Canad. Math. Bull. Vol. **51** (1), 2008 pp. 32–46

On Linear Independence of a Certain Multivariate Infinite Product

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Abstract. Let $q, m, M \geq 2$ be positive integers and $r_1, r_2, \dots, r_m$ be positive rationals and consider the following M multivariate infinite products

$$F_i = \prod_{j=0}^{\infty}(1 + q^{-(Mj+i)}r_1 + q^{-2(Mj+i)}r_2 + \cdots + q^{-m(Mj+i)}r_m)$$

for $i = 0, 1, \dots, M-1$. In this article, we study the linear independence of these infinite products. In particular, we obtain a lower bound for the dimension of the vector space $\mathbb{Q}F_0+\mathbb{Q}F_1+\cdots+\mathbb{Q}F_{M-1}+\mathbb{Q}$ over $\mathbb{Q}$ and show that among these M infinite products, $F_0, F_1, \dots, F_{M-1}$, at least $\sim M/m(m+1)$ of them are irrational for fixed m and $M \to \infty$.

1 Introduction and Result

For any integer $m \geq 1$ and fixed $q \in \mathbb{C}$ with $|q| > 1$, the infinite product

$$\prod_{j=0}^{\infty}(1 + q^{-j}z_1 + q^{-2j}z_2 + \cdots + q^{-mj}z_m)$$

defines an entire function in $\mathbb{C}^m$. In the case where $m = 1$, the one variable version of the above product, $\prod_{j=0}^{\infty}(1 + q^{-j}z)$, has been studied extensively and results on its irrationality have been obtained since 1943 [1, 4–10]. For example, Lototsky [5] showed that for any integer $q \geq 2$ and $r \in \mathbb{Q}$, $r \neq 0, -q^j$ $(j = 1, 2, \dots)$,

$$\prod_{j=0}^{\infty}(1 + q^{-j}r)$$

is irrational. For the cases when $m = 2$ and $m > 2$, there are only few results [2, 3, 12, 13]. Recently, the second author [12] investigated the infinite products for the multivariate case when $m \geq 2$ and showed the following.

Theorem 1.1 *If $q, m, M \geq 2$ are positive integers and $M \geq m^2 - 2$, then for any*

Received by the editors October 14, 2004.
The authors' research is supported by NSERC of Canada.
AMS subject classification: 11J72.

positive rationals $r_1, r_2, \dots, r_m$, at least one of the infinite products

$$F_0 := \prod_{j=0}^{\infty}(1 + q^{-Mj}r_1 + q^{-2Mj}r_2 + \cdots + q^{-mMj}r_m),$$

$$F_1 := \prod_{j=0}^{\infty}(1 + q^{-Mj-1}r_1 + q^{-2Mj-2}r_2 + \cdots + q^{-mMj-m}r_m),$$

$$\vdots$$

$$F_{M-1} := \prod_{j=0}^{\infty}(1 + q^{-Mj-(M-1)}r_1 + q^{-2Mj-2(M-1)}r_2 + \cdots + q^{-mMj-m(M-1)}r_m)$$

is irrational. In particular, when $M = m = 2$, we have that at least one of the two infinite products

$$\prod_{j=0}^{\infty}(1 + q^{-2j}r_1 + q^{-4j}r_2) \quad \textit{and} \quad \prod_{j=0}^{\infty}(1 + q^{-2j-1}r_1 + q^{-4j-2}r_2)$$

is irrational.

Since

$$\prod_{j=0}^{\infty}(1 + q^{-2j}r_1 + q^{-4j}r_2) \times \prod_{j=0}^{\infty}(1 + q^{-2j-1}r_1 + q^{-4j-2}r_2) = \prod_{j=0}^{\infty}(1 + q^{-j}r_1 + q^{-2j}r_2),$$

the last result of Theorem 1.1 also shows that at least one of the two infinite products

$$\prod_{j=0}^{\infty}(1 + q^{-j}r_1 + q^{-2j}r_2) \quad \text{and} \quad \prod_{j=0}^{\infty}(1 + q^{-2j}r_1 + q^{-4j}r_2)$$

is irrational.

Like the one variable infinite product, one should expect all such M infinite products to be irrational. So when M is large, the above result is weak. In this article, we improve this result by considering the linear independence of these infinite products and prove the following theorem.

Theorem 1.2 *If $m, M \geq 2$ are positive integers, let $r_1, r_2, \dots, r_m$ be positive rational numbers and let $q = \alpha/\beta > 1$ be a rational number with $\alpha > \beta > 0$. Then the dimension of the vector space $\mathbb{Q}F_0 + \mathbb{Q}F_1 + \cdots + \mathbb{Q}F_{M-1} + \mathbb{Q}$ over $\mathbb{Q}$ is at least*

$$\left\lceil \frac{1 + M(M + 2 - m^2)}{m(Mm + M + 2)} \frac{\log q}{\log \alpha} - \frac{\log \beta}{\log \alpha} \right\rceil + 1 \tag{1.1}$$

where $\lceil x \rceil$ is the smallest integer $\geq x$. In particular, if q is a positive integer and

$$M \geq \frac{m^2 - 2 + m\sqrt{m^2 - 4}}{2},$$

then the dimension of the vector space $\mathbb{Q}F_0 + \mathbb{Q}F_1 + \cdots + \mathbb{Q}F_{M-1} + \mathbb{Q}$ over $\mathbb{Q}$ is at least

$$\left\lceil \frac{1 + M(M + 2 - m^2)}{m(Mm + M + 2)} \right\rceil + 1 \geq 2.$$

We remark that when m is fixed and $M \to \infty$, the expression in (1.1) is

$$\sim \frac{M}{m(m+1)} \frac{\log q}{\log \alpha}.$$

Hence, at least $\sim \frac{M}{m(m+1)} \frac{\log q}{\log \alpha}$ of these infinite products $F_0, F_1, \dots, F_{M-1}$ are irrational.

