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C^*-ALGEBRAS WITH REAL RANK ZERO AND THE INTERNAL STRUCTURE OF THEIR CORONA AND MULTIPLIER ALGEBRAS PART III

SHUANG ZHANG

0. Introduction. In this part, we shall be concerned with the structure of projections in a simple σ-unital C^*-algebra with the FS property, and in the associated multiplier and corona algebras. We shall also consider the closed ideal structure of the corona algebra. Most of results appear to be new even for separable simple AF algebras, and are technically independent of the previous parts I and II ([**37**] and [**38**]). The whole work develops after finding a new property of a σ-unital (nonunital) simple C^*-algebra with FS, which was not known even for a separable simple AF algebra. We relate this new property to the structure of the multiplier and corona algebras from vairous points of view.

We will try to make this part as self-contained as possible. Let us recall some concepts and known facts first. A C^*-algebra is said to have FS if the set of self-adjoint elements with finite spectra is norm dense in the set of all self-adjoint elements ([**4**, 2.6], [**26**] and [**37**]). The class of C^*-algebras with FS has been recently investigated from various perspectives in [**9**] and in [**33**] to [**41**], which contain a number of interesting subclasses of C^*-algebras, for example, AF algebras (not necessarily σ-unital), von Neumann algebras, AW^*-algebras, the Calkin algebra, the Bunce–Deddens algebras ([**2**] and [**6**]), the Cuntz algebras O_n ($2 \leqq n \leqq +\infty$) and O_M if M is an irreducible matrix ([Part I, 2.1]), more generally all purely infinite simple C^*-algebras ([Part I, 1.3] or [**41**]), the multiplier and corona algebras of nonunital separable matroid algebras ([**9**] and [Part I, 3.4]), the corona algebras of σ-unital nonunital purely infinite simple C^*-algebras, the multiplier algebras of σ-unital purely infinite simple C^*-algebras with trivial K_1-group ([Part I, 3.3]), certain irrational rotation C^*-algebras ([**22**]), and all the above C^*-algebras tensored with K ([**9**]), where K is the C^*-algebra of all compact operators on a separable Hilbert space.

We denote the multiplier algebra of a C^*-algebra A by $M(A)$. For more information on the multiplier algebras of C^*-algebras, the reader is referred to the articles [**1**], [**10**], and, more recently, [**8**], among many others. Many aspects of the multiplier algebras of C^*-algebras, even for separable nonunital AF algebras, are not clear. This series of articles is an attempt to give some structural description for the multiplier and corona algebras of σ-unital nonunital C^*-algebras with FS. The content of this part is arranged in the following way:

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In Section 1, we shall prove a technical lemma (1.1) concerning the relation between two projections in a C^*-algebra with FS, which is one of the key ingredients for the results in the later sections. The main result of this section is that if A is a σ-unital (nonunital) simple C^*-algebra with FS, then A has an approximate identity consisting of a sequence of increasing projections $\{e_n\}$ such that $e_{n+1} - e_n$ is Murray-von Neumann equivalent to a subprojection of $e_n - e_{n-1}$ for each $n \geqq 1$, where $e_0 = 0$. Such an approximate identity is called a fundamental approximate identity or "telescopic approximate identity". Moreover, if a nonzero projection q of A is given, then we can choose the above sequence $\{e_n\}$ such that e_1 is equivalent to a subprojection of q, so that $e_n - e_{n-1}$ is equivalent to a subprojection of q for each $n \geqq 1$. A result of this type was proved recently in [**23**, Theorem 2] for certain special separable simple AF algebras, heavily relying on the traces on the dimension group of an AF algebra. We prove the above general result via a purely C^*-algebraic construction. We establish a necessary and sufficient condition for a σ-unital non-projectionless simple C^*-algebra to be stable, in terms of the telescopic approximate identity.

In Section 2, we consider a short exact sequence of C^*-algebras

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0.$$

We shall prove that if I is a σ-unital simple C^*-algebra with FS such that every projection in A/I lifts (in particular, if $K_1(I) = 0$), and if both I and $\pi(A_0)$ have fundamental approximate identities, then A_0 has a fundamental approximate identity, where A_0 is any σ-unital hereditary subalgebra of A and π is the canonical quotient map. This result generalizes a recent result of [**23**, Theorem 1], which proves a weaker conclusion under the assumption $I = K$. We must overcome some technical obstacles to prove the result at the above level of generality. In later sections, this result proves to be very useful for studying the structure of $M(A)$.

In Section 3, we shall find an application of the lifting theorem in Section 2 to the structure of certain multiplier algebras. We prove that if A is a σ-unital nonunital simple C^*-algebra with FS such that every projection in $M(A)/A$ lifts and $M(A)/A$ is also simple, then any σ-unital hereditary C^*-subalgebra of $M(A)$ has either a unit or a fundamental approximate identity. Under the same assumption, the author proved in [Part I, 3.3] that $M(A)$ has FS, or, equivalently, that any hereditary C^*-subalgebra of $M(A)$ has an approximate identity consisting of projections. Here, we strengthen the earlier result.

In Section 4, we investigate the structure of projections in $M(A) \setminus A$ and in $M(A)/A$, if A is a σ-unital (nonunital) simple C^*-algebra with FS. We prove that for any two projections p and q in $M(A)\setminus A$, p can be decomposed into a sum $p = \sum_{i=1}^{\infty} p_i$ such that $q \gtrsim p_1 \gtrsim p_2 \gtrsim \dots$ and $\pi(p_i) \lesssim \pi(p_j)$, $\pi(p_j) \lesssim \pi(p_i)$ for any pair (i,j) where $p_i \in M(A)\setminus A$. (So any projection p in $M(A) \setminus A$ can be decomposed into either a finite or an infinite sum $p = \sum p_i$ such that $p_i \gtrsim p_2 \gtrsim \dots$ and

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$\pi(p_i) \precsim \pi(p_j), \pi(p_j) \precsim \pi(p_i)$ for each pair (i,j).) Moreover, we prove that if $\{\bar{p}_i\}$ is any sequence of nonzero projections in $M(A)/A$, then there exists a nonzero projection $\bar{p}_0$ in $M(A)/A$ such that $\bar{p}_0 \precsim \bar{p}_i$ for each $i \geqq 1$.

In Section 5, we obtain two sufficient conditions for $M(A)/A$ to be purely infinite, under the assumption that A is a σ-unital (nonunital) C^*-algebra with FS (not necessarily simple). We prove that if every σ-unital nonunital hereditary C^*-subalgebra of A has a fundamental approximate identity, then $M(A)/A$ is purely infinite. This slightly generalizes a previous result of the author in [**34**, 1.3], using a different proof. There, we proved that if A is simple, then $M(A)/A$ is purely infinite. We shall also prove that if no nonzero closed (two-sided) ideal A is contained in pAp for any projection p of A, then $M(A)/A$ is purely infinite.

In Section 6, we provide some new information about the closed ideal lattice of $M(A)$ and $M(A)/A$, assuming that A is a σ-unital (nonunital) simple C^*-algebra with FS. For example, we prove that the intersection of countably many nonzero closed ideals of $M(A)/A$ is nonzero, and that a nonzero closed ideal of $M(A)/A$ has a nonzero intersection with any nonzero hereditary C^*-subalgebra of $M(A)/A$. We also prove that any nonzero closed ideal of $M(A)/A$ is never σ-unital, or equivalently, any proper closed ideal of $M(A)$ strictly containing A is never σ-unital. In addition, if $\bar{B}$ is a nonzero hereditary C^*-subalgebra of $M(A)/A$, then no nontrivial closed ideal of $\bar{B}$ is ever σ-unital, no matter whether $\bar{B}$ is σ-unital or not.

If B is a C^*-algebra, we shall denote two Murray-von Neumann equivalent projections p and q in B by "$p \sim q$". The local semigroup consisting of equivalence classes of projections in B is denoted by $D(B)$. "$\precsim$" means "is equivalent to a subprojection of". π will denote the canonical map from B to B/A if A is a closed two-sided ideal of B. $\{e_{ij}\}$ will denote the matrix units of K, the C^*-algebra consisting of all compact operators on a separable Hilbert space.

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1. A property of simple C^*-algebra with real rank zero. In [**23**, Theorem 2], H. Lin found that a certain special subclass of separable simple AF C^*-algebras has an approximate identity consisting of a sequence of increasing projections, say $\{e_n\}$, such that

$$0 < e_{n+1} - e_n \precsim e_n - e_{n-1} \quad \text{for all } n \geqq 1 \text{ (where } e_0 = 0),$$

which he called a fundamental approximate identity. Intuitively, we may call such an approximate identity a "telescopic approximate identity". Lin's proof relies heavily on the traces over the dimension group of a separable AF algebra. We shall prove the same conclusion for a general σ-unital (nonunital) simple C^*-algebra with FS, via a technical construction not referring to traces at all. Certainly, simple AF algebras have FS, and hence Lin's result is included. We start with the following technical lemma, which is one of the key ingredients of our generalization and proves to be very useful.

LEMMA 1.1. *If A is a C^*-algebra with* FS *and q is a projection in A which generates A as a closed two-sided ideal, and if p a projection in A, then there exist mutually orthogonal projections $r_1, r_2, \ldots, r_n$ in A such that*

$$p = \sum_{i=1}^{n} r_i, \quad \text{and} \quad r_1 \precsim r_2 \precsim \cdots \precsim r_{n-1} \precsim r_n \precsim q.$$

Proof. The proof of this lemma uses a combination of the arguments in the author's two recent papers [**36**, 2.3] and [**35**]. We divide the proof into three steps, as follows.

Step 1. Working in $A \otimes M_n$, we identify A with $A \otimes e_{11}$, p with $p \otimes e_{11}$ and q with $q \otimes e_{11}$. We show that there exists a partial isometry w_0 of $A \otimes M_n$ such that

$$w_0 w_0^* = p \otimes e_{11} \quad \text{and} \quad w_0^* w_0 = q_1 \otimes e_{11} + q_2 \otimes e_{22} + \cdots + q_n \otimes e_{nn},$$

where q_i is a subprojection of q in A for $1 \leqq i \leqq n$.

Since q generates A as a closed ideal, there exist elements x_i and y_i in A such that

$$\left\| \sum_{i=1}^{n} x_i q y_i - p \right\| < 1.$$

It is obvious that there exist partial isometries v_i in $A \otimes M_n$ such that $v_i^* v_i = q \otimes e_{11}$ and $v_i v_i^* = q \otimes e_{ii}$ for each $1 \leqq i \leqq n$. It follows that

$$q \otimes e_{11} = v_i^* (q \otimes e_{ii}) v_i \quad \text{for } 1 \leqq i \leqq n.$$

Set

$$z_1 = (p \otimes e_{11}) \sum_{i=1}^{n} x_i v_i^* (q \otimes e_{ii}) \quad \text{and}$$

$$z_2 = \sum_{j=1}^{n} (q \otimes e_{jj}) v_j y_j (p \otimes e_{11}).$$

Then $\|z_1 z_2 - (p \otimes e_{11})\| < 1$ and so $z_1 z_1^*$ is invertible in $pAp \otimes e_{11}$. Set

$$u = (z_1 z_1^*)^{-1/2} z_1,$$

where of course the inverse is taken in $pAp \otimes e_{11}$. Clearly, u is a partial isometry in $A \otimes M_n$ such that

$$uu^* = p \otimes e_{11} \quad \text{and} \quad u^* u = q_0 \leqq \sum_{i=1}^{n} q \otimes e_{ii} \quad ([\mathbf{13}, \ 1.5]).$$

Since A has FS, $A\otimes M_n$ has FS ([**9**]). By the Riesz decomposition of $D(A\otimes M_n)$ ([**36**, 1.1]), there exists a partial isometry w in $A\otimes M_n$ such that

$$ww^* = q_0 \quad \text{and} \quad w^*w = \sum_{i=1}^{n} q_i \otimes e_{ii}, \quad \text{where } q_i \leqq q \ (1 \leqq i \leqq n).$$

Let $w_0 = uw$. Then w_0 is a partial isometry in $A\otimes M_n$ such that

$$w_0^* w_0 = \sum_{i=1}^{n} q_i \otimes e_{ii} \quad \text{and} \quad w_0 w_0^* = p\otimes e_{11},$$

namely, $p\otimes e_{11} \sim q_1\otimes e_{11} + q_2 \otimes e_{22} + \cdots + q_n \otimes e_{nn}$.

