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IDEALS WITH TRIVIAL CONORMAL BUNDLE

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Throughout this paper all rings considered will be commutative, noetherian with identity. If R is such a ring and M is a finitely generated R-module, we shall use $\nu(M)$ to denote that non-negative integer with the property that M can be generated by $\nu(M)$ elements but not by fewer.

1. Introduction. Since every ideal in a noetherian ring is finitely generated, it is a natural question to ask what $\nu(I)$ is for a given ideal I. Hilbert's Nullstellensatz may be viewed as the first general theorem dealing with this question, answering it when I is a maximal ideal in a polynomial ring over an algebraically closed field.

More recently, it has been noticed that the properties of an R-ideal I are intertwined with those of the R-module I/I^2. In particular, the following folklore theorem shows that this is the case for the number of generators:

THEOREM A. *If $I \subset R$ and the images of $a_1, \ldots, a_n \in I$ generate the R-module I/I^2, then $I = (a_1, \ldots, a_n, e)$ for some $e \equiv e^2 \bmod (a_1, \ldots, a_n)$. Thus*

$$\nu(I/I^2) \leqq \nu(I) \leqq \nu(I/I^2) + 1.$$

Which of these inequalities is an equality then becomes the question of interest.

Several years ago Vasconcelos [**14**] and Ferrand [**7**] independently proved:

THEOREM B. *Let R be a local ring and I an ideal of R. The following are equivalent:*

1) *I is generated by a regular sequence of length n.*

2) *$pd_R I < \infty$ and I/I^2 is a free R/I-module of rank n.*

Seeking a globalization of this result, Vasconcelos proposed in [**14**] that one attempt to prove 2) $\Rightarrow$ 1) for any ring R which has all its projective modules free. Recently M. Boratyński showed [**4**] that this condition on R is not enough to give 2) $\Rightarrow$ 1).

M. P. Murthy has offered another possible globalization, namely: does 2) $\Rightarrow$ 1) when $R = k[x_1, \ldots, x_n]$, k a field? Recent work of N. Mohan Kumar [**9**] has shown this to be the case when $n \leqq 4$ and in some cases when $n > 4$ (we shall see this below). In fact, Mohan Kumar's work has directly inspired this note and led us to consider the following:

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Problem 1. Suppose R is a regular domain and $A = R[x_1, \ldots, x_n]$. Let I be an ideal of A with ht $I > \dim R$. If I/I^2 is a free A/I-module is I generated by a regular sequence?

In the next section we consider this problem. By reorganizing the proof of Mohan Kumar's theorem (referred to above) and using some recent results of Suslin and Quillen, it is possible to generalize Mohan Kumar's result to obtain evidence which affirms Problem 1. We also collect some results from the literature which pertain to this problem.

In the last section we return to a discussion of the inequalities of Theorem A for general noetherian rings. There are two related situations where the equality $\nu(I) = \nu(I/I^2)$ is valid. However, neither implies the other except in special cases. Two possible "ancestors" for these two known results are discussed. We prove a theorem about real maximal ideals in rings of functions on smooth compact manifolds which gives conditions under which the equality $\nu(I) = \nu(I/I^2) + 1$ is valid and use it to provide a counter-example to one of our proposed "ancestors". Problem 2 is stated in this section and refers to the other possible ancestor. As a further application of the main theorem of this section we give another proof for Boratyński's counter-example to Vasconcelos' conjecture.

The ideas of this last section use results from differential topology and we would like to thank J. Milnor for focusing our attention in this direction. Also, we would like to thank the Institute for Advanced Study in Princeton for its hospitality during the preparation of this work.

2. We begin with a simple observation.

LEMMA 1. *Let R be a ring, $I \subset R$ with* $\mathrm{pd}_R\, I < \infty$. *If I/I^2 is a projective R/I-module of rank t then:*

1) *I is locally generated by a regular sequence of length t;*

2) *If R is* C-M *then I is unmixed of height t.*

Proof. 1) Follows immediately from Theorem B in § 1, and 2) is a direct consequence of 1) and the definition of Cohen-Macaulay (C-M) ring.