2 Some Properties of the Infinite Products

In this section, for positive integer m and $q, x_1, x_2, \dots, x_m \in \mathbb{C}$ with $|q| > 1$, we define

$$\begin{aligned} f(\overline{x}) &:= f_q(x_1, x_2, \dots, x_m) \\ &:= \prod_{j=0}^{\infty} (1 + q^{-j} x_1 + q^{-2j} x_1 x_2 + \cdots + q^{-mj} x_1 \cdots x_m). \end{aligned} \tag{2.1}$$

This infinite product defines an entire function in $\mathbb{C}^m$ and we write its Taylor expansion as

$$f(\overline{x}) = \sum_{j_1, \dots, j_m = 0}^{\infty} c_{j_1, \dots, j_m} x_1^{j_1} \cdots x_m^{j_m}, \quad c_{j_1, \dots, j_m} \in \mathbb{C}.$$

Clearly since $f(0, \dots, 0) = 1$, so we have $c_{0,\dots,0} = 1$. Moreover, in view of (2.1), the exponent of x_k in the Taylor expansion of $f(\overline{x})$ is not less than the exponent of x_l if $k \le l$. Hence the non-zero term $x_1^{j_1} \cdots x_m^{j_m}$ appearing in the Taylor expansion must satisfy the condition $j_1 \ge j_2 \ge \cdots \ge j_m$. It then follows that

$$c_{j_1, \dots, j_m} = 0 \quad \text{if } j_1, \dots, j_m \text{ is not in a decreasing order.} \tag{2.2}$$

In view of (2.1), the infinite product $f(\overline{x})$ also has the following functional equation

$$f(q\overline{x}) = (1 + q x_1 + q^2 x_1 x_2 + \cdots + q^m x_1 x_2 \cdots x_m) f(\overline{x}) \tag{2.3}$$

and hence the coefficients $c_{j_1, \dots, j_m}$ satisfy the recurrence relation

$$\begin{aligned} (q^{j_1 + \cdots + j_m} - 1) c_{j_1, \dots, j_m} &= q c_{j_1 - 1, j_2, \dots, j_m} \\ &\quad + q^2 c_{j_1 - 1, j_2 - 1, j_3, \dots, j_m} + \cdots + q^m c_{j_1 - 1, \dots, j_m - 1}. \end{aligned} \tag{2.4}$$

The following estimate to the coefficients is essential in the later sections.

Lemma 2.1 *For $j_1, \dots, j_m \ge 0$, let $N = j_1 + j_2 + \cdots + j_m$. If $N \ge 1$, then*

$$c_{j_1, j_2, \dots, j_m} = \frac{q^N}{\prod_{j=1}^{N} (q^j - 1)} Q(q)$$

where $Q(q) \in \mathbb{Z}[q]$ of degree at most $N(N-1)/2$.

Proof The proof is by induction on N. If $N = 1$, by (2.2), the only non-zero $c_{j_1,j_2,\dots,j_m}$ are

$$c_{j_1,j_2,\dots,j_m} = c_{1,0,\dots,0} = \frac{qc_{0,\dots,0}}{q-1} = \frac{q}{q-1}$$

by (2.4). Suppose the lemma is true for any $j_1,\dots,j_m \geq 0$ such that $j_1+\cdots+j_m \leq N-1$. We now suppose that $j_1 \geq j_2 \geq \cdots \geq j_k > j_{k+1} = \cdots = j_m = 0$ for some $1 \leq k \leq m$ and $j_1+\cdots+j_m = j_1+\cdots+j_k = N$. Clearly, $N \geq k$. Now by the induction assumption and (2.4), we have

$$\begin{aligned}
&(q^N-1)c_{j_1,j_2,\dots,j_m}\\
&\quad= qc_{j_1-1,j_2,\dots,j_m}+\cdots+q^k c_{j_1-1,j_2-1,\dots,j_k-1,0\dots,0}\\
&\quad= q\frac{q^{N-1}}{\prod_{j=1}^{N-1}(q^j-1)}Q_1(q)+\cdots+q^k\frac{q^{N-k}}{\prod_{j=1}^{N-k}(q^j-1)}Q_k(q)\\
&\quad= \frac{q^N}{\prod_{j=1}^{N-1}(q^j-1)}\Big\{Q_1(q)+(q^{N-1}-1)Q_2(q)+\cdots+\Big(\prod_{j=N-k+1}^{N-1}(q^j-1)\Big)Q_k(q)\Big\}\\
&\quad:= \frac{q^N}{\prod_{j=1}^{N-1}(q^j-1)}Q(q),
\end{aligned}$$

where the degree of $Q_j(q) \leq (N-j)(N-j-1)/2$. Therefore the degree of $Q(q)$ is at most

$$\begin{aligned}
(N-k+1)+(N-k+2)+\cdots+(N-1)+(N-k)(N-k-1)/2&\\
&= N(N-1)/2-(N-k)\\
&\leq N(N-1)/2.
\end{aligned}$$

