y^{(k-1)}|.$$

By utilizing Lemma 3.4 and (3.93) and (3.94), we have the following lemma.

Lemma 3.5 *Let the functions $(\widetilde{R}_\eta^{(k)}, \widetilde{S}_\eta^{(k)}, t_\eta^{(k)})$ be defined by the iterative system (EQ.a2) and (EQ.c2). Then, for all $k \ge 0$, the following inequalities*

$$\begin{aligned} |\widetilde{R}_\eta^{(k+1)}(\xi,\eta) - \widetilde{R}_\eta^{(k)}(\xi,\eta)| &\le \overline{M}\xi^2\left(\frac{2}{3}\right)^k, \\ |\widetilde{S}_\eta^{(k+1)}(\xi,\eta) - \widetilde{S}_\eta^{(k)}(\xi,\eta)| &\le \overline{M}\xi^2\left(\frac{2}{3}\right)^k, \\ |t_\eta^{(k+1)}(\xi,\eta) - t_\eta^{(k)}(\xi,\eta)| &\le \overline{M}\xi\left(\frac{2}{3}\right)^k \end{aligned} \tag{3.95}$$

hold in $[0,\widetilde{\delta}]\times\mathbb{R}$.

Proof The proof of the lemma is still based on the inductive method. It is obvious by Lemma 3.4 that all inequalities in (3.95) are true for $k = 0$. We assume that each inequality in (3.95) holds for $n = k - 1$ and then verify all of them valid for $n = k$.

For the term $|t_\eta^{(k+1)}(\xi,\eta) - t_\eta^{(k)}(\xi,\eta)|$, one obtains by (EQ.c2), (3.52), and the induction assumptions

$$\begin{aligned} |t_\eta^{(k+1)}(\xi,\eta) - t_\eta^{(k)}(\xi,\eta)| \le \int_0^\xi \Big\{&|C_{\widetilde{R}}|\cdot|\widetilde{R}_y^{(k)} - \widetilde{R}_y^{(k-1)}| \\ &+ |C_{\widetilde{S}}|\cdot|\widetilde{S}_y^{(k)} - \widetilde{S}_y^{(k-1)}| + |C_t|\cdot|t_y^{(k)} - t_y^{(k-1)}|\Big\}\mathrm{d}\tau \end{aligned}$$

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$$\leq \int_0^\xi \left\{ \frac{\overline{M}}{16}\cdot\overline{M}\tau^2\left(\frac{2}{3}\right)^{k-1} + \frac{\overline{M}}{16}\cdot\overline{M}\tau^2\left(\frac{2}{3}\right)^{k-1} + \overline{K}\widehat{M}_0\cdot\overline{M}\tau\left(\frac{2}{3}\right)^{k-1}\right\} d\tau$$

$$\leq \left(\frac{\overline{M}\widetilde{\delta}^2}{24} + \frac{\overline{K}\widehat{M}_0\widetilde{\delta}}{2}\right)\overline{M}\xi\left(\frac{2}{3}\right)^{k-1} \leq \overline{M}\xi\left(\frac{2}{3}\right)^{k}. \tag{3.96}$$

For the term $|\widetilde{R}_\eta^{(k+1)}(\xi,\eta) - \widetilde{R}_\eta^{(k)}(\xi,\eta)|$, one gets by (EQ.a2)

$$\begin{aligned}&\left|\widetilde{R}_\eta^{(k+1)}(\xi,\eta) - \widetilde{R}_\eta^{(k)}(\xi,\eta)\right| \\ &\leq \int_0^\xi \left\{ T_{10}^{(k)}\left|\frac{\partial y_-^{(k)}(\tau)}{\partial\eta}\right| + T_{11}^{(k)}\left|\frac{\partial y_-^{(k)}(\tau)}{\partial\eta} - \frac{\partial y_-^{(k-1)}(\tau)}{\partial\eta}\right|\right\} d\tau,\end{aligned} \tag{3.97}$$

where

$$\begin{aligned}T_{10}^{(k)} &= \frac{T_9^{(k)}}{2\tau} + |A|\frac{|\widetilde{R}-\widetilde{S}|\cdot T_9^{(k)}}{2\tau} + |A|T_9^{(k)} \\ &\quad + |A_{11}A_{\widetilde{S}}|\cdot|\widetilde{S}_y^{(k)} - \widetilde{S}_y^{(k-1)}| + |A_{11}A_t - Aw_t\tau|\cdot|t_y^{(k)} - t_y^{(k-1)}|,\end{aligned} \tag{3.98}$$

$$\begin{aligned}T_{11}^{(k)} &= \frac{|\widetilde{R}_y^{(k-1)} - \widetilde{S}_y^{(k-1)}|}{2\tau} + |A|\frac{|\widetilde{R}-\widetilde{S}|\cdot|\widetilde{R}_y^{(k-1)} - \widetilde{S}_y^{(k-1)}|}{2\tau} \\ &\quad + |A|\cdot|\widetilde{R}_y^{(k-1)} - \widetilde{S}_y^{(k-1)}| + |A_{11}A_{\widetilde{S}}|\cdot|\widetilde{S}_y^{(k-1)}| \\ &\quad + |A_{11}A_t - Aw_t\tau|\cdot|t_y^{(k-1)}| + |A_{11}A_y| + |A(\theta_0'' - w_y)|\tau.\end{aligned} \tag{3.99}$$

Recalling the estimates in (3.50) yields

$$\begin{aligned}&|A_{11}A_{\widetilde{S}}| \leq \frac{\overline{M}^2}{4}\tau^2 + \frac{\overline{M}}{8}\tau, \quad |A_{11}A_t - Aw_t\tau| \leq \frac{1}{4}, \\ &|A_{11}A_y| \leq \frac{\overline{M}}{4}\tau, \qquad\qquad |A(\theta_0'' - w_y)| \leq \frac{\overline{M}}{8},\end{aligned} \tag{3.100}$$

which, together with (3.86), (3.91), (3.98)–(3.99), and the induction assumptions, arrive at

$$\begin{aligned}T_{10}^{(k)} &\leq \frac{1}{2\tau}\cdot 2\overline{M}\tau^2\left(\frac{2}{3}\right)^{k-1} + \frac{\overline{M}}{16}\cdot\frac{3\overline{M}\tau^2}{2\tau}\cdot 2\overline{M}\tau^2\left(\frac{2}{3}\right)^{k-1} + \frac{\overline{M}^2}{8}\tau^2\left(\frac{2}{3}\right)^{k-1} \\ &\quad + \left(\frac{\overline{M}^2}{4}\tau^2 + \frac{\overline{M}}{8}\tau\right)\cdot\overline{M}\tau^2\left(\frac{2}{3}\right)^{k-1} + \frac{1}{4}\cdot\overline{M}\tau\left(\frac{2}{3}\right)^{k-1} \\ &\leq \left(\frac{5}{4} + \frac{1}{2}(\overline{M}\widetilde{\delta})^2 + \frac{1}{4}(\overline{M}\widetilde{\delta})\right)\overline{M}\tau\left(\frac{2}{3}\right)^{k-1} < \frac{41}{32}\overline{M}\tau\left(\frac{2}{3}\right)^{k-1}\end{aligned} \tag{3.101}$$

and

$$T_{11}^{(k)} \leq \frac{3\overline{M}\tau^2}{2\tau} + \frac{\overline{M}}{16}\cdot\frac{3\overline{M}\tau^2\cdot 3\overline{M}\tau^2}{2\tau} + \frac{\overline{M}}{16}\cdot 3\overline{M}\tau^2$$

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$$+\left(\frac{\overline{M}^2}{4}\tau^2+\frac{\overline{M}}{8}\tau\right)\cdot 3\overline{M}\tau^2+\frac{1}{4}\cdot 3\overline{M}\tau+\frac{\overline{M}}{4}\tau+\frac{\overline{M}}{8}\tau$$

$$\le\left(\frac{21}{8}+\frac{17}{32}(\overline{M}\widetilde{\delta})^2+\frac{7}{32}(\overline{M}\widetilde{\delta})\right)\overline{M}\tau\le 3\overline{M}\tau. \tag{3.102}$$

Moreover, it suggests by (3.94) and the induction assumptions that

$$\left|\frac{\partial y_-^{(k)}(\tau)}{\partial\eta}-\frac{\partial y_-^{(k-1)}(\tau)}{\partial\eta}\right|$$
$$\le e^{\overline{M}\widetilde{\delta}^2}\int_0^\xi s\left\{\frac{\overline{M}}{16}\cdot 2\overline{M}s^2\left(\frac{2}{3}\right)^{k-1}+\frac{1}{16\widetilde{\delta}}\cdot\overline{M}s\left(\frac{2}{3}\right)^{k-1}\right\}\mathrm{d}s$$
$$\le e^{\overline{M}\widetilde{\delta}^2}\overline{M}\xi^2\left(\frac{2}{3}\right)^{k-1}. \tag{3.103}$$

Putting (3.101)–(3.103) into (3.97) and using (3.93) leads to

$$\left|\widetilde{R}_\eta^{(k+1)}(\xi,\eta)-\widetilde{R}_\eta^{(k)}(\xi,\eta)\right|$$
$$\le\int_0^\xi\left\{\frac{41}{32}\overline{M}\tau\left(\frac{2}{3}\right)^{k-1}\cdot e^{\overline{M}\widetilde{\delta}^2}+3\overline{M}\tau\cdot e^{\overline{M}\widetilde{\delta}^2}\overline{M}\xi^2\left(\frac{2}{3}\right)^{k-1}\right\}\mathrm{d}\tau$$
$$\le\left(\frac{41}{64}+\frac{3}{2}\overline{M}\widetilde{\delta}^2\right)e^{\overline{M}\widetilde{\delta}^2}\overline{M}\xi^2\left(\frac{2}{3}\right)^{k-1}$$
$$\le\frac{21}{32}e^{1/64}\overline{M}\xi^2\left(\frac{2}{3}\right)^{k-1}<\overline{M}\xi^2\left(\frac{2}{3}\right)^{k}, \tag{3.104}$$

by the fact $e^{1/64}<64/63$. The estimate of the term $|\widetilde{S}_\eta^{(k+1)}(\xi,\eta)-\widetilde{S}_\eta^{(k)}(\xi,\eta)|$ can be derived similar to (3.104). The proof of the lemma is complete. ■

According to Lemmas 3.4 and 3.5, we know that the sequences $(\widetilde{R}_\eta^{(k)},\widetilde{S}_\eta^{(k)},t_\eta^{(k)})(\xi,\eta)$ are uniformly convergent, which means that the functions $(\widetilde{R}_\eta,\widetilde{S}_\eta,t_\eta)(\xi,\eta)$ are continuous in $[0,\widetilde{\delta}]\times\mathbb{R}$. Furthermore, one also has by (3.91)

$$\begin{aligned}&\left|\widetilde{R}_\eta(\xi,\eta)\right|,\left|\widetilde{S}_\eta(\xi,\eta)\right|\le 3\overline{M}\xi^2,\quad \left|t_\eta(\xi,\eta)\right|\le 3\overline{M}\xi,\\&\left|\widetilde{R}_\eta(\xi,\eta)-\widetilde{S}_\eta(\xi,\eta)\right|\le 3\overline{M}\xi^2.\end{aligned} \tag{3.105}$$

In addition, we differentiate equations (3.19) with respect to ξ to gain

$$\widetilde{R}_\xi(\xi,\eta)=\frac{\widetilde{R}-\widetilde{S}}{2\xi}+A\cdot\frac{(\widetilde{R}-\widetilde{S})^2}{4\xi}-A(\widetilde{R}-\widetilde{S})+A\theta_0'\xi-Aw(t,y)\xi$$
$$+\int_0^\xi\left\{\frac{\widetilde{R}_y-\widetilde{S}_y}{2\tau}+A\frac{(\widetilde{R}-\widetilde{S})(\widetilde{R}_y-\widetilde{S}_y)}{2\tau}-A(\widetilde{R}_y-\widetilde{S}_y)+A_{11}A_{\widetilde{S}}\widetilde{S}_y\right.$$
$$\left.+[A_{11}A_t-Aw_t\tau]t_y+A_{11}A_y+A(\theta_0''-w_y)\tau\right\}\frac{\partial y_-(\tau)}{\partial\xi}(\tau,y_-(\tau))\,\mathrm{d}\tau, \tag{3.106}$$

$$\widetilde{S}_\xi(\xi,\eta) = \frac{\widetilde{S}-\widetilde{R}}{2\xi} + B\cdot\frac{(\widetilde{R}-\widetilde{S})^2}{4\xi} + B(\widetilde{R}-\widetilde{S}) - B\theta_0'\xi + Bw(t,y)\xi$$
$$+ \int_0^\xi \left\{ \frac{\widetilde{S}_y-\widetilde{R}_y}{2\tau} + B\frac{(\widetilde{R}-\widetilde{S})(\widetilde{R}_y-\widetilde{S}_y)}{2\tau} + B(\widetilde{R}_y-\widetilde{S}_y) + B_{11}B_{\widetilde{R}}\widetilde{R}_y\right.$$
$$\left. + [B_{11}B_t + Bw_t\tau]t_y + B_{11}B_y - B(\theta_0''-w_y)\tau\right\}\frac{\partial y_+(\tau)}{\partial\xi}(\tau,y_+(\tau))\,\mathrm{d}\tau, \tag{3.107}$$

and

$$t_\xi(\xi,\eta) = C(\xi,\eta). \tag{3.108}$$

The terms $\partial_\xi y_\pm(\tau)$ in (3.106) and (3.107) are given by

$$\frac{\partial y_\pm(\tau)}{\partial\xi} = -\lambda_\pm\frac{\partial y_\pm(\tau)}{\partial\eta} = -\lambda_\pm\exp\left\{\int_\xi^\tau \frac{\partial\lambda_\pm}{\partial y}(s,y_\pm(s))\,\mathrm{d}s\right\}. \tag{3.109}$$

In view of the expressions of $\widetilde{R}_\xi$ and $\widetilde{S}_\xi$ in (3.106) and (3.107), we employ the estimates in (3.86) and (3.105) to find that the functions $(\widetilde{R}_\tau,\widetilde{S}_\tau,t_\tau)(\tau,y)$ are continuous in $[0,\widetilde{\delta}]\times\mathbb{R}$ and satisfy

$$\widetilde{R}_\tau(0,y) = \widetilde{S}_\tau(0,y) = 0. \tag{3.110}$$

All in all, the functions $(\widetilde{R},\widetilde{S},t)(\tau,y)$ satisfy the integral equations (3.19) and the homogeneous initial conditions (3.87) and (3.110), and they also own the required differentiability properties. Therefore, they are a smooth solution to the singular problem (3.7), (3.8).

In order to verify the uniqueness, we suppose that $(\widetilde{R}_a,\widetilde{S}_a,t_a)(\tau,y)$ and $(\widetilde{R}_b,\widetilde{S}_b,t_b)(\tau,y)$ are two smooth solutions of the problem (3.7), (3.8) and consider their difference. Denote $X=\widetilde{R}_a-\widetilde{R}_b$, $Y=\widetilde{S}_a-\widetilde{S}_b$, and $T=t_a-t_b$. By (3.7), one derives the equations for (X,Y,T) as follows:

$$X_\tau - A_a\tau X_y = (A_a-A_b)\tau\widetilde{R}_{by} + \frac{X-Y}{2\tau}$$
$$+\left\{\frac{(\widetilde{R}_a-\widetilde{S}_a)^2}{4\tau} - (\widetilde{R}_a-\widetilde{S}_a) + \theta_0'\tau - w(t_a,y)\right\}(A_a-A_b)$$
$$+A_b\left\{\frac{(R_a-S_a+R_b-S_b)}{4\tau}(X-Y) - (X-Y) - [w(t_a,y)-w(t_b,y)]\right\}, \tag{3.111}$$

$$Y_\tau + B_a\tau Y_y = -(B_a-B_b)\tau\widetilde{S}_{by} + \frac{Y-X}{2\tau}$$
$$+\left\{\frac{(\widetilde{R}_a-\widetilde{S}_a)^2}{4\tau} + (\widetilde{R}_a-\widetilde{S}_a) - \theta_0'\tau + w(t_a,y)\right\}(B_a-B_b)$$
$$+B_b\left\{\frac{(R_a-S_a+R_b-S_b)}{4\tau}(X-Y) + (X-Y) + [w(t_a,y)-w(t_b,y)]\right\}, \tag{3.112}$$

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and

$$T_\tau = C_a - C_b, \tag{3.113}$$

where

$$A_i = \frac{1}{f_i + \widetilde{S}_i}, \quad B_i = \frac{1}{f_i + \widetilde{R}_i}, \quad C_i = \frac{1}{\frac{\widetilde{R}_i + \widetilde{S}_i}{2} - \tau - \widetilde{v}(t_i, y) + \theta_0(y)},$$

and $f_i = \theta_0(y) - \widetilde{v}(t_i, y) - \tau$ for $i = a, b$. Performing a direct calculation gets

$$\begin{aligned} A_a - A_b &= - A_a A_b \{[\widetilde{v}(t_a, y) - \widetilde{v}(t_b, y)] + Y\}, \\ B_a - B_b &= - B_a B_b \{[\widetilde{v}(t_a, y) - \widetilde{v}(t_b, y)] + X\}, \\ C_a - C_b &= - \frac{1}{2} C_a C_b \{X + Y - 2[\widetilde{v}(t_a, y) - \widetilde{v}(t_b, y)]\}. \end{aligned}$$

Integrating (3.111)–(3.113) from 0 to ξ and utilizing the estimates (3.86) and (3.105) and the mean value theorem, we can find that the functions (X, Y, T) satisfy a homogeneous integral inequality system as the following form

$$\begin{cases} \left|X(\xi,\eta)\right| \le \displaystyle\int_0^\xi \left\{ \frac{|X - Y|}{2\tau} + M^*\big(|X| + |Y| + |T|\big) \right\} \mathrm{d}\tau, \\ \left|Y(\xi,\eta)\right| \le \displaystyle\int_0^\xi \left\{ \frac{|X - Y|}{2\tau} + M^*\big(|X| + |Y| + |T|\big) \right\} \mathrm{d}\tau, \\ \left|T(\xi,\eta)\right| \le \displaystyle\int_0^\xi M^*\big(|X| + |Y| + |T|\big)\, \mathrm{d}\tau, \end{cases} \tag{3.114}$$

for some positive constant M^*. One repeats the insertion of the right side of (3.114) to see that the functions (X, Y, T) must satisfy

$$|X|, |Y|, |T| \le \overline{M}^* \left(\frac{2}{3}\right)^\ell,$$

for arbitrary integer $\ell \ge 1$ and some positive constant $\overline{M}^*$. This implies that $X(\tau, y) = Y(\tau, y) = T(\tau, y) \equiv 0$. Hence, the smooth solution of the singular initial value problem (3.7), (3.8) is unique.

3.5 The problem in the original plane

Based on the unique smooth solution $(\widetilde{R}, \widetilde{S}, t)(\tau, y)$ of problem (3.7), (3.8) obtained in Section 3.4, we establish the existence and uniqueness of smooth solutions for the initial value problem (3.2), (3.3) in this subsection.

By virtue of (3.86), we know that the functions $(\widetilde{R}, \widetilde{S}, t)(\tau, y)$ are in the space $\widetilde{\Sigma}(\widetilde{\delta})$, from which and (3.16) one acquires

$$0 < \frac{1}{2}\underline{\theta} \le \frac{\widetilde{R}(\tau, y) + \widetilde{S}(\tau, y)}{2} - \tau - \widetilde{v}(t(\tau, y), y) + \theta_0(y) \le 2\overline{\theta}, \tag{3.115}$$

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for $(\tau, y) \in [0, \widetilde{\delta}] \times \mathbb{R}$. Then we can use (3.5) and (3.7) to construct the functions (t, x) as follows:

$$x = y, \quad t = t(\tau, y) = \int_0^{\tau} \frac{1}{\frac{\widetilde{R}(s,y)+\widetilde{S}(s,y)}{2} - s - \widetilde{v}(t(s,y),y) + \theta_0(y)}\, ds, \tag{3.116}$$

for any $(\tau, y) \in [0, \widetilde{\delta}] \times \mathbb{R}$. Obviously, by (3.115), the mapping $(\tau, y) \mapsto (t, x)$ is global one-to-one in the region $[0, \widetilde{\delta}] \times \mathbb{R}$. Now, set

$$\overline{\delta} = \frac{\widetilde{\delta}}{2\overline{\theta}}. \tag{3.117}$$

Then, for any point $(t^*, x^*) \in [0, \overline{\delta}] \times \mathbb{R}$, we see that there exists a unique corresponding point $(\tau^*, y^*) \in [0, \widetilde{\delta}] \times \mathbb{R}$ such that

$$x^* = y^*, \quad t^* = t(\tau^*, y^*). \tag{3.118}$$

Thus, one can define the functions $(\theta, \widetilde{R}, \widetilde{S})(t, x)$ as follows:

$$\theta(t^*, x^*) = \tau^*, \quad \widetilde{R}(t^*, x^*) = \widetilde{R}(\tau^*, y^*), \quad \widetilde{S}(t^*, x^*) = \widetilde{S}(\tau^*, y^*). \tag{3.119}$$

In sum, we have obtained the functions $(\theta, \widetilde{R}, \widetilde{S})(t, x)$ in the region $[0, \overline{\delta}] \times \mathbb{R}$.