Step 2. Working in $(qAq)\otimes M_n$, we want to adjust the q_i's to get projections p_i such that

$$p_1 \leqq p_2 \leqq \cdots \leqq p_n \leqq q \quad \text{and}$$

$$p\otimes e_{11} \sim p_1 \otimes e_{11} + p_2\otimes e_{22} + \cdots + p_n \otimes e_{nn}.$$

We use induction on n. If $n=2$,

$$p\otimes e_{11} \sim q_1 \otimes e_{11} + q_2 \otimes e_{22}.$$

Applying the Riesz decomposition of $D(qAq)$ ([**36**, 1.1]), we obtain that $q_1 \sim q'_1 + q'_2$ in qAq, where q'_1 and q'_2 are two projections in qAq such that $q'_1 \leqq q_2$ and $q'_2 \leqq q - q_2$. (By [**35**, 5] we can actually conclude that q_1 is path connected to $q'_1 + q'_2$ in the set of projections of qAq). It follows that

$$p \sim q'_i \otimes e_{11} + (q'_2 + q_2)\otimes e_{22}.$$

Let $p_1 = q'_1$ and $p_2 = q_2 + q'_2$. Clearly, $p_1 \leqq p_2$.

Assume that

$$p \sim \sum_{i=1}^{n} p'_i \otimes e_{ii} \quad \text{such that } p'_2 \leqq p'_3 \leqq \cdots \leqq p'_n.$$

Applying the Riesz decomposition property to p'_1 and p'_n, we have $p'_1 \sim s_n + q'_n$ in qAq, where s_n and q'_n are projections in qAq such that $s_n \leqq q - p'_n$ and $q'_n \leqq p'_n$. Clearly,

$$p \sim q'_n \otimes e_{11} + \sum_{i=2}^{n-1} p'_i \otimes e_{ii} + (p'_n + s_n)\otimes e_{nn}.$$

Repeating this argument to q'_n and p'_{n-1}, we get that $q'_n \sim q'_{n-1} + s_{n-1}$ in qAq, where q'_{n-1} and s_{n-1} are two projections in qAq such that $s_{n-1} \leqq p'_n - p'_{n-1}$ and $q'_{n-1} \leqq p'_{n-1}$. It follows that

$$\begin{aligned} p \sim\ & q'_{n-1} \otimes e_{11} + \sum_{i=2}^{n-2} p'_i \otimes e_{ii} \\ & + (p'_{n-1} + s_{n-1}) \otimes e_{n-1,n-1} + (p'_n + s_n) \otimes e_{nn}. \end{aligned}$$

Proceeding in this way, we can write

$$p'_1 = \sum_{i=1}^{n} s_i$$

for some projections s_i in qAq such that $s_i \leqq p'_{i+1} - p'_i$ for $2 \leqq i \leqq n, s_1 \leqq p'_2$ $(p'_{n+1} = 1)$ and

$$p \sim s_1 \otimes e_{11} + \sum_{i=2}^{n} (p'_i + s_i) \otimes e_{ii}.$$

Let $p_1 = s_1$ and $p_i = p'_i + s_i$ for $2 \leqq i \leqq n$. Then

$$p_1 \leqq p_2 \leqq \cdots \leqq p_n \quad \text{and} \quad p \sim \sum_{i=1}^{n} p_i \otimes e_{ii}.$$

Let v_0 be a partial isometry in $A \otimes M_n$ such that

$$v_0 v_0^* = p \otimes e_{11} \quad \text{and} \quad v_0^* v_0 = p_1 \otimes e_{11} + p_2 \otimes e_{22} + \cdots p_n \otimes e_{nn}.$$

Step 3. Let $w_i = v_0(p_i \otimes e_{ii})v_i$. Then w_i is a partial isometry in $A \otimes e_{11}$ such that

$$w_i w_i^* = r_i \leqq p \otimes e_{11} \quad \text{and} \quad w_i^* w_i = v_i^*(p_i \otimes e_{ii})v_i \leqq q \otimes e_{11} \quad (1 \leqq i \leqq n).$$

It is easy to check that $r_i r_j = 0$ if $i \neq j$, $\sum_{i=1}^{n} r_i = p \otimes e_{11}$ and

$$r_1 \lesssim r_2 \lesssim \cdots \lesssim r_{n-1} \lesssim r_n \lesssim q \quad \text{for each } 1 \lesssim i \lesssim n.$$

Remark. It was pointed out in [**36**] that a projection q is a full projection in a C^*-algebra A with FS (in $M(A)$, resp.) if and only if $[q]$ generates $D(A)$ ($D[M(A)]$, resp.) as an ideal, where a subset D_0 of $D(A)$ is said to be an ideal if D_0 is additively closed and hereditary. It is certainly known to G.A. Elliott if A is a separable AF algebra ([**20**]). The generalized version was established after the Riesz decomposition property of $D(A)$ and of $D[M(A)]$), if

A is a C^*-algebra with FS, was found ([**36**, §1 and §2]). Therefore, we can use the Riesz decomposition property proved in [**36**, §1] and the arguments in [**20**], or the arguments in [**36**, §2] to give two different proofs for the following equivalent version of Lemma (1.1):

LEMMA (1.1)$'$. *If A is a C^*-algebra with* FS, *and if $[q]$ generates $D(A)$ as an ideal, then for any nonzero element $[p]$ in $D(A)$, there exist nonzero elements $[r_1], [r_2], \ldots, [r_n]$ in $D(A)$ such that*

$$[p] = [r_1] + [r_2] + \cdots + [r_n] \quad \textit{and} \quad [r_1] \leqq [r_2] \leqq \cdots \leqq [r_n] \leqq [q].$$

We suggest the reader to read [**20**], [**36**, §1 and §2] and the above proof for Lemma (1.1) in detail to come up with the proofs.

Now we prove the main theorem of this section. This result will be useful later for studying the structure of the multiplier and corona algebras. It includes [**23**, Theorem 2] as a special case.

THEOREM 1.2. *If A is a σ-unital (nonunital) simple C^*-algebra with* FS, *and q is a fixed nonzero projection of A, then A has a sequential increasing approximate identity, $\{e_n\}$, consisting of projections such that*

$$0 < e_{n+1} - e_n \lesssim e_n - e_{n-1} \quad \textit{for } n \geqq 1 \ (e_0 = 0) \textit{ and } e_1 \lesssim q.$$

In other words, A has a fundamental approximate identity.

Proof. Since A is σ-unital with FS, by [**33**, 1.2] A has an approximate identity consisting of an increasing sequence of projections, say $\{r_n\}$. Applying Lemma (1.1) to r_1 and q, we conclude that

$$r_1 = r_{11} + r_{12} + \cdots + r_{1n_1},$$

where the r_{1i}'s are nonzero projections of A such that

$$r_{1n_1} \lesssim \cdots \lesssim r_{12} \lesssim r_{11} \lesssim q.$$

Applying Lemma (1.1) to r_{1n_1} and $r_2 - r_1$, we have

$$r_2 - r_1 = r_{21} + r_{22} + \cdots + r_{2n_2},$$

where the r_{2i}'s are nonzero projections of A such that

$$r_{2n_2} \lesssim \cdots \lesssim r_{22} \lesssim r_{21} \lesssim r_{1n_1}.$$

Repeating Lemma (1.1) recursively in this way to r_{1n_m} and $r_{m+1} - r_m$, we get a double sequence of nonzero projections, $\{r_{ij}\}_{1 \leqq j \leqq n_i}$, such that

$$r_m - r_{m-1} = r_{m1} + r_{m2} + \cdots + r_{mn_m}, \quad m = 1, 2, \ldots,$$

where $r_0 = 0$, and such that

$$r_{mn_m} \precsim \cdots \precsim r_{m2} \precsim r_{m1} \precsim r_{m-1,n_{m-1}}, \qquad m = 1, 2, \ldots,$$

where $r_{0,n_0} = q$.

Set

$$\begin{aligned} &e_1 = r_{11}, e_2 = r_{11} + r_{12}, \ldots, e_{n_1} = r_1, \\ &e_{n_1+1} = r_1 + r_{21}, e_{n_1+2} = r_1 + r_{21} + r_{22}, \ldots, e_{n_1+n_2} = r_2, \ldots, \\ &e_{n_1+\cdots+n_{m-1}+1} = r_{m-1} + r_{m1}, \ldots, e_{n_1+n_2+\ldots n_{m-1}+n_m} = r_m, \ldots. \end{aligned}$$

By the construction, it is routine to check that $\{e_n\}$ constitutes an approximate identity of A consisting of projections satisfying the requirements.

A C^*-subalgebra B of a C^*-algebra A is said to be *hereditary* if $0 \leqq a \leqq b$, $a \in A$ and $b \in B \Rightarrow a \in B$; equivalently, if $(BAB)^- = B$, or if $B = L \cap L^*$ for some closed left ideal L of A. The following corollary will be useful later.

COROLLARY 1.3. *If A is a σ-unital (nonunital) simple C^*-algebra with* FS, *then any σ-unital (nonunital) hereditary C^*-subalgebra of A has a fundamental approximate identity.*

Proof. Let A_0 be any σ-unital (nonunital) hereditary C^*-subalgebra of A. It is easy to see that A_0 is still simple and has FS (see [**21**]). Theorem (1.2) applies to A_0 to reach the conclusion.

COROLLARY 1.4. *If A is a σ-unital simple C^*-algebra with* FS, *and if $p_1, p_2, \ldots, p_m$ are any finitely many nonzero projections of A, then there exists a nonzero projection p_0 in A such that $p_0 \precsim p_i$ for $1 \leqq i \leqq m$. Equivalently, if $[p_1], [p_2], \ldots, [p_n]$ are any finitely many nonzero elements in $D(A)$, then there exists a nonzero element $[p_0]$ in $D(A)$ such that $[p_0] \leqq [p_i]$ for each $1 \leqq i \leqq n$.*

Proof. Applying Lemma (1.1) to p_1 and p_2, we can find a nonzero projection p_1' in A such that $p_1' \precsim p_1$ and $p_1' \precsim p_2$. Applying Lemma (1.1) again to p_1' and p_3, we can find a nonzero projection p_2' in A such that $p_2' \precsim p_1'$ and $p_2' \precsim p_3$. Clearly, $p_2' \precsim p_i$ for $i = 1, 2, 3$. The conclusion follows from the induction.

