Can. J. Math. Vol. **47** (6), 1995 pp. 1317–1328.

IMBEDDING C^1 INTO H_1

JINGZHI TIE

ABSTRACT. This article gives a direct proof of Theorem 7.58 of Greiner [4]. This result implies that the classical Mikhlin-Calderón-Zygmund calculus for the principal value convolution operators on $\mathbb{C}$ is, in a natural way, the limit of the Laguerre calculus for principal value convolution operators on $\mathbb{H}_1 = \mathbb{C} \times \mathbb{R}$.

1. **Introduction.** The simplest noncommutative nilpotent Lie group is the Heisenberg group $\mathbb{H}_1$ with underlying manifold $\mathbb{R}^3$ and with the group law

$$(x_1, x_2, t)(y_1, y_2, s) = (x_1 + y_1, x_2 + y_2, t + s + 2[x_2 y_1 - x_1 y_2]) \tag{1}$$

(1) should be looked upon as the non-commutative analogue of Euclidean translation on $\mathbb{R}^3$. Note that $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$, and by writing $z = x_1 + ix_2$ and $w = y_1 + iy_2$ the Heisenberg group law can be written in the following form

$$(z, t)(w, s) = \big(z + w, t + s + 2\Im m(z\bar{w})\big). \tag{2}$$

The unit of $\mathbb{H}_1$ is $(z, t) = (0, 0)$ and the inverse of (z, t) is $(z, t)^{-1} = (-z, -t)$.

Given functions $\phi(u)$ and $\psi(u)$ in $C_0^\infty(\mathbb{H}_1)$, the H-convolution (Heisenberg convolution) is defined by

$$\phi *_H \psi(u) = \int_{R^3} \phi(v^{-1}u)\psi(v)\,dv,$$

where $u = (z, t)$, $v = (w, s)$ and $dv = dy_1 dy_2 ds$ with $w = y_1 + iy_2$, is the Lebesgue measure on $\mathbb{R}^3$.

It is useful to define the $\mathbb{H}_1$ analogues of Mikhlin-Calderón-Zygmund principal value convolution operators on $\mathbb{R}^2$. Let $r(z, t) = (rz, r^2 t)$, $r > 0$ denote the Heisenberg dilation. F is said to be *H-homogeneous* of degree m on $\mathbb{H}_1$ if

$$\mathbf{F}(rz, r^2 t) = r^m \mathbf{F}(z, t), \quad \forall r > 0.$$

The Korányi norm on $\mathbb{H}_1$ is defined by $\|(z, t)\| = (|z|^4 + t^2)^{\frac{1}{4}}$, which is H-homogeneous of degree 1. The distance $d(u, v)$ of the points u and v in $\mathbb{H}_1$ is $d(v, u) = d(u^{-1}v, 0) = \|u^{-1}v\|$.

Suppose that $\mathbf{F} \in C^\infty(\mathbb{R}^3 \setminus 0)$ is H-homogeneous of degree γ. Then F is locally integrable if and only if $\gamma > -4$. The main result in this paper is concerned with functions which are H-homogeneous of the critical degree -4.

Received by the editors February 2, 1994.
AMS subject classification: 43M80.

DEFINITION 1.1. Let $\mathbf{F} \in C^{\infty}(\mathbb{R}^3 \setminus 0)$ be H-homogeneous of degree -4. $\mathbf{F}$ has mean value zero if

$$\int_{d(u,0)=1} \mathbf{F}\, d\sigma = 0,$$

where $d\sigma$ is the induced measure on the Heisenberg unit ball $d(u,0) = 1$.

The basic result concerning principal value convolution operators on $\mathbb{H}_1$ follows—see [3].

PROPOSITION 1.1. *Let $\mathbf{F} \in C^{\infty}(\mathbb{R}^3 \setminus 0)$ be H-homogeneous of degree -4 with mean value zero. Then $\mathbf{F}$ induces a principal value convolution operator,*

$$\mathbf{F} *_H \phi(u) = \lim_{\epsilon \to 0^+} \int_{d(u,v)>\epsilon} \mathbf{F}(v^{-1}u)\phi(v)\, dv \tag{3}$$

on functions $\phi(u) \in C_0^{\infty}(\mathbb{R}^3)$. F can be extended to a bounded operator $F\colon L^2(\mathbb{H}_1) \longrightarrow L^2(\mathbb{H}_1)$.

In particular principal value convolution operators can be composed, and their composition yields another principal value convolution operator.

The Euclidean Fourier transform $\hat{\phi}$ of $\phi \in C_0^{\infty}(\mathbb{R}^3)$ is given by

$$\hat{\phi}(\xi,\tau) = \int_{R^3} e^{-i\langle \xi,\mathbf{x}\rangle - i\tau t}\phi(\mathbf{x},t)\, d\mathbf{x}\, dt,$$

where $\mathbf{x} = (x_1,x_2)$, $\xi = (\xi_1,\xi_2)$, $d\mathbf{x} = dx_1dx_2$ and $\langle \xi,\mathbf{x}\rangle = \xi_1x_1 + \xi_2x_2$. Its inverse is

$$\phi(\mathbf{x},t) = (2\pi)^{-3} \int_{R^3} e^{i\langle \xi,\mathbf{x}\rangle + i\tau t}\hat{\phi}(\xi,\tau)\, d\xi\, d\tau,$$

with $d\xi = d\xi_1 d\xi_2$. If $\mathbf{F} \in C^{\infty}(\mathbb{R}^3 \setminus 0)$, H-homogeneous of degree -4 with vanishing mean value, then $\hat{\mathbf{F}}(\xi,\tau)$ exists as a tempered distribution. Furthermore $\hat{\mathbf{F}}(\xi,\tau) \in C^{\infty}(\mathbb{R}^3 \setminus 0)$ and H-homogeneous of degree 0 ([9]). The following result can be found in [1] and [4].