We next verify that the functions defined in (3.119) satisfy the initial value conditions (3.3) and equations (3.2). By means of (3.86) and (3.119), one concludes

$$|\widetilde{R}(t,x)|, |\widetilde{S}(t,x)| \le 3\overline{M}\tau^2 = 3\overline{M}\theta^2, \quad \frac{1}{2}\underline{\theta}t \le \theta(t,x) \le 2\overline{\theta}t, \tag{3.120}$$

from which we get $\theta(0, x) = \widetilde{R}(0, x) = \widetilde{S}(0, x) = 0$. Moreover, we calculate

$$\begin{aligned}
&\widetilde{R}_t(t,x) = \widetilde{R}_\tau(\tau,y)\tau_t \\
&= \widetilde{R}_\tau(\tau,y)\left(\frac{\widetilde{R}(\tau,y)+\widetilde{S}(\tau,y)}{2} - \tau - \widetilde{v}(t(\tau,y),y) + \theta_0(y)\right), \\
&\widetilde{S}_t(t,x) = \widetilde{S}_\tau(\tau,y)\tau_t \\
&= \widetilde{S}_\tau(\tau,y)\left(\frac{\widetilde{R}(\tau,y)+\widetilde{S}(\tau,y)}{2} - \tau - \widetilde{v}(t(\tau,y),y) + \theta_0(y)\right),
\end{aligned} \tag{3.121}$$

which along with (3.105) and (3.115) achieve $\widetilde{R}_t(0, x) = \widetilde{S}_t(0, x) = 0$. Thus, the functions $(\theta, \widetilde{R}, \widetilde{S})(t, x)$ satisfy the initial value conditions (3.3). Furthermore, we apply the fact $\tau_x = (\widetilde{R} - \widetilde{S})/2\tau$ by (3.6) to arrive at

$$\widetilde{R}_x = \frac{\widetilde{R} - \widetilde{S}}{2\tau}\widetilde{R}_\tau + \widetilde{R}_y,$$

which combined with (3.121) and (3.7) leads to

$$\begin{aligned}
\widetilde{R}_t - \theta\widetilde{R}_x = &\left(\frac{\widetilde{R}(\tau,y) + \widetilde{S}(\tau,y)}{2} - \tau - \widetilde{v}(t(\tau,y),y) + \theta_0(y)\right)\widetilde{R}_\tau \\
&- \tau \cdot \left(\frac{\widetilde{R} - \widetilde{S}}{2\tau}\widetilde{R}_\tau + \widetilde{R}_y\right)
\end{aligned}$$

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$$
\begin{aligned}
&=(\widetilde{S}-\tau-\widetilde{v}+\theta_0)\widetilde{R}_\tau-\tau\widetilde{R}_y=(\widetilde{S}+f)\left(\widetilde{R}_\tau-\frac{\tau}{\widetilde{S}+f}\widetilde{R}_y\right)\\
&=(\widetilde{S}+f)\left(\frac{\widetilde{R}-\widetilde{S}}{2\tau}+\frac{1}{f+\widetilde{S}}\cdot\frac{(\widetilde{R}-\widetilde{S})^2}{4\tau}-\frac{\widetilde{R}-\widetilde{S}}{f+\widetilde{S}}+\frac{\theta_0'}{f+\widetilde{S}}\tau-\frac{w(t,y)}{f+\widetilde{S}}\tau\right)\\
&=\frac{(\widetilde{S}+f)(\widetilde{R}-\widetilde{S})}{2\tau}+\frac{(\widetilde{R}-\widetilde{S})^2}{4\tau}-(\widetilde{R}-\widetilde{S})+\theta_0'\tau-w(t,y)\tau\\
&=\frac{2(\widetilde{S}+f)(\widetilde{R}-\widetilde{S})+(\widetilde{R}-\widetilde{S})^2}{4\theta}-(\widetilde{R}-\widetilde{S})+\theta_0'\theta-w(t,x)\theta
\end{aligned}
$$

$$
=\frac{\widetilde{R}+\widetilde{S}-2\widetilde{v}+2\theta_0}{4\theta}(\widetilde{R}-\widetilde{S})-(\widetilde{R}-\widetilde{S})+\theta\theta_0'-\theta w, \tag{3.122}
$$

which is the desired equation for $\widetilde{R}$ in (3.2). One can check the other equations in (3.2) in a similar way. The inequalities in (3.4) come from (3.120). Therefore, the proof of Theorem 3.1 is completed.

In addition, we check that the following relation holds:

$$
\theta_x=\frac{\widetilde{R}-\widetilde{S}}{2\theta}. \tag{3.123}
$$

To show (3.123), we set $\Phi=\widetilde{R}-\widetilde{S}-2\theta\theta_x$ and apply (3.2) to compute

$$
\begin{aligned}
\partial_t\Phi&=\widetilde{R}_t-\widetilde{S}_t-2\theta_t\theta_x-2\theta\theta_{tx}\\
&=[\widetilde{R}_t-\theta\widetilde{R}_x]-[\widetilde{S}_t+\theta\widetilde{S}_x]-2\left(\frac{\widetilde{R}+\widetilde{S}}{2}-\theta-\widetilde{v}+\theta_0\right)\theta_x\\
&\quad-2\theta(-\theta_x-\widetilde{v}_x+\theta_0')\\
&=\frac{\widetilde{R}+\widetilde{S}-2\widetilde{v}+2\theta_0}{2\theta}(\widetilde{R}-\widetilde{S})-(\widetilde{R}-\widetilde{S})-[\widetilde{R}+\widetilde{S}-2\theta-2\widetilde{v}+2\theta_0]\theta_x+2\theta\theta_x
\end{aligned}
$$

$$
=\frac{[\widetilde{R}+\widetilde{S}-2\widetilde{v}+2\theta_0-4\theta]t}{2\theta}\cdot\frac{\Phi}{t}. \tag{3.124}
$$

Recalling (3.17) gives

$$
\left|\frac{[\widetilde{R}+\widetilde{S}-2\widetilde{v}+2\theta_0-4\theta]t}{2\theta}\right|\le\frac{24\overline{M}\overline{\theta}t^2+\widehat{M}_0t+2\overline{\theta}+8\overline{\theta}t}{\underline{\theta}}<\infty,
$$

which, along with (3.124) and the initial condition $(\Phi/t)|_{t=0}=0$, arrives at $\Phi\equiv0$. Hence, the relation (3.123) holds.

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Finally, for later applications, we summarize some properties for the functions $(\theta, \widetilde{R}, \widetilde{S})(t,x)$ in the region $[0,\overline{\delta}]\times\mathbb{R}$

$$(3.125)\quad \begin{aligned}
&|\widetilde{R}(t,x)|, |\widetilde{S}(t,x)| \le 3\overline{M}\theta^2 \le 12\overline{M}\overline{\theta}^2t^2, \quad \frac{1}{2}\underline{\theta}t \le \theta(t,x) \le 2\overline{\theta}t,\\
&|\widetilde{R}(t,x)-\widetilde{S}(t,x)| \le 3\overline{M}\theta^2 \le 12\overline{M}\overline{\theta}^2t^2,\\
&|\widetilde{R}_t(t,x)|, |\widetilde{S}_t(t,x)| \le 8\overline{M}\overline{\theta}^2t, \quad |\theta_t(t,x)| \le 2\overline{\theta},\\
&|\widetilde{R}_x(t,x)|, |\widetilde{S}_x(t,x)| \le 24\overline{M}^2\overline{\theta}^2t^2, \quad |\theta_x(t,x)| \le 3\overline{M}\overline{\theta}t,\\
&|\theta_{xx}(t,x)| \le \frac{3}{2}\overline{M}\theta + \frac{9}{4}\overline{M}^2\theta \le 3\overline{M}^2\theta \le 6\overline{M}^2\overline{\theta}t.
\end{aligned}$$

Here, we used the following estimates:

$$\begin{aligned}
&\tau = \theta(t,x) \le 2\overline{\theta}t, \quad |\theta_x(t,x)| = \left|\frac{\widetilde{R}-\widetilde{S}}{2\tau}\right| \le \frac{3}{2}\overline{M}\tau,\\
&|\theta_t(t,x)| = \left|\frac{\widetilde{R}+\widetilde{S}}{2} - \tau - \widetilde{v} + \theta_0\right| \le 3\overline{M}\tau^2 + \tau + \widehat{M}_0\cdot\frac{2}{\underline{\theta}}\tau + \overline{\theta} \le 2\overline{\theta},\\
&|\widetilde{R}_t(t,x)| \le |\widetilde{R}_\tau|\cdot|\theta_t|, \quad |\widetilde{S}_t(t,x)| \le |\widetilde{S}_\tau|\cdot|\theta_t|,\\
&|\widetilde{R}_x(t,x)| \le \frac{3}{2}\overline{M}\tau\cdot|\widetilde{R}_\tau| + |\widetilde{R}_y|, \quad |\widetilde{S}_x(t,x)| \le \frac{3}{2}\overline{M}\tau\cdot|\widetilde{S}_\tau| + |\widetilde{S}_y|,
\end{aligned}$$

and

$$\begin{aligned}
|\widetilde{R}_\tau| &\le |A|\tau|\widetilde{R}_y| + \frac{|\widetilde{R}-\widetilde{S}|}{2\tau} + |A|\frac{|\widetilde{R}-\widetilde{S}|^2}{4\tau} + \frac{|A|}{2}|\widetilde{R}-\widetilde{S}| + |A\theta_0'|\tau + |Aw|\tau\\
&\le \frac{\overline{M}}{16}\tau\cdot 3\overline{M}\tau^2 + \frac{3\overline{M}\tau}{2} + \frac{\overline{M}}{16}\cdot\frac{9\overline{M}^2\tau^3}{4} + \frac{\overline{M}}{32}\cdot 3\overline{M}\tau^2 + \frac{\overline{M}}{16}\cdot\tau + \frac{\overline{M}}{16}\cdot\tau\\
&\le \overline{M}\tau\left\{\frac{3\overline{M}\widetilde{\delta}^2}{16} + \frac{3}{2} + \frac{9(\overline{M}\widetilde{\delta})^2}{64} + \frac{\overline{M}\widetilde{\delta}}{32} + \frac{1}{8}\right\} \le 2\overline{M}\tau.
\end{aligned}$$

4 The hyperbolic–parabolic coupled problem

In this section, we show Theorem 2.1 and then obtain Theorem 1.1 based on the results established in Section 3. The strategy is to solve the parabolic equation in (2.7) to construct an iterative sequence for the variable $\widetilde{v}$ and then verify the uniform convergence of the iterative sequence.

4.1 Preliminary results for heat equation

We first introduce some known results for the heat equation. These results can be found in the many excellent texts (see, e.g., [14]). Let $b(t,x)\in C^1$ be a function defined in the region $[0,\overline{\delta}]\times\mathbb{R}$. Here, the positive number $\overline{\delta}$ is given in Theorem 3.1. Consider the following initial value problem for the heat equation:

$$(4.1)\quad \begin{cases} \dfrac{\partial\widetilde{v}}{\partial t} - \dfrac{\partial^2\widetilde{v}}{\partial x^2} = b(t,x),\\ \widetilde{v}(t,x)|_{t=0} = 0. \end{cases}$$

We know that the fundamental solution of the heat equation is

$$G(t,x) = \frac{1}{\sqrt{4\pi t}} \exp\left\{ -\frac{x^2}{4t} \right\}, \tag{4.2}$$

that is, $G(t,x)$ satisfies

$$\frac{\partial G}{\partial t} - \frac{\partial^2 G}{\partial x^2} = 0, \text{ with } G(0,x) = \delta_0(x), \tag{4.3}$$

where $\delta_0(x)$ is the Dirac function at $x = 0$. Thus, the smooth solution of problem (4.1) can be expressed by the fundamental solution

$$\begin{aligned} \widetilde{v}(t,x) &= \int_0^t \int_{\mathbb{R}} G(t-\varsigma, x-z) b(\varsigma, z)\, \mathrm{d}z\mathrm{d}\varsigma \\ &= \int_0^t \int_{\mathbb{R}} G(t-\varsigma, z) b(\varsigma, x-z)\, \mathrm{d}z\mathrm{d}\varsigma. \end{aligned} \tag{4.4}$$

Furthermore, the functions $\widetilde{v}_x, \widetilde{v}_t, \widetilde{v}_{xx}$, and $\widetilde{v}_{xt}$ can also be stated as follows:

$$\begin{aligned} \widetilde{v}_x(t,x) &= \int_0^t \int_{\mathbb{R}} \frac{\partial G(t-\varsigma, x-z)}{\partial x} b(\varsigma, z)\, \mathrm{d}z\mathrm{d}\varsigma \\ &= \int_0^t \int_{\mathbb{R}} \frac{\partial G(\varsigma, z)}{\partial z} b(t-\varsigma, x-z)\, \mathrm{d}z\mathrm{d}\varsigma \\ &= \int_0^t \int_{\mathbb{R}} \frac{\partial G(t-\varsigma, z)}{\partial z} b(\varsigma, x-z)\, \mathrm{d}z\mathrm{d}\varsigma \\ &= \int_0^t \int_{\mathbb{R}} G(t-\varsigma, z) b_x(\varsigma, x-z)\, \mathrm{d}z\mathrm{d}\varsigma, \end{aligned} \tag{4.5}$$

$$\widetilde{v}_t(t,x) = \int_0^t \int_{\mathbb{R}} G(t-\varsigma, z) b_\varsigma(\varsigma, x-z)\, \mathrm{d}z\mathrm{d}\varsigma + \int_{\mathbb{R}} G(t,z) b(0, x-z)\, \mathrm{d}z, \tag{4.6}$$

$$\begin{aligned} \widetilde{v}_{xx}(t,x) &= \int_0^t \int_{\mathbb{R}} \frac{\partial G(t-\varsigma, z)}{\partial z} b_x(\varsigma, x-z)\, \mathrm{d}z\mathrm{d}\varsigma, \\ &= \int_0^t \int_{\mathbb{R}} \frac{\partial^2 G(t-\varsigma, x-z)}{\partial x^2} [b(\varsigma, z) - b(\varsigma, x)]\, \mathrm{d}z\mathrm{d}\varsigma, \end{aligned} \tag{4.7}$$

and

(4.8)

$$\widetilde{v}_{xt}(t,x) = \int_0^t \int_{\mathbb{R}} \frac{\partial G(t-\varsigma, z)}{\partial z} b_\varsigma(\varsigma, x-z)\, \mathrm{d}z\mathrm{d}\varsigma + \int_{\mathbb{R}} G(t,z) b_x(0, x-z)\, \mathrm{d}z.$$

We set

$$H_\sigma(t,x) = \frac{1}{\sqrt{4\pi}} t^{-\frac{\sigma}{2}} \exp\left\{ -\frac{x^2}{16t} \right\}. \tag{4.9}$$

Then one has

$$\int_0^t \int_{\mathbb{R}} H_\sigma(t-\varsigma, z)\, \mathrm{d}z\mathrm{d}\varsigma = \frac{4}{3-\sigma} t^{\frac{3-\sigma}{2}} \quad \text{for } \sigma < 3, \tag{4.10}$$

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and

$$\begin{aligned}(4.11)\qquad \int_{t-\gamma}^{t}\int_{\mathbb{R}} H_\sigma(t-\varsigma,z)\,\mathrm{d}z\mathrm{d}\varsigma &= \frac{4}{3-\sigma}\gamma^{\frac{3-\sigma}{2}} \quad \text{for } \sigma<3 \text{ and } 0<\gamma\le t,\\ \int_{0}^{t_1-\gamma}\int_{\mathbb{R}} H_\sigma(t-\varsigma,z)\,\mathrm{d}z\mathrm{d}\varsigma &\le \frac{4}{\sigma-3}\gamma^{\frac{3-\sigma}{2}} \quad \text{for } \sigma>3 \text{ and } 0<\gamma\le t_1\le t.\end{aligned}$$

The results in (4.9)–(4.11) can be found in [31]. Thus, we obtain for $\sigma = 1, 2$,

$$(4.12)\qquad \int_0^t\int_{\mathbb{R}} H_1(t-\varsigma,z)\,\mathrm{d}z\mathrm{d}\varsigma = 2t,\qquad \int_0^t\int_{\mathbb{R}} H_2(t-\varsigma,z)\,\mathrm{d}z\mathrm{d}\varsigma = 4\sqrt{t}.$$

In addition, it is clear that the following inequality holds

$$(4.13)\qquad z^\beta \exp\left\{-\frac{z}{4}\right\} \le C_\beta \exp\left\{-\frac{z}{16}\right\},$$

for $z \ge 0$, where $\beta \ge 0$ is an arbitrary number and C_β is a positive constant depending only on β. Then, by (4.2), (4.9), and (4.13), one acquires

$$(4.14)\qquad G(t-\varsigma,z) \le H_1(t-\varsigma,z),\qquad \left|\partial_z G(t-\varsigma,z)\right| \le H_2(t-\varsigma,z),$$

and

$$(4.15)\qquad \left| |z|^\beta \frac{\partial^j G(t-\varsigma,z)}{\partial z^j}\right| \le C_{j,\beta} H_{j+1-\beta}(t-\varsigma,z),$$

where $C_{j,\beta}(j \ge 1)$ are positive constants depending only on β. The inequalities in (4.15) can also be found in [31]. Making use of (4.12) and (4.14), we gain

$$(4.16)\qquad \int_0^t\int_{\mathbb{R}} \left|G(t-\varsigma,z)\right|\,\mathrm{d}z\mathrm{d}\varsigma \le 2t,\qquad \int_0^t\int_{\mathbb{R}} \left|\frac{\partial G(t-\varsigma,z)}{\partial z}\right|\,\mathrm{d}z\mathrm{d}\varsigma \le 4\sqrt{t}.$$

4.2 The iterative sequence for the heat equation

For any $(t,x) \in [0,\overline{\delta}] \times \mathbb{R}$, we set

$$\widetilde{v}^{(0)}(t,x) \equiv 0,$$

which obviously satisfies $\widetilde{v}^{(0)} \in \Sigma(\overline{\delta})$. Here, the space $\Sigma(\overline{\delta})$ is defined in (3.1) but with the number $\overline{\delta}$ replacing δ_0. In view of the results in Section 3, one concludes the functions $(\widetilde{R}^{(0)}, \widetilde{S}^{(0)}, \theta^{(0)})(t,x)$ for $(t,x) \in [0,\overline{\delta}] \times \mathbb{R}$. According to (4.4), we define the function $\widetilde{v}^{(1)}(t,x)$ in the region $[0,\overline{\delta}] \times \mathbb{R}$ as

$$\begin{aligned}\widetilde{v}^{(1)}(t,x) &= \int_0^t\int_{\mathbb{R}} G(t-\varsigma,x-z)b^{(0)}(\varsigma,z)\,\mathrm{d}z\mathrm{d}\varsigma\\ (4.17)\qquad &= \int_0^t\int_{\mathbb{R}} G(t-\varsigma,z)b^{(0)}(\varsigma,x-z)\,\mathrm{d}z\mathrm{d}\varsigma,\end{aligned}$$

where

$$b^{(0)}(t,x) = \frac{\widetilde{R}^{(0)}(t,x)+\widetilde{S}^{(0)}(t,x)}{2} - \theta^{(0)}(t,x) - \widetilde{v}^{(0)}(t,x) + \theta_0(x) + v_0''(x).$$

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After determining the function $\widetilde{v}^{(k)}(t,x)$ in $[0,\overline{\delta}]\times\mathbb{R}$, one can achieve the functions $(\widetilde{R}^{(k)},\widetilde{S}^{(k)},\theta^{(k)})(t,x)$ based on the results in Section 3. Then we define the function $\widetilde{v}^{(k+1)}(t,x)$ in the region $[0,\overline{\delta}]\times\mathbb{R}$ by the following relation:

$$\begin{aligned}\widetilde{v}^{(k+1)}(t,x)&=\int_0^t\int_{\mathbb{R}}G(t-\varsigma,x-z)b^{(k)}(\varsigma,z)\,\mathrm{d}z\mathrm{d}\varsigma\\&=\int_0^t\int_{\mathbb{R}}G(t-\varsigma,z)b^{(k)}(\varsigma,x-z)\,\mathrm{d}z\mathrm{d}\varsigma,\end{aligned}\tag{4.18}$$

where

$$b^{(k)}(t,x)=\frac{\widetilde{R}^{(k)}(t,x)+\widetilde{S}^{(k)}(t,x)}{2}-\theta^{(k)}(t,x)-\widetilde{v}^{(k)}(t,x)+\theta_0(x)+v_0''(x).$$

Therefore, we have constructed an iterative sequence $\{\widetilde{v}^{(k)}(t,x)\}$.

Denote

$$\begin{aligned}\widehat{K}=\max\Big\{&1,\max_{x\in\mathbb{R}}|\theta_0(x)+v_0''(x)|,\ \max_{x\in\mathbb{R}}|\theta_0'(x)+v_0'''(x)|,\\&\max_{x\in\mathbb{R}}|\theta_0''(x)+v_0''''(x)|\Big\},\end{aligned}\tag{4.19}$$

which originates from the last two terms in the expression of $b^{(k)}$. Set

$$\begin{aligned}&\widehat{M}_0=\max\left\{32\widehat{K},\ 12\overline{M}^2(1+\overline{\theta})^2,32\overline{\theta}\Big(C_{2,1}+2C_{3,1/2}+C_{3,0}\Big)\right\},\\&\delta=\min\left\{\overline{\delta},\frac{1}{\widehat{M}_0},\frac{\underline{\theta}^2}{16\widehat{M}_0\overline{\theta}}\right\}.\end{aligned}\tag{4.20}$$

Here, the constant $\overline{M}\ge16$ is given in (3.30), which depends only on $\underline{\theta}$ and the C^3 norm of $\theta_0(y)$, $\overline{\delta}$ is defined in (3.117), and $C_{j,\beta}$ are presented in (4.15). Then we see by (3.125) that if $\widetilde{v}(t,x)\in\Sigma(\delta)$,

$$\begin{aligned}&\big|\widetilde{R}(t,x)\big|,\big|\widetilde{S}(t,x)\big|\le3\overline{M}\theta^2\le\widehat{M}_0t^2,\quad\frac12\underline{\theta}t\le\theta(t,x)\le2\overline{\theta}t\le\frac1{32}\widehat{M}_0t,\\&\big|\widetilde{R}(t,x)-\widetilde{S}(t,x)\big|\le3\overline{M}\theta^2\le\widehat{M}_0t^2,\quad|\theta_t(t,x)|\le\frac1{32}\widehat{M}_0,\\&\big|\widetilde{R}_t(t,x)\big|,\big|\widetilde{S}_t(t,x)\big|\le\widehat{M}_0t,\quad\big|\widetilde{R}_x(t,x)\big|,\big|\widetilde{S}_x(t,x)\big|\le6\overline{M}^2\theta^2\le\widehat{M}_0t^2,\\&|\theta_x(t,x)|\le\frac32\overline{M}\theta\le\frac14\widehat{M}_0t,\quad|\theta_{xx}(t,x)|\le3\overline{M}^2\theta\le\frac14\widehat{M}_0t.\end{aligned}\tag{4.21}$$

For the sequence $\{\widetilde{v}^{(k)}(t,x)\}$, we have the following lemma.

Lemma 4.1 *For all $k\ge0$, there hold $\widetilde{v}^{(k)}(t,x)\in\Sigma(\delta)$, that is, the functions $\widetilde{v}^{(k)}(t,x)(k=1,2,\dots)$ satisfy $\widetilde{v}^{(k)}(t,x)\in C^1$, $\widetilde{v}_x^{(k)}(t,x)\in C^1$, and $\forall\ (t,x)\in[0,\delta]\times\mathbb{R}$*

$$\begin{aligned}&\big|\widetilde{v}^{(k)}(t,x)\big|,\big|\widetilde{v}_x^{(k)}(t,x)\big|\le\widehat{M}_0t,\quad\big|\widetilde{v}_t^{(k)}(t,x)\big|\le\widehat{M}_0,\\&\big|\widetilde{v}_{xt}^{(k)}(t,x)\big|\le\widehat{M}_0,\qquad\qquad\big|\widetilde{v}_{xx}^{(k)}(t,x)\big|\le\widehat{M}_0\sqrt{t}.\end{aligned}\tag{4.22}$$

Proof We mainly verify that all the inequalities in (4.22) are true for any $k\ge0$. The proof is also based on the argument of induction.