The following proposition gives a necessary and sufficient condition for a σ-unital simple C^*-algebra to be stable. The proof uses a recent result of L. G. Brown ([**8**, 4.23]).

THEOREM 1.5. *If A is a σ-unital simple C^*-algebra with a nonzero projection, then A is stable if and only if A has a fundamental approximate identity $\{e_n\}$ such that for some nonzero projection f in A, $f \precsim e_n - e_{n-1}$ for all $n \geqq 1$.*

Proof. If A is stable, then $A \cong A \otimes K \cong pAp \otimes K$ for any nonzero projection p in A ([**7**, 2.8]). Then

$$e_n = \sum_{i=1}^{n} p \otimes e_{ii} \quad (n \geqq 1)$$

constitute a fundamental approximate identity such that

$$f = p \otimes e_{11} \lesssim e_n - e_{n-1} \quad \text{for all } n \geqq 1.$$

If A has a fundamental approximate identity and a nonzero projection f such that $f \sim f_n \leqq e_n - e_{n-1}$ for all $n \geqq 1$, then it is routine to check that $q = \sum_{n=1}^{\infty} f_n$ is a projection in $M(A) \setminus A$. Clearly, qAq is a stable hereditary C^*-subalgebra of A, and qAq generates A as a closed ideal. The conclusion follows from [**8**, 4.23].

2. On lifting problems. H. Lin considered in [**23**] the following extension problem: in the short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0,$$

where A and B are C^*-algebras, if A and B/A have fundamental approximate identities, does B have a fundamental approximate identity?

Lin gave a positive answer for a very special case, assuming $A = K$. We shall provide a positive answer for the problem in considerable generality, by breaking past some technical obstacles if $A \neq K$. We start with some necessary lemmas as follows:

LEMMA 2.1. *If A is a C^*-algebra and I is a σ-unital closed ideal of A, then A is σ-unital if and only if A/I is σ-unital.*

Proof. If a is a strictly positive element of A, i.e., if $(aA)^- = A$, then the image of a in A/I is a strictly positive element of A/I.

If A/I has a strictly positive element $\bar{a}$, choose a positive element a in the preimage of $\bar{a}$. Let b be a strictly positive element of I. Then $h = a + b$ is a strictly positive element of A.

The following lemma is useful for working on the lifting of projections.

LEMMA 2.2. *Assume that A is a C^*-algebra, I is a closed ideal of A with* FS *and every projection in A/I lifts to a projection in A. If A_0 is any hereditary C^*-subalgebra of A (not necessarily containing I), then every projection $\bar{p}$ in $\pi(A_0)$ lifts to a projection in A_0.*

Proof. Since every projection in A/I lifts to a projection in A, there exists a projection p in A such that $\pi(p) = \bar{p}$, but p may not be in A_0. Our job is to find a projection q in A_0 such that $\pi(q) = \bar{p}$.

Let x be any positive element in $\pi^{-1}(\bar{p}) \cap A_0$. Then $x - p = a$ is an element in I. Since I has FS, pIp and $(1-p)I(1-p)$ have FS. It follows that there exist an approximate identity $\{p_\lambda\}$ of pIp and an approximate identity $\{q_\mu\}$ of $(1-p)I(1-p)$ both consisting of projections. Clearly, $\{p_\lambda + q_\mu\}$ constitutes an approximate identity of I consisting of projections. Hence, there exists (λ, μ) such that

$$\|[1 - (p_\lambda + q_\mu)](x - p)[1 - (p_\lambda + q_\mu)]\| < \epsilon < 1.$$

Since

$$\begin{aligned}&\|(p-p_\lambda)x(p-p_\lambda)-(p-p_\lambda)\|\\&\leqq \|[1-(p_\lambda+q_\mu)](x-p)[1-(p_\lambda+q_\mu)]\|,\end{aligned}$$

we see that $(p-p_\lambda)x(p-p_\lambda)$ is invertible in $(p-p_\lambda)A(p-p_\lambda)$. Set

$$v=[(p-p_\lambda)x(p-p_\lambda)]^{-\frac{1}{2}}(p-p_\lambda)x^{\frac{1}{2}}.$$

Then v is a partial isometry of A such that $vv^*=p-p_\lambda$. Set

$$v^*v=x^{\frac{1}{2}}[(p-p_\lambda)x(p-p_\lambda)]^{-1}x^{\frac{1}{2}}=q.$$

Since x is a positive element of A_0 and A_0 is hereditary, the projection q belongs to A_0. Since $\pi(p-p_\lambda)=\pi(p)=\pi(x)=\bar{p}$, it is easily verified that $\pi(q)=\bar{p}$.

The following proposition once more recaptures a recent result in [**9**] by a different proof (see [Part I, 3.2] for another proof). We also have one more different proof of this fact which shall appear in a subsequent paper. We hope that each proof will provide some new information about the lifting problem.

Let us recall that a C^*-algebra is said to have 'HP' if every hereditary C^*-subalgebra has an approximate identity consisting of projections. It is known ([**4**, 2.7] and [**21**], see also [**26**]) that a C^*-algebra has FS if and only if it has HP.

PROPOSITION 2.3. *Suppose that A is a C^*-algebra, and I is a closed ideal of A with* HP. *If every projection in A/I lifts to a projection in A, then A has the* HP *property if and only if A/I has the* HP *property. Equivalently, A has* FS *if and only if A/I has* FS.

Recall that the author proved in [**33**, 2.12] that "$K_1(I)=0$" implies that every projection in A/I lifts.

Proof. If A has HP, of course A/I has HP. We need only prove that if A/I has HP, then A has HP.

Let B be any hereditary C^*-subalgebra of A. We want to show that B has an approximate identity consisting of projections. Since I has HP, any hereditary C^*-subalgebra of I has HP. We can assume that $B\not\subset A$. It suffices to prove that for any positive element x of B and any positive number $e>0$, there exists a projection p in B such that

$$\|(1-p)x\|<\epsilon.$$

Since A/I has HP, we can find a projection $\bar{q}$ in $\pi(B)$ such that

$$\|(\bar{1}-\bar{q})\bar{x}(\bar{1}-\bar{q})\|<\epsilon.$$

Since every projection in A/I lifts to a projection in A, by Lemma (2.2) we can find a projection q in B such that $\pi(q) = \bar{q}$. Since $\pi(B) = B/B \cap I$, by [**1**, 4.3], there exists an element b in $(1-q)(B \cap I)(1-q)$ such that

$$\|(1-q)x(1-q) - b\| = \|(\bar{1} - \bar{q})\bar{x}(\bar{1} - \bar{q})\| < \epsilon.$$

Since I has HP and $B \cap I$ is a hereditary C^*-subalgebra of I, $(1-q)(B \cap I)$ $(1-q)$ has an approximate identity consisting of projections. Choose a projection r in $(1-q)(B \cap I)(1-q)$ such that

$$\|(1-r)b(1-r)\| < \epsilon.$$

Set $p = q + r$. Then p is a projection in B and

$$\begin{aligned}\|(1-p)x(1-p)\| &\leqq \|(1-p)[(1-q)x(1-q) - b](1-p)\| \\ &+ \|(1-p)b(1-p)\| < 2\epsilon.\end{aligned}$$

This completes the proof.

COROLLARY 2.4. *If A is a nonunital C^*-algebra with* FS *and every projection in $M(A)/A$ lifts to a projection in $M(A)$, then $M(A)$ has* HP *if and only if $M(A)/A$ has* HP*; equivalently, $M(A)$ has* FS *if and only if $M(A)/A$ has* FS.

Proof. This is a special case of Proposition (2.3).

The following Lemma (2.5) slightly strengthens a previous result of the author ([**33**, 2.5]). Since this strengthening is sometimes important, we note it here for further reference.

LEMMA 2.5. *Assume that A is a C^*-algebra and I is a closed ideal of A with* FS. *If B is a hereditary C^*-subalgebra of A (not necessarily containing I) and $\bar{p}$ and $\bar{q}$ are two projections in $\pi(B)$ which lift to projections in A (and hence lift to projections in B by Lemma* (2.2)*), then the following hold:*

(i) *If $\bar{p} \perp \bar{q}$ and $\bar{q}$ lifts to a projection q in B, then we can choose a projection p in B such that $p \perp q$ and $\pi(p) = \bar{p}$.*

(ii) *If $\bar{p} \leqq \bar{q}$ and $\bar{q}$ lifts to a projection q in B, then we can choose a projection p in B such that $p \leqq q$ and $\pi(p) = \bar{p}$. If $\bar{p} \leqq \bar{q}$ and $\bar{p}$ lifts to a projection p in B, then we can choose a projection q in B such that $p \leqq q$ and $\pi(q) = \bar{q}$.*

(iii) *If every projection in A/I lifts, then two commuting projections in $\pi(B)$ lift to two commuting projections in B.*

Proof. The proof is the same as that of [**33**, 2.5], except that one works with projections in B rather than in $M(I)$, and applies Lemma (2.2) as needed. We leave it to the reader to check the details.

The following theorem will be very useful in studying the structure of certain multiplier algebras and corona algebras in the next section. It includes [**23**, Theorem 1] as a special case (but with a different proof). We shall state the result in a general setting as follows (note that on replacing A_0 by A and I by K, we get the result in [**23**, Theorem 1]):

THEOREM 2.6. *Assume that A is a C^*-algebra and I is a σ-unital simple closed ideal of A with* FS *such that every projection in A/I lifts to a projection in A (in particular, this holds if $K_1(I) = 0$,* [**33**, 2.12]). *If A_0 is any σ-unital hereditary C^*-subalgebra of A such that $A_0 \not\subset I$, and if both I and $\pi(A_0)$ have fundamental approximate identities, then A_0 has a fundamental approximate identity.*

Proof. Let $\{\bar{e}_n\}$ be a fundamental approximate identity of $\pi(A_0)$. (Note that if $A_0 = A$, then by Lemma (2.1), A must be σ-unital.) By Lemma (2.2), every projection in $\pi(A_0)$ lifts to a projection in A_0. Since $\pi(A_0)$ has a fundamental approximate identity, by Lemma (2.3) and its proof, A_0 has an approximate identity consisting of projections. Since A_0 is σ-unital, A_0 has an approximate identity consisting of an increasing sequence of projections by [**33**, 1.2], say $\{p_n\}$. We divide the remainder of the proof into the following steps:

Step 1. Let $\bar{p}_n$ be the image of p_n in $\pi(A_0)$. It follows that $\{\bar{p}_n\}$ constitutes an approximate identity of $\pi(A_0)$ consisting of projections. Thus, by applying G. A. Elliott's arguments in the proof of [**19**, 2.4], we can choose a unitary $\bar{u}$ in $[\pi(A_0)]^+$, the C^*-algebra obtained by joining an identity to $\pi(A_0)$, such that $\|\bar{u} - \bar{1}\| < \epsilon < 2$ and

$$\bar{u}\bar{e}_1\bar{u}^* \leqq \bar{p}_{n_1} \leqq \cdots \leqq \bar{u}\bar{e}_{m_{i-1}}\bar{u}^* \leqq \bar{p}_{n_i} \leqq \bar{u}\bar{e}_{m_i}\bar{u}^* \leqq \bar{p}_{n_{i+1}} \leqq \cdots (m_0 = 1).$$

Step 2. Repeatedly applying Lemma (2.5) (ii), we can find projections $\{e_{m_i}\}$ in A_0 whose image in $\pi(A_0)$ is $\bar{u}\bar{e}_{m_i}\bar{u}^*$ for each $i \geqq 0$, and

$$e_1 \leqq p_{n_1} \leqq e_{m_1} \leqq \cdots \leqq e_{m_{i-1}} \leqq p_{n_i} \leqq e_{m_i} \leqq p_{n_{i+1}} \leqq \ldots .$$

It is routine to check that $\{e_{m_i}\}$ constitutes an approximate identity of A_0 consisting of projections.