The following proposition is due to Mohan Kumar (see [**9**, Lemma 4] for a more general formulation).

PROPOSITION 2. *Let $I \subset R$ with $\nu(I/I^2) = n$. Let m be an integer such that* $\dim R/\mathscr{P} \leqq m$ *if $\mathscr{P} \not\subset I$, $\mathscr{P}$ prime in R.*

Then $\exists\, a_1, \ldots, a_n \in I$ where

1) *$a_1, \ldots, a_n$ generate I/I^2*

2) *if $\mathscr{P} \supset (a_1, \ldots, a_n)$ ($\mathscr{P}$ prime) and $\mathscr{P} \not\supset I$ then* $\dim R/\mathscr{P} \leqq m - n$.

In particular, if $n > \dim R$ then $\nu(I) = \nu(I/I^2)$.

In trying to decide if an ideal can be generated by n elements a key technique, first used successfully by Serre, is to find a projective module of rank n to map onto the ideal in question and somehow prove this projective is free.

Boratyński [**3**] has found a very general method to get a projective module to map onto an ideal.

PROPOSITION 3. (Boratyński) *Let I be an ideal of A with $\nu(I/I^2) = n$. Suppose there are elements $a_1, \ldots, a_n, s, s' \in A$ such that*

1) $a_1, \ldots, a_n, s' \in I$
2) $sI \subset (a_1, \ldots, a_n)$
3) $(s, s') = A$.

Further suppose

4) *the unimodular row $[a_1, \ldots, a_n]$ over $A_{ss'}$ defines a free module.*

Then there is a projective A-module P of rank n mapping onto I.

We now extract the main ideas of a theorem of Mohan Kumar [**9**, Theorem 5] to obtain the following generalization of his result.

THEOREM 4. *Let I be an ideal in $A = R[T]$ containing a monic polynomial. Suppose $n = \nu(I/I^2) \geqq \dim A/I + 2$. Then there is an A-projective P, of rank n, mapping onto I.*

Proof. We assemble a collection of elements as in Proposition 3.

Let $a_1 \in I$ be part of a minimal generating set for I/I^2. Since I^2 contains monic polynomials of arbitrarily high degree, we may as well assume that a_1 is monic of degree $\geqq 1$. Then $(a_1) \cap R = (0)$ and the natural map $R \to A/(a_1) = B$ is a finite injection. Hence $\dim R = \dim B$.

Set $J = I \cap R$. We intend to pass to the B/J^2-module $I^{**} = I/(a_1, J^2)$ and apply Mohan Kumar's result (Proposition 2). We first show that $\dim B/J^2B = \dim A/I$. Since $a_1 \in A = R[T]$ is monic, the inclusions

$$R/J \to A/I \quad \text{and} \quad R/J^2 \to (R/J^2)[T]/(\bar{a}_1) = B/J^2B$$

are finite maps. This gives $\dim B/J^2B = \dim R/J^2 = \dim R/J = \dim A/I$.

Now $I^{**}/I^{**2} = I/(a_1, I^2)$ is an A-module by restriction of scalars. This yields

$$\nu(I^{**}/I^{**2}) = \nu(I/I^2) - 1 > \dim B/J^2B.$$

By Proposition 2, I^{**} is generated as B/J^2B-module by some $b_2, \ldots, b_n$. Let $a_2^*, \ldots, a_n^*$ be a lifting of the b_i to the ideal $I^* = I/(a_1)$ of B.

Observe that $a_2^*, \ldots, a_n^*$ generate I^*/I^{*2}. This follows since $I^*/I^{*2} = I/(a_1, I^2) = I^{**}/I^{**2}$ as A-modules. In fact $I^{**} = I^*/J^2B$ so that $I^* = (a_2^*, \ldots, a_n^*) + J^2B$ and, a fortiori, $I^* = (a_2^*, \ldots, a_n^*) + JB$.