Due to $\widetilde{v}^{(0)} \equiv 0 \in \Sigma(\delta)$, we see by (4.21) that the functions $(\widetilde{R}^{(0)}, \widetilde{S}^{(0)}, \theta^{(0)})(t,x)$ satisfy

(4.23)
$$\begin{aligned}
&\big|\widetilde{R}^{(0)}(t,x)\big|, \big|\widetilde{S}^{(0)}(t,x)\big| \le \widehat{M}_0 t^2, \quad \frac{1}{2}\underline{\theta} t \le \theta^{(0)}(t,x) \le \frac{1}{32}\widehat{M}_0 t,\\
&\big|\widetilde{R}^{(0)}(t,x) - \widetilde{S}^{(0)}(t,x)\big| \le \widehat{M}_0 t^2, \quad |\theta_t^{(0)}(t,x)| \le \frac{1}{32}\widehat{M}_0, \quad |\theta_x^{(0)}(t,x)| \le \frac{1}{4}\widehat{M}_0 t,\\
&\big|\widetilde{R}_t^{(0)}(t,x)\big|, \big|\widetilde{S}_t^{(0)}(t,x)\big| \le \widehat{M}_0 t, \quad \big|\widetilde{R}_x^{(0)}(t,x)\big|, \big|\widetilde{S}_x^{(0)}(t,x)\big| \le \widehat{M}_0 t^2,
\end{aligned}$$

from which we find that

$$\begin{aligned}
&\big|b^{(0)}(t,x)\big|\\
&\le \frac{|\widetilde{R}^{(0)}(t,x)| + |\widetilde{S}^{(0)}(t,x)|}{2} + |\theta^{(0)}(t,x)| + |\widetilde{v}^{(0)}(t,x)| + |\theta_0(x) + v_0''(x)|\\
&\le \widehat{M}_0 t^2 + \frac{1}{32}\widehat{M}_0 t + 0 + \widehat{K} \le \widehat{M}_0\delta^2 + \frac{1}{32}\widehat{M}_0\delta + \frac{1}{16}\widehat{M}_0 \le \frac{1}{4}\widehat{M}_0
\end{aligned} \tag{4.24}$$

and

$$\begin{aligned}
&\big|b_x^{(0)}(t,x)\big|\\
&\le \frac{|\widetilde{R}_x^{(0)}(t,x)| + |\widetilde{S}_x^{(0)}(t,x)|}{2} + |\theta_x^{(0)}(t,x)| + |\widetilde{v}_x^{(0)}(t,x)| + |\theta_0'(x) + v_0'''(x)|\\
&\le \frac{1}{2}\widehat{M}_0\delta^2 + \frac{1}{4}\widehat{M}_0\delta + 0 + \widehat{K} \le \frac{1}{4}\widehat{M}_0,
\end{aligned} \tag{4.25}$$

$$\begin{aligned}
\big|b_t^{(0)}(t,x)\big| &\le \frac{|\widetilde{R}_t^{(0)}(t,x)| + |\widetilde{S}_t^{(0)}(t,x)|}{2} + |\theta_t^{(0)}(t,x)| + |\widetilde{v}_t^{(0)}(t,x)|\\
&\le \widehat{M}_0\delta + 2\overline{\theta} + 0 \le 3\overline{\theta} \le \frac{3}{32}\widehat{M}_0 \le \frac{1}{4}\widehat{M}_0.
\end{aligned} \tag{4.26}$$

One combines (4.17) and (4.5)–(4.8) and utilizes (4.24)–(4.26) to arrive at

(4.27)
$$\begin{aligned}
\big|\widetilde{v}^{(1)}(t,x)\big| &\le \int_0^t \int_{\mathbb{R}} |G(t-\varsigma,z)| \cdot |b^{(0)}| \,\mathrm{d}z\mathrm{d}\varsigma \le \frac{1}{4}\widehat{M}_0 \int_0^t \int_{\mathbb{R}} |G(t-\varsigma,z)| \,\mathrm{d}z\mathrm{d}\varsigma,\\
\big|\widetilde{v}_x^{(1)}(t,x)\big| &\le \int_0^t \int_{\mathbb{R}} |G(t-\varsigma,z)| \cdot |b_x^{(0)}| \,\mathrm{d}z\mathrm{d}\varsigma \le \frac{1}{4}\widehat{M}_0 \int_0^t \int_{\mathbb{R}} |G(t-\varsigma,z)| \,\mathrm{d}z\mathrm{d}\varsigma,\\
\big|\widetilde{v}_t^{(1)}(t,x)\big| &\le \int_0^t \int_{\mathbb{R}} |G(t-\varsigma,z)| \cdot |b_\varsigma^{(0)}| \,\mathrm{d}z\mathrm{d}\varsigma + \int_{\mathbb{R}} |G(t,z)| \cdot |b^{(0)}(0,x-z)| \,\mathrm{d}z\\
&\le \frac{1}{4}\widehat{M}_0 \int_0^t \int_{\mathbb{R}} |G(t-\varsigma,z)| \,\mathrm{d}z\mathrm{d}\varsigma + \frac{1}{4}\widehat{M}_0 \int_{\mathbb{R}} |G(t,z)| \,\mathrm{d}z,
\end{aligned}$$

and

$$
(4.28)\qquad
\begin{aligned}
\left|\widetilde{v}^{(1)}_{xx}(t,x)\right| &\le \int_0^t \int_{\mathbb{R}} \left|\frac{\partial G(t-\varsigma,z)}{\partial z}\right| \cdot |b^{(0)}_x| \,\mathrm{d}z\mathrm{d}\varsigma \\
&\le \frac{1}{4}\widehat{M}_0 \int_0^t \int_{\mathbb{R}} \left|\frac{\partial G(t-\varsigma,z)}{\partial z}\right| \mathrm{d}z\mathrm{d}\varsigma, \\
\left|\widetilde{v}^{(1)}_{xt}(t,x)\right| &\le \int_0^t \int_{\mathbb{R}} \left|\frac{\partial G(t-\varsigma,z)}{\partial z}\right| \cdot |b^{(0)}_\varsigma| \,\mathrm{d}z\mathrm{d}\varsigma \\
&\quad + \int_{\mathbb{R}} |G(t,z)| \cdot |b^{(0)}_x(0,x-z)| \,\mathrm{d}z \\
&\le \frac{1}{4}\widehat{M}_0 \int_0^t \int_{\mathbb{R}} \left|\frac{\partial G(t-\varsigma,z)}{\partial z}\right| \mathrm{d}z\mathrm{d}\varsigma + \frac{1}{4}\widehat{M}_0 \int_{\mathbb{R}} |G(t,z)| \,\mathrm{d}z.
\end{aligned}
$$

Making use of (4.16), (4.27), and (4.28), we get

$$
(4.29)\qquad
\begin{aligned}
&\left|\widetilde{v}^{(1)}(t,x)\right|, \left|\widetilde{v}^{(1)}_x(t,x)\right| \le \frac{1}{4}\widehat{M}_0 \cdot 2t = \frac{1}{2}\widehat{M}_0 t \le \widehat{M}_0 t, \\
&\left|\widetilde{v}^{(1)}_t(t,x)\right| \le \frac{1}{4}\widehat{M}_0 \cdot 2t + \frac{1}{4}\widehat{M}_0 \cdot 2 \le \frac{1}{2}(1+\delta)\widehat{M}_0 \le \widehat{M}_0, \\
&\left|\widetilde{v}^{(1)}_{xx}(t,x)\right| \le \frac{1}{4}\widehat{M}_0 \cdot 4\sqrt{t} = \widehat{M}_0\sqrt{t}, \\
&\left|\widetilde{v}^{(1)}_{xt}(t,x)\right| \le \frac{1}{4}\widehat{M}_0 \cdot 4\sqrt{t} + \frac{1}{4}\widehat{M}_0 \cdot 2 = \frac{1}{2}(1+2\sqrt{\delta})\widehat{M}_0 \le \widehat{M}_0,
\end{aligned}
$$

from which we gain that (4.22) holds for $k = 1$.

Assume that $\widetilde{v}^{(k)}(t,x) \in \Sigma(\delta)$, that is, (4.22) is true for $n = k$. Thanks to (4.21), the functions $(\widetilde{R}^{(k)}, \widetilde{S}^{(k)}, \theta^{(k)})(t,x)$ obtained in Section 3 satisfy

$$
(4.30)\qquad
\begin{aligned}
&\left|\widetilde{R}^{(k)}(t,x)\right|, \left|\widetilde{S}^{(k)}(t,x)\right| \le \widehat{M}_0 t^2, \quad \frac{1}{2}\underline{\theta} t \le \theta^{(k)}(t,x) \le \frac{1}{32}\widehat{M}_0 t, \\
&|\theta^{(k)}_t(t,x)| \le \frac{1}{32}\widehat{M}_0, \left|\widetilde{R}^{(k)}(t,x) - \widetilde{S}^{(k)}(t,x)\right| \le \widehat{M}_0 t^2, \\
&|\theta^{(k)}_x(t,x)| \le \frac{1}{4}\widehat{M}_0 t, \quad |\theta^{(k)}_{xx}(t,x)| \le \frac{1}{4}\widehat{M}_0 t, \\
&\left|\widetilde{R}^{(k)}_t(t,x)\right|, \left|\widetilde{S}^{(k)}_t(t,x)\right| \le \widehat{M}_0 t, \quad \left|\widetilde{R}^{(k)}_x(t,x)\right|, \left|\widetilde{S}^{(k)}_x(t,x)\right| \le \widehat{M}_0 t^2.
\end{aligned}
$$

It suggests by (4.30) and the induction assumptions that

$$
(4.31)\qquad
\begin{aligned}
&\left|b^{(k)}(t,x)\right| \le \widehat{M}_0\delta^2 + \frac{1}{32}\widehat{M}_0\delta + \widehat{M}_0\delta + \widehat{K} \le \frac{1}{4}\widehat{M}_0, \\
&\left|b^{(k)}_x(t,x)\right| \le \frac{1}{2}\widehat{M}_0^2\delta^2 + \frac{1}{4}\widehat{M}_0\delta + \widehat{M}_0\delta + \widehat{K} \le \frac{1}{4}\widehat{M}_0, \\
&\left|b^{(k)}_t(t,x)\right| \le \widehat{M}_0\delta + 2\overline{\theta} + \widehat{M}_0 \le 3\overline{\theta} + \widehat{M}_0 \le \frac{5}{4}\widehat{M}_0.
\end{aligned}
$$

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Thus, by applying (4.18), (4.5)–(4.8), (4.16), and (4.31), one achieves

$$
\begin{aligned}
&|\widetilde{v}^{(k+1)}(t,x)| \le \int_0^t \int_{\mathbb{R}} |G(t-\varsigma,z)|\cdot|b^{(k)}|\,\mathrm{d}z\mathrm{d}\varsigma \le \frac{1}{4}\widehat{M}_0\cdot 2t \le \widehat{M}_0 t,\\
&|\widetilde{v}_x^{(k+1)}(t,x)| \le \int_0^t \int_{\mathbb{R}} |G(t-\varsigma,z)|\cdot|b_x^{(k)}|\,\mathrm{d}z\mathrm{d}\varsigma \le \frac{1}{4}\widehat{M}_0\cdot 2t \le \widehat{M}_0 t,\\
&|\widetilde{v}_t^{(k+1)}(t,x)|\\
&\le \int_0^t \int_{\mathbb{R}} |G(t-\varsigma,z)|\cdot|b_\varsigma^{(k)}|\,\mathrm{d}z\mathrm{d}\varsigma + \int_{\mathbb{R}} |G(t,z)|\cdot|b^{(k)}(0,x-z)|\,\mathrm{d}z\\
&\le \frac{5}{4}\widehat{M}_0\cdot 2t + \frac{1}{4}\widehat{M}_0\cdot 2 \le \left(\frac{5}{2}\delta + \frac{1}{2}\right)\widehat{M}_0 \le \left(\frac{5}{64}+\frac{1}{2}\right)\widehat{M}_0 \le \widehat{M}_0,
\end{aligned}
\tag{4.32}
$$

and

$$
\begin{aligned}
&|\widetilde{v}_{xx}^{(k+1)}(t,x)| \le \int_0^t \int_{\mathbb{R}} \left|\frac{\partial G(t-\varsigma,z)}{\partial z}\right|\cdot\left|b_x^{(k)}\right|\,\mathrm{d}z\mathrm{d}\varsigma \le \widehat{M}_0\sqrt{t},\\
&|\widetilde{v}_{xt}^{(k+1)}(t,x)|\\
&\le \int_0^t \int_{\mathbb{R}} \left|\frac{\partial G(t-\varsigma,z)}{\partial z}\right|\cdot|b_\varsigma^{(k)}|\,\mathrm{d}z\mathrm{d}\varsigma + \int_{\mathbb{R}} |G(t,z)|\cdot|b_x^{(k)}(0,x-z)|\,\mathrm{d}z\\
&\le \frac{5}{4}\widehat{M}_0\cdot 4\sqrt{t} + \frac{1}{4}\widehat{M}_0\cdot 2 \le \left(5\sqrt{\delta}+\frac{1}{2}\right)\widehat{M}_0 \le \widehat{M}_0.
\end{aligned}
\tag{4.33}
$$

From (4.32) and (4.33), we acquire $\widetilde{v}^{(k+1)}(t,x) \in \Sigma(\delta)$ and then complete the proof of the lemma. ∎

By virtue of Lemma 4.1 and (4.21), we find that for any $k \ge 0$ and $(t,x) \in [0,\delta]\times\mathbb{R}$,

$$
\begin{aligned}
&|\widetilde{R}^{(k)}(t,x)|, |\widetilde{S}^{(k)}(t,x)| \le 3\overline{M}(\theta^{(k)})^2 \le \widehat{M}_0 t^2,\\
&\frac{1}{2}\underline{\theta}t \le \theta^{(k)}(t,x) \le 2\overline{\theta}t \le \frac{1}{32}\widehat{M}_0 t,\\
&|\widetilde{R}^{(k)}(t,x) - \widetilde{S}^{(k)}(t,x)| \le 3\overline{M}(\theta^{(k)})^2 \le \widehat{M}_0 t^2, \quad |\theta_t^{(k)}(t,x)| \le \frac{1}{32}\widehat{M}_0,\\
&|\widetilde{R}_t^{(k)}(t,x)|, |\widetilde{S}_t^{(k)}(t,x)| \le \widehat{M}_0 t,\\
&|\widetilde{R}_x^{(k)}(t,x)|, |\widetilde{S}_x^{(k)}(t,x)| \le 6\overline{M}^2(\theta^{(k)})^2 \le \widehat{M}_0 t^2,\\
&|\theta_x^{(k)}(t,x)| \le \frac{3}{2}\overline{M}\theta^{(k)} \le \frac{1}{4}\widehat{M}_0 t, \quad |\theta_{xx}^{(k)}(t,x)| \le 3\overline{M}^2\theta^{(k)} \le \frac{1}{4}\widehat{M}_0 t.
\end{aligned}
\tag{4.34}
$$

Furthermore, the function $\widetilde{v}_{xx}^{(k)}(t,x)$ owns the following regularity.

Lemma 4.2 *For any $k \ge 0$, the function $\widetilde{v}_{xx}^{(k)}(t,x)$ is uniformly α-Hölder continuous with respect to x, that is, there exists a positive constant $\widehat{C}_\alpha$ independent of k such that for any two points (t,x_1) and (t,x_2) in $[0,\delta]\times\mathbb{R}$, there hold for $\alpha \in (0,1)$*

$$
|\widetilde{v}_{xx}^{(k)}(t,x_1) - \widetilde{v}_{xx}^{(k)}(t,x_2)| \le \widehat{C}_\alpha |x_1 - x_2|^\alpha. \tag{4.35}
$$

Proof The proof of the lemma is similar to that of Lemma 3.3 in [31], but we list it here for the sake of completeness. By means of (4.7) and (4.18), we see that

$$\widetilde{v}_{xx}^{(k+1)}(t,x) = \int_0^t \int_{\mathbb{R}} \frac{\partial^2 G(t-\varsigma, x-z)}{\partial x^2}[b^{(k)}(\varsigma,z) - b^{(k)}(\varsigma,x)]\, \mathrm{d}z\mathrm{d}\varsigma, \tag{4.36}$$

and then for any $x_1 < x_2$,

$$\begin{aligned}
&\widetilde{v}_{xx}^{(k+1)}(t,x_1) - \widetilde{v}_{xx}^{(k+1)}(t,x_2)\\
&= \int_0^t \int_{\mathbb{R}} \frac{\partial^2 G(t-\varsigma, x_1-z)}{\partial x^2}[b^{(k)}(\varsigma,z) - b^{(k)}(\varsigma,x_1)]\, \mathrm{d}z\mathrm{d}\varsigma\\
&\quad - \int_0^t \int_{\mathbb{R}} \frac{\partial^2 G(t-\varsigma, x_2-z)}{\partial x^2}[b^{(k)}(\varsigma,z) - b^{(k)}(\varsigma,x_2)]\, \mathrm{d}z\mathrm{d}\varsigma.
\end{aligned} \tag{4.37}$$

Denote $\gamma = (x_2 - x_1)^2$. The proof is divided into two cases: $\gamma > t$ and $\gamma \le t$.

If $\gamma > t$, we use (4.31), (4.15) with $\beta = 1$, and (4.12) to estimate (4.37) directly

$$\begin{aligned}
&\left|\widetilde{v}_{xx}^{(k+1)}(t,x_1) - \widetilde{v}_{xx}^{(k+1)}(t,x_2)\right| \le \int_0^t \int_{\mathbb{R}} \left|\frac{\partial^2 G(t-\varsigma, x_1-z)}{\partial x^2}\right| \cdot |b_x^{(k)}| \cdot |z - x_1|\, \mathrm{d}z\mathrm{d}\varsigma\\
&\quad + \int_0^t \int_{\mathbb{R}} \left|\frac{\partial^2 G(t-\varsigma, x_2-z)}{\partial x^2}\right| \cdot |b_x^{(k)}| \cdot |z - x_2|\, \mathrm{d}z\mathrm{d}\varsigma\\
&\le \frac{1}{2}\widehat{M}_0 \int_0^t \int_{\mathbb{R}} C_{2,1} H_2(t-\varsigma, z)\, \mathrm{d}z\mathrm{d}\varsigma\\
&\le 2\widehat{M}_0 C_{2,1} \delta^{\frac{1-\alpha}{2}} t^{\frac{\alpha}{2}} \le 2\widehat{M}_0 C_{2,1}|x_2 - x_1|^{\alpha}.
\end{aligned} \tag{4.38}$$

If $\gamma \le t$, the relation (4.37) can be rewritten as

$$\widetilde{v}_{xx}^{(k+1)}(t,x_1) - \widetilde{v}_{xx}^{(k+1)}(t,x_2) = I_1^{(k)} + I_2^{(k)} + I_3^{(k)} + I_4^{(k)}, \tag{4.39}$$

where

$$\begin{aligned}
I_1^{(k)} &= \int_{t-\gamma}^t \int_{\mathbb{R}} \frac{\partial^2 G(t-\varsigma, x_1-z)}{\partial x^2}[b^{(k)}(\varsigma,z) - b^{(k)}(\varsigma,x_1)]\, \mathrm{d}z\mathrm{d}\varsigma,\\
I_2^{(k)} &= -\int_{t-\gamma}^t \int_{\mathbb{R}} \frac{\partial^2 G(t-\varsigma, x_2-z)}{\partial x^2}[b^{(k)}(\varsigma,z) - b^{(k)}(\varsigma,x_2)]\, \mathrm{d}z\mathrm{d}\varsigma,\\
I_3^{(k)} &= \int_0^{t-\gamma} \int_{\mathbb{R}} \frac{\partial^2 G(t-\varsigma, x_2-z)}{\partial x^2}[b^{(k)}(\varsigma,x_2) - b^{(k)}(\varsigma,x_1)]\, \mathrm{d}z\mathrm{d}\varsigma,\\
I_4^{(k)} &= \int_0^{t-\gamma} \int_{\mathbb{R}} \left(\frac{\partial^2 G(t-\varsigma, x_1-z)}{\partial x^2} - \frac{\partial^2 G(t-\varsigma, x_2-z)}{\partial x^2}\right)\\
&\qquad \times [b^{(k)}(\varsigma,z) - b^{(k)}(\varsigma,x_1)]\, \mathrm{d}z\mathrm{d}\varsigma.
\end{aligned}$$

For the term $I_1^{(k)}$, it concludes by (4.31), (4.15) with $\beta = 1$, and (4.11) that

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$$
\begin{aligned}
\left|I_1^{(k)}\right| &\le \int_{t-\gamma}^{t}\int_{\mathbb{R}}\left|\frac{\partial^2 G(t-\varsigma,x_1-z)}{\partial x^2}\right|\cdot|b_x^{(k)}|\cdot|z-x_1|\,\mathrm{d}z\mathrm{d}\varsigma\\
&\le \frac{1}{4}\widehat{M}_0\int_{t-\gamma}^{t}\int_{\mathbb{R}} C_{2,1}H_2(t-\varsigma,z)\,\mathrm{d}z\mathrm{d}\varsigma = \widehat{M}_0 C_{2,1}\gamma^{\frac{1}{2}}\\
&\le \widehat{M}_0 C_{2,1}\delta^{\frac{1-\alpha}{2}}|x_1-x_2|^{\alpha} \le \widehat{M}_0 C_{2,1}|x_1-x_2|^{\alpha}. \qquad (4.40)
\end{aligned}
$$

The estimate (4.40) also holds for the term $I_2^{(k)}$. Furthermore, one acquires by the integration by parts that $I_3^{(k)} = 0$. For the term $I_4^{(k)}$, we use (4.31), (4.15), and (4.11) again to find

$$
\begin{aligned}
\left|I_4^{(k)}\right| &= \left|\int_{x_1}^{x_2}\int_0^{t-\gamma}\int_{\mathbb{R}}\frac{\partial^3 G(t-\varsigma,x-z)}{\partial x^3}[b^{(k)}(\varsigma,z)-b^{(k)}(\varsigma,x_1)]\,\mathrm{d}z\mathrm{d}\varsigma\mathrm{d}x\right|\\
&\le \int_{x_1}^{x_2}\int_0^{t-\gamma}\int_{\mathbb{R}}\left|\frac{\partial^3 G(t-\varsigma,x-z)}{\partial x^3}\right|\cdot\frac{1}{2}\widehat{M}_0\cdot|z-x_1|^{\alpha}\,\mathrm{d}z\mathrm{d}\varsigma\mathrm{d}x\\
&\le \frac{1}{2}\widehat{M}_0\int_{x_1}^{x_2}\int_0^{t-\gamma}\int_{\mathbb{R}}\left|\frac{\partial^3 G(t-\varsigma,x-z)}{\partial x^3}\right|\left(|z-x|^{\alpha}+|x-x_1|^{\alpha}\right)\mathrm{d}z\mathrm{d}\varsigma\mathrm{d}x\\
&\le \frac{1}{2}\widehat{M}_0\int_{x_1}^{x_2}\int_0^{t-\gamma}\int_{\mathbb{R}}\Big\{C_{3,\alpha}H_{4-\alpha}(t-\varsigma,z)\\
&\qquad + C_{3,0}|x-x_1|^{\alpha}H_4(t-\varsigma,z)\Big\}\,\mathrm{d}z\mathrm{d}\varsigma\mathrm{d}x\\
&\le \frac{1}{2}\widehat{M}_0\Big\{C_{3,\alpha}|x_2-x_1|\cdot\frac{4}{(4-\alpha)-3}\gamma^{\frac{3-(4-\alpha)}{2}}\\
&\qquad + C_{3,0}|x_2-x_1|^{1+\alpha}\cdot\frac{4}{4-3}\gamma^{\frac{3-4}{2}}\Big\}\\
&= 2\widehat{M}_0\left(\frac{C_{3,\alpha}}{1-\alpha}+C_{3,0}\right)|x_2-x_1|^{\alpha}. \qquad (4.41)
\end{aligned}
$$

Substituting (4.40) and (4.41) into (4.39) yields

$$
\left|\widetilde{v}_{xx}^{(k+1)}(t,x_1)-\widetilde{v}_{xx}^{(k+1)}(t,x_2)\right| \le 2\widehat{M}_0\left(C_{2,1}+\frac{C_{3,\alpha}}{1-\alpha}+C_{3,0}\right)|x_2-x_1|^{\alpha}. \qquad (4.42)
$$

We denote

$$
\widehat{C}_\alpha = 2\widehat{M}_0\left(C_{2,1}+\frac{C_{3,\alpha}}{1-\alpha}+C_{3,0}\right),
$$

and combine (4.38) and (4.42) to achieve

$$
\left|\widetilde{v}_{xx}^{(k+1)}(t,x_1)-\widetilde{v}_{xx}^{(k+1)}(t,x_2)\right| \le \widehat{C}_\alpha|x_1-x_2|^{\alpha}, \qquad (4.43)
$$

which is the desired inequality (4.35). ■

Remark 1 The restriction $\alpha < 1$ in Lemma 4.2 is only used in the derivation of (4.41). Based on Lemma 4.2, we shall show that the functions $\widetilde{v}_{xx}^{(k)}(t,x)(k=1,2,\dots)$ are uniformly Lipschitz continuous with respect to x by improving the regularity of $b_x^{(k)}$.