Step 3. Consider subprojections of $\bar{u}\bar{e}_{m_1}\bar{u}^*$, namely

$$\bar{u}\bar{e}_1\bar{u}^*, \bar{u}\bar{e}_2\bar{u}^*, \ldots, \bar{u}\bar{e}_{m_1}\bar{u}^*,$$

in $(\bar{u}\bar{e}_{m_1}\bar{u}^*)\pi(A_0)(\bar{u}\bar{e}_{m_1}\bar{u}^*)$. Since $\{\bar{e}_n\}$ is a fundamental approximate identity of $\pi(A_0)$, it is clear that

$$\bar{u}\bar{e}_{n+1}\bar{u}^* - \bar{u}\bar{e}_n\bar{u}^* \precsim \bar{u}\bar{e}_n\bar{u}^* - \bar{u}\bar{e}_{n-1}\bar{u}^* \quad \text{for each } n \geqq 1 (\bar{e}_{n_0} = 0)$$

By an argument in the first paragraph of the proof of [**23**, Theorem 1] or in the proof of [**18**, 9.8], there exist projections $f_1 \leqq f_2 \leqq \cdots \leqq f_{m_1} \leqq e_{m_1}$ in $e_{m_1}A_0e_{m_1}$ such that the image of f_i in $\pi(A_0)$ is $\bar{u}\bar{e}_i\bar{u}^*$ and

$$f_{i+1} - f_i \precsim f_i - f_{i-1} \quad (1 \leqq i \leqq m_1 - 1, f_0 = 0).$$

It follows that $e_{m_1} - f_{m_1} < e_{m_1}$ and $e_{m_1} - f_{m_1}$ is a projection in I. Using the same argument in $(e_{m_2} - e_{m_1})A_0(e_{m_2} - e_{m_1})$, we can find projections $f_{m_1+1} \leqq f_{m_1+2} \leqq \cdots \leqq f_{m_2} \leqq e_{m_2} - e_{m_1}$ in $(e_{m_2} - e_{m_1})A_0(e_{m_2} - e_{m_1})$ such that the image of f_{m_1+i} in $\pi(A_0)$ is $\bar{u}\bar{e}_{m_1+i}\bar{u}^* - \bar{u}\bar{e}_{m_1}\bar{u}^*$ and

$$f_{m_1+1} \lesssim f_{m_1} - f_{m_1-1} \quad \text{and} \quad f_{m_1+(i+1)} - f_{m_1+i} \lesssim f_{m_1+i} - f_{m_1+(i-1)}$$

for $1 \leqq i \leqq m_2 - 1 - m_1$. Moreover, $e_{m_2} - e_{m_1} - f_{m_2}$ is a projection in $(e_{m_2} - e_{m_1})I(e_{m_2} - e_{m_1})$.

Repeating the above arguments recursively, we obtain projections $f_{m_i+1} \leqq f_{m_i+2} \leqq \cdots \leqq f_{m_{i+1}} \leqq e_{m_{i+1}} - e_{m_i}$ in $(e_{m_{i+1}} - e_{m_i})A_0(e_{m_{i+1}} - e_{m_i})$ for each $i \geqq 1$ such that the image of f_{m_i+j} in $\pi(A_0)$ is $\bar{u}\bar{e}_{m_i+j}\bar{u}^* - \bar{u}\bar{e}_{m_i}\bar{u}^*$ and

$$f_{m_i+1} \lesssim f_{m_i} - f_{m_i-1} \quad \text{and} \quad f_{m_i+(j+1)} - f_{m_i+j} \lesssim f_{m_i+j} - f_{m_i+(j-1)}$$

for $1 \leqq j \leqq m_{i+1} - 1 - m_i$. It is clear that $e_{m_{i+1}} - e_{m_i} - f_{m_{i+1}}$ is a projection in $(e_{m_{i+1}} - e_{m_i})I(e_{m_{i+1}} - e_{m_i})$.

By the construction, we conclude that

$$\begin{aligned} & f_1 \gtrsim f_2 - f_1 \gtrsim \cdots \gtrsim f_{m_1} - f_{m_1-1} \gtrsim f_{m_1+1} \\ & \gtrsim f_{m_1+2} - f_{m_1+1} \gtrsim \cdots \gtrsim f_{m_2} - f_{m_2-1} \gtrsim f_{m_2+1} \\ & \gtrsim f_{m_2+2} - f_{m_2+1} \gtrsim \cdots \gtrsim f_{m_i} - f_{m_i-1} \gtrsim f_{m_i+1} \\ & \gtrsim f_{m_i+2} - f_{m_i+1} \gtrsim \cdots \gtrsim f_{m_{i+1}} - f_{m_{i+1}-1} \gtrsim f_{m_{i+1}+1} \gtrsim \cdots . \end{aligned}$$

Set

$$p = \sum_{i=1}^{\infty} \left[f_{m_{i-1}+1} + \sum_{j=1}^{m_i - m_{i-1} - 1} (f_{m_{i-1}+(j+1)} - f_{m_{i-1}+j}) \right], \quad (m_0 = 0).$$

Then it is routine to show that p is a projection in $M(A_0)$. It follows that $1 - p$ is a projection in $M(A_0)$, also. Since $e_{m_{i+1}} - e_{m_i} - f_{m_{i+1}}$ is a projection in $(e_{m_{i+1}} - e_{m_i})I(e_{m_{i+1}} - e_{m_i})$ for each $i \geqq 1$ and $\{e_{m_i}\}$ is an approximate identity of A_0, it follows that

$$1 - p = \sum_{i=1}^{\infty} (e_{m_i} - e_{m_{i-1}} - f_{m_i})$$

can be identified with a projection in $M(I)$ and so can p (actually $p \in M(A_0, A_0 \cap I)$). Moreover, it is easily verified that, by the construction,

$$(1 - p)A_0(1 - p) = (1 - p)(A_0 \cap I)(1 - p).$$

Step 4. Consider the equality

$$(1 - p)A_0(1 - p) = (1 - p)(A_0 \cap I)(1 - p).$$

Since I is σ-unital simple and has FS, $(1-p)A_0(1-p)$ is a σ-unital simple hereditary C^*-subalgebra of both A_0 and I with FS. By Corollary (1.3), $(1-p)A_0(1-p)$ has either a unit r or a fundamental approximate identity, say $\{r_n\}$. We set either

$$q_n = r + \sum_{m=1}^{n}(f_m - f_{m-1}) \quad \text{or}$$
$$q_n = \sum_{m=1}^{n}[(f_m - f_{m-1}) + (r_m - r_{m-1})]$$

for each $n \geqq 1$, respectively, where $r_0 = 0$. It is easily seen that

$$q_{n+1} - q_n \precsim q_n - q_{n-1} \quad \text{for each } n \geqq 1.$$

Therefore, A_0 has a fundamental approximate identity $\{q_n\}$. This completes the proof.

2.7. *Examples.* Many C^*-algebras satisfy the hypotheses of Theorem (2.6) in the place of I. For example, in general all nonunital σ-unital simple C^*-algebras having FS and trivial K_1-group satisfy the hypotheses of Theorem (2.6) in the place of I ([**33**, 2.12]). We give some specific examples as follows:

Example (i). If I is a nonunital σ-unital simple AF algebra, in particular if I is a separable nonunital simple AF algebra, then I has FS and $K_1(I) = 0$ ([**32**]).

Example (ii). If I is the tensor product of a type III factor and K, then $K_1(I) = 0, I$ is σ-unital simple and has FS.

Example (iii). If I is the tensor product of a Cuntz algebra and K, then I is simple, $K_1(I) = 0$ ([**13**]) and I has FS by [Part I, 1.3] or [**41**].

Example (iv). If $I = [M(B)/B] \otimes K$, where B is a nonunital finite matroid algebra, then I is σ-unital simple ([**19**, 3.1]), $K_1(I) = 0$ ([**19**] and [**34**, 2.4]) and I has FS by [**37**, 1.3] or [**41**].

Example (v). If I is a σ-unital (nonunital) simple C^*-algebra with FS such that $K_1(I) = 0$, then $I \otimes K$ is σ-unital (nonunital) simple with FS ([**9**]) and has a trivial K_1-group.

The reader can find more examples in Section 2 of part I ([**37**]) of this series.

3. An application of theorem (2.6). In this section, we shall apply the result of the last section to the multiplier algebras of certain C^*-algebras with FS. Let us recall some previous results relevant to this section first.

3.1 Recall that the author has recently proved the following results:

(i) ([**34**, 1.3]) If A is a σ-unital (nonunital) simple C^*-algebra with FS, then $M(A)/A$ is purely infinite, i.e., every nonzero hereditary C^*-subalgebra contains an infinite projection.

(ii) ([Part I, 1.2]) If B is a σ-unital purely simple C^*-algebra, then B is either unital or stable.

(iii) ([Part I, 1.3]) A C^*-algebra B is purely infinite simple if and only if B is simple with FS and every nonzero projections of B is infinite, and if and only if every nonzero hereditary C^*-subalgebra of B is purely infinite simple, and, also, if and only if $B \otimes K$ is purely infinite simple.

With the aid of Theorem (1.2) and Theorem (2.6), we can now provide some new information on the structure of certain multiplier algebras, beyond [Part I. 3.3]. The author has proved ([**37**, 3.3]) that if A satisfies the hypotheses in the following theorem, then $M(A)$ has FS and hence has HP. The following theorem proves that every σ-unital nonunital hereditary C^*-subalgebra of $M(A)$ actually has a fundamental approximate identity (a property which is stronger than that $M(A)$ has HP).

THEOREM 3.2. *Assume that A is a σ-unital simple C^*-algebra with* FS *such that every projection of $M(A)/A$ lifts to a projection in $M(A)$, and such that $M(A)/A$ is simple. If B is any σ-unital hereditary C^*-subalgebra of $M(A)$, then either B is unital or B has a fundamental approximate identity.*

Proof. We may assume that $B \not\subset A$ by Corollary (1.3), where B is non-unital. By (3.1) (i) and (iii) above, $\pi(B)$ is purely infinite and simple. Since B is σ-unital, $\pi(B)$ is σ-unital. By (3.1) (ii), $\pi(B)$ is either unital or stable.

If $\pi(B)$ is stable, then

$$\pi(B) \cong \pi(B) \otimes K \cong \bar{p}\pi(B)\bar{p} \otimes K$$

by [**7**, 2.8], where $\bar{p}$ is any nonzero projection in $\pi(B)$; the existence of such a nonzero projection is garanteed by [**34**, 1.1]. Certainly, $\pi(B)$ has a fundamental approximate identity. Theorem (2.6) applies to A and B to reach the conclusion.