This proves the lemma. ∎

Lemma 2.2 *Let $q > 1$ be a real number and m be a positive integer. Define the non-negative function $\psi(N)$ recursively by $\psi(N) = 0$ for $N < 0$, $\psi(0) = 1$ and*

$$\psi(N) = q^{-(N-1)}\psi(N-1)+\cdots+q^{-(N-m)}\psi(N-m) \tag{2.5}$$

for $N \geq 1$. Then for $N \geq m$, we have

$$\psi(N) \leq K(m,q)q^{-\frac{N^2}{2m}+\frac{N}{2}}, \tag{2.6}$$

where $K(m,q)$ is an explicit constant defined below and depending only on q and m.

Proof By writing $\psi(N) = q^{-N^2/(2m)+N/2}\chi(N)$, (2.5) becomes

$$\chi(N) = \sum_{i=1}^{m} q^{-(2N-i)(m-i)/(2m)}\chi(N-i).$$

We claim that

$$\chi(N) \le \max\{\chi(0), \dots, \chi(m-1)\} \prod_{j=0}^{N-m} (1 - q^{-(2j+m)/(2m)})^{-1}, \tag{2.7}$$

for $N \ge m$. We prove this claim by induction on N. For $N = m$, we have

$$\begin{aligned}\chi(m) &= \chi(0) + q^{-\frac{m+1}{2m}} \chi(1) + \dots + q^{-\frac{(2m-1)(m-1)}{2m}} \chi(m-1)\\ &\le \max\{\chi(0), \dots, \chi(m-1)\} \times \left\{ 1 + q^{-\frac{1}{2}} + q^{-\frac{2}{2}} + \dots + q^{-\frac{m-1}{2}} \right\}\\ &\le \max\{\chi(0), \dots, \chi(m-1)\}(1 - q^{-\frac{1}{2}})^{-1}.\end{aligned}$$

This proves (2.7) for $N = m$. By induction assumption, we have

$$\begin{aligned}\chi(N) &\le \max\{\chi(0), \dots, \chi(m-1)\} \sum_{i=1}^{m} q^{-\frac{(2N-i)(m-i)}{2m}} \prod_{j=0}^{N-m-i} \left(1 - q^{-\frac{2j+m}{2m}}\right)^{-1}\\ &\le \max\{\chi(0), \dots, \chi(m-1)\} \prod_{j=0}^{N-m-1} \left(1 - q^{-\frac{2j+m}{2m}}\right)^{-1} \left\{ \sum_{i=1}^{m} q^{-\frac{2N-i}{2m}(m-i)} \right\}.\end{aligned}$$

Here we understand that the empty product is 1. Now (2.7) follows from

$$\sum_{i=1}^{m} q^{-\frac{2N-i}{2m}(m-i)} \le \sum_{i=0}^{\infty} q^{-\frac{2N-m}{2m} i} = \left(1 - q^{-\frac{2N-m}{2m}}\right)^{-1},$$

and this proves the claim. We next observe that

$$\begin{aligned}\prod_{j=0}^{N-m} \left(1 - q^{-\frac{2j+m}{2m}}\right)^{-1} &\le \left(1 - q^{-\frac{1}{2}}\right)^{-1} \prod_{j=1}^{N-m} \left(1 - q^{-\frac{j}{m}}\right)^{-1}\\ &\le \left(1 - q^{-\frac{1}{2}}\right)^{-1} \prod_{j=1}^{\infty} \left(1 - q^{-\frac{j}{m}}\right)^{-1}.\end{aligned}$$

Hence this proves (2.6) for

$$K(m, q) = \max\{\chi(0), \dots, \chi(m-1)\} \left(1 - q^{-\frac{1}{2}}\right)^{-1} \prod_{j=1}^{\infty} \left(1 - q^{-\frac{j}{m}}\right)^{-1}. \qquad \blacksquare$$

Corollary 2.3 *Let $j_1, \dots, j_m \ge 0$ and $N = j_1 + \dots + j_m$. For $N \ge 1$, we have*

$$c_{j_1,\dots,j_m} \le K_1(m, q) q^{-\frac{N^2}{2m} + \frac{N}{2}},$$

where $K_1(m, q)$ is an explicit constant defined below and depending only on q and m.

https://doi.org/10.4153/CMB-2008-005-7 Published online by Cambridge University Press

Proof We will show that

$$c_{j_1,\dots,j_m} \le \varphi(j_1 + \cdots + j_m) \tag{2.8}$$

for any $j_1, \dots, j_m \ge 0$ where $\varphi(N)$ is the non-negative function defined recursively by $\varphi(N) = 0$ for $N < 0$, $\varphi(0) = 1$ and

$$(q^N - 1)\varphi(N) = q\varphi(N-1) + q^2\varphi(N-2) + \cdots + q^m\varphi(N-m). \tag{2.9}$$