PROPOSITION 1.2. *Let $\mathbf{F}$ induce a left-invariant principal value convolution operator on $\mathbb{H}_1$. Then $\mathbf{F} *_H \phi$ has the following representation as a pseudo-differential operator*

$$\mathbf{F} *_H \phi(\mathbf{x},t) = (2\pi)^{-3} \int_{R^3} e^{i\langle \xi,\mathbf{x}\rangle + i\tau t}\hat{\mathbf{F}}(\xi_1 - 2x_2\tau, \xi_2 + 2x_1\tau, \tau)\hat{\phi}(\xi,\tau)\, d\xi\, d\tau,$$

where $\phi(\mathbf{x},t) \in C_0^{\infty}(\mathbb{R}^3)$.

The best known example of a left invariant principal value convolution operator on $\mathbb{H}_1$ is induced by the singular Cauchy-Szegö kernel:

$$S(z,t) = \frac{1}{\pi^2(|z|^2 - it)^2}.$$

If $\mathbb{H}_1$ is viewed as the boundary of the generalized upper half-plane $\mathfrak{D} = \{(z_1,z_2) : \Im m\, z_2 > |z_1|^2\}$ via the identification $(z,t) \leftrightarrow (z, t + i|z|^2)$, then $S*_H$ may be viewed as the orthogonal projection of $L^2(\mathbb{H}_1)$ onto its subspace of boundary values of the Hardy space $H_2(\mathfrak{D})$ of holomorphic functions in $\mathfrak{D}$. A simple calculation yields the Fourier transform, $\hat{S}$ of S, namely

$$\hat{S}(\xi,\tau) = \begin{cases} 2\exp\left(-\frac{|\xi|^2}{4\tau}\right), & \text{if } \tau > 0 \\ 0, & \text{if } \tau < 0. \end{cases}$$

See details in [2],[4] and [6]. S turns out to be the simplest of a large number of basic operators on $\mathbb{H}_1$ induced by Laguerre functions. Laguerre functions have been used in the study of the twisted convolution for several decades [1] [4] [10].

One defines the generalized Laguerre polynomials, $L_n^{(p)}(x), n, p = 0, 1, 2, \ldots$ via the following generating function formula

$$\sum_{n=0}^{\infty} L_n^{(p)}(x) z^n = \frac{1}{(1-z)^{p+1}} \exp\left(-\frac{zx}{1-z}\right).$$

Then

$$\ell_n^{(p)}(x) = \left[\frac{\Gamma(n+1)}{\Gamma(n+p+1)}\right]^{1/2} x^{p/2} L_n^{(p)}(x) e^{-\frac{x}{2}}$$

are known as the Laguerre functions, where $x \geq 0$. It is well known that $\{\ell_n^{(p)}(x) : n = 0, 1, 2, \ldots\}$ is a complete orthonormal set of functions in $L^2\big([0,\infty)\big)$ for each $p = 0, 1, 2, \ldots$. (See [11]).

DEFINITION 1.2 ([4], [5]). The exponential Laguerre functions $\hat{\mathfrak{L}}_n^{(p)}(\xi)$ are defined on $\mathbb{C}$ by

$$\hat{\mathfrak{L}}_n^{(p)}(\xi) = 2(-i)^p(-1)^n \ell_n^{(p)}(|\xi|^2) e^{ip\theta}$$

and

$$\hat{\mathfrak{L}}_n^{(-p)}(\xi) = \overline{\hat{\mathfrak{L}}_n^{(p)}(\xi)} = 2i^p(-1)^n \ell_n^{(p)}(|\xi|^2) e^{-ip\theta},$$

where $n, p = 0, 1, 2, \ldots$ and $\xi = \xi_1 + i\xi_2 = |\xi| e^{i\theta}$

Note that $\hat{S}(\xi,\tau) = \hat{\mathfrak{L}}_0^{(0)}(\xi/\sqrt{2|\tau|})$ for $\tau > 0$. Now suppose that we are given $\hat{\mathbf{F}}(\xi,\tau)$, H-homogeneous of degree 0, *i.e.*

$$\hat{\mathbf{F}}(r\xi, r^2\tau) = \hat{\mathbf{F}}(\xi,\tau), \ (\xi,\tau) \in \mathbb{C} \times \mathbb{R} \quad \text{for } r > 0,$$

then the homogeneity permits us to write $\hat{\mathbf{F}}(\xi,\tau)$ as a direct sum of two functions on $\mathbb{C}$:

$$\hat{\mathbf{F}}(\xi,\tau) = \hat{\mathbf{F}}\left(\frac{\xi}{\sqrt{2|\tau|}}, \frac{1}{2}\operatorname{sgn}(\tau)\right) = \hat{\mathbf{F}}_+\left(\frac{\xi}{\sqrt{2|\tau|}}\right) \oplus \hat{\mathbf{F}}_-\left(\frac{\xi}{\sqrt{2|\tau|}}\right), \tag{4}$$

where

$$\hat{\mathbf{F}}_+\left(\frac{\xi}{\sqrt{2|\tau|}}\right) = \begin{cases} \hat{\mathbf{F}}\left(\frac{\xi}{\sqrt{2|\tau|}}, \frac{1}{2}\right), & \text{if } \tau > 0 \\ 0, & \text{if } \tau < 0 \end{cases}$$

and

$$\hat{\mathbf{F}}_-\left(\frac{\xi}{\sqrt{2|\tau|}}\right) = \begin{cases} \hat{\mathbf{F}}\left(\frac{\xi}{\sqrt{2|\tau|}}, -\frac{1}{2}\right), & \text{if } \tau < 0 \\ 0, & \text{if } \tau > 0. \end{cases}$$

In particular one has, formally, the decomposition $\mathbf{F} = \mathbf{F}_+ \oplus \mathbf{F}_-$, where

$$\mathbf{F}_+(z,t) = (2\pi)^{-3} \int_0^\infty e^{it\tau}\, d\tau \int_{R^2} e^{i\langle \xi, \mathbf{x}\rangle} \hat{\mathbf{F}}_+\Big(\frac{\xi}{\sqrt{2|\tau|}}\Big)\, d\xi,$$

and

$$\mathbf{F}_-(z,t) = (2\pi)^{-3} \int_{-\infty}^0 e^{it\tau}\, d\tau \int_{R^2} e^{i\langle \xi, \mathbf{x}\rangle} \hat{\mathbf{F}}_-\Big(\frac{\xi}{\sqrt{2|\tau|}}\Big)\, d\xi,$$

with $z = x_1 + ix_2$. Proposition 1.2 implies that $\mathbf{F}_+ *_H \mathbf{F}_- = 0$, see [4] for a proof.

