ON SETS OF ARCS CONTAINING NO CYCLES IN A TOURNAMENT*

K.B. Reid

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1. A tournament T_n with n nodes is a complete asymmetric digraph [2]. A set S of arcs of a tournament is called consistent if the tournament contains no oriented cycles composed entirely of arcs of S [1]. The object of this note is to provide a new lower bound for $f(n)$, the greatest integer k such that every tournament with n nodes contains a set of k consistent arcs. Erdös and Moon [1] showed that $[\frac{n}{2}][\frac{n+1}{2}] \leqq f(n) \leqq \left(\frac{1+\epsilon}{\epsilon}\right)\binom{n}{2}$, where $[x]$ denotes the largest integer not exceeding x, and the second inequality holds for any fixed $\epsilon > 0$ and all sufficiently large n.

Consistent arcs are of interest, for example, in consistency of paired comparison experiments [3]. The problem of finding largest sets of consistent arcs in a tournament is an extension of the problem of finding largest transitive subtournaments [4].

2. A T_{2m+1} is regular if the outdegree of each node is m. A T_{2m} is almost regular if m of the nodes have outdegree $m-1$ and m of the nodes have outdegree m.

LEMMA 1. *If T_n is neither regular nor almost regular and S is a consistent set of arcs in T_n such that $|S|$ is a maximum, then $|S| \geq f(n-1) + [\frac{n}{2}] + 1$.*

Proof. Since T_n is neither regular nor almost regular, T_n has a node v with outdegree not less than $[\frac{n}{2}] + 1$ or a node w with indegree not less than $[\frac{n}{2}] + 1$. The T_{n-1} obtained from T_n by deleting v (or w) and its adjacent arcs contains a consistent set of at least

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$f(n-1)$ arcs, which with the $[\frac{n}{2}] + 1$ arcs in T_n directed away from v (or directed towards w) forms a consistent set of at least $f(n-1) + [\frac{n}{2}] + 1$ arcs in T_n. Thus, if S is as stated, $|S| \geqq f(n-1) + [\frac{n}{2}] + 1$.

LEMMA 2. <u>Every almost regular</u> T_{2m} <u>contains two nodes</u> u <u>and</u> v <u>such that the outdegree of</u> u <u>is</u> $m-1$, <u>the outdegree of</u> v <u>is</u> m, <u>and</u> T_{2m} <u>contains an arc from</u> u <u>to</u> v.

<u>Proof</u>. For $m = 1$, the result is clear. If there were m^2 arcs directed from the m nodes of outdegree m to the m nodes of outdegree $m-1$, then there could be no arc joining any two nodes of outdegree m for then one of these nodes would have outdegree greater than m. This is impossible when $m \geqq 2$. Thus, such a u and v are guaranteed in every almost regular T_{2m}.

LEMMA 3. <u>If</u> S <u>is a consistent set of arcs in an almost regular</u> T_{2m} $(m \geqq 2)$ <u>such that</u> $|S|$ <u>is a maximum</u>, <u>then</u> $|S| \geqq f(2m-2) + 2m$.

<u>Proof</u>. Let u and v be as in Lemma 2. Let A be the m arcs directed away from v, and let B be the m arcs directed towards u. Then $A \cap B = \phi$ since u is directed towards v. The T_{2m-2} obtained from T_{2m} by deleting nodes u and v and all their adjacent arcs contains a consistent set of at least $f(2m-2)$ arcs, which with $A \cup B$ forms a consistent set of at least $f(2m-2) + 2m$ arcs in T_{2m}. Thus, if S is as stated, $|S| \geqq f(2m-2) + 2m$.

THEOREM 1. $f(n) \geqq [\frac{n}{2}][\frac{n+3}{2}] - 1$ <u>for all integers</u> $n \geqq 2$.