We claim that for $N \ge 1$, $\varphi(N) \le \psi(N)\prod_{j=1}^{N}(1-q^{-j})^{-1}$ where ψ is defined in Lemma 2.2. The claim is clearly true for $N = 1$ because $\varphi(1) = q/(q-1) = (1-q^{-1})^{-1}$ and $\psi(1) = 1$. From (2.5), (2.9) and the induction assumption, we have

$$\begin{aligned}
\varphi(N) &= \sum_{i=1}^{m} \frac{q^i}{q^N - 1}\varphi(N-i) \\
&\le \sum_{i=1}^{m} \frac{q^i}{q^N - 1}\psi(N-i)\prod_{j=1}^{N-i}(1-q^{-j})^{-1} \\
&\le \Big\{\sum_{i=1}^{m} q^{-(N-i)}\psi(N-i)\Big\}\prod_{j=1}^{N}(1-q^{-j})^{-1} \\
&= \psi(N)\prod_{j=1}^{N}(1-q^{-j})^{-1}.
\end{aligned}$$

This proves the claim. By the recurrence relation (2.4), the inequality (2.8) can be proved by induction on $j_1+\cdots+j_m$ in the same way as above. Thus, from Lemma 2.2, we have

$$\begin{aligned}
c_{j_1,\dots,j_m} &\le K(m,q)q^{-\frac{N^2}{2m}+\frac{N}{2}}\prod_{j=1}^{N}(1-q^{-j})^{-1} \\
&\le K(m,q)q^{-\frac{N^2}{2m}+\frac{N}{2}}\prod_{j=1}^{\infty}(1-q^{-j})^{-1} \\
&=: K_1(m,q)q^{-\frac{N^2}{2m}+\frac{N}{2}}.
\end{aligned}$$

This completes the proof of Corollary 2.3. ∎

3 Padé Approximation

We need here the standard q analogue of factorial and binomial coefficients. Define the q-factorial to be

$$[n]_q! := [n]! := \frac{(q^n-1)(q^{n-1}-1)\cdots(q-1)}{(q-1)^n},$$

where $[0]_q! := 1$ and the q-binomial coefficient to be

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]![n-k]!}.$$

Then we have see [11])

$$\prod_{k=0}^{n} (t - q^{-k})^{-1} = (-1)^{n+1} q^{n(n+1)/2} \sum_{l=0}^{\infty} \begin{bmatrix} n+l \\ l \end{bmatrix} t^l. \tag{3.1}$$

Lemma 3.1 *If $q > 1$, then we have*

(i) $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = \left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right] + q^k \left[\begin{smallmatrix} n-1 \\ k \end{smallmatrix}\right]$, *for $n > k \geq 1$;*

(ii) $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ *is a monic polynomial in q over $\mathbb{Z}$ of degree $k(n-k)$ for $n \geq k \geq 0$;*

(iii) *all the coefficients of $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ in q are positive with sum at most 2^n;*

(iv) $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] \leq 2^n q^{k(n-k)}$.

Proof The lemma follows easily from the identity

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}$$

and the induction on n and k. ■

From now on, we assume $q > 1$ and integers $M, m \geq 2$ and consider

$$F(\overline{x}) := f_{q^M}(\overline{x}) = \prod_{j=0}^{\infty} (1 + q^{-Mj} x_1 + q^{-2Mj} x_1 x_2 + \cdots + q^{-mMj} x_1 \cdots x_m)$$

and write

$$F(\overline{x}) = \sum_{j_1, \ldots, j_m = 0}^{\infty} d_{j_1, \ldots, j_m} x_1^{j_1} \ldots x_m^{j_m}. \tag{3.2}$$

Similar to (2.3), the entire function $F(\overline{x})$ satisfies the functional equation

$$\begin{aligned} F(q^{-Mk}\overline{x}) &= \prod_{j=0}^{\infty} (1 + q^{-Mj-Mk} x_1 + \cdots + q^{-mMj-mMk} x_1 \cdots x_m) \\ &= \Big(\prod_{j=0}^{k-1} (1 + q^{-Mj} x_1 + \cdots + q^{-mMj} x_1 \cdots x_m) \Big)^{-1} F(\overline{x}) \\ &= R_k(\overline{x})^{-1} F(\overline{x}), \end{aligned} \tag{3.3}$$

where

$$R_k(\overline{x}) := \prod_{j=0}^{k-1} (1 + q^{-Mj} x_1 + \cdots + q^{-mMj} x_1 \cdots x_m)$$

and $R_0(\overline{x}) := 1$. Note that $R_k(\overline{x})$ is a polynomial in $x_1, x_1x_2, \ldots, x_1x_2 \cdots x_m$ of degree k.

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Lemma 3.2 *Let*

$$R(\overline{x}) := R_n(\overline{x}) \prod_{j=1}^{M-1} R_{n-1}(q^{-j}\overline{x}) \in \mathbb{Z}[q^{-1}, x_1, x_1x_2, \dots, x_1x_2\cdots x_m]$$

be a polynomial in $x_1, x_1x_2, \dots, x_1x_2\cdots x_m$ *of degree at most* nM. *Then for* $0 \le x_i \le 1$, $1 \le k \le n$ *and* $j = 0, 1, \dots, M-1$, *we have*

$$\Big| \frac{R(\overline{x})}{R_k(q^{-j}\overline{x})} \Big| \le u_q,$$

where $u_q := \prod_{j=0}^{\infty}(1 + q^{-Mj} + \cdots + q^{-mMj})^M$ *is a constant depending on* q, m *and* M.