Let S denote the multiplicative set $1 + J$ in R (or A) and also denote its image in B by S. As J lies in the Jacobson radicals of $S^{-1}R$ and $S^{-1}B$ we have, by Nakayama, that $I^* = (a_2^*, \ldots, a_n^*)$ in $S^{-1}B$. Thus $sI^* \subset (a_2^*, \ldots, a_n^*)B$ for some $s \in S$. Lifting back to A we get $sI \subset (a_1, \ldots, a_n)A$ and $s' = s - 1 \in J \subset I$.

Finally, we observe that the unimodular row $[a_1, \ldots, a_n]$ defines a free module over $A_{ss'} = R_{ss'}[T]$ because a_1 is monic [**11**]. All ingredients have now been assembled to use Proposition 3 and hence complete the proof.

COROLLARY. *Let* $A = R[x_1, \dots, x_n]$, *where* R *is regular and* $K_0(R) = \mathbf{Z}$. *If* I *is an ideal of* A *and*

$$\operatorname{ht} I > \max \{\dim R, (1 + \dim A)/2\}$$

then $\nu(I) = \nu(I/I^2)$.

If in addition I/I^2 *is a free* A/I*-module, then* I *is generated by a regular sequence.*

Proof. Let $t = \nu(I/I^2)$ so $t \geqq \operatorname{ht} I$. Since $\operatorname{ht} I > \dim R$ a result of Suslin (see [**2**]) states that I contains a monic polynomial. We claim that the corollary will be proven if we can show that there is a projective module of rank t mapping onto I. To see this recall that, by [**11**], any projective A-module of rank $> \dim R$ is extended from R. However, since $K_0(R) = \mathbf{Z}$ and $t > \dim R$, the projective mapping onto I must be extended from a free module and hence be free. Thus I can be generated by t elements. If I/I^2 is free then $\nu(I/I^2) = \operatorname{ht} I$. In this case, I can be generated by $\operatorname{ht} I$ elements which form a regular sequence.

We use Theorem 4 to get a projective module of rank t mapping onto I. We have already mentioned that I contains a monic polynomial. We also deduce that $2 + \dim A/I \leqq \nu(I/I^2)$ from the computation

$$1 + \dim A/I \leqq (1 + \dim A) - \operatorname{ht} I < \operatorname{ht} I \leqq t.$$

This completes the proof.

Remarks. 1) By the corollary we obtain an affirmative answer to Problem 1 when $K_0(R) = \mathbf{Z}$, $A = R[x_1, \dots, x_n]$ and $n \leqq \dim R$.

2) It is easy to proliferate examples of regular rings, R, for which $K_0(R) = \mathbf{Z}$. Regular local rings have this property as do polynomial extensions of them. Also, if

$$R = \mathbf{C}[x_0, \dots, x_{2n}]/\left(\sum_{i=0}^{2n} x_i^2 - 1\right)$$

then $K_0(R) = \mathbf{Z}$. ([**5**]).

3) We now consider Problem 1 for $A = R[x_1, \dots, x_n]$, R a regular local ring of dimension d. A is a U.F.D. so if $\operatorname{ht} I = 1$ then $\nu(I) = 1$ also (recall that we are assuming I/I^2 is free as an A/I-module). If $\operatorname{ht} I = 2$ then, by a lemma of Serre [**12**], there is a projective of rank two mapping onto I. If $d \leqq 2$ it is known that A-projectives are all free and so $\nu(I) = 2$ in this case. Coupling these statements with the Corollary to Theorem 4 we obtain an affirmative solution to Problem 1 whenever $n + d \leqq 4$. (This was observed in [**9**] for the case $d = 0$.)

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Thus, in this context, the first unsettled cases arise when $d = 0$, 1 or 2, $\mathrm{ht}\, I = 3$ and $n + d = 5$.

We now give an example which shows that the hypothesis on $K_0(R)$ in the Corollary to Theorem 4 may sometimes be weakened.

Let $A = D[x_1, \ldots, x_n]$, D a Dedekind domain not a P.I.D. (so $K_0(A) \neq \mathbf{Z}$) and let $I \subset A$ be an ideal with I/I^2 a free A/I-module. If $\mathrm{ht}\, I = n + 1$ then we know by Lemma 1 that I is locally generated by a regular sequence. Mohan Kumar's solution of the Eisenbud-Evans conjecture [**9**, Theorem 2] gives $\nu(I) = n + 1$.

