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PARANORMAL OPERATORS ON BANACH SPACES

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In this note we show that a paranormal operator T on a Banach space satisfies Weyl's theorem. This is accomplished by showing that

(i) every isolated point of its spectrum is an eigenvalue and the corresponding eigenspace has invariant complement,

(ii) for $\alpha \neq 0$, $\operatorname{Ker}(T-\alpha) \perp \operatorname{Ker}(T-\beta)$ (in the sense of Birkhoff) whenever $\beta \neq \alpha$.

1. Introduction and notations

X will denote a Banach space and $B(X)$ the Banach algebra of bounded linear operators on X. $T \in B(X)$ will be called paranormal if

$$\|Tx\|^2 \leq \|T^2x\|\|x\| , \quad \forall x \in X .$$

We note that an isometry is always paranormal. Also the restriction to an invariant subspace, any scalar multiple and the inverse (if it exists) of a paranormal operator are paranormal. Further, every paranormal operator is normaloid. (By normaloid we mean those operators T for which $\|T\| = r(T)$, the spectral radius of T.) For proofs refer to [7], where T is taken to be a paranormal operator in a Hilbert space.

Let M and N be linear subspaces of X. Then M is said to be orthogonal to N (in the sense of Birkhoff) and we write $M \perp N$ if $\|x+y\| \geq \|x\|$ for all $x \in M$ and $y \in N$. This is a nonsymmetric relation in a Banach space; but it is equivalent to the usual concept of

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orthogonality in a Hilbert space ([4], Theorem 2). Let $\sigma(T)$ denote the spectrum of T, and $R(T)$ and $N(T)$ its range and null space, respectively. The nullity of T is denoted by $n(T)$ while $\sigma_p(T)$ denotes the point spectrum of T and $\zeta(T)$ the complement of $\sigma(T)$.

Let $P_T\{\lambda\}$ denote the algebraic eigenprojection associated with $\{\lambda\}$ whenever λ is an isolated point of $\sigma(T)$. If $T' \in B(X^*)$ is the conjugate of $T \in B(X)$, then

$$\left(P_T\{\lambda\}\right)' = P_{T'}\{\lambda\} .$$

The corresponding result for a Hilbert space operator T is

$$\left(P_T\{\lambda\}\right)^* = P_{T^*}\{\overline{\lambda}\} ,$$

where $*$ denotes the adjoint and $-$ is complex conjugate.

An operator $T \in B(X)$ is said to be isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T. The numerical range of $T \in B(X)$ is given by

$$V\left(B(X), T\right) = \{F(T) : F \in B(X)^*, \|F\| = F(1) = 1\}$$

and its numerical radius by

$$v(T) = \operatorname{Sup}\left\{|\lambda| \mid \lambda \in V\left(B(X), T\right)\right\} .$$

If $r(T)$ denotes the spectral radius, then it is known that

$$r(T) \leq v(T) \leq \|T\| .$$

The operator T is said to satisfy the single valued extension property if $f(\lambda) \equiv 0$ for any X-valued analytic function f defined on an open set of the complex plane with $(T-\lambda I)f(\lambda) \equiv 0$.

Denote by $\Pi_{00}(T)$ the set of isolated points λ of $\sigma(T)$ for which $0 < n(T-\lambda I) < \infty$, and by $\Pi_{0A}(T)$ the set of isolated points λ of $\sigma(T)$ for which $R\left(P_T\{\lambda\}\right)$ is finite dimensional. Note that $\Pi_{0A}(T) \subseteq \Pi_{00}(T)$.