Proof For $j > 0$, we have

$$\begin{aligned}\Big| \frac{R(\overline{x})}{R_k(q^{-j}\overline{x})} \Big| &= \Big| \frac{R_{n-1}(q^{-j}\overline{x})}{R_k(q^{-j}\overline{x})} \Big| |R_n(\overline{x})| \prod_{\substack{l=1\\ l\ne j}}^{M-1} |R_{n-1}(q^{-l}\overline{x})| \\ &\le \prod_{l=k}^{n-2} (1 + q^{-Ml-j} + \cdots + q^{-mMl-mj}) \prod_{l=0}^{n-1} (1 + q^{-Ml} + \cdots + q^{-mMl}) \\ &\quad \times \prod_{\substack{l=1\\ l\ne j}}^{M-1} \prod_{r=0}^{n-2} (1 + q^{-Mr-l} + \cdots + q^{-mMr-ml}) \\ &\le \Big\{ \prod_{l=0}^{\infty} (1 + q^{-Ml} + \cdots + q^{-mMl}) \Big\}^M = u_q.\end{aligned}$$

The case $j = 0$ can be proved similarly. ■

For $n \ge 1$ a fixed integer, we let

$$I(\overline{x}) := \frac{1}{2\pi i} \int_\Gamma \frac{F(t\overline{x})dt}{\big(\prod_{k=0}^{Mn}(t - q^{-k})\big) t^{n+1}},$$

where Γ is a circle centered at origin with radius > 1.

As in [12], we define

$$a_k(q) := (-1)^k \begin{bmatrix} Mn \\ k \end{bmatrix} q^{k(k+1)/2+nk}, \quad k = 0, 1, 2, \dots, Mn,$$

$$A_0(\overline{x}) := \frac{q^{Mn(Mn+1)/2}}{(q-1)^{Mn}[Mn]!} \sum_{k=0}^{n} a_{Mk}(q) R_k^{-1}(\overline{x}),$$

$$A_j(\overline{x}) := \frac{q^{Mn(Mn+1)/2}}{(q-1)^{Mn}[Mn]!} \sum_{k=0}^{n-1} a_{Mk+j}(q) R_k^{-1}(q^{-j}\overline{x}), \quad j = 1, 2, \dots, M-1,$$

and

$$B(\overline{x}) := \frac{1}{n!}\frac{d^n}{dt^n}\Big\{\frac{F(t\overline{x})}{\prod_{k=0}^{Mn}(t-q^{-k})}\Big\}_{t=0}. \tag{3.4}$$

It is proved in [12] that

$$I(\overline{x}) = A_0(\overline{x})F(\overline{x}) + A_1(\overline{x})F(q^{-1}\overline{x}) + \cdots + A_{M-1}(\overline{x})F(q^{-M+1}\overline{x}) + B(\overline{x}). \tag{3.5}$$

However, $A_j(\overline{x})$ and $B(\overline{x})$ are not integral over q and $\overline{x}$. For $j = 0, 1, 2, \ldots, M-1$, we let

$$A_j^*(\overline{x}) := q^{mM^2n(n-1)/2}\Big(\prod_{j=1}^{Mn}(1-q^{-j})\Big)R(\overline{x})A_j(\overline{x}), \tag{3.6}$$

$$B^*(\overline{x}) := q^{mM^2n(n-1)/2}\Big(\prod_{j=1}^{Mn}(1-q^{-j})\Big)R(\overline{x})B(\overline{x}). \tag{3.7}$$

From [12], we know that for $j = 0, 1, 2, \ldots, M-1$,

$$A_j^*(\overline{x}), B^*(\overline{x}) \in \mathbb{Z}[q, x_1, x_1x_2, \ldots, x_1x_2\cdots x_m].$$

and the degrees of $A_j^*(\overline{x})$ and $B^*(\overline{x})$ in $x_1, x_1x_2, \ldots, x_1x_2\cdots x_m$ are at most Mn and $(M+1)n$ respectively.

Lemma 3.3 *For integers $q > 1, M, m \geq 2$ and positive real numbers $1 \geq x_1, \ldots, x_m > 0$, we have*

$$\deg_q(A_j^*(\overline{x})) \leq \frac{1}{2}M(Mm+M+2)n^2 + O(n)$$

and

$$\Big|A_j^*(\overline{x})\Big| \leq q^{\frac{1}{2}M(Mm+M+2)n^2+O(n)}$$

for $j = 0, 1, 2, \ldots, M-1$.