Now suppose $\mathrm{ht}\, I = 2$. In this case $\mathrm{pd}\, I = 1$ and, since I is locally generated by 2 elements,

$$\mathrm{Ext}^1(I, A) \simeq \mathrm{Hom}_{A/I}(\Lambda^2(I/I^2), A/I) \quad \text{(see [}\mathbf{1}\text{, Theorem 4.5]).}$$

Hence $\mathrm{Ext}^1(I, A)$ is cyclic. Using Serre's lemma we obtain an exact sequence

$$(*) \qquad 0 \to A \to P \xrightarrow{f} I \to 0$$

with P projective. On the other hand, the Koszul complex over f, being locally exact, gives a resolution of I

$$(**) \qquad 0 \to \Lambda^2 P \to P \xrightarrow{f} I \to 0.$$

From (*) and (**) we conclude that $\Lambda^2 P \simeq A$. It is a consequence of [**11**] that $P \simeq A \oplus J$, where $\mathrm{rk}\, J = 1$. Hence $\Lambda^2 P \simeq J$. Thus $P \simeq A^2$ and so $\nu(I) = 2$. We have thus shown:

PROPOSITION 5. *Problem* 1 *has an affirmative solution for* $D[x_1, x_2]$, *D a Dedekind domain.*

Here is another special case where we can settle Problem 1.

PROPOSITION 6. *If M is a maximal ideal in $A = R[x_1, \ldots, x_n]$ and* $\mathrm{ht}\, M >$ $\dim R$ (*R regular*), *then M is generated by a regular sequence.*

Proof. This is a special case of results in [**6**], for if $n \geqq 2$ the conclusion is valid with no restrictions on $\mathrm{ht}\, M$. If $n = 1$ and $\mathrm{ht}\, M > \dim R$ then $M \cap R$ is maximal in R and the result follows from [**6**, Theorem 2]. Note that M/M^2 is always a free A/M-module.

3. This section is devoted to a study of the inequalities of Theorem A for I an ideal in a noetherian $\mathbf{R}$-algebra.

Before stating the main result of this section we recall some terminology and notation from differential topology. The reader is referred to [**8**] for any unexplained terms.

Let M denote a smooth, compact (boundaryless) manifold of dimension l. The $\mathbf{R}$-algebra of smooth functions $M \to \mathbf{R}$ is denoted $C^\infty(M)$. If $F : M \to \mathbf{R}^l$

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is a smooth map, the points $x \in M$ for which $dF_x : (TM)_x \to \mathbf{R}^l$ is non-singular are called *regular* points of F; $y \in \mathbf{R}^l$ is a regular value if $F^{-1}(y)$ contains only regular points. We denote the number of points in $F^{-1}(y)$ by $\#F^{-1}(y)$. If y is a regular value then $\#F^{-1}(y)$ is a finite number and this number, modulo 2, is independent of the regular value y. (See [**8**] for proofs.) This number (mod 2) is called the mod 2 *degree* of F.

THEOREM 7. *Let M be a compact smooth boundaryless manifold, smoothly embedded in $\mathbf{R}^N$. Let A be a noetherian $\mathbf{R}$-algebra with an $\mathbf{R}$-algebra homomorphism $\varphi : A \to C^\infty(M)$. Suppose also that there are $x_1, \ldots, x_n \in A$ such that the $\varphi(x_i)$ are the coordinate functions of $\mathbf{R}^N$, restricted to M. Finally, let $p = (\alpha_1, \ldots, \alpha_N)$ be a point on M.*

If $\mathscr{M}$ is a maximal ideal of A containing $(x_1 - \alpha_1, \ldots, x_N - \alpha_N)$ for which $A_{\mathscr{M}}$ is regular of dimension = dim (M), *then*

$$\nu(\mathscr{M}) = 1 + \nu(\mathscr{M}/\mathscr{M}^2) = 1 + \dim (M).$$

Proof. As $A_{\mathscr{M}}$ is regular, $\nu(\mathscr{M}/\mathscr{M}^2) = l = \dim (M)$. We know that $\nu(\mathscr{M}) = l$ or $l + 1$, so assume $\mathscr{M}$ is generated by $f_1, \ldots, f_l$.