If $\pi(B)$ is unital, with unit $\bar{p}$, by Lemma (2.2) there exists a projection p in B such that $\pi(p) = \bar{p}$. It follows that

$$\pi(B) = \pi(pBp) = \pi(pM(A)p).$$

Thus,

$$(1-p)Bp \cup pB(1-p) \cup (1-p)B(1-p) \subset A.$$

It follows from Corollary (1.3) and

$$(1-p)B(1-p) = (1-p)(B \cap A)(1-p)$$

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that $(1-p)B(1-p)$ has a fundamental approximate identity, say $\{p_n\}$. Set $q_n = p + p_n$ for each $n \geqq 1$. It is routine to check that $\{q_n\}$ is an approximate identity of B consisting of projections. Since

$$q_{n+1} - q_n = p_{n+1} - p_n \lesssim p_n - p_{n-1} = q_n - q_{n-1} \quad \text{for each } n \geqq 1,$$

$\{q_n\}$ is, moreover, a fundamental approximate identity.

Remark 3.3. In [Part II, 4.7], the author proved that if A is a σ-unital (nonunital) purely infinite simple C^*-algebra, then every essentially non-unital hereditary C^*-subalgebra of $M(A)$ is stable. Theorem (3.2) can be regarded as a complementary result to [Part II, 4.7] for certain C^*-algebras which have FS but are not necessarily purely infinite. Both results describe the set of hereditary C^*-subalgebras of the multiplier algebra, and there are common and noncommon parts in the conclusions. The reader is invited to compare these results.

4. On the structure of projections in $M(A)$ and in $M(A)/A$. In this section, we shall apply the results of Section 1 to obtain results concerning projections in $M(A)$, if A is a σ-unital simple C^*-algebra with FS.

If A is a σ-unital simple C^*-algebra with FS, Lemma (1.1) gives a relation between two nonzero projections in A. The following theorem gives a relation between two projections in $M(A) \setminus A$.

THEOREM 4.1. *Suppose that A is a σ-unital (nonunital) simple C^*-algebra with* FS. *If p and q are two projections in $M(A) \setminus A$, then there exist countably many mutually orthogonal subprojectijons of p in $M(A)\setminus A$, say $\{p_n\}$, such that*

$$\sum_{n=1}^{\infty} p_n = p \quad \text{and} \quad p_{n+1} \lesssim p_n \lesssim q \quad \textit{for } n \geqq 1,$$

where the sum converges in the strict topology of $M(A)$. Equivalently, if $[p]$ and $[q]$ are any two nonzero elements in $D[M(A)]$, then there exists a decreasing sequence of elements $\{[p_n]\} \subset D(M(A))\setminus D(A)$ such that

$$[p] = \sum_{n=1}^{\infty} [p_n] \quad \textit{and} \quad [q] \geqq [p_1] \geqq [p_2] \geqq \cdots \geqq [p_{n-1}] \geqq [p_n] \geqq \ldots.$$

Moreover, $\pi(p_n) \lesssim \pi(p_m)$ and $\pi(p_m) \lesssim \pi(p_n)$ for $m, n \geqq 1$.

Proof. Since A is σ-unital with FS, by [**33**, 1.2], we conclude that

$$p = \sum_{i=1}^{\infty} e_i \quad \text{and} \quad q = \sum_{i=1}^{\infty} f_i,$$

where the e_i's are mutually orthogonal nonzero projections in pAp, the f_i's are mutually orthogonal nonzero projections in qAq, and the sums converge in the strict topology of $M(A)$. Applying Lemma (1.1) to e_1 and f_1, we can decompose e_1 into a sum of mutually orthogonal nonzero subprojections as follows:

$$e_1 = f_{11} + f_{12} + \cdots + f_{1n_1},$$

where

$$f_{1n_1} \lesssim f_{1,n_1-1} \lesssim \cdots \lesssim f_{12} \lesssim f_{11} \lesssim f_1.$$

By Corollary (1.4), there exists a nonzero subprojection f_2' of f_2 such that $f_2' \lesssim f_{1n_1}$. Applying Lemma (1.1) to f_2' and e_2, we can decompose e_2 into a sum of mutually orthogonal nonzero subprojections as follows:

$$e_2 = f_{21} + f_{22} + \cdots + f_{2n_2},$$

where

$$f_{2n_2} \lesssim \cdots \lesssim f_{22} \lesssim f_{21} \lesssim f_2'.$$

Proceeding in this way by repeatedly using Corollary (1.4) recursively, for each $i \geqq 2$ we can find a nonzero subprojection f_i' of f_i such that

$$f_i' \lesssim f_{i-1,n_{i-1}}.$$

Then we can decompose e_i into a sum of mutually orthogonal nonzero subprojections as follows:

$$e_i = f_{i1} + f_{i2} + \cdots + f_{in_i},$$

where

$$f_{in_i} \lesssim f_{i,n_i-1} \lesssim \cdots \lesssim f_{i2} \lesssim f_{i1} \lesssim f_i';$$

here f_{ij} $(1 \leqq j \leqq n_i)$ are mutually orthogonal nonzero projections of e_iAe_i.

Set

$$p_n = \sum_{i=1}^{\infty} f_{in} \quad \text{for each } n \geqq 1,$$

where $f_{in} = 0$ if $n \geqq n_i + 1$. Since

$$f_{i,n+1} \lesssim f_{in} \lesssim f_i' \quad \text{for each pair } (i, n)$$

by the construction (where $f_1 = f_1'$), it follows that

$$p_{n+1} \lesssim p_n \lesssim q \quad \text{for } n \geqq 1 \text{ and } \sum_{n=1}^{\infty} p_n = p.$$

If

$$\sup_{i \geqq 1}\{n_i\} = +\infty,$$

then the sum $\sum_{n=1}^{\infty} p_n$ has infinitely many nonzero terms. Otherwise, the sum has only finitely many nonzero terms. (If A does not have a minimal projection, we can always choose f_{ij}'s so that the sum has infinitely many nonzero terms.) It is routine to check that, by the construction, the sum converges in the strict topology. To see that $\pi(p_n) \lesssim \pi(p_m)$ and $\pi(p_m) \lesssim \pi(p_n)$ for each pair (n, m), we can assume $n < m$. Then, by the construction, $p_m \lesssim p_n$, and hence $\pi(p_m) \lesssim \pi(p_n)$. On the other hand,

$$p_n - \sum_{i=1}^{m+1} f_{in} \lesssim p_m.$$

Since $\sum_{i=1}^{m+1} f_{in}$ is a projection in A, $\pi(p_n) \lesssim \pi(p_m)$.

The following theorem gives another decomposition of a projection in $M(A)\backslash A$ into a sum, either finite or infinite, of projections in $M(A) \backslash A$ with a "telescopic property".

THEOREM 4.2. *If A is a σ-unital simple C^*-algebra with* FS, *and if p is any projection in $M(A)\backslash A$, then for any $1 \leqq n \leqq +\infty$ there exist mutually orthogonal nonzero subprojections $\{p_i : 1 \leqq i \leqq n\}$ of p in $M(A) \backslash A$ such that*

$$p = \sum_{i=1}^{n} p_i \quad \textit{and} \quad p_1 \gtrsim p_2 \gtrsim \cdots \gtrsim p_{i-1} \gtrsim p_i \gtrsim \cdots.$$

Equivalently, if $[p]$ is any nonzero element in $D[M(A)]$, then for any $1 \leqq n \leqq +\infty$, there exist n elements $[p_i]$ $(1 \leqq i \leqq n)$ such that

$$[p] = \sum_{i=1}^{n} [p_i] \quad \textit{and} \quad [p_1] \geqq [p_2] \geqq \cdots \geqq [p_{i-1}] \geqq [p_i] \geqq \cdots.$$

Moreover, $\pi(p_i) \lesssim \pi(p_j)$ and $\pi(p_j) \lesssim \pi(p_i)$ for $i, j \geqq 1$.

Proof. By Corollary (1.3), pAp has a fundamental approximate identity, and hence

$$p = \sum_{i=1}^{\infty} e_i,$$

where the e_i's are mutually orthogonal nonzero projections of pAp such that

$$e_1 \gtrsim e_2 \gtrsim e_3 \gtrsim \cdots \gtrsim e_{n-1} \gtrsim e_n \gtrsim \cdots.$$

If n is finite, then we set

$$p_i = \sum_{k=1}^{\infty} e_{kn+i} \quad \text{for each } 1 \leqq i \leqq n.$$

It is routine to show that the p_i's are mutually orthogonal subprojections of p such that

$$\sum_{i=1}^{n} p_i = p \quad \text{and} \quad p_1 \gtrsim p_2 \gtrsim \cdots \gtrsim p_{n-1} \gtrsim p_n.$$

If n is infinite, we write p as a sum converging in the strict topology as follows:

$$\begin{array}{lllllll}
p = e_1 & +e_2 & +e_6 & +e_7 & +e_{15} & +e_{16} & +\dots \\
+e_3 & +e_5 & +e_8 & +e_{14} & +e_{17} & +e_{27} & +\dots \\
+e_4 & +e_9 & +e_{13} & +e_{18} & +e_{26} & +e_{31} & +\dots \\
+e_{10} & +e_{12} & +e_{19} & +e_{25} & +e_{32} & +e_{42} & +\dots \\
+e_{11} & +e_{20} & +e_{24} & +e_{33} & +e_{41} & +e_{50} & +\dots \\
+e_{21} & +e_{23} & +e_{34} & +e_{40} & +e_{51} & +\dots & +\dots \\
+e_{22} & +e_{35} & +e_{39} & +e_{52} & +\dots & +\dots & +\dots \\
+\dots & \dots & \dots & \dots & \dots & \dots & \dots .
\end{array}$$

Set p_1 equal to the sum of the first row in the above list, p_2 equal to the sum of the second row, $\dots$, and p_n equal to the sum of the nth row. Then it is routine to check that the p_n's are projections in $M(A) \setminus A$, and

$$p_1 \gtrsim p_2 \gtrsim \cdots \gtrsim p_{n-1} \gtrsim p_n \gtrsim \cdots.$$

Using similar arguments in the proof for Theorem (4.1), we can prove that $\pi(p_i) \lesssim \pi(p_j)$ and $\pi(p_j) \lesssim \pi(p_i)$ for each pair (i,j), for the cases either the decomposition is finite or infinite.

The following proposition establishes a property of projections in $M(A) \setminus A$ which we have already shown for A (see Corollary (1.4)).