$\hat{\mathbf{F}}_+$ and $\hat{\mathbf{F}}_-$ can be expanded in an exponential Laguerre series, namely, with $\xi \in \mathbb{C}$,

$$\hat{\mathbf{F}}_\pm(\xi) = \sum_{p=-\infty}^{\infty} \sum_{n=0}^{\infty} \hat{\mathbf{F}}^{(p)}_{\pm,n} \hat{\mathfrak{L}}^{(p)}_n(\xi) \tag{5}$$

This expansion is unique. In [4], Greiner studied and gave an outline of a proof of the following result:

THEOREM [GREINER]. *Let* $\mathbf{F}$ *induce a principal value convolution operator on* $\mathbb{H}_1$ *with exponential Laguerre expansion (5). Assume that* $\hat{\mathbf{F}}(\xi,\tau)$ *is Lipschitz of order* $\alpha > 1/2$ *with respect to the* τ*-variable at* $\tau = 0$*, i.e., there exist* $0 < \delta < 1$ *and* $C(\theta) \in L^1[0, 2\pi]$*, such that*

$$|\hat{\mathbf{F}}(e^{i\theta},\tau) - \hat{\mathbf{F}}(e^{i\theta},0)| \le C(\theta)|\tau|^\alpha \quad \textit{for } |\tau| < \delta. \tag{6}$$

Then $\hat{\mathbf{F}}(\tau = 0) = \hat{\mathbf{F}}_+(\tau = 0) = \hat{\mathbf{F}}_-(\tau = 0)$ *has the Fourier series expansion*

$$\hat{\mathbf{F}}(\tau = 0) = \sum_{p=-\infty}^{\infty} a_p e^{ip(\theta - \frac{\pi}{2})} \quad \textit{with} \quad a_p = \lim_{n\to\infty} \hat{\mathbf{F}}^{(p)}_{\pm,n}. \tag{7}$$

Note that $\hat{\mathbf{F}}(r\xi, r^2\tau) = \hat{\mathbf{F}}(\xi,\tau)$ for $r > 0$, so if we set $\tau = 0$ and $r = |\xi|^{-1}$, we have

$$\hat{\mathbf{F}}(\tau = 0) = \hat{\mathbf{F}}(\xi, 0) = \hat{\mathbf{F}}(|\xi| e^{i\theta}, 0) = \hat{\mathbf{F}}(e^{i\theta}, 0),$$

hence the Fourier series (7) for $\hat{\mathbf{F}}(\xi, 0)$ makes sense in the theorem. This result implies that the Laguerre calculus for the principal value convolution operators on $\mathbb{H}_1$ is the natural extension of the classical Mikhlin-Calderón-Zygmund calculus for the principal value convolution operators on $\mathbb{C}^2$. The purpose of this article is to give a direct proof of this theorem.

This result was stated in [4] without the Lipschitz condition (6). Some such condition is essential. We shall discuss this at the end.

There is an analogue result for $\mathbb{H}_n$, see Greiner [5] for details.

2. **Proof of Greiner's theorem.** We only consider the case $\tau \to 0^+$, the case of $\tau \to 0^-$ can be proved similarly. First we write $\hat{\mathbf{F}}^{(p)}_{+,n}$ and a_p in the integral form.

LEMMA 2.1.

$$4\pi \hat{\mathbf{F}}^{(p)}_{+,n} = i^p \int_0^{2\pi} e^{-ip\theta} \int_0^\infty \hat{\mathbf{F}}\Big(e^{i\theta}, \frac{1}{2r}\Big)(-1)^n \ell^{(|p|)}_n(r)\, dr\, d\theta \tag{8}$$

$$2\pi a_p = i^p \int_0^{2\pi} \hat{\mathbf{F}}(e^{i\theta}, 0) e^{-ip\theta}\, d\theta. \tag{9}$$

PROOF. Since

$$\int_{R^2} \hat{\mathfrak{L}}_n^{(p)}(\xi)\overline{\hat{\mathfrak{L}}_m^{(q)}(\xi)}\,d\xi = 4(-i)^p(-1)^{n+m}i^q \int_{R^2} \ell_n^{(|p|)}(|\xi|^2)\ell_m^{(|q|)}(|\xi|^2)e^{i(p-q)\theta}\,d\xi$$
$$= 4i^{q-p}(-1)^{n+m}\int_0^{2\pi} e^{i(p-q)\theta}\,d\theta \int_0^\infty \ell_n^{(|p|)}(r^2)\ell_m^{(|q|)}(r^2)r\,dr$$
$$= 4\pi i^{q-p}(-1)^{n+m}\delta_{p,q}\int_0^\infty \ell_n^{(|p|)}(r)\ell_m^{(|q|)}(r)\,dr$$
$$= 4\pi\delta_{p,q}\delta_{n,m}$$

and

$$\hat{\mathbf{F}}\Big(\xi, \frac{1}{2}\Big) = \sum_{p=-\infty}^{\infty}\sum_{n=0}^{\infty} \hat{\mathbf{F}}_{+,n}^{(p)}\hat{\mathfrak{L}}_n^{(p)}(\xi),$$