<u>Proof</u>. It is easy to see that equality holds for $2 \leqq n \leqq 4$. Let $n \geqq 5$ and assume $f(k) \geqq [\frac{k}{2}][\frac{k+3}{2}] - 1$ for all k such that $4 \leqq k \leqq n-1$. Let S be a consistent set of arcs in T_n such that $|S|$ is a maximum. If T_n is neither regular nor almost regular, then by Lemma 1 and the induction hypothesis, $|S| \geqq f(n-1) + [\frac{n}{2}] + 1 \geqq [\frac{n-1}{2}][\frac{n+2}{2}] - 1 + [\frac{n}{2}] + 1 = [\frac{n}{2}]([\frac{n-1}{2}] + 1) + [\frac{n-1}{2}] = [\frac{n}{2}][\frac{n+1}{2}] + [\frac{n+1}{2}] - 1 = [\frac{n+1}{2}][\frac{n+2}{2}] - 1 \geqq [\frac{n}{2}][\frac{n+3}{2}] - 1$. If T_n is regular (i.e. $n = 2m + 1$ some $m \geqq 2$) and v is a node of T_n, then the T_{n-1} obtained from T_n by deleting v and its adjacent arcs contains a consistent set of at least $f(n-1) = f(2m)$ arcs, which with the m arcs in T_n directed

away from v forms a consistent set of at least $f(2m) + m$ arcs in T_n. Thus, if T_n is regular $|S| \geqq f(2m) + m \geqq m(m+2) - 1 = [\frac{n}{2}][\frac{n+3}{2}] - 1$, by the induction hypothesis. If T_n is almost regular (i.e. $n = 2m$ for some $m \geqq 3$), then by Lemma 3 and the induction hypothesis, $|S| \geqq f(2m-2) + 2m \geqq m(m+1) - 1 = [\frac{n}{2}][\frac{n+3}{2}] - 1$. Thus, $f(n) \geqq [\frac{n}{2}][\frac{n+3}{2}] - 1$. By induction, the result follows.

3. For $2 \leqq n \leqq 7$, $f(n) = [\frac{n}{2}][\frac{n+3}{2}] - 1$ as can be seen by considering "extremal" tournaments where the lower bound in Theorem 1 is assumed for each such n. This is done in [5], but the arguments are very special for each case so that only the results are given here. To show $f(2) = 1$, $f(3) = 2$, and $f(4) = 5$ simply consider the one T_2, the strong T_3, and the strong T_4. To show $f(5) = 7$, consider the regular T_5. To show $f(6) = 11$, consider the unique T_6 containing no transitive T_4 as a subtournament [4]. To show $f(7) = 14$, consider the unique T_7 containing no transitive T_4 as a subtournament [4].

While Theorem 1 yields $f(8) \geqq 19$, the exact value of $f(8)$ is 20 [5]. To show $f(8) \geqq 20$, the following result given in [5] is of help: any T_8 without a consistent set of 20 arcs is almost regular, contains no regular T_7 as a subtournament, but for every pair of nodes x and y with outdegrees 4, the T_6 obtained from T_8 by deleting x and y and all their adjoining arcs is almost regular. But, on the other hand, among the four nodes of outdegree 4 of such a T_8 there is a transitive T_3, so that deleting from T_8 the transmitter and carrier [2] of this T_3 results in a T_6 which is not almost regular. Thus, $f(8) \leqq 19$, is impossible so that $f(8) \geqq 20$. To show $f(8) \leqq 20$, consider the T_8 obtained from the unique T_7 containing no transitive T_4 by adding a new node x and seven new arcs adjacent to x such that the nodes joined by arcs directed towards x form a strong T_3.

While $f(9) \geqq 23$ by Theorem 1, the exact value of $f(9)$ is 24. That $f(9) \geqq 24$ follows easily from $f(8) = 20$; to show $f(9) \leqq 24$, a certain regular T_9 (the composition or lexicographic product of the two strong T_3's) has no consistent set of 25 arcs.

Since $f(8) = 20$, we can use Lemmas 1 and 3 to proceed as in Theorem 1 to obtain

THEOREM 2. $f(n) \geq [\frac{n}{2}][\frac{n+3}{2}]$ <u>for integers</u> $n \geq 8$.

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University of Illinois, Urbana

Louisiana State University, Baton Rouge