Now the $(d\varphi(x_i - \alpha_i))_p$ are $\mathbf{R}$-linear combinations of the $(d\varphi(f_j))_p$ and span the l-dimensional cotangent bundle of M at p. This shows that the $(d\varphi(f_j))_p$ are linearly independent, i.e., p is a regular point of the smooth map

$$F = (\varphi(f_1), \ldots, \varphi(f_l)) : M \to \mathbf{R}^l.$$

As $F^{-1}(\mathbf{0}) = \{p\}$ we find that $\mathbf{0}$ is a regular value of F and that the mod 2 degree of F is 1. Since M is compact, there is some $y \in \mathbf{R}^l \backslash F(M)$. Such a y is a regular value of F, yet $\#F^{-1}(y) \equiv 0 \pmod 2$. This contradiction yields the result $\nu(\mathscr{M}) = l + 1$.

Our main applications of Theorem 7 involve the following case. Let A be an affine $\mathbf{R}$-algebra, so that we can embed the real points X of Max spec (A) in $\mathbf{R}^N$ with the classical topology. If some component M of X is a smooth compact manifold, $\dim(M) = \dim(A)$, we are in the situation described by the Theorem.

When A is a domain, the requirement $\dim (M) = \dim (A)$ is equivalent to the injectivity of the natural map $\varphi : A \to C^\infty(M)$.

Example 8. Let $g = x_0^2 + \ldots + x_n^2 - 1$, $M = S^n$ the graph of $g = 0$ in $\mathbf{R}^{n+1}$, and $A = \mathbf{R}[x_0, \ldots, x_n]/(g)$. A is a regular domain and if $p = (\alpha_0, \ldots, \alpha_n) \in S^n$ the maximal ideal $\mathscr{M}_p = (x_0 - \alpha_0, \ldots, x_n - \alpha_n)$ requires $n + 1$ generators in A although $\nu(\mathscr{M}_p/\mathscr{M}_p^2) = n$.

This example was first stated in [**5**], but the proof given there has a gap. M. R. Gabel gave a proof for S^1 and S^2 and E. D. Davis extended his proof to S^n. Gabel's idea relies on the contractibility of $S^n - \{p\}$.

Example 9. (Boratyński). Let A, M be as in Example 8 and let $B = S^{-1}A$ where S is either $\{1 + f^2 | f \in A\}$ or the set of all $s \in A$ for which $\varphi(s)$ never

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vanishes on M. The maximal ideal $m_p = (x_0 - \alpha_0, \ldots, x_n - \alpha_n)$ of B, $p = (\alpha_0, \ldots, \alpha_n) \in S^n$, cannot be n-generated either. The same is true for $B = \hat{A}$, the ring of convergent power series on S^n.

We can now give the example of Boratynski mentioned in § 1. Let B be as above with $n = 3$, i.e., $M = S^3$. It is well-known that all (real) vector bundles on S^3 are trivial and thus, by [**13**, Theorem 11.1] it follows that all projective B-modules are free. But, for m_p as above, $m_p/m_p{}^2$ is free of rank 3 over $B/m_p = \mathbf{R}$, yet $\nu(m_p) = 4$. Thus m_p cannot be generated by a regular sequence.

Remarks. 1) To obtain his example Boratyński used a result of Eisenbud-Levine on the degree of an analytic map germ. This is based on Milnor's notion of degree for a map of oriented manifolds. Our approach is somewhat simpler and does not require orientability.

2) C. Giffen has informed us that S^3 is unique in the following sense: Let M be a closed topological manifold. All vector bundles on M are trivial if and only if M is a point or a homology 3-sphere. This means that the trick of Example 9 is very limited in its applicability.