we have

$$4\pi\hat{\mathbf{F}}_{+,n}^{(p)} = \int_{R^2}\hat{\mathbf{F}}\Big(\xi,\frac{1}{2}\Big)\overline{\hat{\mathfrak{L}}_n^{(p)}(\xi)}\,d\xi = \int_{R^2}\hat{\mathbf{F}}\Big(\xi,\frac{1}{2}\Big)\hat{\mathfrak{L}}_n^{(-p)}(\xi)\,d\xi$$
$$= \int_0^{2\pi}\int_0^\infty \hat{\mathbf{F}}\Big(re^{i\theta},\frac{1}{2}\Big)2i^pe^{-ip\theta}(-1)^n\ell_n^{(|p|)}(r^2)r\,dr\,d\theta$$
$$= 2i^p\int_0^{2\pi} e^{-ip\theta}\int_0^\infty \hat{\mathbf{F}}\Big(e^{i\theta},\frac{1}{2r^2}\Big)(-1)^n\ell_n^{(|p|)}(r^2)r\,dr\,d\theta$$
$$= i^p\int_0^{2\pi} e^{-ip\theta}\int_0^\infty \hat{\mathbf{F}}\Big(e^{i\theta},\frac{1}{2r}\Big)(-1)^n\ell_n^{(|p|)}(r)\,dr\,d\theta.$$

We used the homogeneity of $\hat{\mathbf{F}}$ in the third equality. This proves (8). (9) yields the coefficients of the Fourier series expression of $\hat{\mathbf{F}}(e^{i\theta}, 0)$. ■

LEMMA 2.2. *For $p \geq 0$,*

$$\int_0^\infty (-1)^n\ell_n^{(p)}(x)\,dx = \left[\frac{\Gamma(n+1)}{\Gamma(n+p+1)}\right]^{\frac{1}{2}}\frac{2^{\frac{p}{2}+1}\Gamma(\frac{p}{2}+\lfloor\frac{n}{2}\rfloor+1)}{\Gamma(\lfloor\frac{n}{2}\rfloor+1)}$$

where $\lfloor x\rfloor$ = integer part of x.

PROOF. We will calculate this integral by the generating function formula of Laguerre polynomials:

$$\sum_{n=0}^{\infty} L_n^{(p)}(x)z^n = \frac{1}{(1-z)^{p+1}}\exp\Big(-\frac{zx}{1-z}\Big).$$

The definition of $\ell_n^{(p)}(r)$ yields

$$\int_0^\infty (-1)^n\ell_n^{(p)}(x)\,dx = (-1)^n\left[\frac{\Gamma(n+1)}{\Gamma(n+p+1)}\right]^{\frac{1}{2}}\int_0^\infty e^{-\frac{x}{2}}x^{\frac{p}{2}}L_n^{(p)}(x)\,dx.$$

Let

$$R_n^{(p)} = (-1)^n\int_0^\infty e^{-\frac{x}{2}}x^{\frac{p}{2}}L_n^{(p)}(x)\,dx.$$

Then

$$\sum_{n=0}^{\infty} R_n^{(p)} z^n = \sum_{n=0}^{\infty} \int_0^{\infty} x^{\frac{p}{2}} e^{-\frac{x}{2}} L_n^{(p)}(x)(-z)^n\, dx \tag{10}$$
$$= \int_0^{\infty} x^{\frac{p}{2}} e^{-\frac{x}{2}} \sum_{n=0}^{\infty} L_n^{(p)}(x)(-z)^n\, dx$$
$$= \frac{1}{(1+z)^{p+1}} \int_0^{\infty} x^{\frac{p}{2}} e^{-\frac{x}{2}} \exp\Big(\frac{zx}{1+z}\Big)\, dx$$
$$= \frac{1}{(1+z)^{p+1}} \int_0^{\infty} x^{\frac{p}{2}} \exp\Big(-\frac{x}{2}\frac{1-z}{1+z}\Big)\, dx$$
$$= \frac{1}{(1+z)^{p+1}} \frac{\Gamma(\frac{p}{2}+1)}{[\frac{1}{2}(\frac{1-z}{1+z})]^{\frac{p}{2}+1}}$$
$$= \frac{2^{\frac{p}{2}+1}\Gamma(\frac{p}{2}+1)}{(1+z)^{\frac{p}{2}}(1-z)^{\frac{p}{2}+1}}$$
$$= \frac{2^{\frac{p}{2}+1}\Gamma(\frac{p}{2}+1)(1+z)}{(1-z^2)^{\frac{p}{2}+1}}.$$

The binomial formula yields:

$$(1-z^2)^{-(\frac{p}{2}+1)} = \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{p}{2}-1}{n} z^{2n}.$$

Since

$$\binom{-\frac{p}{2}-1}{n} = \frac{(-\frac{p}{2}-1)(-\frac{p}{2}-2)(-\frac{p}{2}-3)\cdots(-\frac{p}{2}-n)}{n!}$$
$$= (-1)^n \frac{(\frac{p}{2}+1)(\frac{p}{2}+2)\cdots(\frac{p}{2}+n)}{n!} = (-1)^n \frac{\Gamma(\frac{p}{2}+n+1)}{\Gamma(\frac{p}{2}+1)n!},$$

we have

$$(1-z^2)^{-(\frac{p}{2}+1)} = \sum_{n=0}^{\infty} \frac{\Gamma(\frac{p}{2}+n+1)}{\Gamma(\frac{p}{2}+1)n!} z^{2n}$$

and

$$(1+z)(1-z^2)^{-(\frac{p}{2}+1)} = \sum_{n=0}^{\infty} \frac{\Gamma(\frac{p}{2}+n+1)}{\Gamma(\frac{p}{2}+1)n!} (z^{2n}+z^{2n+1}).$$

Therefore

$$\sum_{n=0}^{\infty} R_n^{(p)} z^n = 2^{\frac{p}{2}+1}\Gamma\Big(\frac{p}{2}+1\Big) \sum_{n=0}^{\infty} \frac{\Gamma(\frac{p}{2}+n+1)}{\Gamma(\frac{p}{2}+1)n!} (z^{2n}+z^{2n+1}).$$