Denote by $W(T)$ Weyl's essential spectrum of T (as defined, for example, in [9], p. 206); let

$$\sigma_l(T) = W(T) \cup \{\lambda \mid \lambda \text{ is a limit point of } \sigma(T)\} .$$

Note that

$$\sigma_l(T) = \sigma(T) \sim \Pi_{0A}(T)$$

[6] (here $\sim$ denotes set difference). The operator $T \in B(X)$ is said to satisfy Weyl's theorem if

$$(*) \qquad W(T) = \sigma(T) \sim \Pi_{00}(T) .$$

The famous result of Weyl, that self adjoint operators satisfy (*), has been extended to several classes of operators (see [5] for an account). It is noted in [5] that, among operators related to normal operators, Weyl's theorem does not extend appreciably beyond the seminormal ones. Below we show that for any paranormal operator T, T and T' (T and T^* in the case of a Hilbert space operator T) satisfy Weyl's theorem, thus extending the theorem beyond the seminormal ones. Also we get a class of operators on Banach space, including isometries, which satisfy Weyl's theorem. In the process we get some results regarding eigenspaces of a paranormal operator.

2. Main results

We start with the observation that if T is a paranormal operator on X and $\sigma(T)$ lies on the unit circle then T is an invertible isometry. In fact, T being invertible, both T and T^{-1} are normaloid, being paranormal. Hence $\|T\| = \|T^{-1}\| = 1$ and $\|x\| = \|T^{-1}Tx\| \le \|Tx\| \le \|x\|$, for all $x \in X$. This shows that T is an invertible isometry.

THEOREM 2.1. *If T is a paranormal operator in $B(X)$, then every isolated point of $\sigma(T)$ is a pole of the resolvent $R_\lambda(T)$ of order 1 and the corresponding eigenspace has an invariant complement.*

Proof. Suppose λ_1 is an isolated point of $\sigma(T)$. If $\lambda_1 = 0$, consider the paranormal operator $T/R(P_T\{0\})$. Since $\sigma(T/R(P_T\{0\})) = \{0\}$, $T/R(P_T\{0\}) = 0$. Thus 0 is a pole of the resolvent $R_\lambda(T)$ of order 1 ([11], p. 306). If $\lambda_1 \ne 0$, consider $T_1 = (1/\lambda_1)(T/R(P_T\{\lambda_1\}))$. Then T_1 is a paranormal operator with $\sigma(T_1) = \{1\}$. Thus T_1 and T_1^{-1} both are isometries and $\|T_1^n\| = 1$ for $n = \pm 1, \pm 2, \pm 3, \ldots$. Also we have

$T_1 = I + Q$, where Q is some quasinilpotent operator. It follows from ([3], Section 5, Theorem 3) that $T_1 = I$. So again $(T-\lambda_1)R(P_T(\lambda_1)) = 0$ and λ_1 is a pole of the resolvent $R_\lambda(T)$ of order 1 . Thus, using [11], Theorem 5.8A, $R(T-\lambda_1)$ is closed and $X = R(T-\lambda_1) \oplus N(T-\lambda_1)$. This completes the proof of the theorem.

COROLLARY 2.2. *If $T \in B(X)$ is paranormal, then T is isoloid.*

COROLLARY 2.3. *If $T \in B(X)$ is an isometry, then every isolated point of $\sigma(T)$ is an eigenvalue of T and the corresponding eigenspace has an invariant complement.*

REMARK 2.4. The above result was proved in [8] for invertible isometries on a normed linear space.

COROLLARY 2.5. *If $T \in B(X)$ is paranormal, then $\Pi_{00}(T) = \Pi_{0A}(T)$ and $\Pi_{00}(T') = \Pi_{0A}(T')$. In the case of a Hilbert space paranormal operator T we have*

$$\Pi_{00}(T^*) = \Pi_{0A}(T^*) .$$

Proof. If λ_1 is an isolated point of $\sigma(T)$, then λ_1 is a pole of the resolvent $R_\lambda(T)$ of order 1 . Thus $(T-\lambda_1)P_T\{\lambda_1\} = 0$, so that $R(P_T\{\lambda_1\}) \subseteq N(T-\lambda_1)$, which implies that $\Pi_{00}(T) \subseteq \Pi_{0A}(T)$. If λ_1 is an isolated point of $\sigma(T')$, then λ_1 is a pole of the resolvent $R_\lambda(T)$ of order 1 . Thus $(T-\lambda_1)P_T\{\lambda_1\} = 0$. Taking conjugates, $(P_T\{\lambda_1\})'(T-\lambda_1)' = 0$. Now $(P_T\{\lambda_1\})' = P_{T'}\{\lambda_1\}$ and $P_{T'}\{\lambda_1\}$ commutes with T' . Hence $(T'-\lambda_1)P_{T'}(\lambda_1) = 0$, so that $\Pi_{00}(T') \subseteq \Pi_{0A}(T')$. The corresponding result for T^* can be similarly proved by taking adjoints.