Example 10. Consider the ring $A = \mathbf{R}[x, y]/(f)$, where

$$f = \lambda^2(x + 1)y^2 + \lambda x(2x + 1) - 2x^3, \ \lambda = \sqrt{2} - 1.$$

It is a simple matter to check that f is irreducible (even in $\mathbf{C}[x, y]$) and that f defines a smooth 1-dimensional manifold X with one bounded component and three unbounded components. (See Fig. 1.) In fact, the projective completion C of $f = 0$ in $\mathbf{P}^2(\mathbf{C})$ is an elliptic curve.

The proof of Theorem 7 shows that any maximal ideal of A corresponding to a point on the bounded component of X requires two generators. However, the maximal ideal corresponding to $Q = (1/\sqrt{2}, 0)$ is generated by either of

$$L_{\pm} = y \pm (2 + \sqrt{2})(x - 1/\sqrt{2}).$$

In fact, Q is the only real point of X corresponding to a principal maximal ideal of A. This follows from the fact that Q is the only point of Max spec $(A \otimes \mathbf{C})$ corresponding to a principal maximal ideal of $A \otimes \mathbf{C}$.

To see this, let $g \in A$ generate the ideal corresponding to $P \in X \subset C$. This means that $(g) - P$ is a divisor with support at infinity. Fix $(0 : 1 : 0)$ as the identity of C, and let Γ be the subgroup generated by the points at infinity. Γ is cyclic of order 4 and Q is the only finite point of Γ. As we have seen that $P \in \Gamma$ we must have $P = Q$.

This example raises the following interesting question. Suppose A is the coordinate ring of a smooth affine curve, defined over $\mathbf{R}$. What can we say about the points of this curve corresponding to principal maximal ideals? We intend to deal with this question in a future paper.

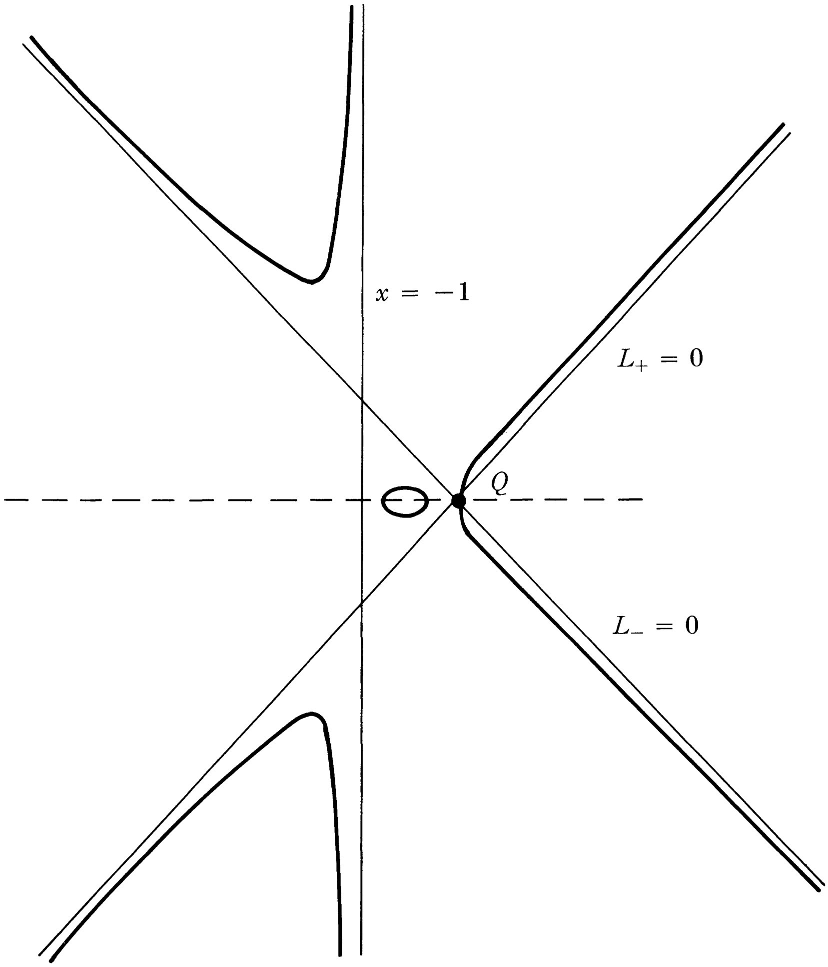

FIG. 1

Example 11. Proposition 2 gives one general instance where $\nu(I) = \nu(I/I^2)$. In [**6**] the authors proved: If I is a maximal ideal in a noetherian ring R and $\nu(I/I^2) > \operatorname{ht} I$ then $\nu(I) = \nu(I/I^2)$. This is a special case of Proposition 2 only when $\operatorname{ht} I = \dim R$.