The last equation implies

$$\mathbb{R}_{2n}^{(p)} = \mathbb{R}_{2n+1}^{(p)} = \frac{2^{\frac{p}{2}+1}\Gamma(\frac{p}{2}+n+1)}{n!}. \tag{11}$$

We can write (11) as follows:

$$\mathbb{R}_n^{(p)} = \frac{2^{\frac{p}{2}+1}\Gamma(\frac{p}{2} + \lfloor\frac{n}{2}\rfloor + 1)}{\Gamma(\lfloor\frac{n}{2}\rfloor + 1)},$$

and we obtain

$$\begin{aligned}\int_0^\infty (-1)^n \ell_n^{(p)}(x)\,dx &= \left[\frac{\Gamma(n+1)}{\Gamma(n+p+1)}\right]^{\frac{1}{2}} R_n^{(p)} \\ &= \left[\frac{\Gamma(n+1)}{\Gamma(n+p+1)}\right]^{\frac{1}{2}} \frac{2^{\frac{p}{2}+1}\Gamma(\frac{p}{2} + \lfloor\frac{n}{2}\rfloor + 1)}{\Gamma(\lfloor\frac{n}{2}\rfloor + 1)}.\end{aligned}$$

This proves Lemma 2.2. ■

LEMMA 2.3.

$$\lim_{n\to\infty} \int_0^\infty (-1)^n \ell_n^{(p)}(r)\,dr = 2, \quad \textit{for } p = 0, 1, 2, \ldots.$$

PROOF. From Lemma 2.2, we have

$$\int_0^\infty (-1)^n \ell_n^{(p)}(r)\,dr = \left[\frac{\Gamma(n+1)}{\Gamma(n+p+1)}\right]^{\frac{1}{2}} \frac{2^{\frac{p}{2}+1}\Gamma(\frac{p}{2} + \lfloor\frac{n}{2}\rfloor + 1)}{\Gamma(\lfloor\frac{n}{2}\rfloor + 1)}$$

From p. 12 of [7] we know that

$$\frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)} \approx n^{\alpha-\beta}\left[1 + \frac{1}{2n}(\alpha-\beta)(\alpha+\beta-1) + O(n^{-2})\right] \tag{12}$$

for large integer n, which leads to

$$\begin{aligned}\lim_{n\to\infty} \int_0^\infty (-1)^k \ell_n^{(p)}(r)\,dr &= 2^{\frac{p}{2}+1} \lim_{n\to\infty}\left[\frac{\Gamma(n+1)}{\Gamma(n+p+1)}\right]^{\frac{1}{2}} \frac{\Gamma(\frac{p}{2} + \lfloor\frac{n}{2}\rfloor + 1)}{\Gamma(\lfloor\frac{n}{2}\rfloor + 1)} \\ &= 2^{\frac{p}{2}+1} \lim_{n\to\infty} n^{-\frac{p}{2}}\left(\left\lfloor\frac{n}{2}\right\rfloor\right)^{\frac{p}{2}} = 2.\end{aligned}$$

This proves the lemma. ■

In the final stage of the proof of the theorem, we need certain norm estimate for the Laguerre functions (See Lemma 2.5). To show this estimate, we need the following asymptotic properties of Laguerre polynomials and functions.

LEMMA 2.4. *Let $M \geq 1$ be fixed constant, $\mu = 4n + 2p + 1$ and $p \geq 0$, then*

$$|L_n^{(p)}(x)| = x^{-\frac{p}{2}-\frac{1}{4}} O(n^{\frac{p}{2}-\frac{1}{4}}) \quad \textit{for } 0 \leq x \leq M, \tag{13}$$

$$|\ell_n^{(p)}(x)| \leq C \begin{cases} (x\mu)^{\frac{p}{2}} & 0 \leq x \leq \frac{1}{\mu} \\ (x\mu)^{-\frac{1}{4}} & \frac{1}{\mu} \leq x \leq \frac{\mu}{2} \\ \mu^{-\frac{1}{4}}(\mu^{\frac{1}{3}} + |\mu - x|)^{-\frac{1}{4}} & \frac{\mu}{2} \leq x \leq \frac{3\mu}{2} \\ e^{-\gamma x} & x \geq \frac{3\mu}{2}, \end{cases} \tag{14}$$

for some positive constants C and γ.

https://doi.org/10.4153/CJM-1995-067-5 Published online by Cambridge University Press

See Szegö [11] Theorem 7.6.4 for the proof of (13) and Section 8.22 for the proof of (14).

LEMMA 2.5. *Let $M \geq 1$ and $\alpha > 0$ be fixed positive constants, $p \geq 0$ and $\beta = \max(1/2 - \alpha, -1/4)$, $\mu = 4n + 2p + 1$ as in Lemma 2.4. Then we have the following norm estimates for the Laguerre functions:*

$$\int_0^M |\ell_n^{(p)}(x)|\,dx = O(n^{-\frac{1}{4}}) \tag{15}$$

$$\int_M^\infty x^{-\alpha}|\ell_n^{(p)}(x)|\,dx \leq C\mu^\beta \tag{16}$$

PROOF. First, formula (13) of Lemma 2.4 implies that as $n \longrightarrow \infty$,

$$\begin{aligned}
\int_0^M |\ell_n^{(p)}(x)|\,dx &= \left[\frac{\Gamma(n+1)}{\Gamma(n+p+1)}\right]^{\frac{1}{2}} \int_0^M x^{\frac{p}{2}} e^{-\frac{x}{2}} L_n^{(p)}(x)\,dx \\
&= \left[\frac{\Gamma(n+1)}{\Gamma(n+p+1)}\right]^{\frac{1}{2}} \int_0^M x^{\frac{p}{2}} e^{-\frac{x}{2}} x^{-\frac{p}{2}-\frac{1}{4}} O(n^{\frac{p}{2}-\frac{1}{4}})\,dx \\
&= \left[\frac{\Gamma(n+1)}{\Gamma(n+p+1)}\right]^{\frac{1}{2}} O(n^{\frac{p}{2}-\frac{1}{4}}) \int_0^M \mathbf{x}^{-\frac{1}{4}} e^{-\frac{x}{2}}\,dx = O(n^{-\frac{1}{4}}).
\end{aligned}$$

This proves (15). As to the proof of (16), we estimate the integrand in the various intervals as in Lemma 2.4. To keep track of the constants in the following estimates would be both wasteful and confusing. Thus C denotes a given constant which may change during the argument.