Proof By Lemma 3.2, we have

$$\begin{aligned}|A_0^*(\overline{x})| &= q^{mM^2n(n-1)/2}\Big(\prod_{j=1}^{Mn}(1-q^{-j})\Big)|R(\overline{x})A_0(\overline{x})| \\ &= q^{mM^2n(n-1)/2}\Big|\sum_{k=0}^{n} a_{Mk}(q)\frac{R(\overline{x})}{R_k(\overline{x})}\Big| \\ &\leq u_q q^{mM^2n(n-1)/2}\sum_{k=0}^{n}|a_{Mk}(q)|.\end{aligned}$$

It then follows from Lemma 3.1(iv) that

$$
\begin{aligned}
(3.8) \qquad |A_0^*(\overline{x})| &\le u_q q^{mM^2 n(n-1)/2} \sum_{k=0}^{n} \begin{bmatrix} Mn \\ Mk \end{bmatrix} q^{Mk(Mk+1)/2+nMk} \\
&\le 2^{nM} u_q q^{mM^2 n(n-1)/2} \sum_{k=0}^{n} q^{Mk(Mn-Mk)+Mk(Mk+1)/2+nMk} \\
&\le 2^{nM}(n+1) u_q q^{\frac{1}{2}M(Mm+M+2)n^2+\frac{1}{2}M(1-Mm)n} \\
&\le q^{\frac{1}{2}M(Mm+M+2)n^2+O(n)}.
\end{aligned}
$$

Since $A_0^*(\overline{x}) \in \mathbb{Z}[q, x_1, \dots, x_m]$ and $u_q \ll 1$ as $q \to +\infty$, so from (3.8) we have

$$\deg_q(A_0^*(\overline{x})) \le \frac{1}{2} M(Mm + M + 2)n^2 + O(n).$$

The case $j \ge 1$ can be proved in the same way. ■

Lemma 3.4 *For integers $q > 1, M, m \ge 2$ and positive real numbers $1 \ge x_1, \dots, x_m > 0$, we have*

$$
\begin{aligned}
\deg_q(B^*(\overline{x})) &\le \frac{1}{2} M(Mm + M + 2)n^2 + O(n), \\
|B^*(\overline{x})| &\le q^{M(Mm+M+2)n^2/2+O(n)}.
\end{aligned}
$$

Proof In view of (3.1), (3.2) and (3.4), we have

$$
\begin{aligned}
B(\overline{x}) &= (-1)^{Mn+1} q^{Mn(Mn+1)/2} \sum_{\substack{j_1,\dots,j_m,l\ge 0 \\ j_1+\dots+j_m+l=n}} d_{j_1,ldots,j_m} x_1^{j_1} \cdots x_m^{j_m} \begin{bmatrix} Mn+l \\ l \end{bmatrix} \\
&= (-1)^{Mn+1} q^{Mn(Mn+1)/2} \sum_{\mu=0}^{n} \begin{bmatrix} (M+1)n-\mu \\ n-\mu \end{bmatrix} \sum_{j_1+\dots+j_m=\mu} d_{j_1,\dots,j_m} x_1^{j_1} \cdots x_m^{j_m}.
\end{aligned}
$$

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Now using Corollary 2.3 and Lemma 3.1(iv), we have

$$\begin{aligned}|B(\overline{x})| &\le q^{Mn(Mn+1)/2} \sum_{\mu=0}^{n} \begin{bmatrix} (M+1)n-\mu \\ n-\mu \end{bmatrix} \sum_{j_1+\cdots+j_m=\mu} K_1(m,q^M) q^{-M\mu^2/(2m)+M\mu/2} \\ &\le q^{Mn(Mn+1)/2} K_1(m,q^M) \sum_{\mu=0}^{n} q^{-M\mu^2/(2m)+M\mu/2} 2^{(M+1)n-\mu} \\ &\quad \times q^{(n-\mu)((M+1)n-\mu-(n-\mu))} \sum_{j_1+\cdots+j_m=\mu} 1 \\ &\le K_1(m,q^M) 2^{(M+1)n} q^{Mn(Mn+1)/2} \sum_{\mu=0}^{n} q^{-M\mu^2/(2m)+M\mu/2+Mn(n-\mu)} \sum_{j_1+\cdots+j_m=\mu} 1 \\ &\le K_1(m,q^M) 2^{(M+1)n} q^{Mn(Mn+1)/2+Mn^2} \sum_{\mu=0}^{n} \sum_{j_1+\cdots+j_m=\mu} 1 \\ &\le (n+1)^m K_1(m,q^M) 2^{(M+1)n} q^{Mn(Mn+1)/2+Mn^2}.\end{aligned}$$

It follows from (3.7) and the fact that $|R(\overline{x})| \le u_q$ that

$$\begin{aligned}|B^*(\overline{x})| &\le (n+1)^m K_1(m,q^M) u_q 2^{(M+1)n} q^{mM^2n(n-1)/2+Mn(Mn+1)/2+Mn^2} \\ &\le q^{\frac{1}{2}M(Mm+M+2)n^2+O(n)}.\end{aligned}$$

The degree $B^*(\overline{x})$ in q can be estimated as before. ■

Lemma 3.5 *For integers $q > 1, M, m \ge 2$ and positive real numbers $1 \ge x_1, \ldots, x_m > 0$, we have*

$$\log I(\overline{x}) = -\frac{1}{2m} M(M+1)^2 n^2 \log q + O(n),$$

where the implicit constant depends on q, m, M and x_i.