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4.3 The convergence of the iterative sequence (I)

In order to establish the uniform convergence of the iterative sequence $\{\widetilde{v}^{(k)}(t,x)\}$, we need to derive its series of properties in the space $\Sigma(\delta)$.

By direct calculations, one applies (4.18) and (4.5) to get

$$\begin{aligned}&\left|\widetilde{v}^{(k+1)}(t,x)-\widetilde{v}^{(k)}(t,x)\right|\\ &\le\int_0^t\int_{\mathbb{R}}G(t-\varsigma,z)\left\{\frac{1}{2}(T_{12}^{(k)}+T_{13}^{(k)})+T_{14}^{(k)}+T_{15}^{(k)}\right\}(\varsigma,x-z)\,\mathrm{d}z\mathrm{d}\varsigma\end{aligned}\tag{4.44}$$

and

$$\begin{aligned}\left|\widetilde{v}_x^{(k+1)}(t,x)-\widetilde{v}_x^{(k)}(t,x)\right|&\le\int_0^t\int_{\mathbb{R}}\left|\frac{\partial G(t-\varsigma,z)}{\partial z}\right|\\ &\times\left\{\frac{1}{2}(T_{12}^{(k)}+T_{13})^{(k)}+T_{14}^{(k)}+T_{15}^{(k)}\right\}(\varsigma,x-z)\,\mathrm{d}z\mathrm{d}\varsigma,\end{aligned}\tag{4.45}$$

where

$$\begin{aligned}T_{12}^{(k)}(t,x)&=\left|\widetilde{R}^{(k)}(t,x)-\widetilde{R}^{(k-1)}(t,x)\right|,\ T_{13}^{(k)}(t,x)=\left|\widetilde{S}^{(k)}(t,x)-\widetilde{S}^{(k-1)}(t,x)\right|,\\ T_{14}^{(k)}(t,x)&=\left|\theta^{(k)}(t,x)-\theta^{(k-1)}(t,x)\right|,\ T_{15}^{(k)}(t,x)=\left|\widetilde{v}^{(k)}(t,x)-\widetilde{v}^{(k-1)}(t,x)\right|.\end{aligned}$$

To estimate $T_{12-14}^{(k)}$, we first recall (3.2) to gain the equations for $(\widetilde{R}^{(k)},\widetilde{S}^{(k)},\theta^{(k)})(t,x)$ in $[0,\delta]\times\mathbb{R}$

$$\begin{cases}\widetilde{R}_t^{(k)}-\theta^{(k)}\widetilde{R}_x^{(k)}=(\Psi^{(k)}-1)(\widetilde{R}^{(k)}-\widetilde{S}^{(k)})+\theta^{(k)}\theta_0'-\theta^{(k)}\widetilde{v}_x^{(k)}=:F_1^{(k)}(t,x),\\ \widetilde{S}_t^{(k)}+\theta^{(k)}\widetilde{S}_x^{(k)}=(\Psi^{(k)}-1)(\widetilde{S}^{(k)}-\widetilde{R}^{(k)})-\theta^{(k)}\theta_0'+\theta^{(k)}\widetilde{v}_x^{(k)}=:F_2^{(k)}(t,x),\\ \theta_t^{(k)}=\dfrac{\widetilde{R}^{(k)}+\widetilde{S}^{(k)}}{2}-\theta^{(k)}-\widetilde{v}^{(k)}+\theta_1=:F_3^{(k)}(t,x),\end{cases}\tag{4.46}$$

where

$$\Psi^{(k)}=\frac{\widetilde{R}^{(k)}+\widetilde{S}^{(k)}-2\widetilde{v}^{(k)}+2\theta_0}{4\theta^{(k)}}.\tag{4.47}$$

It follows by (4.46) that

$$\begin{cases}\left(\widetilde{R}^{(k)}-\widetilde{R}^{(k-1)}\right)_t-\theta^{(k)}\left(\widetilde{R}^{(k)}-\widetilde{R}^{(k-1)}\right)_x\\ \quad=F_1^{(k)}-F_1^{(k-1)}+(\theta^{(k)}-\theta^{(k-1)})\widetilde{R}_x^{(k-1)},\\ \left(\widetilde{S}^{(k)}-\widetilde{S}^{(k-1)}\right)_t+\theta^{(k)}\left(\widetilde{S}^{(k)}-\widetilde{S}^{(k-1)}\right)_x\\ \quad=F_2^{(k)}-F_2^{(k-1)}-(\theta^{(k)}-\theta^{(k-1)})\widetilde{S}_x^{(k-1)},\\ \left(\theta^{(k)}-\theta^{(k-1)}\right)_t=F_3^{(k)}-F_3^{(k-1)},\end{cases}$$

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from which one acquires for any $(\xi,\eta)\in[0,\delta]\times\mathbb{R}$,

$$(4.48)\qquad \begin{cases} T_{12}^{(k)}(\xi,\eta)\le\int_0^{\xi}\Big\{\big|F_1^{(k)}-F_1^{(k-1)}\big|+|\widetilde{R}_x^{(k-1)}|T_{14}^{(k)}\Big\}(t,x_-^{(k)}(t))\,\mathrm{d}t,\\ T_{13}^{(k)}(\xi,\eta)\le\int_0^{\xi}\Big\{\big|F_2^{(k)}-F_2^{(k-1)}\big|+|\widetilde{S}_x^{(k-1)}|T_{14}^{(k)}\Big\}(t,x_+^{(k)}(t))\,\mathrm{d}t,\\ T_{14}^{(k)}(\xi,\eta)\le\int_0^{\xi}\big|F_3^{(k)}-F_3^{(k-1)}\big|(t,\eta)\,\mathrm{d}t, \end{cases}$$

where $x_\pm^{(k)}(t)=x_\pm^{(k)}(t;\xi,\eta)(t\in[0,\xi])$ are provided by

$$\begin{cases} \dfrac{\mathrm{d}x_\pm^{(k)}(t;\xi,\eta)}{\mathrm{d}t}=\pm\theta^{(k)}(t,x_\pm^{(k)}(t;\xi,\eta)),\\ x_\pm^{(k)}(\xi;\xi,\eta)=\eta. \end{cases}$$

Performing direct calculations achieves

$$(4.49)\qquad \begin{aligned} &\big|F_1^{(k)}-F_1^{(k-1)}\big|,\big|F_2^{(k)}-F_2^{(k-1)}\big|\\ &\le\big(|\Psi^{(k)}|+1\big)(T_{12}^{(k)}+T_{13}^{(k)})+|\theta^{(k-1)}|T_{16}^{(k)}\\ &\quad+|\widetilde{R}^{(k-1)}-\widetilde{S}^{(k-1)}|T_{17}^{(k)}+\big(|\theta_0'|+|\widetilde{v}_x^{(k)}|\big)T_{14}^{(k)},\\ &\big|F_3^{(k)}-F_3^{(k-1)}\big|\le\frac{1}{2}(T_{12}^{(k)}+T_{13}^{(k)})+T_{14}^{(k)}+T_{15}^{(k)}, \end{aligned}$$

where

$$T_{16}^{(k)}=\big|\widetilde{v}_x^{(k)}-\widetilde{v}_x^{(k-1)}\big|,\qquad T_{17}^{(k)}=\big|\Psi^{(k)}-\Psi^{(k-1)}\big|.$$

Moreover, thanks to (4.34) and Lemma 4.1, we can get a more precise estimate for $\theta^{(k)}$

$$\theta_0-\left(\widehat{M}_0t^2+\frac{1}{32}\widehat{M}_0t+\widehat{M}_0t\right)\le\theta_t^{(k)}\le\theta_0+\left(\widehat{M}_0t^2+\frac{1}{32}\widehat{M}_0t+\widehat{M}_0t\right),$$

and thus

$$\theta_0-\frac{17}{16}\widehat{M}_0t\le\theta_t^{(k)}\le\theta_0+\frac{17}{16}\widehat{M}_0t,$$

and then

$$(4.50)\qquad \theta_0t-\frac{17}{32}\widehat{M}_0t^2\le\theta^{(k)}\le\theta_0t+\frac{17}{32}\widehat{M}_0t^2.$$

It suggests by (4.34) and (4.50) that

$$(4.51)\qquad \begin{aligned} \big|\Psi^{(k)}\big|&\le\frac{2\theta_0+2\widehat{M}_0t^2+2\widehat{M}_0t}{4t\big\{\theta_0-\frac{17}{32}\widehat{M}_0t\big\}}=\frac{1}{2t}+\frac{\frac{17}{32}\widehat{M}_0+\widehat{M}_0t+\widehat{M}_0}{2(\theta_0-\widehat{M}_0t)}\\ &\le\frac{1}{2t}+\frac{2\widehat{M}_0}{\underline{\theta}}. \end{aligned}$$

Furthermore, for the term $T_{17}^{(k)}$, we obtain

$$\begin{aligned} T_{17}^{(k)} &\le \frac{T_{12}^{(k)} + T_{13}^{(k)} + 2T_{15}^{(k)}}{4\theta^{(k)}} + \frac{\left|\widetilde{R}^{(k-1)} + \widetilde{S}^{(k-1)} - 2\widetilde{\nu}^{(k-1)} + 2\theta_0\right|}{4\theta^{(k)}\theta^{(k-1)}} T_{14}^{(k)} \\ &\le \frac{1}{2\underline{\theta} t}\left(T_{12}^{(k)} + T_{13}^{(k)} + 2T_{15}^{(k)}\right) + \frac{2\widehat{M}_0\delta^2 + 2\widehat{M}_0\delta + 2\overline{\theta}}{\underline{\theta}^2 t^2} T_{14}^{(k)} \\ &\le \frac{1}{2\underline{\theta} t}\left(T_{12}^{(k)} + T_{13}^{(k)} + 2T_{15}^{(k)}\right) + \frac{3\overline{\theta}}{\underline{\theta}^2 t^2} T_{14}^{(k)}, \end{aligned} \tag{4.52}$$

by $8\widehat{M}_0\delta \le \underline{\theta} < \overline{\theta}$. Putting (4.51) and (4.52) into (4.49) and applying (4.34) again yield

$$\begin{aligned} &\left|F_1^{(k)} - F_1^{(k-1)}\right| + |\widetilde{R}_x^{(k-1)}| T_{14}^{(k)}, \quad \left|F_2^{(k)} - F_2^{(k-1)}\right| + |\widetilde{S}_x^{(k-1)}| T_{14}^{(k)} \\ &\le \frac{T_{12}^{(k)} + T_{13}^{(k)}}{2t} + \left(\frac{2\widehat{M}_0}{\underline{\theta}} + 1 + \frac{\widehat{M}_0\delta}{2\underline{\theta}}\right)\left(T_{12}^{(k)} + T_{13}^{(k)}\right) + \frac{1}{32}\widehat{M}_0 t T_{16}^{(k)} \\ &\quad + \frac{\widehat{M}_0 t}{\underline{\theta}} T_{15}^{(k)} + \left(\frac{3\widehat{M}_0\overline{\theta}}{\underline{\theta}^2} + \overline{\theta}_1 + \frac{1}{6}(\widehat{M}_0\delta)^2 + \widehat{M}_0\delta\right) T_{14}^{(k)} \\ &\le \frac{T_{12}^{(k)} + T_{13}^{(k)}}{2t} + \frac{3\widehat{M}_0}{\underline{\theta}}\left(T_{12}^{(k)} + T_{13}^{(k)}\right) + \frac{4\widehat{M}_0\overline{\theta}}{\underline{\theta}^2} T_{14}^{(k)} \\ &\quad + \left(\frac{\widehat{M}_0\delta}{32} + \frac{\widehat{M}_0\delta}{\underline{\theta}}\right)\left(T_{15}^{(k)} + T_{16}^{(k)}\right) \\ &\le \frac{T_{12}^{(k)} + T_{13}^{(k)}}{2t} + \frac{1}{4\delta}\left(T_{12}^{(k)} + T_{13}^{(k)} + T_{14}^{(k)}\right) + \frac{1}{4}\left(T_{15}^{(k)} + T_{16}^{(k)}\right). \end{aligned} \tag{4.53}$$

We insert (4.53) and (4.49) into (4.48) to see that

$$\begin{aligned} &T_{12}^{(k)}(\xi,\eta), T_{13}^{(k)}(\xi,\eta) \\ &\le \int_0^\xi \left\{\frac{T_{12}^{(k)} + T_{13}^{(k)}}{2t} + \frac{1}{4\delta}\left(T_{12}^{(k)} + T_{13}^{(k)} + T_{14}^{(k)}\right) + \frac{1}{4}\left(T_{15}^{(k)} + T_{16}^{(k)}\right)\right\} \mathrm{d}t \end{aligned} \tag{4.54}$$

and

$$T_{14}^{(k)}(\xi,\eta) \le \int_0^\xi \left\{\left(T_{12}^{(k)} + T_{13}^{(k)} + T_{14}^{(k)}\right) + T_{15}^{(k)}\right\}(t,\eta)\, \mathrm{d}t. \tag{4.55}$$

Based on (4.54) and (4.55), we can show the following lemma.

Lemma 4.3 *For any $k \ge 1$ and $(\xi,\eta) \in [0,\delta] \times \mathbb{R}$, if $T_{15}^{(k)}(\xi,\eta)$ and $T_{16}^{(k)}(\xi,\eta)$ satisfy*

$$T_{15}^{(k)}(\xi,\eta), T_{16}^{(k)}(\xi,\eta) \le \widehat{M}_0\xi\left(\frac{2}{3}\right)^{k-1}, \tag{4.56}$$

then there hold

$$T_{12}^{(k)}(\xi,\eta), T_{13}^{(k)}(\xi,\eta), T_{14}^{(k)}(\xi,\eta) \le 2\widehat{M}_0\xi^2\left(\frac{2}{3}\right)^{k-1}. \tag{4.57}$$

Proof By (4.34) and the definitions of $T^{(k)}_{12-14}$ in (4.45), we first find that

$$T^{(k)}_{12}(\xi,\eta),\, T^{(k)}_{13}(\xi,\eta) \le 2\widehat{M}_0\xi^2, \quad T^{(k)}_{14}(\xi,\eta) \le \frac{1}{16}\widehat{M}_0\xi. \tag{4.58}$$

Substituting (4.58) into (4.55) and utilizing the assumptions in (4.56) give

$$\begin{aligned} T^{(k)}_{14}(\xi,\eta) &\le \int_0^\xi \left\{4\widehat{M}_0t^2 + \frac{1}{16}\widehat{M}_0t + \widehat{M}_0t\left(\frac{2}{3}\right)^{k-1}\right\}(t,\eta)\,\mathrm{d}t \\ &\le \left[\frac{4\delta}{3} + \frac{1}{32} + \frac{1}{2}\left(\frac{2}{3}\right)^{k-1}\right]\widehat{M}_0\xi^2 \le 2\widehat{M}_0\xi^2. \end{aligned} \tag{4.59}$$

We now put the estimates (4.58) and (4.59) into (4.54) and apply the assumptions in (4.56) to deduce

$$\begin{aligned} &T^{(k)}_{12}(\xi,\eta),\, T^{(k)}_{13}(\xi,\eta) \\ &\le \int_0^\xi \left\{2\widehat{M}_0t + \frac{1}{4\delta}\cdot 3\cdot 2\widehat{M}_0t^2 + \frac{1}{4}\cdot 2\widehat{M}_0t\left(\frac{2}{3}\right)^{k-1}\right\}\mathrm{d}t \\ &\le \widehat{M}_0\xi^2 + \frac{1}{2\delta}\widehat{M}_0\xi^3 + \frac{1}{4}\widehat{M}_0\xi^2\left(\frac{2}{3}\right)^{k-1} \le \left[\frac{3}{4}\cdot 2 + \frac{1}{2}\left(\frac{2}{3}\right)^{k-1}\right]\widehat{M}_0\xi^2. \end{aligned} \tag{4.60}$$

One also has by (4.59)

$$\begin{aligned} T^{(k)}_{14}(\xi,\eta) &\le \int_0^\xi \left\{6\widehat{M}_0t^2 + \widehat{M}_0t\left(\frac{2}{3}\right)^{k-1}\right\}(t,\eta)\,\mathrm{d}t \\ &\le \left[2\delta + \frac{1}{2}\left(\frac{2}{3}\right)^{k-1}\right]\widehat{M}_0\xi^2 \le \left[\frac{3}{4}\cdot 2 + \frac{1}{2}\left(\frac{2}{3}\right)^{k-1}\right]\widehat{M}_0\xi^2. \end{aligned} \tag{4.61}$$

Inserting the estimates of $T^{(k)}_{12-14}$ in (4.60) and (4.61) into (4.54) and (4.55) and using the assumptions in (4.56) again arrive at

$$\begin{aligned} &T^{(k)}_{12}(\xi,\eta),\, T^{(k)}_{13}(\xi,\eta) \\ &\le \int_0^\xi \left\{\left[\frac{3}{4}\cdot 2 + \frac{1}{2}\left(\frac{2}{3}\right)^{k-1}\right]\widehat{M}_0t + \frac{3}{4\delta}\left[\frac{3}{4}\cdot 2 + \frac{1}{2}\left(\frac{2}{3}\right)^{k-1}\right]\widehat{M}_0t^2 \right. \\ &\qquad \left. + \frac{1}{2}\widehat{M}_0t\left(\frac{2}{3}\right)^{k-1}\right\}\mathrm{d}t \\ &\le \left[\frac{3}{4}\cdot 2 + \frac{1}{2}\left(\frac{2}{3}\right)^{k-1}\right]\cdot\frac{1}{2}\widehat{M}_0\xi^2 + \left\{\frac{3}{8} + \frac{1}{8}\left(\frac{2}{3}\right)^{k-1}\right\}\widehat{M}_0\xi^2 + \frac{1}{4}\widehat{M}_0\xi^2\left(\frac{2}{3}\right)^{k-1} \\ &\le \left[\frac{3}{4}\cdot 2 + \frac{1}{2}\left(\frac{2}{3}\right)^{k-1}\right]\cdot\frac{3}{4}\widehat{M}_0\xi^2 + \frac{1}{2}\widehat{M}_0\xi^2\left(\frac{2}{3}\right)^{k-1} \\ &= \left[\left(\frac{3}{4}\right)^2\cdot 2 + \frac{1}{2}\sum_{j=0}^{1}\left(\frac{3}{4}\right)^j\left(\frac{2}{3}\right)^{k-1}\right]\widehat{M}_0\xi^2 \end{aligned} \tag{4.62}$$

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and

$$T_{14}^{(k)}(\xi,\eta) \le \int_0^{\xi} \left\{ 3 \cdot \left[\frac{3}{4} \cdot 2 + \frac{1}{2}\left(\frac{2}{3}\right)^{k-1} \right] \widehat{M}_0 t^2 + \widehat{M}_0 t \left(\frac{2}{3}\right)^{k-1} \right\} \mathrm{d}t$$
$$= \left[\frac{3}{4} \cdot 2 + \frac{1}{2}\left(\frac{2}{3}\right)^{k-1} \right] \widehat{M}_0 \xi^3 + \frac{1}{2} \widehat{M}_0 \xi^2 \left(\frac{2}{3}\right)^{k-1}$$
$$\le \left[\left(\frac{3}{4}\right)^2 \cdot 2 + \frac{1}{2} \sum_{j=0}^{1} \left(\frac{3}{4}\right)^j \left(\frac{2}{3}\right)^{k-1} \right] \widehat{M}_0 \xi^2. \tag{4.63}$$

We next repeatedly substitute the new obtained estimates of $T_{12-14}^{(k)}$ into (4.54) and (4.55) to get for arbitrary integer $\ell \ge 1$,

$$T_{12}^{(k)}(\xi,\eta), T_{13}^{(k)}(\xi,\eta), T_{14}^{(k)}(\xi,\eta)$$
$$\le \left[\left(\frac{3}{4}\right)^{\ell} \cdot 2 + \frac{1}{2} \sum_{j=0}^{\ell} \left(\frac{3}{4}\right)^j \left(\frac{2}{3}\right)^{k-1} \right] \widehat{M}_0 \xi^2. \tag{4.64}$$

It follows by the arbitrariness of ℓ that

$$T_{12}^{(k)}(\xi,\eta), T_{13}^{(k)}(\xi,\eta), T_{14}^{(k)}(\xi,\eta)$$
$$\le \frac{1}{2} \sum_{j=0}^{\infty} \left(\frac{3}{4}\right)^j \left(\frac{2}{3}\right)^{k-1} \widehat{M}_0 \xi^2 \le 2\widehat{M}_0 \xi^2 \left(\frac{2}{3}\right)^{k-1}, \tag{4.65}$$

which finishes the proof of the lemma. ■

By virtue of Lemma 4.3 and (4.44) and (4.45), we can achieve the following lemma.

Lemma 4.4 *Let the iterative sequence $\{\widetilde{v}^{(k)}\}$ be defined by (4.18). Then, for all $k \ge 0$, the following inequalities*

$$\left|\widetilde{v}^{(k+1)}(t,x) - \widetilde{v}^{(k)}(t,x)\right|, \left|\widetilde{v}_x^{(k+1)}(t,x) - \widetilde{v}_x^{(k)}(t,x)\right| \le \widehat{M}_0 t \left(\frac{2}{3}\right)^k \tag{4.66}$$

hold for $(t,x) \in [0,\delta] \times \mathbb{R}$.

Proof We show the lemma by using the argument of induction again. Obviously, the results of Lemma 4.1 and the fact $\widetilde{v}^{(0)}(t,x) \equiv 0$ indicate that (4.66) is valid for $k = 0$.