THEOREM 2.6. *If T is a paranormal operator, then $N(T-\alpha) \perp N(T-\beta)$ for distinct complex numbers α and β provided $\alpha \neq 0$.*

Proof. Suppose $|\alpha| \geq |\beta|$. Let $Tx = \alpha x$ and $Ty = \beta y$. We will show that $\|x\| \leq \|x+y\|$. Let M be the subspace generated by x and y and $T_1 = T/M$. Then $\sigma(T_1) = (\alpha, \beta)$ and T_1 being normaloid, $\|T_1\| = r(T_1) = |\alpha|$. Hence, $v(T_1) = |\alpha|$. Thus, $\alpha \in bd(V(B(M), T_1))$.

Now, by [10], Proposition 1, $N(T_1-\alpha) \perp R(T_1-\alpha)$. As α and β are poles of the resolvent $R_\lambda(T_1)$ of order 1 ,

$$R(T_1-\alpha) = R\left(I-P_{T_1}\{\alpha\}\right) = R\left(P_{T_1}\{\beta\}\right) = N(T_1-\beta)$$

([11], Theorem 5.8 (A)). Now, $x \in N(T_1-\alpha)$ and $y \in N(T_1-\beta)$. Hence $\|x+y\| \geq \|x\|$. If $|\alpha| < |\beta|$, $\beta \neq 0$ as $\alpha \neq 0$. So T_1 is invertible and $\sigma\left(T_1^{-1}\right) = \left(\alpha^{-1}, \beta^{-1}\right)$ and $|\alpha^{-1}| > |\beta^{-1}|$. Being paranormal, T_1^{-1} is also normaloid. As in the first case we see that $N\left(T_1^{-1}-\alpha^{-1}\right) \perp N\left(T_1^{-1}-\beta^{-1}\right)$. As $x \in N\left(T_1^{-1}-\alpha^{-1}\right)$ and $y \in N\left(T_1^{-1}-\beta^{-1}\right)$, the proof is complete.

QUESTION 2.7. Is $N(T) \perp N(T-\alpha)$?

REMARK 2.8. It is shown in [8], Corollary 3, that, for an isometry T on a normal linear space, $N(T-\alpha) \perp N(T-\beta)$ for distinct scalars α and β . This can be shown by completing the normal linear space, extending the isometry to the completed space and using Theorem 2.6.

COROLLARY 2.9. *If T is a paranormal operator on a Hilbert space, then $N(T-\alpha) \perp N(T-\beta)$ for all distinct complex numbers α and β .*

COROLLARY 2.10. *If X is a separable Banach space and $T \in B(X)$ is paranormal, then T has single valued extension property.*

Proof. We will show that $\sigma_p(T)$ is countable. Then, by [1], p. 22, T has single valued extension property. If $\sigma_p(T)$ is not countable, we would have an uncountable set of unit vectors such that $\|x_i-x_j\| \geq 1$. Since the space is separable, this is not possible.

In [8] it is pointed out that for an arbitrary Banach space, it is not known whether an eigenspace of an isometry has an invariant complement. In [2], it is shown that the answer is affirmative for an orthogonally complemented isometry on a smooth reflexive Banach space. We improve this result by using results in [10].

COROLLARY 2.11. *If T is an isometry on a reflexive Banach space, then every eigenspace of T has an invariant complement.*

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Proof. If T is an isometry and λ is an eigenvalue of T, then $|\lambda| = 1$ and $\|T\| = r(T) = v(T) = 1$. Thus

$$\lambda \in bd\big(V\big(B(X), T\big)\big) .$$

As X is reflexive, by Remark 4 of [10], $X = N(T-\lambda) \oplus R(T-\lambda)$. This completes the proof.