PROPOSITION 4.3. *Suppose that A is a σ-unital simple C^*-algebra with* FS. *If $q_1, q_2, \dots, q_m$ are any finite number of projections in $M(A) \setminus A$, then there exists a projection q_0 in $M(A) \setminus A$ such that*

$$q_0 \leqq q_1 \quad \textit{and} \quad q_0 \lesssim q_i \quad \textit{for } 2 \leqq i \leqq m.$$

Proof. Since A is σ-unital with FS, by [**33**, 1.2], for each $1 \leqq i \leqq m$ we can write

$$q_i = \sum_{j=1}^{\infty} q_{ij}$$

where $\{q_{ij} : j \geqq 1\}$ are mutually orthogonal nonzero projections of q_iAq_i. For each $j \geqq 1$, applying Corollary (1.4) to

$$q_{1j}, q_{2j}, \dots, q_{mj},$$

we can choose a nonzero projection q_{0j} in A such that

$$q_{0j} \leqq q_{1j} \quad \text{and} \quad q_{0j} \precsim q_{ij} \quad \text{for } 2 \leqq i \leqq m.$$

Set

$$q_0 = \sum_{j=1}^{\infty} q_{0j}.$$

It is routine to show that

$$q_0 \leqq q_1 \quad \text{and} \quad q_0 \precsim q_i \quad \text{for } 2 \leqq i \leqq m.$$

COROLLARY 4.4. *If A is a σ-unital simple C^*-algebra with* FS, *and if* $[\bar{q}_i]$ $(1 \leqq i \leqq m)$ *are any finite number of nonzero elements in* $D[M(A)/A]$, *there exists a nonzero element* $[\bar{q}_0]$ *in* $D[M(A)/A]$ *such that* $[\bar{q}_0] \leqq [\bar{q}_i]$ *for* $1 \leqq i \leqq m$. *(We shall see in Remark* (6.4) *that the same conclusion holds if we replace the words "finite number" by "countable number".)*

Proof. Let x_i be a positive element in the preimage of $\bar{q}_i$ in $M(A)$ for each $1 \leqq i \leqq m$. Then the hereditary C^*-subalgebra B_i of $M(A)$ generated by x_i is not contained in A. By [**34**, 1.1], we can find a projection q_i in $B_i \setminus A$ for each $1 \leqq i \leqq m$. Applying Proposition (4.3) to $q_1, q_2, \dots, q_m$, we obtain a projection q_0 in $M(A) \setminus A$ such that $q_0 \precsim q_i$ for each $1 \leqq i \leqq m$. Since $[\pi(q_i)] \leqq [\bar{q}_i]$,

$$[\pi(q_0)] \leqq [\bar{q}_i] \quad \text{for each } 1 \leqq i \leqq m.$$

The conclusion follows.

5. Purely infinite property of $M(A)/A$. In [**34**, 1.3], the author proved that if A is a σ-unital (nonunital) simple C^*-algebra with FS, then every nonzero projection in $M(A)/A$ is infinite, or in other words, $M(A)/A$ is purely infinite. The following theorem, combined with Corollary (1.3), recaptures this result and gives a generalization to the non-simple case.

THEOREM 5.1. *If A is a σ-unital (nonunital) C^*-algebra with* FS, *and if pAp has a fundamental approximate identity for each projection $p \in M(A) \setminus A$, then every nonzero projection in $M(A)/A$ is infinite.*

Proof. Let $\bar{q}$ be any nonzero projection in $M(A)/A$. By [**34**, 1.1], there exists a projection p in $M(A) \setminus A$ such that $\pi(p) \leqq \bar{q}$. It suffices to show that $\pi(p)$ is infinite. By the hypotheses, pAp has a fundamental approximate identity $\{p_n\}$.

If $p_{n+1} - p_n \sim q_n < p_n - p_{n-1}$ for only finitely many n's, then there exists n_0 such that

$$p_{n+1} - p_n \sim p_n - p_{n-1} \quad \text{for all } n \geqq n_0.$$

For all $n \geqq n_0$, let v_n be a partial isometry in A such that

$$v_n v_n^* = p_n - p_{n-1} \quad \text{and} \quad v_n^* v_n = p_{2n} - p_{2n-1}.$$

Set

$$v = \sum_{n=n_0}^{\infty} v_n.$$

Then it is routine to show that v is a partial isometry in $M(A)$, and

$$vv^* = \sum_{n=n_0}^{\infty} (p_n - p_{n-1}) \quad \text{and} \quad v^*v = \sum_{n=n_0}^{\infty} (p_{2n} - p_{2n-1}) < vv^*.$$

Clearly, $\pi\left[\sum_{n=n_0}^{\infty}(p_n - p_{n-1})\right]$ is an infinite projection of $M(A)/A$. Since $\pi\left[\sum_{n=n_0}^{\infty}(p_n - p_{n-1})\right]$ is a subprojection of $\pi(p)$, hence $\pi(p)$ is infinite.

If $p_{n+1} - p_n \sim q_n < p_n - p_{n-1}$ for infinitely many n's, say

$$p_{n_i+1} - p_{n_i} \sim q_i < p_{n_i} - p_{n_i-1} \quad \text{for } i \geqq 1.$$

For each $i \geqq 1$, let v_i be a partial isometry of A such that

$$v_i v_i^* = p_{n_i+1} - p_{n_i} \quad \text{and} \quad v_i^* v_i = q_i.$$

Set

$$v = \sum_{i=1}^{\infty} v_i.$$

It is routine to check that v is a partial isometry of $M(A)$ such that

$$vv^* = \sum_{i=1}^{\infty} (p_{n_i+1} - p_{n_i}) = p_0 \quad \text{and} \quad v^*v = \sum_{i=1}^{\infty} q_i.$$

Clearly, p_0 is a subprojection of p. Since

$$p_{n_i} - p_{n_{i-1}} - q_i \neq 0 \quad \text{for all } i \geqq 1,$$

$\sum_{i=1}^{\infty}(p_{n_i} - p_{n_{i-1}} - q_i)$ is a projection in $M(A) \setminus A$ and hence

$$\pi(p_0) \sim \pi(v)^*\pi(v) < \pi(p_0);$$

namely $\pi(p_0)$, as a subprojection of $\pi(p)$, is infinite. It follows that $\pi(p)$ is infinite.

Remark 5.2. In the following theorem, we use this hypothesis: every nonzero closed ideal is not contained in pAp for any projection p in A. This condition, in case A is separable, is equivalent to that the lifting of closed ideals of A from A to $M(A)/A$ is nonzero, namely, $M(A,I)$ is a closed ideal of $M(A)$ not contained in A whenever I is a nonzero closed ideal of A (see the proof of [**34**, 3.5] for details). The condition, roughly speaking, avoids the situation that a closed ideal of A is contained in a direct summand of A, and so is weaker than the condition that A be simple.

THEOREM 5.3. *Suppose that A is a σ-unital C^*-algebra with* FS *(not necessarily simple). If any nonzero closed ideal of A is not contained in pAp for any nonzero projection p in A, then every nonzero projection in $M(A)/A$ is infinite.*

Proof. If $\bar{q}$ is a nonzero projection, take a positive element x in $M(A)$ such that $\pi(x) = \bar{q}$. By [**34**, 1.1], there exists a projection q in the hereditary C^*-subalgebra of $M(A)$ generated by x but not in A. Then $\pi(q) \leqq \bar{q}$. It suffices to show that $\pi(q)$ is infinite.

Since A is σ-unital with FS, by [**33**, 1.2] there exists an increasing sequence of projctions $\{f_n\}$ in qAq such that f_n converges to q in the strict topology. Let $I(f_1)$ be the closed ideal of A generated by f_1. By the hypotheses, there exists $n_1 > 1$ such that

$$(f_{n_1} - f_1)I(f_1)(f_{n_1} - f_1) \neq \{0\}.$$

Take a nonzero projection e_1 in $(f_{n_1} - f_1)I(f_1)(f_{n_1} - f_1)$ such that $e_1 < f_{n_1} - f_1$. By Lemma (1.1), there exist projections $e_{11}, e_{12}, \ldots, e_{1m_1}$ in e_1Ae_1 such that

$$e_1 = e_{11} + e_{12} + \cdots + e_{1m_1},$$

where

$$e_{1m_1} \lesssim \cdots \lesssim e_{12} \lesssim e_{11} \lesssim f_1.$$

We can assume that $e_{11} \sim r_1 < f_1$. It follows that the closed ideal $I(e_1)$ of A generated by e_1 is contained in $I(f_1)$, moreoever, $I(e_1)$ is equal to $I(e_{11})$, the closed ideal of A generated by e_{11}. By the hypotheses again, there exists $n_2 > n_1$ such that

$$(f_{n_2} - f_{n_1})I(e_{11})(f_{n_2} - f_{n_1}) \neq \{0\}.$$

Choose a nonzero projection e_2 in $(f_{n_2} - f_{n_1})I(e_{11})(f_{n_2} - f_{n_1})$ such that $e_2 < f_{n_2} - f_{n_1}$. Applying Lemma (1.1) again to e_2 and e_{11}, we obtain projections $e_{21}, e_{22}, \ldots e_{2m_2}$ in e_2Ae_2 such that

$$e_2 = e_{21} + e_{22} + \cdots + e_{2m_2},$$

where

$$e_{2m_2} \lesssim \cdots \lesssim e_{22} \lesssim e_{21} \lesssim e_{11}.$$

We can assume that $e_{21} \sim r_2 < e_{11}$. Proceeding in this way recursively, we find a sequence of projections $\{e_{i1}\}$ such that

$$e_{11} \gtrsim e_{21} \gtrsim \cdots \gtrsim e_{i-1,1} \gtrsim e_{i1} \gtrsim e_{i2} \gtrsim \cdots \gtrsim e_{i+1,1} \gtrsim e_{i+1,2} \gtrsim \cdots$$

such that $e_{i1} \sim r_i < e_{i-1,1}$ (where $e_{01} = f_1$) and $e_{i1} < f_{n_i} - f_{n_{i-1}}$ for each $i \geqq 1$. Set

$$q_0 = \sum_{i=1}^{\infty} e_{i1}.$$

It is routine to show that q_0 is a projection in $M(A) \setminus A$ and $q_0 < q$. For each $i \geqq 2$, we can choose a partial isometry v_i in A such that

$$v_i v_i^* = e_{i1} \quad \text{and} \quad v_i^* v_i = r_i.$$

Set

$$v = \sum_{i=2}^{\infty} v_i.$$

Then

$$vv^* = q_0 - e_{11} \quad \text{and} \quad v^*v = \sum_{i=2}^{\infty} r_i.$$

Since $e_{i-1,1} - r_i \neq 0$ for each $i \geqq 1$, $\sum_{i=2}^{\infty}(e_{i-1,1} - r_i)$ is a projection in $M(A)\setminus A$. It follows that

$$\pi(q_0) = \pi(vv^*) \sim \pi(v^*v) < \pi(q_0),$$

and hence $\pi(q_0)$ is infinite. On the other hand, $\pi(q_0) < \pi(q)$. It follows that $\pi(q)$ is infinite also.

We certainly do not expect that every nonzero projection in a general corona algebra is infinite. We give the following trivial examples for completeness.

Examples 5.4. (i) If A is a nonunital commutative C^*-algebra, then every nonzero projection in $M(A)/A$ is finite.

(ii) If $A = C_0(X) \otimes K$, where X is a locally compact Hausdorff space, then $\pi(1 \otimes e_{11})$ is a finite projection in $M(A)/A$.

Questions 5.5. Two quite natural questions come to mind:

(i) Is every nonzero projection in a simple corona algebra infinite?

(ii) Does a simple corona algebra with only two projections, namely 0 and 1, exist?

6. On the ideal structure of $M(A)/A$. In this section, we shall find some consequences of the property of a σ-unital simple C^*-algebras with FS found in Section 1 to the structure of the closed ideal lattice of the multiplier and corona algebras. Lemma (1.1) and Theorem (1.2) do give a lot of new information.

We say a closed ideal I of a C^*-algebra is *nontrivial* if I is neither the zero ideal nor the C^*-algebra itself. We say a closed ideal I is proper if I is not the C^*-algebra itself. If A is a nonunital simple C^*-algebra, the proper closed ideals of $M(A)$ strictly containing A correspond bijectively to the nontrivial closed ideals of $M(A)/A$, by passing to the preimage. In other words, to consider the set of closed ideals of $M(A)$ strictly containing A but not $M(A)$ itself, it is equivalent to consider the set of nontrivial closed ideals of $M(A)/A$.