Now, suppose that all inequalities in (4.66) are true for $n = k$. Recalling the definitions of $T_{15}^{(k)}$ and $T_{16}^{(k)}$, the induction assumptions mean that

$$\begin{aligned} T_{15}^{(k)}(t,x) &= \left|\widetilde{v}^{(k)}(t,x) - \widetilde{v}^{(k-1)}(t,x)\right| \le \widehat{M}_0 t \left(\frac{2}{3}\right)^{k-1}, \\ T_{16}^{(k)}(t,x) &= \left|\widetilde{v}_x^{(k)}(t,x) - \widetilde{v}_x^{(k-1)}(t,x)\right| \le \widehat{M}_0 t \left(\frac{2}{3}\right)^{k-1}, \end{aligned} \tag{4.67}$$

from which and Lemma 4.3 one has

$$T_{12}^{(k)}(t,x), T_{13}^{(k)}(t,x), T_{14}^{(k)}(t,x) \le 2\widehat{M}_0 t^2 \left(\frac{2}{3}\right)^{k-1}, \tag{4.68}$$

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for any $(t,x) \in [0,\delta] \times \mathbb{R}$. Inserting (4.67) and (4.68) into (4.44) and (4.45) and making use of (4.16) give

$$\begin{aligned} &\left|\widetilde{v}^{(k+1)}(t,x) - \widetilde{v}^{(k)}(t,x)\right| \\ &\le \int_0^t \int_{\mathbb{R}} G(t-\varsigma,z)\left\{4\widehat{M}_0\varsigma^2\left(\frac{2}{3}\right)^{k-1} + \widehat{M}_0\varsigma\left(\frac{2}{3}\right)^{k-1}\right\}(\varsigma,x-z)\,\mathrm{d}z\mathrm{d}\varsigma \\ &\le (4t^2+t)\widehat{M}_0\left(\frac{2}{3}\right)^{k-1}\int_0^t \int_{\mathbb{R}} G(t-\varsigma,z)\,\mathrm{d}z\mathrm{d}\varsigma \\ &\le (4t^2+t)\widehat{M}_0\left(\frac{2}{3}\right)^{k-1}\cdot 2t \le (8\delta^2+2\delta)\widehat{M}_0 t\left(\frac{2}{3}\right)^{k-1} \le \widehat{M}_0 t\left(\frac{2}{3}\right)^{k} \end{aligned} \tag{4.69}$$

and

$$\begin{aligned} &\left|\widetilde{v}_x^{(k+1)}(t,x) - \widetilde{v}_x^{(k)}(t,x)\right| \\ &\le \int_0^t \int_{\mathbb{R}} \left|\frac{\partial G(t-\varsigma,z)}{\partial z}\right|\left\{4\widehat{M}_0\varsigma^2\left(\frac{2}{3}\right)^{k-1} + \widehat{M}_0\varsigma\left(\frac{2}{3}\right)^{k-1}\right\}(\varsigma,x-z)\,\mathrm{d}z\mathrm{d}\varsigma \\ &\le (4t^2+t)\widehat{M}_0\left(\frac{2}{3}\right)^{k-1}\int_0^t \int_{\mathbb{R}} \left|\frac{\partial G(t-\varsigma,z)}{\partial z}\right|\mathrm{d}z\mathrm{d}\varsigma \\ &\le (4t^2+t)\widehat{M}_0\left(\frac{2}{3}\right)^{k-1}\cdot 4\sqrt{t} \le (16\delta\sqrt{\delta}+4\sqrt{\delta})\widehat{M}_0 t\left(\frac{2}{3}\right)^{k-1} \\ &\le \widehat{M}_0 t\left(\frac{2}{3}\right)^{k}, \end{aligned} \tag{4.70}$$

by the choice of δ in (4.20). We combine (4.69) and (4.70) to complete the proof of the lemma. ∎

4.4 The improved regularity of the iterative sequence

In this subsection, we improve the regularity of the function $\widetilde{v}_{xx}^{(k)}(t,x)$ with respect to x, which will be used to establish the uniform convergence of sequences $\{(\widetilde{v}_t^{(k)}, \widetilde{v}_{xt}^{(k)}, \widetilde{v}_{xx}^{(k)})(t,x)\}$ in the space $\Sigma(\delta)$.

Integrating system (4.46) along the characteristic curves $x_i^{(k)}(t)(i=\pm,0)$ and differentiating the resulting with respect to η, we deduce for any $(\xi,\eta) \in [0,\delta] \times \mathbb{R}$,

$$\begin{cases} \widetilde{R}_\eta^{(k)}(\xi,\eta) = \displaystyle\int_0^\xi \frac{\partial F_1^{(k)}}{\partial x}\cdot\frac{\partial x_-^{(k)}}{\partial \eta}(t,x_-^{(k)}(t))\,\mathrm{d}t, \\ \widetilde{S}_\eta^{(k)}(\xi,\eta) = \displaystyle\int_0^\xi \frac{\partial F_2^{(k)}}{\partial x}\cdot\frac{\partial x_+^{(k)}}{\partial \eta}(t,x_+^{(k)}(t))\,\mathrm{d}t, \\ \theta_\eta^{(k)}(\xi,\eta) = \displaystyle\int_0^\xi \frac{\partial F_3^{(k)}}{\partial \eta}(t,\eta)\,\mathrm{d}t, \end{cases} \tag{4.71}$$

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where $x_\pm^{(k)}(t) = x_\pm^{(k)}(t;\xi,\eta)$ are determined by (4.48), and

$$\frac{\partial x_\pm^{(k)}}{\partial \eta} = \exp\left\{ \int_\xi^t \theta_x^{(k)}(s, x_\pm^{(k)}(s))\, \mathrm{d}s \right\}. \tag{4.72}$$

From (4.71), one obtains for any $(\xi,\eta_1),(\xi,\eta_2) \in [0,\delta]\times\mathbb{R}$,

$$\begin{aligned} I_5^{(k)} &:= \left|\widetilde{R}_\eta^{(k)}(\xi,\eta_1) - \widetilde{R}_\eta^{(k)}(\xi,\eta_2)\right| \le \int_0^\xi I_7^{(k)}\left|\frac{\partial x_-^{(k)}}{\partial \eta}\right| + I_8^{(k)}\left|\frac{\partial F_1^{(k)}}{\partial x}\right| \mathrm{d}t, \\ I_6^{(k)} &:= \left|\widetilde{S}_\eta^{(k)}(\xi,\eta_1) - \widetilde{S}_\eta^{(k)}(\xi,\eta_2)\right| \le \int_0^\xi I_9^{(k)}\left|\frac{\partial x_+^{(k)}}{\partial \eta}\right| + I_{10}^{(k)}\left|\frac{\partial F_2^{(k)}}{\partial x}\right| \mathrm{d}t, \end{aligned} \tag{4.73}$$

where

$$\begin{aligned} I_7^{(k)} &= \left|\frac{\partial F_1^{(k)}}{\partial x}(t, x_-^{(k)}(t;\xi,\eta_1)) - \frac{\partial F_1^{(k)}}{\partial x}(t, x_-^{(k)}(t;\xi,\eta_2))\right|, \\ I_8^{(k)} &= \left|\frac{\partial x_-^{(k)}}{\partial \eta}(t, x_-^{(k)}(t;\xi,\eta_1)) - \frac{\partial x_-^{(k)}}{\partial \eta}(t, x_-^{(k)}(t;\xi,\eta_2))\right|, \\ I_9^{(k)} &= \left|\frac{\partial F_2^{(k)}}{\partial x}(t, x_+^{(k)}(t;\xi,\eta_1)) - \frac{\partial F_2^{(k)}}{\partial x}(t, x_+^{(k)}(t;\xi,\eta_2))\right|, \\ I_{10}^{(k)} &= \left|\frac{\partial x_+^{(k)}}{\partial \eta}(t, x_+^{(k)}(t;\xi,\eta_1)) - \frac{\partial x_+^{(k)}}{\partial \eta}(t, x_+^{(k)}(t;\xi,\eta_2))\right|. \end{aligned}$$

We point out by the estimate $\theta_{\eta\eta}^{(k)}$ in (4.34) that there is no need to consider the difference between $\theta_\eta^{(k)}(\xi,\eta_1)$ and $\theta_\eta^{(k)}(\xi,\eta_2)$. One performs direct calculations and uses (4.34) and (4.72) to achieve

$$\left|\frac{\partial x_\pm^{(k)}}{\partial \eta}\right| \le \exp\left(\int_0^\xi \frac{1}{4}\widehat{M}_0 s\, \mathrm{d}s \right) \le e^{\widehat{M}_0\delta^2} \tag{4.74}$$

and

$$\begin{aligned} &\left|\frac{\partial x_\pm^{(k)}}{\partial \eta}(t, x_\pm^{(k)}(t;\xi,\eta_1)) - \frac{\partial x_\pm^{(k)}}{\partial \eta}(t, x_\pm^{(k)}(t;\xi,\eta_2))\right| \\ &\le \exp\left(\int_0^\xi \frac{1}{4}\widehat{M}_0 s\, \mathrm{d}s \right) \int_t^\xi |\theta_{xx}^{(k)}| \cdot \left|x_\pm^{(k)}(s;\xi,\eta_1) - x_\pm^{(k)}(s;\xi,\eta_2)\right| \mathrm{d}s \\ &\le \frac{1}{4} e^{\widehat{M}_0\delta^2} \overline{M}\widehat{M}_0 \xi^2 D_\pm^{(k)}, \end{aligned} \tag{4.75}$$

where

$$D_\pm^{(k)} = \max_{t\in[0,\xi]} \left|x_\pm^{(k)}(t;\xi,\eta_1) - x_\pm^{(k)}(t;\xi,\eta_2)\right|.$$

Recalling the definitions of $x_\pm^{(k)}(t;\xi,\eta)$ in (4.48) and utilizing (4.34) get

$$
\begin{aligned}
&\left|x_\pm^{(k)}(t;\xi,\eta_1)-x_\pm^{(k)}(t;\xi,\eta_2)\right|\\
&\le|\eta_1-\eta_2|+\int_t^\xi\left|\theta^{(k)}(s,x_\pm^{(k)}(s;\xi,\eta_1))-\theta^{(k)}(s,x_\pm^{(k)}(s;\xi,\eta_2))\right|\mathrm{d}s\\
&\le|\eta_1-\eta_2|+\int_t^\xi|\theta_x^{(k)}|\cdot\left|x_\pm^{(k)}(s;\xi,\eta_1)-x_\pm^{(k)}(s;\xi,\eta_2)\right|\mathrm{d}s\\
&\le|\eta_1-\eta_2|+\frac{1}{8}\widehat{M}_0\xi^2D_\pm^{(k)},
\end{aligned}
$$

for any $t\in[0,\xi]$, from which one finds that

$$
D_\pm^{(k)}\le 2|\eta_1-\eta_2|. \tag{4.76}
$$

We put (4.76) into (4.75) to gain

$$
I_8^{(k)},I_{10}^{(k)}\le\frac{1}{2}e^{\widehat{M}_0\delta^2}\overline{M}\widehat{M}_0\xi^2|\eta_1-\eta_2|\le\frac{1}{12}\widehat{M}_0^2\xi^2|\eta_1-\eta_2|. \tag{4.77}
$$

Moreover, by applying (4.20), (4.34), (4.51), and Lemma 4.1, one computes

$$
\begin{aligned}
\left|\frac{\partial F_1^{(k)}}{\partial x}\right|,\left|\frac{\partial F_2^{(k)}}{\partial x}\right|&\le|\Psi^{(k)}|\left(|\widetilde{R}_x^{(k)}-\widetilde{S}_x^{(k)}|+\frac{|(\widetilde{R}^{(k)}-\widetilde{S}^{(k)})\theta_x^{(k)}|}{\theta^{(k)}}\right)\\
&\quad+|\widetilde{R}_x^{(k)}-\widetilde{S}_x^{(k)}|+\left|\frac{\widetilde{R}_x^{(k)}+\widetilde{S}_x^{(k)}-2\widetilde{v}_x^{(k)}+2\theta_0'}{4\theta^{(k)}}(\widetilde{R}^{(k)}-\widetilde{S}^{(k)})\right|\\
&\quad+|\theta_x^{(k)}||\theta_0'+\widetilde{v}_x^{(k)}|+\theta^{(k)}|\theta_0''-\widetilde{v}_{xx}^{(k)}|\\
&\le\left(\frac{1}{2t}+\frac{2\widehat{M}_0}{\underline{\theta}}\right)\left(2\widehat{M}_0t^2+2\cdot(3\overline{M}\overline{\theta}t)^2\right)+\frac{2\widehat{M}_0t^2+2\widehat{M}_0t+2\overline{\theta}_1}{2}\cdot3\overline{M}\overline{\theta}t\\
&\quad+2\widehat{M}_0t^2+3\overline{M}\overline{\theta}t(\overline{\theta}_0+\widehat{M}_0t)+2\overline{\theta}t(\overline{\theta}_0+\widehat{M}_0\sqrt{t})\\
&\le\left(\frac{1}{2}+\frac{2\widehat{M}_0\delta}{\underline{\theta}}\right)\left(2\widehat{M}_0t+\frac{18}{12}\widehat{M}_0t\right)+\frac{3}{16}\overline{M}^2\overline{\theta}t+2\delta\widehat{M}_0t\\
&\quad+3\overline{M}\overline{\theta}t\left(\frac{1}{16}\overline{M}+\widehat{M}_0\delta\right)+2\overline{\theta}t\left(\frac{1}{16}\overline{M}+\widehat{M}_0\sqrt{\delta}\right)\\
&\le\left(\frac{1}{2}+\frac{1}{8}\right)4\widehat{M}_0t+\frac{1}{16}\widehat{M}_0t+\frac{1}{16}\widehat{M}_0t+\frac{3\widehat{M}_0t}{8\times12}+\frac{\widehat{M}_0t}{8\times16}+\frac{1}{8}\widehat{M}_0t\\
&\le3\widehat{M}_0t.
\end{aligned} \tag{4.78}
$$

To estimate $I_7^{(k)}$, we rewrite the term $F_{1x}^{(k)}$ as

$$
\frac{\partial F_1^{(k)}}{\partial x}=(\Psi^{(k)}-1)(\widetilde{R}_x^{(k)}-\widetilde{S}_x^{(k)})+\frac{1}{2}\theta_x^{(k)}(\widetilde{R}_x^{(k)}+\widetilde{S}_x^{(k)})-\theta^{(k)}\widetilde{v}_{xx}^{(k)}+I_{11}^{(k)}, \tag{4.79}
$$

where $\Psi^{(k)}$ is defined in (4.47) and

$$
I_{11}^{(k)}=-2\Psi^{(k)}(\theta_x^{(k)})^2+\theta^{(k)}\theta_0''+2\theta_x^{(k)}\theta_0'-2\theta_x^{(k)}\widetilde{v}_x^{(k)}.
$$

Here, we used the relation $\theta_x^{(k)}=(\widetilde{R}^{(k)}-\widetilde{S}^{(k)})/2\theta^{(k)}$ by (3.123). Note that the term $I_{11}^{(k)}$ is differentiable with respect to x. Doing a direct calculation and employing (4.20),

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(4.34), (4.51), and Lemma 4.1 arrive at

$$
\begin{aligned}
\left|\Psi_x^{(k)}\right| &\le \left|\frac{\widetilde{R}_x^{(k)}+\widetilde{S}_x^{(k)}-2\widetilde{v}_x^{(k)}+2\theta_0'}{4\theta^{(k)}}\right| + \left|\Psi^{(k)}\right|\cdot\left|\frac{\theta_x^{(k)}}{\theta^{(k)}}\right| \\
&\le \frac{2M_0t^2+2M_0t+2\overline{\theta}_0}{4\cdot\frac{1}{2}\underline{\theta}t} + \left(\frac{1}{2t}+\frac{2\widehat{M}_0}{\underline{\theta}}\right)\cdot\frac{3}{2}\overline{M} \\
&\le \frac{1}{t}\left(\frac{\widehat{M}_0\delta^2}{\underline{\theta}}+\frac{\widehat{M}_0\delta}{\underline{\theta}}+\frac{\overline{\theta}_0}{\underline{\theta}}+\frac{3}{4}\overline{M}+\frac{3\widehat{M}_0\delta}{\underline{\theta}}\overline{M}\right) \le \frac{2\overline{M}}{t} \le \frac{\widehat{M}_0}{8t}
\end{aligned}
\tag{4.80}
$$

and

$$
\begin{aligned}
\left|\frac{\partial I_{11}^{(k)}}{\partial x}\right| &= \Big| -2\Psi_x^{(k)}(\theta_x^{(k)})^2 - 4\Psi^{(k)}\theta_x^{(k)}\theta_{xx}^{(k)} + \theta^{(k)}\theta_0''' \\
&\quad + 3\theta_x^{(k)}\theta_0'' + 2\theta_{xx}^{(k)}\theta_0' - 2\theta_{xx}^{(k)}\widetilde{v}_x^{(k)} - 2\theta_x^{(k)}\widetilde{v}_{xx}^{(k)}\Big| \\
&\le \frac{4\overline{M}}{t}(3\overline{M}\overline{\theta}t)^2 + 4\left(\frac{1}{2t}+\frac{2\widehat{M}_0}{\underline{\theta}}\right)\cdot 3\overline{M}\overline{\theta}t\cdot 6\overline{M}^2\overline{\theta}t + \frac{1}{32}\widehat{M}_0\overline{\theta}_0 t \\
&\quad + 9\overline{\theta}_0\overline{M}\overline{\theta}t + 6\overline{\theta}_0\overline{M}^2\overline{\theta}t + 12\overline{M}^2\overline{\theta}\widehat{M}_0t^2 + 6\overline{M}\overline{\theta}\widehat{M}_0t\sqrt{t} \\
&\le 72\overline{M}^3\overline{\theta}^2 t + \frac{16\widehat{M}_0\delta}{\underline{\theta}}\cdot 9\overline{M}^3\overline{\theta}^2 t + \frac{1}{32}\widehat{M}_0\overline{\theta}_1 t + 15\overline{\theta}_1\overline{M}^2\overline{\theta}t + 18\overline{M}^2\overline{\theta}\widehat{M}_0t\sqrt{t} \\
&\le \frac{1}{8}\widehat{M}_0^2t + \frac{1}{64}\widehat{M}_0^2t + \frac{1}{16^2}\widehat{M}_0^2t + \frac{1}{24}\widehat{M}_0^2t + \frac{1}{24}\widehat{M}_0^2t \le \frac{1}{4}\widehat{M}_0^2t,
\end{aligned}
\tag{4.81}
$$

by the choice $\widehat{M}_0 \ge 24\overline{M}^2\overline{\theta}$ in (4.20). We combine (4.79)–(4.81) and apply (4.34), (4.76), and Lemmas 4.1 and 4.2 to achieve

$$
\begin{aligned}
I_7^{(k)} &\le \left(|\Psi^{(k)}| + 1 + \frac{1}{2}|\theta_x^{(k)}|\right)(I_5^{(k)}+I_6^{(k)}) \\
&\quad + |\theta^{(k)}|\cdot\left|\widetilde{v}_{xx}^{(k)}(t,x_-^{(k)}(t;\xi,\eta_1)) - \widetilde{v}_{xx}^{(k)}(t,x_-^{(k)}(t;\xi,\eta_2))\right| \\
&\quad + \left|\Psi^{(k)}(t,x_-^{(k)}(t;\xi,\eta_1)) - \Psi^{(k)}(t,x_-^{(k)}(t;\xi,\eta_2))\right|\cdot|\widetilde{R}_x^{(k)} - \widetilde{S}_x^{(k)}| \\
&\quad + \frac{1}{2}\left|\theta_x^{(k)}(t,x_-^{(k)}(t;\xi,\eta_1)) - \theta_x^{(k)}(t,x_-^{(k)}(t;\xi,\eta_2))\right|\cdot|\widetilde{R}_x^{(k)} + \widetilde{S}_x^{(k)}| \\
&\quad + \left|\theta^{(k)}(t,x_-^{(k)}(t;\xi,\eta_1)) - \theta^{(k)}(t,x_-^{(k)}(t;\xi,\eta_2))\right|\cdot|\widetilde{v}_{xx}^{(k)}| \\
&\quad + \left|I_{11}^{(k)}(t,x_-^{(k)}(t;\xi,\eta_1)) - I_{11}^{(k)}(t,x_-^{(k)}(t;\xi,\eta_2))\right| \\
&\le \left(\frac{1}{2t}+\frac{3\widehat{M}_0}{\underline{\theta}}\right)(I_5^{(k)}+I_6^{(k)}) + 2\overline{\theta}\widehat{C}_{1/2}t\sqrt{|\eta_1-\eta_2|} \\
&\quad + \left(\frac{1}{4}\widehat{M}_0^2t + \frac{1}{2}\widehat{M}_0^2t^3 + \frac{1}{2}\widehat{M}_0^2t\sqrt{t} + \frac{1}{2}\widehat{M}_0^2t\right)|\eta_1-\eta_2|.
\end{aligned}
\tag{4.82}
$$

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Here, we chosen $\alpha = \frac{1}{2}$ in Lemma 4.2 and then

$$2\overline{\theta}\widehat{C}_{1/2} = 4\overline{\theta}\widehat{M}_0\left(C_{2,1} + 2C_{3,\frac{1}{2}} + C_{3,0}\right) \le \frac{1}{8}\widehat{M}_0^2.$$

If $|\eta_1 - \eta_2| \le 1$, then one obtains by (4.82)

$$\begin{aligned} I_7^{(k)}, I_9^{(k)} &\le \left(\frac{1}{2t} + \frac{3\widehat{M}_0}{\underline{\theta}}\right)(I_5^{(k)} + I_6^{(k)}) + \widehat{M}_0^2 t\sqrt{|\eta_1 - \eta_2|} \\ &\le \frac{3}{4t}(I_5^{(k)} + I_6^{(k)}) + \widehat{M}_0^2 t\sqrt{|\eta_1 - \eta_2|}. \end{aligned} \tag{4.83}$$

Now, we put (4.74), (4.77), (4.78), and (4.83) into (4.73) to find that if $|\eta_1 - \eta_2| \le 1$,

$$\begin{aligned} I_5^{(k)}, I_6^{(k)} &\le \int_0^\xi \left\{\frac{3}{4t}e^{\widehat{M}_0\delta^2}(I_5^{(k)} + I_6^{(k)}) + \left(e^{\widehat{M}_0\delta^2} + \frac{1}{4}\widehat{M}_0\delta^2\right)\widehat{M}_0^2 t\sqrt{|\eta_1 - \eta_2|}\right\} \mathrm{d}t \\ &\le \int_0^\xi \frac{4}{5t}(I_5^{(k)} + I_6^{(k)})\, \mathrm{d}t + \frac{1}{2}\left(\frac{16}{15} + \frac{1}{4}\widehat{M}_0\delta^2\right)\widehat{M}_0^2\xi^2\sqrt{|\eta_1 - \eta_2|} \\ &\le \int_0^\xi \frac{4}{5t}(I_5^{(k)} + I_6^{(k)})\, \mathrm{d}t + \widehat{M}_0^2\xi^2\sqrt{|\eta_1 - \eta_2|}, \end{aligned} \tag{4.84}$$

by the fact $15e^{\widehat{M}_0\delta^2} \le 16$.