Proof We first have from [12, (2.19)] that

$$I(\overline{x}) \le c(q,M,m) q^{-M(M+1)^2n^2/(2m)-M(M+1)n/2}.$$

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We now consider the lower bound for $I(\overline{x})$. Note that

$$\begin{aligned}
I(\overline{x}) &= \frac{1}{2\pi i}\int_{\Gamma}\frac{F(t\overline{x})dt}{\left(\prod_{k=0}^{Mn}(t-q^{-k})\right)t^{n+1}}\\
&= \frac{1}{2\pi i}\int_{\Gamma}\frac{F(t\overline{x})}{t^{(M+1)n+2}}\Big(\prod_{k=0}^{Mn}\Big(\frac{1}{1-1/(q^k t)}\Big)\Big)\,dt\\
&= \frac{1}{2\pi i}\int_{\Gamma}\frac{F(t\overline{x})}{t^{(M+1)n+2}}\Big(\prod_{k=0}^{Mn}\Big(\sum_{j=0}^{\infty}(\frac{1}{q^k t})^j\Big)\Big)\,dt\\
&= \frac{1}{2\pi i}\int_{\Gamma}\frac{F(t\overline{x})}{t^{(M+1)n+2}}\Big(\sum_{j_0,\dots,j_{Mn}\ge 0}\prod_{k=0}^{Mn}\Big(\frac{1}{q^k t}\Big)^{j_k}\Big)\,dt\\
&= \sum_{j_0,\dots,j_{Mn}\ge 0} q^{-\sum_{k=0}^{Mn} k j_k}\cdot\frac{1}{2\pi i}\int_{\Gamma}\frac{F(t\overline{x})dt}{t^{(M+1)n+2+(j_0+\cdots+j_{Mn})}}\\
&= \sum_{j_0,\dots,j_{Mn}\ge 0} q^{-\sum_{k=0}^{Mn} k j_k}\sum_{\substack{l_1,l_2,\dots,l_m\ge 0\\ l_1+l_2+\cdots+l_m=(M+1)n+1+j_0+\cdots+j_{Mn}}} d_{l_1 l_2\dots l_m}x_1^{l_1}x_2^{l_2}\cdots x_m^{l_m}\\
&\ge \sum_{\substack{l_1,l_2,\dots,l_m\ge 0\\ l_1+l_2+\cdots+l_m=(M+1)n+1}} d_{l_1 l_2\dots l_m}x_1^{l_1}x_2^{l_2}\cdots x_m^{l_m}\\
&\ge d_{a+1,a+1,\dots,a+1,a,\dots,a}x_1^{a+1}\cdots x_b^{a+1}x_{b+1}^{a}\cdots x_m^{a},
\end{aligned}$$

where a and b are given by $(M+1)n+1 = am+b$ with $0 \le a$ and $0 \le b \le m-1$. Now using the recursion formula for $d_{l_1 l_2 \dots l_m}$, we get

$$d_{a+1,a+1,\dots,a+1,a,\dots,a} = \frac{(q^M)^b d_{a,a,\dots,a}}{q^{M(am+b)}-1}$$

and

$$d_{a,a,\dots,a} = \frac{q^{mM}d_{a-1,a-1,\dots,a-1}}{(q^{Mma}-1)} = \cdots = \frac{q^{amM}}{(q^{Mma}-1)(q^{Mm(a-1)}-1)\cdots(q^{Mm}-1)}.$$

It follows that

$$\begin{aligned}
d_{a+1,a+1,\dots,a+1,b,\dots,b} &= \frac{q^{(am+b)M}}{(q^{(am+b)M}-1)(q^{Mma}-1)(q^{Mm(a-1)}-1)\cdots(q^{Mm}-1)}\\
&\ge q^{-Mm(1+\cdots+a)}\\
&= q^{-Mma(a+1)/2}\\
&\ge q^{-Mm((M+1)n+1)/m(((M+1)n+1)/m+1)/2}\\
&= q^{-\frac{M(M+1)^2n^2}{2m}-\frac{M(M+1)(m+2)n}{2m}-\frac{M(m+1)}{2m}},
\end{aligned}$$

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because $a = [\frac{(M+1)n+1}{m}] \le \frac{(M+1)n+1}{m}$. Therefore

$$q^{-\frac{M(M+1)^2n^2}{2m}-\frac{M(M+1)(m+2)n}{2m}-\frac{M(m+1)}{2m}}(x_1\cdots x_m)^{((M+1)n+1)/m}(x_1\cdots x_b)$$
$$\le I(\overline{x}) \le c(q,M,m)q^{-M(M+1)^2n^2/(2m)-M(M+1)n/2}$$

and hence

$$\log I(\overline{x}) = -\frac{M(M+1)^2n^2}{2m}\log q + O(n)$$

where the implicit constant depends only on q, m, M and x_i. ■

4 Proof of the Theorem

To prove our theorem, we will apply the following result due to Nesterenko [7].

Lemma 4.1 *Suppose $\overline{w} = (w_1,\dots,w_k) \in \mathbb{R}^k\setminus\{\overline{0}\}$. If there exist $n_0 \in \mathbb{N}$ and $\tau > 0$ and an unbounded, monotonically increasing function $G\colon \mathbb{N} \to (0,\infty)$ with $\limsup_{n\to\infty} G(n+1)/G(n) \le 1$, and a sequence $(L_n)_{n\ge n_0}$ of integral linear forms satisfying*

$$\log|L_n(\overline{w})| + \tau G(n) = o(G(n)), \text{ and } \log\|L_n\| \le G(n), \tag{4.1}$$

where $\|L_n\|$ is the usual Euclidean norm, then

$$\dim_{\mathbb{Q}}(\mathbb{Q}w_1 + \mathbb{Q}w_2 + \cdots + \mathbb{Q}w_k) \ge 1+\tau.$$

We now come to the proof of our theorem. Our aim is to construct an integral linear form satisfying (4.1).