6.1. *Previous results.* (i) ([**19**]) If A is a separable nonunital infinite matroid algebra, then $M(A)/A$ has only one nontrivial closed ideal.

(ii) ([**24**]) If A is a separable nonunital simple AF algebra without a continuous scale, then, roughly speaking, $M(A)/A$ has many nontrivial closed ideals corresponding to subsets of extremal traces on $D(A)$ in certain way; $M(A)/A$ may even have (uncountably) infinitely many nontrivial closed ideals. Moreover, $M(A)/A$ has a smallest nontrivial closed ideal $\bar{J}_0$ in the sense that each nontrivial closed ideal of $M(A)/A$ contains $\bar{J}_0$.

(iii) ([**20**]) If A is a separable nonunital AF algebra, then the closed ideal lattice of $M(A)$ is isomorphic to the lattice of additively closed hereditary subsets, or ideals, of $D[M(A)]$. A subset S of $D(.)$ is said to be *hereditary* if for any two elements x and y in $D(.)$ with $0 \leqq x \leqq y, y \in S$ implies $x \in S$.

(iv) ([**36**, 2.3]) If A is a σ-unital C^*-algebra with FS, then the closed ideal lattice of $M(A)$ is isomorphic to the lattice of ideals of $D[M(A)]$. (Proof is the same as in [**20**], once $D(A)$ is known to have the Riesz decomposition property. But the Riesz decomposition property of $D(A)$ has been proved in [**36**]).

(v) ([**36**]) If A is a σ-unital stable simple non-elementary C^*-algebra with FS and there exists a faithful trace defined on $D(A)$, then $M(A)$ has a largest closed ideal J strictly containing A, called the shell ideal, in the sense that each proper closed ideal of $M(A)$ is contained in J. If the set of traces on $D(A)$ is faithful, then $M(A)/A$ has a smallest nontrivial closed ideal $\bar{J}_0$. Here, the set of traces on $D(A)$ is said to be faithful if $\tau([p]) < \tau([q])$ for any trace τ defined on $D(A) \Rightarrow [p] \leqq [q]$ in $D(A)$.

THEOREM 6.2. *If A is a σ-unital simple C^*-algebra with* FS, *then the intersection of any countable number of nonzero closed ideals of $M(A)/A$ is a nonzero closed ideal; equivalently, the intersection of countably many closed ideals of $M(A)$ strictly containing A is still a closed ideal of $M(A)$ strictly containing A.*

Proof. Let $\{I_n\}$ be countably many closed ideals of $M(A)$ strictly containing A. By [**34**, 1.1], there exists a projection p_n in I_n but not in A. Since A is σ-

unital with FS, p_nAp_n is σ-unital with FS also for each $n \geqq 1$. Let $\{q_n\}$ be an approximate identity of A consisting of an increasing sequence of projections of A. The existence of such a sequence of projections is guaranteed by [**33**, 1.2]. Set

$$e_n = q_n - q_{n-1} \quad \text{for all } n \geqq 1, \text{ where } q_0 = 0.$$

Hence, $1 = \sum_{n=1}^{\infty} e_n$, where the sum converges in the strict topology. For the same reason, for each $n \geqq 1$ there exists a sequence of mutually orthogonal nonzero projections, say $\{f_{nm} : m \geqq 1\}$, in p_nAp_n such that

$$\sum_{m=1}^{\infty} f_{nm} = p_n \quad (n \geqq 1).$$

Applying Lemma (1.1) to two nonzero projections e_m and f_{1m} for each $m \geqq 1$, we can write f_{1m} as a sum of mutually orthogonal nonzero subprojections of e_m as follows:

$$f_{1m} = e_{1m}(1) + e_{1m}(2) + \cdots + e_{1m}(n_{1m})$$

for $m \geqq 1$ such that

$$0 \neq e_{1m}(n_{1m}) \precsim \cdots \precsim e_{1m}(2) \precsim e_{1m}(1) \precsim e_m.$$

Applying Lemma (1.1) to f_{2m} and $e_{1m}(n_{1m})$ for each $m \geqq 1$, we have

$$f_{2m} = e_{2m}(1) + e_{2m}(2) + \cdots + e_{2m}(n_{2m})$$

such that

$$0 \neq e_{2m}(n_{2m}) \precsim \cdots \precsim e_{2m}(2) \precsim e_{2m}(1) \precsim e_{1m}(n_{1m}),$$

Applying Lemma (1.1) recursively, to f_{im} and $e_{i-1,m}(n_{i-1,m})$, we have

$$f_{im} = e_{im}(1) + e_{im}(2) + \cdots + e_{im}(n_{im})$$

such that

$$0 \neq e_{im}(n_{im}) \precsim \cdots \precsim e_{im}(2) \precsim e_{im}(1) \precsim e_{i-1,m}(n_{i-1,m}),$$

where $e_{im}(j)$ $(1 \leqq j \leqq n_{im})$ are all nonzero projections of $f_{im}Af_{im}$ for each $m \geqq 1$ and each $i \geqq 1$.

By the construction, it is easily verified that

$$e_{mm}(1) \precsim \cdots \precsim e_{m-1,m}(1) \precsim \cdots \precsim e_{2m}(1) \precsim \cdots \precsim e_{1m}(1) \precsim e_m$$

for each $m \geqq 1$. Let $e_{mm}(1) \sim e'_{mm}(1) < e_m$ for each $m \geqq 1$. Set

$$p = e'_{11}(1) + e'_{22}(1) + \cdots + e'_{mm}(1) + \ldots.$$

Since $e'_{ii}(1)e'_{jj}(1) = 0$ if $i \neq j$, it is routine to show that p is a projection in $M(A) \setminus A$. Moreover, it is clear that

$$p \lesssim p_1,\ p - e'_{11}(1) \lesssim p_2,$$

$$p - e'_{11}(1) - e'_{22}(1) \lesssim p_3, \ldots, p - \sum_{m=1}^{n-1} e'_{mm}(1) \lesssim p_n, \quad (n \geqq 2).$$

Since $A \subset I_n$ and $p_n \in I_n$, it follows that p is a projection in $I_n \setminus A$ for all $n \geqq 1$. Therefore, p is in the intersection of the I_n's but not in A.

The equivalence comes from the one to one corresponding between the set of nonzero closed ideals of $M(A)/A$ and the set of closed ideals of $M(A)$ strictly containing A via the canonical lifting from $M(A)/A$ to $M(A)$.

COROLLARY 6.3. *Suppose that A is a σ-unital simple C^*-algebra with* FS. *Then we have the following conclusions:*

(i) *The intersection of finitely many closed ideals of $M(A)$ strictly containing A strictly contains A; equivalently, the intersection of finitely many nonzero closed ideals of $M(A)/A$ is again a nonzero closed ideal of $M(A)/A$.*

(ii) *If $M(A)$ has only countably many closed ideals, then $M(A)/A$ has a smallest nonzero closed ideal, i.e., the intersection of all nonzero closed ideals of $M(A)/A$.*

Proof. This is an easy consequence of Theorem (6.2).

Remarks 6.4. (i) In the above Corollary (6.3), if the intersection $\bar{J}_0$ of all nonzero closed ideals is nonzero, then, by [**34**, 1.3], $\bar{J}_0$ is a purely infinite simple C^*-algebra. It follows from [Part I, 1.3] that $\bar{J}_0$ has FS. If, in addition, every projection in $\bar{J}_0$ lifts to a projection in $\pi^{-1}(\bar{J}_0) = J_0$, then J_0 has FS by Proposition (2.3).

(ii) The combination of Theorem (6.2) and [**36**, 2.3] has proved that the intersection of countably many nonzero hereditary subsets of $D[M(A)/A]$ is a nonzero hereditary subset of $D[M(A)/A]$. Actually, we have proved in Theorem (6.3) that for countably many nonzero elements $\{[p_i]\}$ of $D[M(A)/A]$, there exists a nonzero element $[p]$ in $D[M(A)/A]$ such that

$$[p] < [p_i] \quad \text{for all } i \geqq 1,$$

where $[p] < [p_i]$ if and only if $p \sim p_0 < p_i$.

THEOREM 6.5. *If A is a σ-unital simple C^*-algebra with* FS, *then*

(i) *Any proper nonzero closed ideal of $M(A)/A$ is not σ-unital. Equivalently, any proper closed ideal of $M(A)$ strictly containing A is not σ-unital.*

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(ii) *Any closed ideal of $M(A)$ strictly containing A does not have a sequential approximate identity consisting of projections.*

Proof. (ii) easily follows from (i). The equivalence stated in (i) follows from Lemma (2.1). We have two proofs for (i), a direct one and an indirect one. Here we present the direct proof, and will give the indirect one later.

Suppose that a closed ideal I of $M(A)$ strictly containing A were σ-unital. By a standard construction, we could get a countable approximate identity $\{e_n\}$ of I such that

$$e_n e_m = e_n \quad \text{if } n < m.$$

Clearly, we could then find a subsequence $\{e_{n_i}\}$ such that

$$[(e_{n_{i+1}} - e_{n_i})I(e_{n_{i+1}} - e_{n_i})]^- = B_i$$

is a nonzero hereditary C^*-subalgebra of I for each $i \geqq 1$, and $B_iB_j = B_jB_i = \{0\}$ if $i \neq j$. Since I is a closed ideal of $M(A)$, I is hereditary. It follows that

$$B_i = [(e_{n_{i+1}} - e_{n_i})M(A)(e_{n_{i+1}} - e_{n_i})]^- \quad (i \geqq 1),$$

namely, $\{B_i\}$ are mutually orthogonal hereditary C^*-subalgebras of $M(A)$. By [**34**, 1.1], there is a nonzero projection p_i in B_i for each $i \geqq 1$ such that

$$p_i(e_{n_{i+1}} - e_{n_i}) = (e_{n_{i+1}} - e_{n_i})p_i = p_i \quad \text{for } i \geqq 1$$

(see the proof of [Part, I, 1.2] if this is not clear). It follows that $p_ip_j = 0$ if $i \neq j$. It is routine that $\sum_{i=1}^{\infty} p_i = p$ is a projection in the multiplier algebra of I.

Let q be any nonzero projection in $I \setminus A$. We can write

$$q = \sum_{i=1}^{\infty} f_i,$$

where f_i are mutually orthogonal nonzero projections of qAq and the sum converges in the strict topology of $M(A)$. By Lemma (1.1), we can find nonzero projections r_i in p_iAp_i and r_i' in f_iAf_i such that $r_i \sim r_i'$ for each $i \geqq 1$. Set

$$r = \sum_{i=1}^{\infty} r_i \quad \text{and} \quad r' = \sum_{i=1}^{\infty} r_i'.$$

It is routine to check that r is a projection in $M(I)$ since $p \in M(I)$, and r' is a projection in $I \setminus A$ since $\sum_{i=1}^{\infty} f_i$ converges in the strict topology of $M(A)$. Clearly, there exists a partial isometry v in $M(I)$ such that

$$vv^* = r \quad \text{and} \quad v^*v = r'.$$

Hence, r is a projection in $I \setminus A$, since r' is in $I \setminus A$ and I is a closed ideal of $M(I)$. Since $r_i \leqq p_i$ hence

$$r_i(e_{n_{i+1}} - e_{n_i}) = (e_{n_{i+1}} - e_{n_i})r_i = r_i,$$

we would conclude that

$$\|(1 - e_{n_i})r\| \geqq \|(1 - e_{n_i})r_i\| \geqq 1 \quad \text{for } i \geqq 1.$$

This contradicts the fact that $\{e_n\}$ is an approximate identity of I.