In view of (4.84), we have the following lemma.

Lemma 4.5 *The functions $\widetilde{R}_x^{(k)}(t,x)$, $\widetilde{S}_x^{(k)}(t,x)$, and $\widetilde{v}_{xx}^{(k)}(t,x)(k = 0,1,\dots)$ are uniformly Lipschitz continuous with respect to x. More precisely, there hold for any $k \ge 0$ and any two points $(t,x_1), (t,x_2) \in [0,\delta] \times \mathbb{R}$,*

$$\begin{aligned} \left|\widetilde{R}_x^{(k)}(t,x_1) - \widetilde{R}_x^{(k)}(t,x_2)\right|, \left|\widetilde{S}_x^{(k)}(t,x_1) - \widetilde{S}_x^{(k)}(t,x_2)\right| &\le 5\widehat{M}_0^2 t^2|x_1 - x_2|, \\ \left|\widetilde{v}_{xx}^{(k)}(t,x_1) - \widetilde{v}_{xx}^{(k)}(t,x_2)\right| &\le \widehat{M}_0 t^{\frac{1}{4}}|x_1 - x_2|. \end{aligned} \tag{4.85}$$

Proof We first show that the functions $\widetilde{R}_\eta^{(k)}(\xi,\eta)$ and $\widetilde{S}_\eta^{(k)}(\xi,\eta)(k = 0,1,\dots)$ are uniformly 1/2-Hölder continuous with respect to η. For any two numbers η_1, η_2, if $|\eta_1 - \eta_2| > 1$, then one acquires by (4.34)

$$\begin{aligned} &\left|\widetilde{R}_\eta^{(k)}(\xi,\eta_1) - \widetilde{R}_\eta^{(k)}(\xi,\eta_2)\right|, \left|\widetilde{S}_\eta^{(k)}(\xi,\eta_1) - \widetilde{S}_\eta^{(k)}(\xi,\eta_2)\right| \\ &\le 2\widehat{M}_0\xi^2 \le 2\widehat{M}_0\xi^2\sqrt{|\eta_1 - \eta_2|}. \end{aligned} \tag{4.86}$$

If $|\eta_1 - \eta_2| \le 1$, then the inequality (4.84) is valid. Due to the definitions of $I_5^{(k)}$ and $I_6^{(k)}$ in (4.73), we get

$$I_5^{(k)}, I_6^{(k)} \le 2\widehat{M}_0\xi^2 \le \widehat{M}_0^2\xi^2.$$

Inserting the above into (4.84) leads to

$$\begin{aligned} I_5^{(k)}, I_6^{(k)} &\le \int_0^\xi \frac{4}{5} \cdot 2\widehat{M}_0^2 t\, \mathrm{d}t + \widehat{M}_0^2\xi^2\sqrt{|\eta_1 - \eta_2|} \\ &\le \frac{4}{5}\widehat{M}_0^2\xi^2 + \widehat{M}_0^2\xi^2\sqrt{|\eta_1 - \eta_2|}. \end{aligned} \tag{4.87}$$

We put (4.87) into (4.84) again to arrive at

$$I_5^{(k)}, I_6^{(k)} \le \left(\frac{4}{5}\right)^2 \widehat{M}_0^2 \xi^2 + \sum_{j=0}^{1} \left(\frac{4}{5}\right)^j \widehat{M}_0^2 \xi^2 \sqrt{|\eta_1 - \eta_2|}. \tag{4.88}$$

Repeating the above insertion process yields

$$I_5^{(k)}, I_6^{(k)} \le \sum_{j=0}^{\infty} \left(\frac{4}{5}\right)^j \widehat{M}_0^2 \xi^2 \sqrt{|\eta_1 - \eta_2|} = 5\widehat{M}_0^2 \xi^2 \sqrt{|\eta_1 - \eta_2|}. \tag{4.89}$$

One combines (4.86) and (4.89) to achieve for any $(\xi, \eta_1), (\xi, \eta_2) \in [0, \delta] \times \mathbb{R}$,

$$\begin{aligned} &\left|\widetilde{R}_\eta^{(k)}(\xi, \eta_1) - \widetilde{R}_\eta^{(k)}(\xi, \eta_2)\right|, \left|\widetilde{S}_\eta^{(k)}(\xi, \eta_1) - \widetilde{S}_\eta^{(k)}(\xi, \eta_2)\right| \\ &\le 5\widehat{M}_0^2 \xi^2 \sqrt{|\eta_1 - \eta_2|}. \end{aligned} \tag{4.90}$$

Next, we prove that $\widetilde{v}_{xx}^{(k)}(t,x) (k = 0, 1, \dots)$ are uniformly Lipschitz continuous with respect to x. Recalling (4.7) and (4.18) gives

$$\widetilde{v}_{xx}^{(k)}(t,x) = \int_0^t \int_{\mathbb{R}} \frac{\partial G(t-\varsigma, x-z)}{\partial x} b_x^{(k)}(\varsigma, z)\, \mathrm{d}z \mathrm{d}\varsigma, \tag{4.91}$$

from which one has

$$\begin{aligned} \frac{\partial}{\partial x} \widetilde{v}_{xx}^{(k)}(t,x) &= \int_0^t \int_{\mathbb{R}} \frac{\partial^2 G(t-\varsigma, x-z)}{\partial x^2} b_x^{(k)}(\varsigma, z)\, \mathrm{d}z \mathrm{d}\varsigma \\ &= \int_0^t \int_{\mathbb{R}} \frac{\partial^2 G(t-\varsigma, x-z)}{\partial x^2} [b_x^{(k)}(\varsigma, z) - b_x^{(k)}(\varsigma, x)]\, \mathrm{d}z \mathrm{d}\varsigma. \end{aligned} \tag{4.92}$$

Here, the term $b_x^{(k)}$ is

$$\begin{aligned} b_x^{(k)}(t,x) = &\frac{\widetilde{R}_x^{(k)}(t,x) + \widetilde{S}_x^{(k)}(t,x)}{2} - \theta_x^{(k)}(t,x) \\ &- \widetilde{v}_x^{(k)}(t,x) + \theta_0'(x) + v_0'''(x), \end{aligned} \tag{4.93}$$

which satisfies by (4.34) and (4.90)

$$\begin{aligned} &\left|b_x^{(k)}(t,x_1) - b_x^{(k)}(t,x_2)\right| \\ &\le \sqrt{2|\theta_x^{(k)}|} \cdot \sqrt{|\theta_{xx}^{(k)}| \cdot |x_1 - x_2|} + \sqrt{2|\widetilde{v}_x^{(k)}|} \cdot \sqrt{|\widetilde{v}_{xx}^{(k)}| \cdot |x_1 - x_2|} \\ &\quad + 5\widehat{M}_0^2 t^2 \sqrt{|x_1 - x_2|} + \sqrt{2|\theta_0'(x) + v_0'''(x)|} \cdot \sqrt{|\theta_0''(x) + v_0''''(x)| \cdot |x_1 - x_2|} \\ &\le \left(5\widehat{M}_0^2 \delta^2 + \frac{\sqrt{2}}{4}\widehat{M}_0 \delta + \sqrt{2}\widehat{M}_0 \delta^{\frac{3}{4}} + 2\widehat{K}\right) \sqrt{|x_1 - x_2|} \\ &\le \frac{1}{8}\widehat{M}_0 \sqrt{|x_1 - x_2|}. \end{aligned} \tag{4.94}$$

Putting (4.94) into (4.92) and making use of (4.10) yield

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$$\left|\frac{\partial}{\partial x}\widetilde{v}_{xx}^{(k)}(t,x)\right| \le \frac{1}{8}\widehat{M}_0\int_0^t\int_{\mathbb{R}}\left|\frac{\partial^2 G(t-\varsigma,x-z)}{\partial x^2}\right|\cdot\sqrt{|x-z|}\,\mathrm{d}z\mathrm{d}\varsigma$$

$$\le \frac{1}{8}\widehat{M}_0\int_0^t\int_{\mathbb{R}}H_{\frac{5}{2}}(t-\varsigma,z)\,\mathrm{d}z\mathrm{d}\varsigma = \frac{1}{8}\widehat{M}_0\cdot\frac{4}{3-\frac{5}{2}}t^{\frac{3-\frac{5}{2}}{2}} = \widehat{M}_0 t^{\frac{1}{4}}, \tag{4.95}$$

which implies that the function $\widetilde{v}_{xx}^{(k)}(t,x)$ satisfies

$$\left|\widetilde{v}_{xx}^{(k)}(t,x_1)-\widetilde{v}_{xx}^{(k)}(t,x_2)\right| \le \widehat{M}_0 t^{\frac{1}{4}}|x_1-x_2|. \tag{4.96}$$

Finally, we verify that the functions $\widetilde{R}_{\eta}^{(k)}(\xi,\eta)$ and $\widetilde{S}_{\eta}^{(k)}(\xi,\eta)(k=0,1,\ldots)$ are uniformly Lipschitz continuous with respect to η. By using (4.96), we re-estimate the terms $I_7^{(k)}$ and $I_9^{(k)}$ in (4.82) and (4.83) to see that for $|\eta_1-\eta_2|\le 1$,

$$\begin{aligned} I_7^{(k)}, I_9^{(k)} &\le \left(\frac{1}{2t}+\frac{3\widehat{M}_0}{\underline{\theta}}\right)(I_5^{(k)}+I_6^{(k)}) + \frac{1}{32}\widehat{M}_0 t\cdot\widehat{M}_0 t^{\frac{1}{4}}|\eta_1-\eta_2| \\ &\quad + \left(\frac{1}{4}\widehat{M}_0^2 t + \frac{1}{2}\widehat{M}_0^2 t^3 + \frac{1}{2}\widehat{M}_0^2 t\sqrt{t} + \frac{1}{2}\widehat{M}_0^2 t\right)|\eta_1-\eta_2| \\ &\le \frac{3}{4t}(I_5^{(k)}+I_6^{(k)}) + \widehat{M}_0^2 t|\eta_1-\eta_2|, \end{aligned} \tag{4.97}$$

from which the terms $I_5^{(k)}$ and $I_6^{(k)}$ in (4.84) can be improved to

$$I_5^{(k)}, I_6^{(k)} \le \int_0^{\xi}\frac{4}{5t}(I_5^{(k)}+I_6^{(k)})\,\mathrm{d}t + \widehat{M}_0^2\xi^2|\eta_1-\eta_2|. \tag{4.98}$$

Based on (4.98), one employs the same argument as (4.89) to gain

$$I_5^{(k)}, I_6^{(k)} \le \sum_{j=0}^{\infty}\left(\frac{4}{5}\right)^j\widehat{M}_0^2\xi^2|\eta_1-\eta_2| = 5\widehat{M}_0^2\xi^2|\eta_1-\eta_2|, \tag{4.99}$$

for $|\eta_1-\eta_2|\le 1$. Combining (4.90), (4.99), and (4.96) finishes the proof of the lemma. ∎

4.5 The convergence of the iterative sequence (II)

In this subsection, we continue the discussion of Section 4.3 to establish the properties for the sequences $\{(\widetilde{v}_t^{(k)},\widetilde{v}_{xt}^{(k)},\widetilde{v}_{xx}^{(k)})(t,x)\}$ in the space $\Sigma(\delta)$.

We first apply (4.18), (4.6)–(4.8), and (4.34) to deduce

$$\begin{aligned} &\left|\widetilde{v}_t^{(k+1)}(t,x)-\widetilde{v}_t^{(k)}(t,x)\right| \\ &\le \int_0^t\int_{\mathbb{R}}G(t-\varsigma,z)\left\{\frac{1}{2}(T_{18}^{(k)}+T_{19}^{(k)})+T_{20}^{(k)}+T_{21}^{(k)}\right\}(\varsigma,x-z)\,\mathrm{d}z\mathrm{d}\varsigma, \end{aligned} \tag{4.100}$$

$$\begin{aligned} &\left|\widetilde{v}_{xt}^{(k+1)}(t,x)-\widetilde{v}_{xt}^{(k)}(t,x)\right| \\ &\le \int_0^t\int_{\mathbb{R}}\left|\frac{\partial G(t-\varsigma,z)}{\partial z}\right|\left\{\frac{1}{2}(T_{18}^{(k)}+T_{19}^{(k)})+T_{20}^{(k)}+T_{21}^{(k)}\right\}(\varsigma,x-z)\,\mathrm{d}z\mathrm{d}\varsigma \end{aligned} \tag{4.101}$$

and

$$\begin{aligned}&\left|\widetilde{v}_{xx}^{(k+1)}(t,x)-\widetilde{v}_{xx}^{(k)}(t,x)\right|\\ &\le \int_0^t\int_{\mathbb{R}}\left|\frac{\partial G(t-\varsigma,z)}{\partial z}\right|\left\{\frac{1}{2}\left(T_{22}^{(k)}+T_{23}^{(k)}\right)+T_{24}^{(k)}+T_{16}^{(k)}\right\}(\varsigma,x-z)\,\mathrm{d}z\mathrm{d}\varsigma,\end{aligned}\tag{4.102}$$

where

$$\begin{aligned}&T_{18}^{(k)}(t,x)=\left|\widetilde{R}_t^{(k)}(t,x)-\widetilde{R}_t^{(k-1)}(t,x)\right|,\ T_{19}^{(k)}(t,x)=\left|\widetilde{S}_t^{(k)}(t,x)-\widetilde{S}_t^{(k-1)}(t,x)\right|,\\ &T_{20}^{(k)}(t,x)=\left|\theta_t^{(k)}(t,x)-\theta_t^{(k-1)}(t,x)\right|,\ T_{21}^{(k)}(t,x)=\left|\widetilde{v}_t^{(k)}(t,x)-\widetilde{v}_t^{(k-1)}(t,x)\right|,\\ &T_{22}^{(k)}(t,x)=\left|\widetilde{R}_x^{(k)}(t,x)-\widetilde{R}_x^{(k-1)}(t,x)\right|,\ T_{23}^{(k)}(t,x)=\left|\widetilde{S}_x^{(k)}(t,x)-\widetilde{S}_x^{(k-1)}(t,x)\right|,\\ &T_{24}^{(k)}(t,x)=\left|\theta_x^{(k)}(t,x)-\theta_x^{(k-1)}(t,x)\right|.\end{aligned}$$

Recalling the system (4.46) arrives at

$$\begin{aligned}T_{18}^{(k)}(t,x)&\le\left|F_1^{(k)}-F_1^{(k-1)}\right|+\left|\theta^{(k)}\widetilde{R}_x^{(k)}-\theta^{(k-1)}\widetilde{R}_x^{(k-1)}\right|\\ &\le\left|F_1^{(k)}-F_1^{(k-1)}\right|+|\theta^{(k)}|T_{22}^{(k)}+|\widetilde{R}_x^{(k-1)}|T_{14}^{(k)},\\ T_{19}^{(k)}(t,x)&\le\left|F_2^{(k)}-F_2^{(k-1)}\right|+\left|\theta^{(k)}\widetilde{S}_x^{(k)}-\theta^{(k-1)}\widetilde{S}_x^{(k-1)}\right|\\ &\le\left|F_2^{(k)}-F_2^{(k-1)}\right|+|\theta^{(k)}|T_{23}^{(k)}+|\widetilde{S}_x^{(k-1)}|T_{14}^{(k)},\\ T_{20}^{(k)}(t,x)&=\left|F_3^{(k)}-F_3^{(k-1)}\right|,\end{aligned}\tag{4.103}$$

from which and (4.53) and (4.49), we get

$$\begin{aligned}T_{18}^{(k)}(t,x)&\le\frac{T_{12}^{(k)}+T_{13}^{(k)}}{2t}+\frac{1}{4\delta}\left(T_{12}^{(k)}+T_{13}^{(k)}+T_{14}^{(k)}\right)\\ &\quad+\frac{1}{4}\left(T_{15}^{(k)}+T_{16}^{(k)}\right)+|\theta^{(k)}|T_{22}^{(k)},\\ T_{19}^{(k)}(t,x)&\le\frac{T_{12}^{(k)}+T_{13}^{(k)}}{2t}+\frac{1}{4\delta}\left(T_{12}^{(k)}+T_{13}^{(k)}+T_{14}^{(k)}\right)\\ &\quad+\frac{1}{4}\left(T_{15}^{(k)}+T_{16}^{(k)}\right)+|\theta^{(k)}|T_{23}^{(k)},\\ T_{20}^{(k)}(t,x)&\le\frac{1}{2}\left(T_{12}^{(k)}+T_{13}^{(k)}\right)+T_{14}^{(k)}+T_{15}^{(k)}.\end{aligned}\tag{4.104}$$

One utilizes Lemmas 4.3 and 4.4, (4.104), and (4.34) to acquire

$$\begin{aligned}T_{18}^{(k)}(t,x)&\le 4\widehat{M}_0t\left(\frac{2}{3}\right)^{k-1}+\frac{\widehat{M}_0t}{32}T_{22}^{(k)},\\ T_{19}^{(k)}(t,x)&\le 4\widehat{M}_0t\left(\frac{2}{3}\right)^{k-1}+\frac{\widehat{M}_0t}{32}T_{23}^{(k)},\\ T_{20}^{(k)}(t,x)&\le(4\delta+1)\widehat{M}_0t\left(\frac{2}{3}\right)^{k-1}\le 2\widehat{M}_0t\left(\frac{2}{3}\right)^{k-1},\\ T_{16}^{(k)}(t,x)&\le\widehat{M}_0t\left(\frac{2}{3}\right)^{k-1}.\end{aligned}\tag{4.105}$$

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We have by putting (4.105) into (4.100) and (4.102)

$$
\begin{aligned}
&\left|\widetilde{v}_t^{(k+1)}(t,x)-\widetilde{v}_t^{(k)}(t,x)\right| \le \int_0^t \int_{\mathbb{R}} G(t-\varsigma,z)\left\{6\widehat{M}_0\varsigma\left(\frac{2}{3}\right)^{k-1}\right.\\
&\quad\left.+T_{21}^{(k)}+\frac{\widehat{M}_0\varsigma}{64}\left(T_{22}^{(k)}+T_{23}^{(k)}\right)\right\}(\varsigma,x-z)\,\mathrm{d}z\mathrm{d}\varsigma,\\
&\left|\widetilde{v}_{xt}^{(k+1)}(t,x)-\widetilde{v}_{xt}^{(k)}(t,x)\right| \le \int_0^t \int_{\mathbb{R}} \left|\frac{\partial G(t-\varsigma,z)}{\partial z}\right|\left\{6\widehat{M}_0\varsigma\left(\frac{2}{3}\right)^{k-1}\right.\\
&\quad\left.+T_{21}^{(k)}+\frac{\widehat{M}_0\varsigma}{64}\left(T_{22}^{(k)}+T_{23}^{(k)}\right)\right\}(\varsigma,x-z)\,\mathrm{d}z\mathrm{d}\varsigma,\\
&\left|\widetilde{v}_{xx}^{(k+1)}(t,x)-\widetilde{v}_{xx}^{(k)}(t,x)\right| \le \int_0^t \int_{\mathbb{R}} \left|\frac{\partial G(t-\varsigma,z)}{\partial z}\right|\left\{2\widehat{M}_0\varsigma\left(\frac{2}{3}\right)^{k-1}\right.\\
&\quad\left.+\frac{1}{2}\left(T_{22}^{(k)}+T_{23}^{(k)}\right)+T_{24}^{(k)}\right\}(\varsigma,x-z)\,\mathrm{d}z\mathrm{d}\varsigma.
\end{aligned}
\tag{4.106}
$$

In order to proceed with the verification, it is necessary to estimate the terms $T_{22,23,24}^{(k)}$. Recalling system (4.71) leads to

$$
\begin{aligned}
T_{22}^{(k)}(\xi,\eta) &\le \int_0^\xi \left\{T_{25}^{(k)}\left|\frac{\partial x_-^{(k)}}{\partial \eta}\right| + \left|\frac{\partial F_1^{(k-1)}}{\partial x}\right| T_{28}^{(k)}\right\}\mathrm{d}t,\\
T_{23}^{(k)}(\xi,\eta) &\le \int_0^\xi \left\{T_{26}^{(k)}\left|\frac{\partial x_+^{(k)}}{\partial \eta}\right| + \left|\frac{\partial F_2^{(k-1)}}{\partial x}\right| T_{29}^{(k)}\right\}\mathrm{d}t,\\
T_{24}^{(k)}(\xi,\eta) &\le \int_0^\xi T_{27}^{(k)}\,\mathrm{d}t,
\end{aligned}
\tag{4.107}
$$

where

$$
\begin{aligned}
T_{25}^{(k)} &= \left|\frac{\partial F_1^{(k)}}{\partial x}(t,x_-^{(k)}(t)) - \frac{\partial F_1^{(k-1)}}{\partial x}(t,x_-^{(k-1)}(t))\right|,\\
T_{26}^{(k)} &= \left|\frac{\partial F_2^{(k)}}{\partial x}(t,x_-^{(k)}(t)) - \frac{\partial F_2^{(k-1)}}{\partial x}(t,x_-^{(k-1)}(t))\right|,\\
T_{27}^{(k)} &= \left|\frac{\partial F_3^{(k)}}{\partial \eta}(t,\eta) - \frac{\partial F_3^{(k-1)}}{\partial \eta}(t,\eta)\right|,\\
T_{28}^{(k)} &= \left|\frac{\partial x_-^{(k)}}{\partial \eta}(t,x_-^{(k)}(t)) - \frac{\partial x_-^{(k-1)}}{\partial \eta}(t,x_-^{(k-1)}(t))\right|,\\
T_{29}^{(k)} &= \left|\frac{\partial x_+^{(k)}}{\partial \eta}(t,x_+^{(k)}(t)) - \frac{\partial x_+^{(k-1)}}{\partial \eta}(t,x_+^{(k-1)}(t))\right|.
\end{aligned}
$$

We first remember (4.74) and (4.78) to obtain

$$
\left|\frac{\partial x_\pm^{(k)}}{\partial \eta}\right| \le e^{\widehat{M}_0\delta^2}, \qquad \left|\frac{\partial F_1^{(k-1)}}{\partial x}\right|, \left|\frac{\partial F_2^{(k-1)}}{\partial x}\right| \le 3\widehat{M}_0 t.
\tag{4.108}
$$

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Next, we are going to estimate the terms $T^{(k)}_{25-29}$. For the term $T^{(k)}_{27}$, it follows directly by Lemma 4.4 that

$$
\begin{aligned}
T^{(k)}_{27} =& \Bigg| \left(\frac{\widetilde{R}^{(k)}_\eta + \widetilde{S}^{(k)}_\eta}{2} - \theta^{(k)}_\eta - \widetilde{v}^{(k)}_\eta + \theta'_0 \right) \\
& - \left(\frac{\widetilde{R}^{(k-1)}_\eta + \widetilde{S}^{(k-1)}_\eta}{2} - \theta^{(k-1)}_\eta - \widetilde{v}^{(k-1)}_\eta + \theta'_0 \right) \Bigg| \\
\le& \frac{T^{(k)}_{22}(t,\eta) + T^{(k)}_{23}(t,\eta)}{2} + T^{(k)}_{24}(t,\eta) + |\widetilde{v}^{(k)}_\eta(t,\eta) - \widetilde{v}^{(k-1)}_\eta(t,\eta)| \\
\le& \frac{T^{(k)}_{22}(t,\eta) + T^{(k)}_{23}(t,\eta)}{2} + T^{(k)}_{24}(t,\eta) + \widehat{M}_0 t \left(\frac{2}{3}\right)^{k-1}.
\end{aligned}
\tag{4.109}
$$