For $M \geq 1$ and n sufficiently large, we only need to consider the integral in (16) in three intervals *i.e.* $[M, \mu/2]$, $[\mu/2, 3\mu/2]$ and $[3\mu/2, \infty)$. Let I_1, I_2 and I_3 be the integrals on these intervals respectively. We will estimate I_1, I_2 and I_3 separately, then compare the bounds to obtain the final estimate.

$$\begin{aligned}
I_1 &= \int_M^{\frac{\mu}{2}} x^{-\alpha}|\ell_n^{(p)}(x)|\,dx \leq C\int_M^{\frac{\mu}{2}} x^{-\alpha}(x\mu)^{-\frac{1}{4}}\,dx \\
&= C\mu^{-\frac{1}{4}} \int_M^{\frac{\mu}{2}} x^{-\alpha-\frac{1}{4}}\,dx = \frac{4C}{3-4\alpha}\mu^{-\frac{1}{4}} x^{\frac{3}{4}-\alpha}\Big|_M^{\mu/2} \\
&= \frac{4C}{3-4\alpha}\mu^{-\frac{1}{4}}\left[\left(\frac{\mu}{2}\right)^{\frac{3}{4}-\alpha} - M^{\frac{3}{4}-\alpha}\right] \leq C\mu^\beta.
\end{aligned}$$

$$\begin{aligned}
I_2 &= \int_{\frac{\mu}{2}}^{\frac{3\mu}{2}} x^{-\alpha}|\ell_n^{(p)}(x)|\,dx \leq C\int_{\frac{\mu}{2}}^{\frac{3\mu}{2}} x^{-\alpha}\mu^{-\frac{1}{4}}(\mu^{\frac{1}{3}} + |\mu - x|)^{-\frac{1}{4}}\,dx \\
&\leq C\mu^{-\frac{1}{4}-\alpha} \int_{\frac{\mu}{2}}^{\frac{3\mu}{2}} (\mu^{\frac{1}{3}} + |\mu - x|)^{-\frac{1}{4}}\,dx \\
&\leq C\mu^{-\frac{1}{4}-\alpha} \int_{\frac{\mu}{2}}^{\frac{3\mu}{2}} |\mu - x|^{-\frac{1}{4}}\,dx \\
&= 2C\mu^{-\frac{1}{4}-\alpha} \int_0^{\frac{\mu}{2}} x^{-\frac{1}{4}}\,dx = \frac{8}{3}C\mu^{-\frac{1}{4}-\alpha}\left(\frac{\mu}{2}\right)^{\frac{3}{4}} \leq C\mu^{\frac{1}{2}-\alpha}.
\end{aligned}$$

$$I_3 = \int_{\frac{3\mu}{2}}^{\infty} x^{-\alpha}|\ell_n^{(p)}(x)|\,dx \le C\int_{\frac{3\mu}{2}}^{\infty} x^{-\alpha}e^{-\gamma x}\,dx \le C\int_{\frac{3\mu}{2}}^{\infty} e^{-\gamma x}\,dx$$
$$= \frac{C}{\gamma}e^{-\frac{3}{2}\gamma\mu} \le Ce^{-\frac{3}{2}\gamma\mu}.$$

Therefore

$$\int_M^{\infty} x^{-\alpha}|\ell_n^{(p)}(x)|\,dx \le C(\mu^{\beta} + \mu^{\frac{1}{2}-\alpha} + e^{-\frac{3}{2}\gamma\mu}).$$

But $\beta = \max(1/2 - \alpha, -1/4)$, hence

$$\int_M^{\infty} x^{-\alpha}|\ell_n^{(p)}(x)|\,dx \le C\mu^{\beta}.$$

This proves Lemma 2.5. ∎

Now we are ready to prove the theorem. By Lemma 2.3, it suffices to show that

$$\lim_{n\to\infty}\left|4\pi\hat{\mathbf{F}}_{+,n}^{(p)} - 2\pi a_p\int_0^{\infty}(-1)^n\ell_n^{(|p|)}(r)\,dr\right| = 0 \tag{17}$$

in order to prove that

$$\lim_{n\to\infty}\hat{\mathbf{F}}_{+,n}^{(p)} = a_p.$$

Apply Lemma 2.1 to write (17) in the integral form:

$$\begin{aligned}(18)\quad &\left|4\pi\hat{\mathbf{F}}_{+,n}^{(p)} - 2\pi a_p\int_0^{\infty}(-1)^n\ell_n^{(|p|)}(r)\,dr\right| \\ &= \left|\int_0^{2\pi} e^{-ip\theta}\int_0^{\infty}\left[\hat{\mathbf{F}}\left(e^{i\theta},\frac{1}{2r}\right) - \hat{\mathbf{F}}(e^{i\theta},0)\right](-1)^n\ell_n^{(|p|)}(r)\,dr\,d\theta\right|.\end{aligned}$$