An indirect proof of Theorem (6.5) (i) comes as a corollary of the following proposition.

PROPOSITION 6.6. *If A is a σ-unital simple C^*-algebra with* FS, *and if $\bar{I}$ is a nontrivial closed of $M(A)/A$, then*

$$\bar{I}^\perp = \{\bar{x} \in M(A)/A : \bar{x}\bar{I} = \bar{I}\bar{x} = \{\bar{0}\}\} = \{\bar{0}\}.$$

Consequently, $\bar{I}$ is not σ-unital.

Proof. If $\bar{I}^\perp \neq \{\bar{0}\}$, then clearly there exists a nonzero hereditary C^*-subalgebra $\bar{B}$ of $M(A)/A$ orthogonal to $\bar{I}$. By [**34**, 1.1], there exists a projection p in $\pi^{-1}(\bar{B}) = B$ but not in A such that $pI = Ip \subset A$. By [**34**, 1.1] again, there exists a nonzero projection q in I but not in A such that $qB = Bq \subset A$. By Remark (6.4) or Corollary (4.4), there exists a projection r in $M(A)$ but not in A such that $r \sim r_0 < p$ and $r \lesssim q$. Since I is a closed ideal of $M(A)$ and $r \lesssim q$, r is a projection of I because $q \in I$. Since $r \sim r_0, r_0$ is a projection in I but not in A. On the other hand, $r_0 I = r_0 I \subset A$. This is a contradiction. Hence, $\bar{I}^\perp = \{\bar{0}\}$. If $\bar{I}$ were σ-unital, by [**27**, 15], we would reach

$$M(A)/A = \bar{I}^{\perp\perp} = \bar{I}.$$

This is contrary to hypothesis.

Theorem (6.5) and Proposition (6.6) and the following Corollary (6.7) all tell us that every nonzero closed ideal of $M(A)/A$ is rather "spread out" if A is a σ-unital (nonunital) simple C^*-algebra with FS.

COROLLARY 6.7. *Suppose that A is a σ-unital simple C^*-algebra with* FS. *If $\bar{I}$ is a nonzero closed ideal of $M(A)/A$ and $\bar{B}$ is any nonzero hereditary C^*-subalgebra of $M(A)/A$, then $\bar{I} \cap \bar{B} \neq \{\bar{0}\}$.*

Proof. If A_1 is any C^*-algebra, and I_1 is a nonzero closed ideal of A_1 and B_1 is a nonzero hereditary C^*-subalgebra A_1, then I_1 and B_1 do not intersect if and only if B_1 and I_1 are orthogonal.

In fact, if $I_1 \cap B_1 \neq \{0\}$, then certainly B_1 and I_1 are not orthogonal. Conversely, if B_1 and I_1 are not orthogonal, then there exist $b \in B_1$ and $a \in I_1$ such that $ab \neq 0$. It follows that b^*a^*ab is a nonzero element of $B_1I_1B_1 \subset I_1 \cap B_1$, and hence $B_1 \cap I_1 \neq \{0\}$.

By Proposition (6.6), $\bar{I}^\perp = \{\bar{0}\}$. Hence, $\bar{B}$ is not orthogonal to $\bar{I}$, and so the intersection of $\bar{B}$ and $\bar{I}$ is not $\{\bar{0}\}$.

A proper closed ideal of a C^*-algebra is said to be a *shell ideal* if it contains any proper closed ideal of the C^*-algebra. Clearly, if the shell ideal exists, then it is unique.

PROPOSITION 6.8. *If A is a σ-unital C^*-algebra with* FS, *then $M(A)$ has a shell ideal if and only if $I_1 + I_2 \neq M(A)$ for any two proper closed ideals I_1 and I_2 of $M(A)$.*

Proof. If $M(A)$ has the shell ideal J, then any two proper closed ideals I_1 and I_2 are contained in J. Hence, $I_1 + I_2 \subset J \neq M(A)$. Let us prove the converse.

Set J equal to the closed ideal of $M(A)$ generated by the union of all proper closed ideal of $M(A)$. We shall show that $J \neq M(A)$. It is sufficient to show that the identity is not in J. If 1 were in J, then, by the definition of J, there would exist finitely many projections p_i and elements x_i, y_i's in J such that

$$\left\| 1 - \sum_{i=1}^{n} x_i p_i y_j \right\| < 1.$$

It follows from this that $\sum_{i=1}^n x_i p_i y_i = z$ is invertible in $M(A)$. By the proof of [**36**, 2.3], $1 = r_1 + r_2 + \cdots + r_n$ for some mutually orthogonal projections r_i, where $r_i \lesssim p_i$ for each $1 \leqq i \leqq n$. On the other hand, the r_i's can come from at most n proper closed ideals of $M(A)$, say $I_1, I_2, \ldots, I_n$, by the definition of J and the fact that each projection of J belongs to a proper closed ideal of $M(A)$. Since the sum of any two proper closed ideals of $M(A)$ is again a proper closed ideal of $M(A)$, it is easily verified, by induction, that $I = I_1 + I_2 + \cdots + I_n$ is a proper closed ideal of $M(A)$. We would conclude that the identity of $M(A)$ were in a proper closed ideal of $M(A)$. This is not true.

COROLLARY 6.9. *If A is a C^*-algebra with* FS, *and if the closed ideal lattice of $M(A)$ is linearly ordered, then $M(A)$ has a shell ideal.*

Proof. Since the closed ideal lattice of $M(A)$ is linearly ordered, the sum of any two proper closed ideals I_1 and I_2 is either I_1, if $I_2 \subset I_1$, or I_2 if $I_1 \subset I_2$. The conclusion follows from Proposition (6.8).

LEMMA 6.10. *If A is a C^*-algebra and B is any hereditary C^*-subalgebra of A, then any closed ideal I_B of B has the form $B \cap J_A$, where J_A is a closed ideal of A.*

Proof. Let I_B be any closed ideal of B. Let J_A be the closed ideal of A generated by I_B, i.e., $J_A = (AI_BA)^-$. Clearly,

$$J_A \cap B = (BJ_AB)^- = I_B.$$

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The converse is trivial.

PROPOSITION 6.11. *If A is a σ-unital (nonunital) simple C^*-algebra with* FS, *and if $\bar{B}$ is a hereditary C^*-subalgebra of $M(A)/A$ generating $M(A)/A$ as a closed ideal, then the closed ideal lattice of $\bar{B}$ is isomorphic to the closed ideal lattice of $M(A)/A$.*

Proof. By Lemma (6.10), we need only show that if $\bar{I}$ is a nontrivial closed ideal of $M(A)/A$, then $\bar{I} \cap \bar{B}$ is a nontrivial closed ideal of $\bar{B}$. $\bar{I} \cap \bar{B} \neq \{\bar{0}\}$ by Corollary (6.7). $\bar{I} \cap \bar{B} \neq \bar{B}$ since $\bar{B}$ generates $M(A)/A$ as a closed ideal.

The following theorem reveals a unusual aspect of certain corona algebras.

THEOREM 6.12. *If A is a σ-unital simple C^*-algebra with* FS, *and if $\bar{B}$ is a nonzero hereditary C^*-subalgebra of $M(A)/A$, then every nontrivial ideal of $\bar{B}$ is not σ-unital (no matter whether $\bar{B}$ is σ-unital or not).*

Proof. Let $\bar{I}$ be any nonzero proper closed ideal of $\bar{B}$. Assuming that $\bar{I}$ were σ-unital, we would reach a contradiction.

Since $\bar{I}$ is hereditary in $\bar{B}$ and $\bar{B}$ is hereditary in $M(A)/A$, we conclude that

$$\{\bar{I}[M(A)/A]\bar{I}\}^- = \{\bar{I}\bar{B}[M(A)/A]\bar{B}\bar{I}\}^- = (\bar{I}\bar{B}\bar{I})^- = \bar{I},$$

namely, $\bar{I}$ is a hereditary C^*-subalgebra of $M(A)/A$. By [**27**, 15], we have $\bar{I}^{\perp\perp} = \bar{I}$, where "$\perp$" is taken in $M(A)/A$. Since $\bar{B} \neq \bar{I}$, there exists a nonzero positive element $\bar{b}$ in $\bar{B}\backslash\bar{I}$. Let $\bar{B}_0$ be the hereditary C^*-subalgebra of $\bar{B}$ generated by $\bar{b}$ and $\bar{I}$. The same argument as above applies to show that $\bar{B}_0$ is hereditary C^*-subalgebra of $M(A)/A$. Clearly, $\bar{B}_0$ would be σ-unital if $\bar{I}$ were σ-unital. By [**27**, 15] again, $\bar{B}_0^{\perp\perp} = \bar{B}_0$. Since $\bar{B}_0$ strictly contains $\bar{I}$, under the assumption that $\bar{I}$ were σ-unital we would conclude that there exists a nonzero positive element $\bar{x}$ in $\bar{B}_0$ such that $\bar{x}$ is orthogonal to $\bar{I}$.

In fact, if we let $\bar{z}$ be a strictly positive element of $\bar{B}_0$, then for any element $\bar{y} \in \bar{I}^\perp, \bar{z}\bar{y}\bar{y}^*\bar{z}$ is an element of $\bar{B}_0 \cap \bar{I}^\perp$. This is because for all $\bar{a} \in \bar{I}, \bar{a}\bar{z}$ and $\bar{z}\bar{a}$ are elements of $\bar{I}$, and hence

$$\bar{z}\bar{y}\bar{y}^*\bar{z}\bar{a} = \bar{a}\bar{z}\bar{y}\bar{y}^*\bar{z} = \bar{0}.$$

If $\bar{z}\bar{y}\bar{y}^*\bar{z} = \bar{0}$ for all self-adjoint element $\bar{y} \in \bar{I}^\perp$, then $\bar{I}^\perp \subset \bar{B}_0^\perp$. Thus,

$$\bar{B}_0 = \bar{B}_0^{\perp\perp} \subset \bar{I}^{\perp\perp} = \bar{I}.$$

But $\bar{B}_0$ strictly contains $\bar{I}$. Therefore, there exists a nonzero element $\bar{y}$ in $\bar{I}^\perp$ such that $\bar{x} = \bar{z}\bar{y}\bar{y}^*\bar{z} \neq \bar{0}$, which is an element in $\bar{B}_0 \cap \bar{I}^\perp$.

By [**34**, 1.1], we can find a projection p in $M(A) \setminus A$ such that $\pi(p)$ is in the hereditary C^*-subalgebra of $M(A)/A$ generated by $\bar{x}$, denoted by $\bar{B}_{\bar{x}}$. Consequently, $\bar{B}_{\bar{x}}$ is orthogonal to $\bar{I}$. Let q be any projection in $M(A) \setminus A$ such that $\pi(q)$ is in $\bar{I}$. By Corollary (4.4), there exists a projection $p_0 \leqq p$ in $M(A) \setminus A$

such that $p_0 \lesssim q$. It would follow that $\bar{I}$ and $\bar{B}_{\bar{x}}$ were not orthogonal. This is a contradiction.

Remark 6.13. In the part IV of this series ([**40**]), we shall prove various equivalent versions of the generalized Weyl–von Neumann theorem in $M(A)$, assuming that A is a σ-unital C^*-algebra with FS (not necessarily simple).

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University of Kansas,
Lawrence, Kansas