For the terms $T^{(k)}_{28-29}$, one recalls the definitions of $\partial_\eta x^{(k)}_\pm$ in (4.72) to achieve

$$
\begin{aligned}
T^{(k)}_{28}, T^{(k)}_{29} &\le e^{\widehat{M}_0 \delta^2} \left| \int_t^\xi \left(\theta^{(k)}_x(s, x^{(k)}_\pm(s)) - \theta^{(k-1)}_x(s, x^{(k-1)}_\pm(s)) \right) \mathrm{d}s \right| \\
&\le e^{\widehat{M}_0 \delta^2} \int_0^\xi \left\{ T^{(k)}_{24}(s, x^{(k)}_\pm(s)) + |\theta^{(k-1)}_x(s, x^{(k)}_\pm(s)) - \theta^{(k-1)}_x(s, x^{(k-1)}_\pm(s))| \right\} \mathrm{d}s \\
&\le e^{\widehat{M}_0 \delta^2} \int_0^\xi \left\{ T^{(k)}_{24}(s, x^{(k)}_\pm(s)) + |\theta^{(k-1)}_{xx}| \cdot |x^{(k)}_\pm(s) - x^{(k-1)}_\pm(s)| \right\} \mathrm{d}s.
\end{aligned}
\tag{4.110}
$$

Moreover, it concludes by the definitions of $x^{(k)}_\pm(t;\xi,\eta)$ in (4.48) that

$$
\begin{aligned}
& x^{(k)}_\pm(t) \pm \int_t^\xi \theta^{(k)}(s, x^{(k)}_\pm(s;\xi,\eta)) \,\mathrm{d}s \\
=& \eta = x^{(k-1)}_\pm(t) \pm \int_t^\xi \theta^{(k-1)}(s, x^{(k-1)}_\pm(s;\xi,\eta)) \,\mathrm{d}s,
\end{aligned}
$$

from which and (4.34) and Lemma 4.3 one has

$$
\begin{aligned}
\left| x^{(k)}_\pm(t) - x^{(k-1)}_\pm(t) \right| &\le \int_0^\xi |\theta^{(k)}(s, x^{(k)}_\pm(s)) - \theta^{(k-1)}(s, x^{(k-1)}_\pm(s))| \,\mathrm{d}s \\
&\le \int_0^\xi \left\{ T^{(k)}_{14}(s, x^{(k)}_\pm(s)) + |\theta^{(k-1)}_x| \cdot |x^{(k)}_\pm(s) - x^{(k-1)}_\pm(s)| \right\} \mathrm{d}s \\
&\le \int_0^\xi \left\{ 2\widehat{M}_0 s^2 \left(\frac{2}{3}\right)^{k-1} + \frac{1}{4}\widehat{M}_0 s |x^{(k)}_\pm(s) - x^{(k-1)}_\pm(s)| \right\} \mathrm{d}s.
\end{aligned}
\tag{4.111}
$$

Thus, we find by (4.111) that

$$
\Delta^{(k)}_\pm := \max_{t\in[0,\xi]} \left| x^{(k)}_\pm(t) - x^{(k-1)}_\pm(t) \right| \le \widehat{M}_0 \xi^3 \left(\frac{2}{3}\right)^{k-1}.
\tag{4.112}
$$

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Putting (4.112) into (4.110) and using (4.34) again yield

$$\begin{aligned}T_{28}^{(k)}, T_{29}^{(k)} &\le e^{\widehat{M}_0\delta^2}\int_0^{\xi}\left\{T_{24}^{(k)}(s, x_{\pm}^{(k)}(s)) + \frac{1}{4}\widehat{M}_0 s\cdot\Delta_{\pm}^{(k)}\right\}\mathrm{d}s\\ &\le e^{\widehat{M}_0\delta^2}\int_0^{\xi} T_{24}^{(k)}(s, x_{\pm}^{(k)}(s))\,\mathrm{d}s + \frac{1}{4}\widehat{M}_0^2\xi^5\left(\frac{2}{3}\right)^{k-1}.\end{aligned}\tag{4.113}$$

Finally, we estimate $T_{25}^{(k)}$ by re-expressing it as

$$\begin{aligned}T_{25}^{(k)} &\le \left|\frac{\partial F_1^{(k)}}{\partial x}(t, x_-^{(k)}(t)) - \frac{\partial F_1^{(k-1)}}{\partial x}(t, x_-^{(k)}(t))\right|\\ &\quad+\left|\frac{\partial F_1^{(k-1)}}{\partial x}(t, x_-^{(k)}(t)) - \frac{\partial F_1^{(k-1)}}{\partial x}(t, x_-^{(k-1)}(t))\right|\\ &=: T_{30}^{(k)} + T_{31}^{(k-1)}.\end{aligned}\tag{4.114}$$

Recalling the expression of $\partial_x F_1^{(k)}$ in (4.79) and making use of (4.34) and (4.51) arrive at

$$\begin{aligned}T_{30}^{(k)} &\le \left(|\Psi^{(k)}| + 1 + \frac{1}{2}|\theta_x^{(k)}|\right)(T_{22}^{(k)} + T_{23}^{(k)}) + \frac{1}{2}|\widetilde{R}_x^{(k-1)} + \widetilde{S}_x^{(k-1)}|T_{24}^{(k)}\\ &\quad+|\widetilde{R}_x^{(k-1)} - \widetilde{S}_x^{(k-1)}|\cdot|\Psi^{(k)} - \Psi^{(k-1)}| + |\theta^{(k)}|\cdot|\widetilde{v}_{xx}^{(k)} - \widetilde{v}_{xx}^{(k-1)}|\\ &\quad+|\widetilde{v}_{xx}^{(k-1)}|\cdot|\theta^{(k)} - \theta^{(k-1)}| + |I_{11}^{(k)} - I_{11}^{(k-1)}|\\ &\le \left(\frac{1}{2t} + \frac{2\widehat{M}_0}{\underline{\theta}} + 1 + \frac{1}{8}\widehat{M}_0 t\right)(T_{22}^{(k)} + T_{23}^{(k)}) + 2\widehat{M}_0 t^2|\Psi^{(k)} - \Psi^{(k-1)}|\\ &\quad+\widehat{M}_0 t^2 T_{24}^{(k)} + 2\overline{\theta}t|\widetilde{v}_{xx}^{(k)} - \widetilde{v}_{xx}^{(k-1)}| + \widehat{M}_0\sqrt{t}T_{14}^{(k)} + |I_{11}^{(k)} - I_{11}^{(k-1)}|.\end{aligned}\tag{4.115}$$

By the expressions of $\Psi^{(k)}$ in (4.47) and $I_{11}^{(k)}$ in (4.79), we employ Lemmas 4.3 and 4.4 and (4.22), (4.34), and (4.51) again to gain

$$\begin{aligned}&|\Psi^{(k)} - \Psi^{(k-1)}|\\ &=\left|\frac{\widetilde{R}^{(k)} + \widetilde{S}^{(k)} - 2\widetilde{v}^{(k)} + 2\theta_0}{4\theta^{(k)}} - \frac{\widetilde{R}^{(k-1)} + \widetilde{S}^{(k-1)} - 2\widetilde{v}^{(k-1)} + 2\theta_0}{4\theta^{(k-1)}}\right|\\ &\le \frac{T_{12}^{(k)} + T_{13}^{(k)} + 2T_{15}^{(k)}}{4\theta^{(k)}} + |\Psi^{(k-1)}|\frac{T_{14}^{(k)}}{\theta^{(k)}}\\ &\le \frac{4\widehat{M}_0 t^2\left(\frac{2}{3}\right)^{k-1} + 2\widehat{M}_0 t\left(\frac{2}{3}\right)^{k-1}}{2\underline{\theta}t} + \left(\frac{1}{2t} + \frac{2\widehat{M}_0}{\underline{\theta}}\right)\frac{2\widehat{M}_0 t^2\left(\frac{2}{3}\right)^{k-1}}{\frac{1}{2}\underline{\theta}t}\\ &\le \left(\frac{3+2\delta}{\underline{\theta}} + \frac{8\widehat{M}_0\delta}{\underline{\theta}^2}\right)\widehat{M}_0\left(\frac{2}{3}\right)^{k-1} \le \frac{4}{\underline{\theta}}\widehat{M}_0\left(\frac{2}{3}\right)^{k-1}\end{aligned}\tag{4.116}$$

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and

$$
\begin{aligned}
|I_{11}^{(k)} - I_{11}^{(k-1)}| &\le 2|\theta_x^{(k)}|^2|\Psi^{(k)} - \Psi^{(k-1)}| + 2|\Psi^{(k-1)}| \cdot |(\theta_x^{(k)})^2 - (\theta_x^{(k-1)})^2| \\
&\quad + |\theta_0''| \cdot |\theta^{(k)} - \theta^{(k-1)}| + 2(|\theta_0'| + |\widetilde{v}_x^{(k)}|) \cdot |\theta_x^{(k)} - \theta_x^{(k-1)}| \\
&\quad + 2|\theta_x^{(k-1)}| \cdot |\widetilde{v}_x^{(k)} - \widetilde{v}_x^{(k-1)}| \\
&\le 2(3\overline{M}\overline{\theta}t)^2 \cdot \frac{4}{\underline{\theta}}\widehat{M}_0\left(\frac{2}{3}\right)^{k-1} + 2\left(\frac{1}{2t} + \frac{2\widehat{M}_0}{\underline{\theta}}\right) \cdot 6\overline{M}\overline{\theta}tT_{24}^{(k)} \\
&\quad + \overline{\theta}_0 T_{14}^{(k)} + 2(\overline{\theta}_0 + \widehat{M}_0 t)T_{24}^{(k)} + 6\overline{M}\overline{\theta}t \cdot T_{15}^{(k)} \\
&\le \frac{6\widehat{M}_0^2 t^2}{\underline{\theta}}\left(\frac{2}{3}\right)^{k-1} + \left(\frac{1}{2} + \frac{2\widehat{M}_0\delta}{\underline{\theta}}\right)\frac{\widehat{M}_0}{\overline{M}}T_{24}^{(k)} + \overline{\theta}_0 \cdot \widehat{M}_0 t^2\left(\frac{2}{3}\right)^{k-1} \\
&\quad + 2\left(\frac{\overline{M}}{16} + 1\right)T_{24}^{(k)} + \frac{1}{4}\widehat{M}_0 t \cdot \widehat{M}_0 t\left(\frac{2}{3}\right)^{k-1} \\
&\le \frac{1}{16}\widehat{M}_0 T_{24}^{(k)} + \frac{3}{4}\widehat{M}_0 t\left(\frac{2}{3}\right)^{k-1}. \qquad (4.117)
\end{aligned}
$$

One inserts (4.116) and (4.117) into (4.115) to get

$$
\begin{aligned}
T_{30}^{(k)} &\le \left(\frac{1}{2t} + \frac{2\widehat{M}_0}{\underline{\theta}} + 1 + \frac{1}{8}\widehat{M}_0 t\right)(T_{22}^{(k)} + T_{23}^{(k)}) \\
&\quad + 2\widehat{M}_0 t^2 \cdot \frac{4}{\underline{\theta}}\widehat{M}_0\left(\frac{2}{3}\right)^{k-1} + \widehat{M}_0 t^2 T_{24}^{(k)} + 2\overline{\theta}t|\widetilde{v}_{xx}^{(k)} - \widetilde{v}_{xx}^{(k-1)}| \\
&\quad + \widehat{M}_0\sqrt{t} \cdot \widehat{M}_0 t^2\left(\frac{2}{3}\right)^{k-1} + \frac{1}{16}\widehat{M}_0 T_{24}^{(k)} + \frac{3}{4}\widehat{M}_0 t\left(\frac{2}{3}\right)^{k-1} \\
&\le \left(\frac{1}{2t} + \frac{3\widehat{M}_0}{\underline{\theta}}\right)(T_{22}^{(k)} + T_{23}^{(k)}) + \frac{1}{15}\widehat{M}_0 T_{24}^{(k)} + \frac{3}{2}\widehat{M}_0 t\left(\frac{2}{3}\right)^{k-1} \\
&\quad + 2\overline{\theta}t|\widetilde{v}_{xx}^{(k)} - \widetilde{v}_{xx}^{(k-1)}|. \qquad (4.118)
\end{aligned}
$$

In addition, for the term $T_{31}^{(k-1)}$, we also use the expression of $\partial_x F_1^{(k-1)}$ as in (4.79) to derive

$$
\begin{aligned}
T_{31}^{(k-1)} &\le \left(|\Psi^{(k-1)}| + 1 + \frac{1}{2}|\theta_x^{(k-1)}|\right) \\
&\quad \times \Big\{ \left|\widetilde{R}_x^{(k-1)}(t, x_-^{(k)}(t)) - \widetilde{R}_x^{(k-1)}(t, x_-^{(k-1)}(t))\right| \\
&\quad + \left|\widetilde{S}_x^{(k-1)}(t, x_-^{(k)}(t)) - \widetilde{S}_x^{(k-1)}(t, x_-^{(k-1)}(t))\right| \Big\} \\
&\quad + |\widetilde{R}_x^{(k-1)} - \widetilde{S}_x^{(k-1)}| \cdot \left|\Psi^{(k-1)}(t, x_-^{(k)}(t)) - \Psi^{(k-1)}(t, x_-^{(k-1)}(t))\right| \\
&\quad + \frac{1}{2}|\widetilde{R}_x^{(k-1)} + \widetilde{S}_x^{(k-1)}| \cdot \left|\theta_x^{(k-1)}(t, x_-^{(k)}(t)) - \theta_x^{(k-1)}(t, x_-^{(k-1)}(t))\right|
\end{aligned}
$$

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$$+\,|\theta^{(k-1)}|\cdot\big|\widetilde{v}_{xx}^{(k-1)}(t,x_-^{(k)}(t))-\widetilde{v}_{xx}^{(k-1)}(t,x_-^{(k-1)}(t))\big|$$
$$+\,|\widetilde{v}_{xx}^{(k-1)}|\cdot\big|\theta^{(k-1)}(t,x_-^{(k)}(t))-\theta^{(k-1)}(t,x_-^{(k-1)}(t))\big|$$
$$+\,\big|I_{11}^{(k-1)}(t,x_-^{(k)}(t))-I_{11}^{(k-1)}(t,x_-^{(k-1)}(t))\big|. \tag{4.119}$$

Thus, it suggests by Lemma 4.5, (4.34), (4.80), (4.81), and (4.112) that

$$\begin{aligned}
T_{31}^{(k-1)} &\le \left(\frac{1}{2t}+\frac{3\widehat{M}_0}{\underline{\theta}}\right)\cdot 10\widehat{M}_0^2t^2\Delta_\pm^{(k)}+2\widehat{M}_0t^2\cdot|\Psi_x^{(k-1)}|\Delta_\pm^{(k)}\\
&\quad+\widehat{M}_0t^2\cdot|\theta_{xx}^{(k-1)}|\Delta_\pm^{(k)}+\frac{1}{32}\widehat{M}_0t\cdot\widehat{M}_0t^{\frac14}\Delta_\pm^{(k)}\\
&\quad+\frac14\widehat{M}_0t\cdot|\theta_x^{(k-1)}|\Delta_\pm^{(k)}+\left|\frac{\partial I_{11}^{(k-1)}}{\partial x}\right|\Delta_\pm^{(k)}\\
&\le\left\{5\widehat{M}_0^2t+\frac{30\widehat{M}_0^3t^2}{\underline{\theta}}+2\widehat{M}_0t^2\cdot\frac{\widehat{M}_0}{8t}+\widehat{M}_0t^2\cdot\frac14\widehat{M}_0t\right.\\
&\quad\left.+\frac{1}{32}\widehat{M}_0^2tt^{\frac14}+\frac14\widehat{M}_0t\cdot\frac14\widehat{M}_0t+\frac14\widehat{M}_0^2t\right\}\Delta_\pm^{(k)}\\
&\le\left\{5+\frac{30}{16}+\frac14+\frac14\delta^2+\frac{1}{32}\sqrt{\delta}+\frac{1}{16}\delta+\frac14\right\}\widehat{M}_0^2t\cdot\widehat{M}_0\xi^3\left(\frac23\right)^{k-1}\\
&\le 8\widehat{M}_0^3\xi^3t\left(\frac23\right)^{k-1}\le\frac14\widehat{M}_0t\left(\frac23\right)^{k-1}.
\end{aligned}\tag{4.120}$$

One puts (4.118) and (4.120) into (4.114) to deduce

$$\begin{aligned}
T_{25}^{(k)} &\le\left(\frac{1}{2t}+\frac{3\widehat{M}_0}{\underline{\theta}}\right)(T_{22}^{(k)}+T_{23}^{(k)})+\frac{1}{15}\widehat{M}_0T_{24}^{(k)}+\frac74\widehat{M}_0t\left(\frac23\right)^{k-1}\\
&\quad+2\overline{\theta}t|\widetilde{v}_{xx}^{(k)}-\widetilde{v}_{xx}^{(k-1)}|.
\end{aligned}\tag{4.121}$$

The above estimate is also true for the term $T_{26}^{(k)}$.

We now combine (4.107)–(4.109), (4.113), and (4.121) and apply the fact $15e^{\widehat{M}_0\delta^2}\le 16$ to find that

$$\begin{aligned}
T_{22}^{(k)}(\xi,\eta),T_{23}^{(k)}(\xi,\eta) &\le\int_0^\xi e^{\widehat{M}_0\delta^2}\left\{\left(\frac{1}{2t}+\frac{3\widehat{M}_0}{\underline{\theta}}\right)(T_{22}^{(k)}+T_{23}^{(k)})+\frac{1}{15}\widehat{M}_0T_{24}^{(k)}\right.\\
&\quad\left.+\frac74\widehat{M}_0t\left(\frac23\right)^{k-1}+2\overline{\theta}t|\widetilde{v}_{xx}^{(k)}-\widetilde{v}_{xx}^{(k-1)}|\right\}\mathrm{d}t\\
&\quad+\int_0^\xi 3\widehat{M}_0t\left\{e^{\widehat{M}_0\delta^2}\int_0^\xi T_{24}^{(k)}\,\mathrm{d}s+\frac14\widehat{M}_0^2\xi^5\left(\frac23\right)^{k-1}\right\}\mathrm{d}t\\
&\le\frac{16}{15}\int_0^\xi\left(\frac{1}{2t}+\frac{3}{16t}\right)(T_{22}^{(k)}+T_{23}^{(k)})\,\mathrm{d}t+\frac{16}{15}\left(\frac{1}{15}\widehat{M}_0+\frac32\widehat{M}_0\xi^2\right)\int_0^\xi T_{24}^{(k)}\,\mathrm{d}t
\end{aligned}$$

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$$
\begin{aligned}
&+4\overline{\theta}\int_0^\xi t|\widetilde{v}_{xx}^{(k)}-\widetilde{v}_{xx}^{(k-1)}|\,\mathrm{d}t+\left(\frac{14}{15}+\widehat{M}_0^2\xi^5\right)\widehat{M}_0\xi^2\left(\frac{2}{3}\right)^{k-1}\\
&\le\int_0^\xi\left\{\frac{4}{5t}(T_{22}^{(k)}+T_{23}^{(k)})+\frac{1}{10}\widehat{M}_0T_{24}^{(k)}+4\overline{\theta}t|\widetilde{v}_{xx}^{(k)}-\widetilde{v}_{xx}^{(k-1)}|\right\}\mathrm{d}t\\
&+\widehat{M}_0\xi^2\left(\frac{2}{3}\right)^{k-1}
\end{aligned}
\tag{4.122}
$$

and

$$
\begin{aligned}
T_{24}^{(k)}(\xi,\eta)&\le\int_0^\xi\left\{\frac{T_{22}^{(k)}(t,\eta)+T_{23}^{(k)}(t,\eta)}{2}+T_{24}^{(k)}(t,\eta)+\widehat{M}_0t\left(\frac{2}{3}\right)^{k-1}\right\}\mathrm{d}t\\
&\le\int_0^\xi\left\{\frac{T_{22}^{(k)}(t,\eta)+T_{23}^{(k)}(t,\eta)}{2}+T_{24}^{(k)}(t,\eta)\right\}\mathrm{d}t+\widehat{M}_0\xi^2\left(\frac{2}{3}\right)^{k-1}.
\end{aligned}
\tag{4.123}
$$

Based on (4.122) and (4.123), we have the following lemma.