Lipschitz condition (6) implies that there exists a positive function $C(\theta)\in L^1[0,2\pi]$ and constant $M\ge 1$ such that

$$\left|\hat{\mathbf{F}}\left(e^{i\theta},\frac{1}{2r}\right) - \hat{\mathbf{F}}(e^{i\theta},0)\right| < C(\theta)|2r|^{-\alpha} \tag{19}$$

for $r\ge M$. Fix this M and rewrite (18) as the sum of two integrals:

$$\left|4\pi\hat{\mathbf{F}}_{+,n}^{(p)} - 2\pi a_p\int_0^{\infty}(-1)^n\ell_n^{(|p|)}(r)\,dr\right| \le A_1 + A_2$$

with

$$A_1 = \left|\int_0^{2\pi} e^{-ip\theta}\int_0^{M}\left[\hat{\mathbf{F}}\left(e^{i\theta},\frac{1}{2r}\right) - \hat{\mathbf{F}}(e^{i\theta},0)\right](-1)^n\ell_n^{(|p|)}(r)\,dr\,d\theta\right|$$

and

$$A_2 = \left|\int_0^{2\pi} e^{-ip\theta}\int_M^{\infty}\left[\hat{\mathbf{F}}\left(e^{i\theta},\frac{1}{2r}\right) - \hat{\mathbf{F}}(e^{i\theta},0)\right](-1)^n\ell_n^{(|p|)}(r)\,dr\,d\theta\right|.$$

Let $N = \max|\hat{\mathbf{F}}(\xi,\frac{1}{2})|$ with $|\xi|\le\sqrt{M}$, then Lemma 2.5 leads to

$$A_1 \le 4\pi N\cdot\int_0^M|\ell_n^{(|p|)}(r)|\,dr = 4\pi N\cdot O(n^{-\frac{1}{4}}) \longrightarrow 0 \quad\text{as } n\longrightarrow\infty,$$

and

$$\begin{aligned}
A_2 &\leq \int_0^{2\pi} \int_M^\infty \left| \hat{\mathbf{F}}\Big(e^{i\theta}, \frac{1}{2r}\Big) - \hat{\mathbf{F}}(e^{i\theta}, 0) \right| \, |\ell_n^{(|p|)}(r)| \, dr \, d\theta \\
&\leq \int_0^{2\pi} C(\theta) \, d\theta \int_M^\infty (2r)^{-\alpha} |\ell_n^{(|p|)}(r)| \, dr \\
&\leq 2^{-\alpha} C \Big[\int_0^{2\pi} C(\theta) \, d\theta \Big] \mu^\beta \longrightarrow 0, \quad \text{as } n \longrightarrow \infty,
\end{aligned}$$

where we used (19) in the second inequality, (16) in the third one, C is the constant from (16), and $\mu = 4n + 2p + 1$ and $\beta = \max(\frac{1}{2} - \alpha, -\frac{1}{4}) < 0$ for $\alpha > \frac{1}{2}$ for the limit. This proves (17), and therefore Greiner's Theorem.

As we have noted in the introduction, this theorem was stated in [4] without the Lipschitz condition (6) which we used in the estimation of A_2. A few remarks are in order about this condition.

REMARK 1. Since the homogeneity of $\hat{\mathbf{F}}(\xi, \tau)$ and $\hat{\mathbf{F}}(\xi, 0)$ implies

$$\hat{\mathbf{F}}(\xi, \tau) = \hat{\mathbf{F}}(|\xi| e^{i\theta}, \tau) = \hat{\mathbf{F}}\Big(e^{i\theta}, \frac{\tau}{|\xi|^2} \Big),$$
$$\hat{\mathbf{F}}(\xi, 0) = \hat{\mathbf{F}}(|\xi| e^{i\theta}, 0) = \hat{\mathbf{F}}(e^{i\theta}, 0),$$

the condition

$$|\hat{\mathbf{F}}(e^{i\theta}, \tau) - \hat{\mathbf{F}}(e^{i\theta}, 0)| \leq C(\theta) |\tau|^\alpha \quad \text{for } |\tau| < \delta$$

is equivalent to

$$|\hat{\mathbf{F}}(\xi, \tau) - \hat{\mathbf{F}}(\xi, 0)| \leq C(\theta) \frac{|\tau|^\alpha}{|\xi|^{2\alpha}} \quad \text{for } |\tau| < \delta \cdot |\xi|^2.$$

REMARK 2. We provide some evidence for the conclusion that the continuity of $\hat{\mathbf{F}}(e^{i\theta}, \tau)$ at $\tau = 0$

(20) $$\lim_{\tau \to 0} \hat{\mathbf{F}}(e^{i\theta}, \tau) = \hat{\mathbf{F}}(e^{i\theta}, 0)$$

alone is not strong enough to imply that A_2 is small when n is sufficiently large. (20) implies that $\forall \epsilon > 0$, $\exists M \geq 1$ such that

$$\left| \hat{\mathbf{F}}\Big(e^{i\theta}, \frac{1}{2r}\Big) - \hat{\mathbf{F}}(e^{i\theta}, 0) \right| < \epsilon, \quad \text{for } r \geq M.$$

If we choose this M as in A_2, we have

$$\begin{aligned}
A_2 &= \left| \int_0^{2\pi} e^{-ip\theta} \int_M^\infty \Big[\hat{\mathbf{F}}\Big(e^{i\theta}, \frac{1}{2r}\Big) - \hat{\mathbf{F}}(e^{i\theta}, 0) \Big] (-1)^n \ell_n^{(|p|)}(r) \, dr \, d\theta \right| \\
&\leq 2\pi\epsilon \cdot \int_M^\infty |\ell_n^{(|p|)}(r)| \, dr.
\end{aligned}$$

Meanwhile Lemma 2.5 implies that:

$$\int_M^\infty |\ell_n^{(|p|)}(r)| \, dr \leq C \cdot n^{\frac{1}{2}}.$$

In fact, this can be refined to:

$$C^{-1} \cdot n^{\frac{1}{2}} \leq \int_M^\infty |\ell_n^{(|p|)}(r)|\, dr \leq C \cdot n^{\frac{1}{2}}.$$

for some constant C, see Markett [8] Lemma 1 for proof. Hence we obtain $A_2 \leq 2\pi\epsilon C \cdot n^{1/2}$, this upper bound for A_2 is far too large, since we require that A_2 be small when n sufficiently large.