Let the notation be as in Theorem 1.2. Let $r_i = \frac{a_i}{b_i}$ with $a_i, b_i > 0$ and $\gcd(a_i,b_i) = 1$. Let $B := \operatorname{lcm}\{b_1,b_2,\dots,b_m\}$ and $x_1 = r_1, x_j = \frac{r_j}{r_{j-1}}$, $j = 2,3,\dots,m$. In view of (3.3), we can see that the irrationality of $F(\overline{x})$ is equivalent to the irrationality of $F(q^{-Mk}\overline{x})$ for any integer $k \ge 0$. Thus, we may assume that $1 \ge r_1 \ge r_2 \ge \cdots \ge r_m > 0$ so that $0 < x_i \le 1$ for $1 \le i \le m$.

Let $q = \frac{\alpha}{\beta} > 1$ with $\alpha, \beta > 0$. Consider the linear form

$$L_n(\overline{\omega}) := \beta^{\frac{1}{2}M(Mm+M+2)n^2+O(n)}B^{(M+1)n}\left(A_0^*(\overline{x})\omega_0 + \cdots + A_{M-1}^*(\overline{x})\omega_{M-1} + B^*(\overline{x})\omega_M\right).$$

Then since the degrees of $A_j^*(\overline{x})$ and $B^*(\overline{x})$ in $x_1, x_1x_2, \dots, x_1x_2\cdots x_m$ are at most $(M+1)n$ and their degrees in q are at most $\frac{1}{2}M(Mm+M+2)n^2 + O(n)$ by Lemmas 3.3 and 3.4, so the linear form $L_n(\overline{\omega})$ indeed has integer coefficients.

Let $\omega_j = F(q^{-j}\overline{x}), 0 \le j \le M-1$ and $\omega_M = 1$. Then in view of (3.5), (3.6) and (3.7),

$$L_n(\overline{\omega}) = \beta^{\frac{1}{2}M(Mm+M+2)n^2+O(n)}B^{Mn}q^{\frac{mM^2n(n-1)}{2}}R(\overline{x})I(\overline{x})\prod_{j=1}^{Mn}(1-q^{-j}).$$

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Therefore,

$$\log|L_n(\overline{\omega})| = \frac{1}{2}M(Mm+M+2)n^2\log\beta + \frac{mM^2n^2}{2}\log q + O(n) + \log|I(\overline{x})|,$$

because $\prod_{j=1}^{\infty}(1-q^{-j}) < \prod_{j=1}^{Mn}(1-q^{-j}) < 1$ and $\log|R(\overline{x})| \le \log u_q$. Hence

$$\log|L_n(\overline{\omega})| = \frac{1}{2}M(Mm+M+2)n^2\log\beta - \frac{M(1+M(M+2-m^2))}{2m}n^2\log q + O(n),$$

by Lemma 3.5. On the other hand,

$$\begin{aligned}\log\|L_n\| &= \frac{1}{2}\log\left\{|A_0^*(\overline{x})|^2 + \cdots + |A_{M-1}^*(\overline{x})|^2 + |B^*(\overline{x})|^2\right\}\\ &\quad + \frac{1}{2}M(Mm+M+2)n^2\log\beta + O(n)\\ &\le \frac{1}{2}M(Mm+M+2)n^2\log\alpha + O(n)\end{aligned}$$

by Lemmas 3.3 and 3.4.

Let

$$G(n) = \frac{1}{2}M(Mm+M+2)n^2\log\alpha + O(n).$$

Then $\log|L_n(\overline{\omega})| + \tau G(n) = o(G(n))$ and $\log\|L_n\| \le G(n)$, where

$$\tau = \frac{1+M(M+2-m^2)}{m(Mm+M+2)}\frac{\log q}{\log\alpha} - \frac{\log\beta}{\log\alpha}.$$

Therefore, by Lemma 4.1

$$\dim_{\mathbb{Q}}\left(\mathbb{Q}F(\overline{x}) + \mathbb{Q}F(q^{-1}\overline{x}) + \cdots + \mathbb{Q}F(q^{-(M-1)}\overline{x}) + \mathbb{Q}\right) \ge 1+\tau.$$

This proves Theorem 1.1.

If q is an integer, *i.e.*, $\beta = 1$ and $\alpha = q$, then

$$\tau = \frac{1+M(M+2-m^2)}{m(Mm+M+2)}.$$

We note that $\tau > 0$ if and only if

$$M \ge \frac{m^2-2+m\sqrt{m^2-4}}{2}.$$

In particular, if $m \ge 2$ and

$$M \ge \frac{m^2-2+m\sqrt{m^2-4}}{2},$$

then $\dim_{\mathbb{Q}}(\mathbb{Q}F(\overline{x}) + \mathbb{Q}F(q^{-1}\overline{x}) + \cdots + \mathbb{Q}F(q^{-(M-1)}\overline{x}) + \mathbb{Q}) \ge 2$. This completes the proof of Theorem 1.2.

Acknowledgment The authors wish to express their gratitude to the referee for the thoughtful and gracious comments and suggestions, especially for the proof of Lemma 2.2.

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