THEOREM 2.12. *If T is a paranormal operator on a Banach space, then T and T' (T and T^* in the case of a Hilbert space operator T) both satisfy Weyl's theorem.*

Proof. In view of Theorem 3.3 of [6] and Corollary 2.5, it is enough to show that $W(T) = \sigma_l(T)$. This would prove the result for T. Since $W(T) = W(T')$ and $\sigma_l(T) = \sigma_l(T')$, the result would follow for T' and the relations $\overline{W(T)} = W(T^*)$ and $\overline{\sigma_l(T)} = \sigma_l(T^*)$ would yield the result for T^*. We would show that every λ in $\sigma(T) \sim W(T)$ satisfies the eigenspace gap condition of [5]. Then by Theorem 1 (4i) of [5], we get $\sigma(T) \sim W(T) \subseteq \Pi_{00}(T)$. From this it follows easily that $\sigma_l(T) = W(T)$. Note that $\sigma(T) \sim W(T) = \Pi_{0A}(T) \cup \{\sigma_l(T) \sim W(T)\}$. If $\lambda \in \Pi_{0A}(T)$, then we can find a sequence $\{\lambda_n\}$ such that $\lambda_n \in \zeta(T)$ for all n as λ is an isolated point of $\sigma(T)$. Thus the gap condition is satisfied here. Given λ in $\sigma_l(T) \sim W(T)$, we can find a sequence $\{\lambda_n\}$ of nonzero eigenvalues of T converging to λ. Now $N(T-\lambda_n) \perp N(T-\lambda)$ for all n by Theorem 2.6. So $d\big(x_{\lambda_n}, N(T-\lambda)\big) \geq 1$ for all x_{λ_n} in $N(T-\lambda_n)$ such that $\|x_{\lambda_n}\| = 1$. Thus

$$\delta(\lambda_n, \lambda) \equiv \operatorname{Sup}\big\{d\big(x_{\lambda_n}, N(T-\lambda)\big) \mid x_{\lambda_n} \in N(T-\lambda_n) \text{ and } \|x_{\lambda_n}\| = 1\big\}$$
$$\geq 1 .$$

This will be true for all n. Hence

$$|\lambda_n-\lambda|/\delta(\lambda_n, \lambda) \to 0 \quad \text{as} \quad n \to \infty .$$

This completes the proof.

PROPOSITION 2.13. *If T is a paranormal operator with finite spectrum, then there exists a basis of X consisting of eigenvectors of*

T ; that is, X is a direct sum of the eigenspaces of T .

Proof. Using Theorem 5.7 (A) of [11], we see that

$$X = \oplus \sum_{i=1}^{n} R\left(P_T(\{\lambda_i\})\right) ,$$

where $\sigma(T) = \{\lambda_1, \ldots, \lambda_n\}$, since $\sigma(T) = \bigcup_{i=1}^{n} \{\lambda_i\}$ and each $\{\lambda_i\}$ is an open closed subset of $\sigma(T)$, with $\sigma(T)$ being finite. But

$$R\left(P_T\{\lambda_i\}\right) = N\left(T-\lambda_i\right) \quad (i = 1, \ldots, n)$$

since each λ_i is a pole of resolvent of order 1 by Theorem 2.1. Thus

$$X = \oplus \sum_{i=1}^{n} N\left(T-\lambda_i\right) .$$

This completes the proof.

COROLLARY 2.14. *If T is a paranormal operator on a Hilbert space with finite spectrum, then T is normal.*

Proof. $X = \oplus \sum_{i=1}^{n} N\left(T-\lambda_i\right)$ and $N\left(T-\lambda_i\right) \perp N\left(T-\lambda_j\right)$ for $i \neq j$ give the result.

COROLLARY 2.15 ([8], Corollary 4). *If T is an isometry on a finite dimensional normed linear space X , then X has a basis consisting of eigenvectors of T .*

Proof. This follows directly from Proposition 2.12 because a finite dimensional normed linear space is a Banach space.

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