REMARK 3. The Lipschitz condition is sufficient but not necessary. We have the following example to illustrate this point. Let

$$\hat{\mathbf{F}}\Big(e^{i\theta}, \frac{1}{2r}\Big) = \frac{1}{\sqrt{r}} \quad \text{for } r > 0,$$

then the homogeneity of $\hat{\mathbf{F}}$ yields

$$\hat{\mathbf{F}}\Big(\sqrt{r}e^{i\theta}, \frac{1}{2}\Big) = \hat{\mathbf{F}}\Big(e^{i\theta}, \frac{1}{2r}\Big) = \frac{1}{\sqrt{r}}.$$

Substitute r by r^2 in the above formula, we obtain

$$\hat{\mathbf{F}}\Big(re^{i\theta}, \frac{1}{2}\Big) = \frac{1}{r}, \quad \text{for } r > 0.$$

Let $\xi = re^{i\theta}$, then $r = |\xi|$ and

$$\hat{\mathbf{F}}\Big(\xi, \frac{1}{2}\Big) = \frac{1}{|\xi|} \Longrightarrow \hat{\mathbf{F}}(\xi, \tau) = \hat{\mathbf{F}}\Big(\frac{\xi}{\sqrt{2|\tau|}}, \frac{1}{2}\Big) = \frac{\sqrt{2|\tau|}}{|\xi|}, \quad \text{for } \tau > 0.$$

$\hat{\mathbf{F}}(e^{i\theta}, \tau)$ does not satisfy the condition (6) since it is only Lipschitz of order $1/2$. Now we calculate $\hat{\mathbf{F}}_{+,n}^{(p)}$ and a_p. Apply Lemma 2.1, we obtain

$$\begin{aligned} 4\pi\hat{\mathbf{F}}_{+,n}^{(p)} &= i^p \int_0^{2\pi} e^{-ip\theta} \int_0^\infty \hat{\mathbf{F}}\Big(e^{i\theta}, \frac{1}{2r}\Big)(-1)^n \ell_n^{(|p|)}(r)\, dr\, d\theta \\ &= i^p \int_0^{2\pi} e^{-ip\theta} \int_0^\infty r^{-\frac{1}{2}}(-1)^n \ell_n^{(|p|)}(r)\, dr\, d\theta \end{aligned}$$

and

$$2\pi a_p = i^p \int_0^{2\pi} \hat{\mathbf{F}}(e^{i\theta}, 0) e^{-ip\theta}\, d\theta = 0.$$

The definition of $\ell_n^{(|p|)}(r)$ yields

$$\hat{\mathbf{F}}_{+,n}^{(p)} = 0, \quad \text{for } p \neq 0,$$

and

$$\hat{\mathbf{F}}_{+,n}^{(0)} = \frac{(-1)^n}{2} \int_0^\infty r^{-\frac{1}{2}} e^{-\frac{r}{2}} L_n^{(0)}(r)\, dr.$$

A simple calculation similar to that in the proof of Lemma 2.2 yields

$$\hat{\mathbf{F}}^{(0)}_{+,2n} = \frac{1}{\sqrt{2}} \cdot \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \quad \text{and} \quad \hat{\mathbf{F}}^{(0)}_{+,2n-1} = 0.$$

Finally, Formula (12) implies

$$\lim_{n\to\infty} F^{(0)}_{+,2n} = \frac{1}{\sqrt{2}} \lim_{n\to\infty} n^{-\frac{1}{2}} = 0 \Longrightarrow \lim_{n\to\infty} F^{(0)}_{+,n} = 0 = a_0.$$

Therefore $\hat{\mathbf{F}}$ satisfies the conclusion of the theorem. This example also yields the following interesting identity of the Laguerre polynomials:

$$\frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} L^{(0)}_{2n}(x) = \frac{e^{\frac{x^2}{2}}}{x} \quad \text{for } x > 0.$$

ACKNOWLEDGEMENTS. I wish to express my thanks to my supervisor, Professor Peter C. Greiner, who gave me the problem and made valuable suggestions that have been incorporated in the paper.

REFERENCES

1. R. Beals, B. Gaveau, P. C. Greiner and J. Vauthier, *The Laguerre Calculus on the Heisenberg group: II*, Bull. Sci. Math. **110**(1986), 225–288.
2. R. Beals and P. C. Greiner, *Calculus on Heisenberg Manifolds*, Ann. of Math. Stud. **119**, Princeton University Press, Princeton, 1988.
3. G. B. Folland and E. M. Stein, *Estimates for the $\bar{\partial}_b$-Complex and Analysis on the Heisenberg Group*, Comm. Pure Appl. Math. **27**(1974), 429–522.
4. P. C. Greiner, *On the Laguerre Calculus of Left-Invariant Convolution (Pseudo-Differential) Operators on Heisenberg Group*, Seminaire Goulaouic-Meyer-Schwartz **XI**, 1980–1981, 1–39,
5. ———, *Imbeding C^n into H_n*, Proc. Sympos. Pure Math. **43**(1985), 133–147.
6. P. C. Greiner and E. M. Stein, *Estimates for the $\bar{\partial}$-Neumann Problem*, Math. Notes **19**, Princeton University Press, Princeton, 1977.
7. W. Magnus, F. Oberhettinger and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Die Grundlehren Math. Wiss. **52**, Springer-Verlag, New York Inc., New York, 1966.
8. C. Markett, *Mean Cesáro summability of Laguerre expansion and norm estimates with shifted parameter*, Anal. Math. **8**(1982), 19–37.
9. A. Nagel and E. M. Stein, *Lectures on Pseudo-differential operators: Regularity Theorems and Applications to Non-elliptic Problems*, Math. Notes **24**, Princeton University Press, Princeton, 1979.
10. J. Peetre, *The Weyl transform and Laguerre polynomials*, Matematiche **27**(1972), 301–323,
11. G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ. **23**, Amer. Math. Soc., Providence, Rhode Island, 1939.

Department of Mathematics
University of Toronto
Toronto, Ontario
M5S 1A1
e-mail: jingzhi@math.toronto.edu