Lemma 4.6 *For any $k\ge1$ and $(\xi,\eta)\in[0,\delta]\times\mathbb{R}$, if $\widetilde{v}_{\eta\eta}^{(k)}(\xi,\eta)$ satisfies*

$$
\left|\widetilde{v}_{\eta\eta}^{(k)}(\xi,\eta)-\widetilde{v}_{\eta\eta}^{(k-1)}(\xi,\eta)\right|\le\widehat{M}_0\sqrt{\xi}\left(\frac{2}{3}\right)^{k-1},
\tag{4.124}
$$

then there hold

$$
T_{22}^{(k)}(\xi,\eta),T_{23}^{(k)}(\xi,\eta),T_{24}^{(k)}(\xi,\eta)\le12\widehat{M}_0\xi^2\left(\frac{2}{3}\right)^{k-1}.
\tag{4.125}
$$

Proof The proof of the lemma is similar to that of Lemma 4.3. According to the definitions of $T_{22-24}^{(k)}$, we first see by (4.34) that

$$
T_{22}^{(k)}(\xi,\eta),T_{23}^{(k)}(\xi,\eta)\le2\widehat{M}_0\xi^2,\quad T_{24}^{(k)}(\xi,\eta)\le\frac{1}{2}\widehat{M}_0\xi,
\tag{4.126}
$$

which along with (4.123) gets

$$
T_{24}^{(k)}(\xi,\eta)\le\int_0^\xi\left\{2\widehat{M}_0t^2+\frac{1}{2}\widehat{M}_0t\right\}\mathrm{d}t+\widehat{M}_0\xi^2\le2\widehat{M}_0\xi^2.
$$

Thus,

$$
T_{22}^{(k)}(\xi,\eta),T_{23}^{(k)}(\xi,\eta),T_{24}^{(k)}(\xi,\eta)\le2\widehat{M}_0\xi^2.
\tag{4.127}
$$

Next, in view of (4.122) and the assumption (4.124), one has

$$
\begin{aligned}
T_{22}^{(k)}(\xi,\eta),T_{23}^{(k)}(\xi,\eta)&\le\int_0^\xi\left\{\frac{4}{5t}(T_{22}^{(k)}+T_{23}^{(k)})+\frac{1}{10}\widehat{M}_0T_{24}^{(k)}\right.\\
&\left.+4\overline{\theta}t\cdot\widehat{M}_0\sqrt{t}\left(\frac{2}{3}\right)^{k-1}\right\}\mathrm{d}t+\widehat{M}_0\xi^2\left(\frac{2}{3}\right)^{k-1}\\
&\le\int_0^\xi\left\{\frac{4}{5t}(T_{22}^{(k)}+T_{23}^{(k)})+\frac{1}{10}\widehat{M}_0T_{24}^{(k)}\right\}\mathrm{d}t+2\widehat{M}_0\xi^2\left(\frac{2}{3}\right)^{k-1},
\end{aligned}
\tag{4.128}
$$

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by the fact $\overline{\theta}\sqrt{\delta} \leq 1$. We substitute (4.127) into (4.128) and (4.123) to obtain by $\widehat{M}_0\delta \leq 1$ that

$$\begin{aligned} T_{22}^{(k)}(\xi,\eta), T_{23}^{(k)}(\xi,\eta) &\leq \int_0^\xi \left\{\frac{4}{5t}\cdot 4\widehat{M}_0 t^2 + \frac{1}{10}\widehat{M}_0 \cdot 2\widehat{M}_0 t^2\right\} \mathrm{d}t + 2\widehat{M}_0\xi^2\left(\frac{2}{3}\right)^{k-1} \\ &\leq \left(\frac{4}{5} + \frac{\widehat{M}_0\delta}{30}\right)\cdot 2\widehat{M}_0\xi^2 + 2\widehat{M}_0\xi^2\left(\frac{2}{3}\right)^{k-1} \\ &\leq \frac{5}{6}\cdot 2\widehat{M}_0\xi^2 + 2\widehat{M}_0\xi^2\left(\frac{2}{3}\right)^{k-1} \end{aligned} \tag{4.129}$$

and

$$\begin{aligned} T_{24}^{(k)}(\xi,\eta) &\leq \int_0^\xi \left\{2\widehat{M}_0 t^2 + 2\widehat{M}_0 t^2\right\} \mathrm{d}t + \widehat{M}_0\xi^2\left(\frac{2}{3}\right)^{k-1} \\ &= \frac{4}{3}\widehat{M}_0\xi^3 + \widehat{M}_0\xi^2\left(\frac{2}{3}\right)^{k-1} \leq \frac{5}{6}\cdot 2\widehat{M}_0\xi^2 + 2\widehat{M}_0\xi^2\left(\frac{2}{3}\right)^{k-1}. \end{aligned} \tag{4.130}$$

Hence, it follows by (4.129) and (4.130) that

$$T_{22}^{(k)}(\xi,\eta), T_{23}^{(k)}(\xi,\eta), T_{24}^{(k)}(\xi,\eta) \leq 2\widehat{M}_0\xi^2\left[\frac{5}{6} + \left(\frac{2}{3}\right)^{k-1}\right]. \tag{4.131}$$

Moreover, putting (4.131) into (4.128) and (4.123) again yields

$$\begin{aligned} &T_{22}^{(k)}(\xi,\eta), T_{23}^{(k)}(\xi,\eta) \\ &\leq \int_0^\xi \left\{\frac{4}{5t}\cdot 4\widehat{M}_0 t^2\left[\frac{5}{6} + \left(\frac{2}{3}\right)^{k-1}\right] + \frac{\widehat{M}_0}{10}\cdot 2\widehat{M}_0 t^2\left[\frac{5}{6} + \left(\frac{2}{3}\right)^{k-1}\right]\right\} \mathrm{d}t \\ &+ 2\widehat{M}_0\xi^2\left(\frac{2}{3}\right)^{k-1} \\ &\leq \left(\frac{4}{5} + \frac{\widehat{M}_0\delta}{30}\right)\cdot 2\widehat{M}_0\xi^2\left[\frac{5}{6} + \left(\frac{2}{3}\right)^{k-1}\right] + 2\widehat{M}_0\xi^2\left(\frac{2}{3}\right)^{k-1} \\ &\leq 2\widehat{M}_0\xi^2\left[\left(\frac{5}{6}\right)^2 + \sum_{j=0}^{1}\left(\frac{5}{6}\right)^j\left(\frac{2}{3}\right)^{k-1}\right], \end{aligned} \tag{4.132}$$

and

$$\begin{aligned} T_{24}^{(k)}(\xi,\eta) &\leq \int_0^\xi 4\widehat{M}_0 t^2\left[\frac{5}{6} + \left(\frac{2}{3}\right)^{k-1}\right] \mathrm{d}t + \widehat{M}_0\xi^2\left(\frac{2}{3}\right)^{k-1} \\ &= \frac{4}{3}\widehat{M}_0\xi^3\left[\frac{5}{6} + \left(\frac{2}{3}\right)^{k-1}\right] + \widehat{M}_0\xi^2\left(\frac{2}{3}\right)^{k-1} \\ &\leq 2\widehat{M}_0\xi^2\left[\left(\frac{5}{6}\right)^2 + \sum_{j=0}^{1}\left(\frac{5}{6}\right)^j\left(\frac{2}{3}\right)^{k-1}\right]. \end{aligned} \tag{4.133}$$

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One combines (4.132) and (4.133) to acquire

$$
\begin{aligned}
&T_{22}^{(k)}(\xi,\eta), T_{23}^{(k)}(\xi,\eta), T_{24}^{(k)}(\xi,\eta)\\
&\le 2\widehat{M}_0\xi^2\left[\left(\frac{5}{6}\right)^2+\sum_{j=0}^{1}\left(\frac{5}{6}\right)^j\left(\frac{2}{3}\right)^{k-1}\right].
\end{aligned}
\tag{4.134}
$$

Therefore, by repeating the above process, we can achieve

$$
\begin{aligned}
&T_{22}^{(k)}(\xi,\eta), T_{23}^{(k)}(\xi,\eta), T_{24}^{(k)}(\xi,\eta)\\
&\le 2\widehat{M}_0\xi^2\left[\left(\frac{5}{6}\right)^\ell+\sum_{j=0}^{\ell}\left(\frac{5}{6}\right)^j\left(\frac{2}{3}\right)^{k-1}\right],
\end{aligned}
\tag{4.135}
$$

for arbitrary integer $\ell \ge 1$. Due to the arbitrariness of ℓ, it concludes by (4.135) that

$$
\begin{aligned}
&T_{22}^{(k)}(\xi,\eta), T_{23}^{(k)}(\xi,\eta), T_{24}^{(k)}(\xi,\eta)\\
&\le 2\widehat{M}_0\xi^2\sum_{j=0}^{\infty}\left(\frac{5}{6}\right)^j\left(\frac{2}{3}\right)^{k-1} = 12\widehat{M}_0\xi^2\left(\frac{2}{3}\right)^{k-1},
\end{aligned}
\tag{4.136}
$$

which ends the proof of the lemma. ∎

By virtue of (4.106) and Lemma 4.6, we have the following lemma.

Lemma 4.7 *Let the iterative sequence $\{\widetilde{v}^{(k)}\}$ be defined by (4.18). There hold*

$$
\begin{aligned}
\left|\widetilde{v}_t^{(k+1)}(t,x)-\widetilde{v}_t^{(k)}(t,x)\right|, \left|\widetilde{v}_{xt}^{(k+1)}(t,x)-\widetilde{v}_{xt}^{(k)}(t,x)\right| &\le \widehat{M}_0\left(\frac{2}{3}\right)^k,\\
\left|\widetilde{v}_{xx}^{(k+1)}(t,x)-\widetilde{v}_{xx}^{(k)}(t,x)\right| &\le \widehat{M}_0\sqrt{t}\left(\frac{2}{3}\right)^k,
\end{aligned}
\tag{4.137}
$$

for all $k \ge 0$ and $(t,x) \in [0,\delta]\times\mathbb{R}$.

Proof The proof of the lemma is still based on the method of induction. Noting the initial iterative function $\widetilde{v}^{(0)}(t,x) \equiv 0$, it is easy to see by Lemma 4.1 that all inequalities in (4.137) are true for $k = 0$.

Now, assume that (4.137) holds for $n = k$. Thus,

$$
\begin{aligned}
\left|\widetilde{v}_t^{(k)}(t,x)-\widetilde{v}_t^{(k-1)}(t,x)\right|, \left|\widetilde{v}_{xt}^{(k)}(t,x)-\widetilde{v}_{xt}^{(k-1)}(t,x)\right| &\le \widehat{M}_0\left(\frac{2}{3}\right)^{k-1},\\
\left|\widetilde{v}_{xx}^{(k)}(t,x)-\widetilde{v}_{xx}^{(k-1)}(t,x)\right| &\le \widehat{M}_0\sqrt{t}\left(\frac{2}{3}\right)^{k-1},
\end{aligned}
\tag{4.138}
$$

from which and Lemma 4.6 one has for $n = k$ and $(t,x) \in [0,\delta]\times\mathbb{R}$ that

$$
T_{22}^{(k)}(t,x), T_{23}^{(k)}(t,x), T_{24}^{(k)}(t,x) \le 12\widehat{M}_0 t^2\left(\frac{2}{3}\right)^{k-1}.
\tag{4.139}
$$

We insert (4.139) into (4.106) and apply (4.16) to get

$$
\begin{aligned}
\left|\widetilde{v}_t^{(k+1)}(t,x)-\widetilde{v}_t^{(k)}(t,x)\right| &\le \int_0^t\int_{\mathbb{R}} G(t-\varsigma,z)\Big\{6\widehat{M}_0\varsigma\Big(\frac{2}{3}\Big)^{k-1} \\
&\quad + \widehat{M}_0\Big(\frac{2}{3}\Big)^{k-1} + \frac{\widehat{M}_0\varsigma}{64}\cdot 24\widehat{M}_0\varsigma^2\Big(\frac{2}{3}\Big)^{k-1}\Big\}\,\mathrm{d}z\mathrm{d}\varsigma \\
&\le \int_0^t\int_{\mathbb{R}} G(t-\varsigma,z)\,\mathrm{d}z\mathrm{d}\varsigma\cdot\Big\{6\widehat{M}_0\delta+\widehat{M}_0+\widehat{M}_0^2\delta^3\Big\}\Big(\frac{2}{3}\Big)^{k-1} \\
&\le 2t\cdot\frac{3}{2}\widehat{M}_0\Big(\frac{2}{3}\Big)^{k-1} \le 3\delta\cdot\widehat{M}_0\Big(\frac{2}{3}\Big)^{k-1} \le \widehat{M}_0\Big(\frac{2}{3}\Big)^{k}, \qquad (4.140)
\end{aligned}
$$

$$
\begin{aligned}
\left|\widetilde{v}_{xt}^{(k+1)}(t,x)-\widetilde{v}_{xt}^{(k)}(t,x)\right| &\le \int_0^t\int_{\mathbb{R}} \left|\frac{\partial G(t-\varsigma,z)}{\partial z}\right|\Big\{6\widehat{M}_0\varsigma\Big(\frac{2}{3}\Big)^{k-1} \\
&\quad + \widehat{M}_0\Big(\frac{2}{3}\Big)^{k-1} + \frac{\widehat{M}_0\varsigma}{64}\cdot 24\widehat{M}_0\varsigma^2\Big(\frac{2}{3}\Big)^{k-1}\Big\}\,\mathrm{d}z\mathrm{d}\varsigma \\
&\le \int_0^t\int_{\mathbb{R}} \left|\frac{\partial G(t-\varsigma,z)}{\partial z}\right|\mathrm{d}z\mathrm{d}\varsigma\cdot\Big\{6\widehat{M}_0\delta+\widehat{M}_0+\widehat{M}_0^2\delta^3\Big\}\Big(\frac{2}{3}\Big)^{k-1} \\
&\le 4\sqrt{t}\cdot\frac{5}{4}\widehat{M}_0\Big(\frac{2}{3}\Big)^{k-1} \le 5\sqrt{\delta}\cdot\widehat{M}_0\Big(\frac{2}{3}\Big)^{k-1} \\
&\le \frac{1}{2}\cdot\widehat{M}_0\Big(\frac{2}{3}\Big)^{k-1} \le \widehat{M}_0\Big(\frac{2}{3}\Big)^{k}, \qquad (4.141)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\widetilde{v}_{xx}^{(k+1)}(t,x)-\widetilde{v}_{xx}^{(k)}(t,x)\right| \\
&\le \int_0^t\int_{\mathbb{R}} \left|\frac{\partial G(t-\varsigma,z)}{\partial z}\right|\Big\{2\widehat{M}_0\varsigma\Big(\frac{2}{3}\Big)^{k-1} + 24\widehat{M}_0\varsigma^2\Big(\frac{2}{3}\Big)^{k-1}\Big\}\,\mathrm{d}z\mathrm{d}\varsigma \\
&\le \int_0^t\int_{\mathbb{R}} \left|\frac{\partial G(t-\varsigma,z)}{\partial z}\right|\mathrm{d}z\mathrm{d}\varsigma\cdot\Big\{2\widehat{M}_0\delta+24\widehat{M}_0\delta^2\Big\}\Big(\frac{2}{3}\Big)^{k-1} \\
&\le 4\sqrt{t}\cdot 3\widehat{M}_0\delta\Big(\frac{2}{3}\Big)^{k-1} = 12\delta\cdot\widehat{M}_0\sqrt{t}\Big(\frac{2}{3}\Big)^{k-1} \\
&\le \frac{3}{8}\cdot\widehat{M}_0\sqrt{t}\Big(\frac{2}{3}\Big)^{k-1} \le \widehat{M}_0\sqrt{t}\Big(\frac{2}{3}\Big)^{k}. \qquad (4.142)
\end{aligned}
$$

By (4.140)–(4.142), the proof of the lemma is finished. ∎

4.6 The solutions of the hyperbolic–parabolic coupled problem

In this subsection, we complete the proof of Theorem 2.1 and then obtain Theorem 1.1. Thanks to Lemmas 4.4 and 4.7, one finds that the iterative sequence $\{\widetilde{v}^{(k)}\}$ defined by

(4.18) satisfies concurrently

$$\begin{aligned} &\left|\widetilde{v}^{(k+1)}(t,x)-\widetilde{v}^{(k)}(t,x)\right|,\left|\widetilde{v}_x^{(k+1)}(t,x)-\widetilde{v}_x^{(k)}(t,x)\right|\le \widehat{M}_0 t\left(\frac{2}{3}\right)^k,\\ &\left|\widetilde{v}_t^{(k+1)}(t,x)-\widetilde{v}_t^{(k)}(t,x)\right|,\left|\widetilde{v}_{xt}^{(k+1)}(t,x)-\widetilde{v}_{xt}^{(k)}(t,x)\right|\le \widehat{M}_0\left(\frac{2}{3}\right)^k,\\ &\left|\widetilde{v}_{xx}^{(k+1)}(t,x)-\widetilde{v}_{xx}^{(k)}(t,x)\right|\le \widehat{M}_0\sqrt{t}\left(\frac{2}{3}\right)^k, \end{aligned} \tag{4.143}$$

for all $k\ge 0$ and $(t,x)\in[0,\delta]\times\mathbb{R}$. It follows by (4.143) that the iterative sequence $\{\widetilde{v}^{(k)}(t,x)\}$ is uniformly convergent in the space $\Sigma(\delta)$. We denote the limit function by $\widetilde{v}(t,x)$, which satisfies the following properties by Lemma 4.1:

$$\begin{aligned} &\left|\widetilde{v}(t,x)\right|,\left|\widetilde{v}_x(t,x)\right|\le \widehat{M}_0 t,\ \left|\widetilde{v}_t(t,x)\right|\le \widehat{M}_0,\\ &\left|\widetilde{v}_{xt}(t,x)\right|\le \widehat{M}_0,\ \left|\widetilde{v}_{xx}(t,x)\right|\le \widehat{M}_0\sqrt{t}, \end{aligned}\quad \forall\ (t,x)\in[0,\delta]\times\mathbb{R}. \tag{4.144}$$

From (4.144), we know that the limit function $\widetilde{v}(t,x)$ is in the space $\Sigma(\delta)$. Based on this limit function $\widetilde{v}(t,x)\in\Sigma(\delta)$, the degenerate hyperbolic problem (3.2), (3.3) is solved by Section 3 to get the variables $(\widetilde{R},\widetilde{S},\theta)(t,x)$. It is clear that the functions $(\widetilde{v},\widetilde{R},\widetilde{S},\theta)(t,x)$ satisfy the initial value conditions (2.6) and the last three equations in (2.7). Furthermore, due to the construction of the sequence $\{\widetilde{v}^{(k)}\}$ in (4.18), the limit function $\widetilde{v}(t,x)$ obviously satisfies the integral equation

$$\begin{aligned} &\widetilde{v}(t,x)\\ &=\int_0^t\int_{\mathbb{R}} G(t-\varsigma,x-z)\left\{\frac{\widetilde{R}+\widetilde{S}}{2}-\theta-\widetilde{v}+\theta_0+v_0''\right\}(\varsigma,z)\,dzd\varsigma. \end{aligned} \tag{4.145}$$

By combining the integral equation (4.145) and the regularity of $\widetilde{v}(t,x)$ in (4.144), we see that the function $\widetilde{v}(t,x)$ satisfies the first differential equation in (2.7). Therefore, the functions $(\widetilde{v},\widetilde{R},\widetilde{S},\theta)(t,x)$ are the classical solution to the singular Cauchy problem (2.7), (2.6). In addition, we set

$$\widehat{M}=\frac{2}{\underline{\theta}}\widehat{M}_0,\quad \widetilde{M}=3\overline{M},$$

then apply (4.144) and Theorem 3.1 to acquire

$$\begin{aligned} &|\widetilde{v}(t,x)|\le \widehat{M}\theta(t,x),\quad |\widetilde{R}(t,x)|,|\widetilde{S}(t,x)|\le \widetilde{M}\theta^2(t,x),\\ &\tfrac{1}{2}\underline{\theta}t\le\theta(t,x)\le 2\overline{\theta}t, \end{aligned} \tag{4.146}$$

for any $(t,x)\in[0,\delta]\times\mathbb{R}$, which are the desired inequalities in (2.9). This ends the existence proof of Theorem 2.1.

Next, we consider the uniqueness by estimating the difference of solutions. Assume that $\widetilde{v}_a,\widetilde{v}_b$ are any two elements in the space $\Sigma(\delta)$, and $(\widetilde{v}_a,\widetilde{R}_a,\widetilde{S}_a,\theta_a)(t,x)$, $(\widetilde{v}_b,\widetilde{R}_b,\widetilde{S}_b,\theta_b)(t,x)$ are two solutions of (2.7). Set $(\widehat{v},\widehat{R},\widehat{S},\widehat{\theta})(t,x)=(\widetilde{v}_a-\widetilde{v}_b,\widetilde{R}_a-\widetilde{R}_b,\widetilde{S}_a-\widetilde{S}_b,\theta_a-\theta_b)(t,x)$. Performing direct calculations, the functions $(\widehat{v},\widehat{R},\widehat{S},\widehat{\theta})$

satisfy the following homogeneous integral inequality system:

(4.147)

$$\begin{cases} |\widehat{v}(\xi,\eta)| \le \int_0^t \int_{\mathbb{R}} G(\xi-\varsigma,z)\Big\{\frac{1}{2}(|\widehat{R}|+|\widehat{S}|)+|\widehat{\theta}|+|\widehat{v}|\Big\}(\varsigma,\eta-z)\,\mathrm{d}z\mathrm{d}\varsigma, \\ |\widehat{v}_x(\xi,\eta)| \le \int_0^t \int_{\mathbb{R}} \left|\frac{\partial G(\xi-\varsigma,z)}{\partial z}\right|\Big\{\frac{1}{2}(|\widehat{R}|+|\widehat{S}|)+|\widehat{\theta}|+|\widehat{v}|\Big\}(\varsigma,\eta-z)\,\mathrm{d}z\mathrm{d}\varsigma, \\ |\widehat{R}(\xi,\eta)| \le \int_0^\xi \Big\{\frac{|\widehat{R}|+|\widehat{S}|}{2t}+\frac{1}{4\delta}\big(|\widehat{R}|+|\widehat{S}|+|\widehat{\theta}|\big)+\frac{1}{4}\big(|\widehat{v}|+|\widehat{v}_x|\big)\Big\}\,\mathrm{d}t, \\ |\widehat{S}(\xi,\eta)| \le \int_0^\xi \Big\{\frac{|\widehat{R}|+|\widehat{S}|}{2t}+\frac{1}{4\delta}\big(|\widehat{R}|+|\widehat{S}|+|\widehat{\theta}|\big)+\frac{1}{4}\big(|\widehat{v}|+|\widehat{v}_x|\big)\Big\}\,\mathrm{d}t, \\ |\widehat{\theta}(\xi,\eta)| \le \int_0^\xi \Big\{|\widehat{R}|+|\widehat{S}|+|\widehat{\theta}|+|\widehat{v}|\Big\}(t,\eta)\,\mathrm{d}t. \end{cases}$$

Here, we omit the derivation process of system (4.147), since it is very similar to that of equations (4.44) and (4.45), and (4.54) and (4.55). According to the facts $\widetilde{v}_a, \widetilde{v}_b \in \Sigma(\delta)$, one employs (4.144) and (4.34) to achieve that the difference functions $(\widehat{v},\widehat{R},\widehat{S},\widehat{\theta})$ satisfy

$$\begin{aligned} \big|\widehat{v}(t,x)\big|, \big|\widehat{v}_x(t,x)\big| &\le 2\widehat{M}_0 t, \\ \big|\widehat{R}(t,x)\big|, \big|\widehat{S}(t,x)\big|, \big|\widehat{\theta}(t,x)\big| &\le 2\widehat{M}_0 t^2, \end{aligned} \quad \forall\ (t,x)\in[0,\delta]\times\mathbb{R}. \tag{4.148}$$

Making use of (4.148) at the first step and then repeating the insertion of the right side of (4.147), we can find that the functions $(\widehat{v},\widehat{R},\widehat{S},\widehat{\theta})$ must satisfy the inequalities of the forms

$$\big|\widehat{v}(\xi,\eta)\big|, \big|\widehat{v}_x(\xi,\eta)\big|, \big|\widehat{R}(\xi,\eta)\big|, \big|\widehat{S}(\xi,\eta)\big|, \big|\widehat{\theta}(\xi,\eta)\big| \le \widehat{M}^*\left(\frac{2}{3}\right)^{\ell}, \tag{4.149}$$

for arbitrary integer $\ell \ge 1$ and some positive constant $\widehat{M}^*$. It is obvious by (4.149) that there holds $\widehat{v} = \widehat{R} = \widehat{S} = \widehat{\theta} \equiv 0$, which yields the uniqueness of classical solutions of Cauchy problem (2.7), (2.6). Hence, the proof of Theorem 2.1 is completed.

Based on Theorem 2.1 and the transformation (2.5), one obtains the existence and uniqueness of classical solutions for the Cauchy problem (2.3), (2.4). Finally, by the relation $\theta_x = (\widetilde{R}-\widetilde{S})/2\theta$ in (3.123) and the transformation (2.1), it concludes that the two problems (2.3), (2.4) and (1.7), (1.8) are equivalent. Therefore, we complete the proof of Theorem 1.1.

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