However, the proofs for both facts are very similar; one builds a generating set by prime avoidance. We would like a common ancestor to both facts. We give an example to show that one conjectured antecedent (at least for prime ideals) is false, namely:

If I is prime in A and $\nu(I/I^2) > \operatorname{ht} I + \dim (A/I)$ then $\nu(I) = \nu(I/I^2)$.

To see this, consider $A = \mathbf{R}[x, y, z, w]$ with the relations $wx = wy = wz = 0$,

$x^2 + (y - 1)^2 + z^2 = 1$. (The picture of this is an S^2 attached to a line.) It is easy to check that A has two minimal primes: wA and $I = (x, y, z)$. Now $A/I \simeq \mathbf{R}[w]$ so ht I + dim $(A/I) = 1$, while dim $A = 2$ as A/wA is the coordinate ring of a 2-sphere. As $y = (x^2 + y^2 + z^2)/2 \in I^2$, we have $\nu(I/I^2) = 2$. However, $\nu(I) = 3$ because in A/wA the ideal $I + wA/wA = (x, y, z)$ requires three generators, by Example 8.

We leave as a problem another possible ancestor.

Problem 2. Let I be an ideal of A for which

$$\nu(I/I^2) > \text{ht}\,(\mathscr{P}) + \dim\,(A/\mathscr{P}),\ \forall \mathscr{P} \supseteq I.$$

Then, is $\nu(I) = \nu(I/I^2)$?

References

1. A. Altman and S. Kleiman, *Introduction to Grothendieck duality theory*, Springer Lecture Notes in Mathematics, Vol. *146* (Springer-Verlag, Berlin-Heidelberg-New York, 1970).
2. H. Bass, *Liberation des modules projectifs sur certains anneaux de polynomes*, Springer Lecture Notes in Mathematics, Vol. *431*, 228–254 (Springer-Verlag, Berlin-Heidelberg-New York, 1975).
3. M. Boratyński, *A note on the set-theoretic complete intersection ideals*, J. of Algebra *54* (1978), 1–5.
4. ——— *When is an ideal generated by a regular sequence?* J. of Algebra *57* (1979), 236–241.
5. L. Claborn and R. Fossum, *Generalization of the notion of the class group*, Ill. J. of Math. *12* (1968), 228–253.
6. E. D. Davis and A. V. Geramita, *Efficient generation of maximal ideals in polynomial rings*, T.A.M.S., Vol. *231* (1977), 497–504.
7. D. Ferrand, *Suites régulières et intersection complète*, C. R. Acad. Sci., Paris *264* (1967), A427–A428.
8. J. Milnor, *Topology from the differentiable viewpoint* (The University Press of Virginia, Charlottesville, 1965).
9. N. Mohan Kunar, *On two conjectures about polynomial rings*, Inventiones Mathematicae *46* (1978), 225–236.
10. M. P. Murthy, *Complete intersections*, Conference on Commutative Algebra, Queen's Papers in Pure and Applied Mathematics, No. *42* (1975), 137–154.
11. D. Quillen, *Projective modules over polynomial rings*, Inv. Math. *36* (1976), 166–172.
12. J.-P. Serre, *Sur les modules projectif*, Sem. Dubreil-Pisot (1960/61), No. 2.
13. R. Swan, *Topological examples of projective modules*, T.A.M.S. Vol. *230* (1977), 201–234.
14. W. Vasconcelos, *Ideals generated by R-sequences*, J. Alg. *6* (1967), 309–316.

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