A CENSUS OF HAMILTONIAN POLYGONS

W. T. TUTTE

Summary. In this paper we deal with trivalent planar maps in which the boundary of each country (or "face") is a simple closed curve. One vertex is distinguished as the *root* and its three incident edges are distinguished as the first, second, and third *major edges*. We determine the average number of Hamiltonian polygons, passing through the first and second major edges, in such a "rooted map" of $2n$ vertices. Next we consider the corresponding problem for 3-connected rooted maps. In this case we obtain a functional equation from which the average can be computed for small values of n.

1. Rooted maps. For the purposes of this paper a *planar map* M is a representation of the 2-sphere (or closed plane) as a union of a finite number of disjoint point-sets called *cells*. The cells are of three kinds, *vertices*, *edges*, and *faces*, said to have dimension 0, 1, and 2 respectively. Each vertex consists of a single point. Each edge is an open arc whose ends are distinct vertices. Each face is a simply connected domain whose boundary is a simple closed curve made up of edges and vertices. We denote the numbers of cells, vertices, edges, and faces of M by $C(M)$, $V(M)$, $E(M)$, and $F(M)$ respectively.

Two cells whose dimensions differ are *incident* with one another if one is contained in the boundary of the other. We note that each edge must be incident with just two faces.

Let M and N be planar maps. An *isomorphism* of M onto N is a 1-1 mapping f of $C(M)$ onto $C(N)$ with the following properties.

(i) *f preserves dimension,*

(ii) *Both f and f^{-1} preserve incidence relations.*

If such a mapping exists we say that M and N are *isomorphic*.

If X and Y are complementary non-null subsets of $V(M)$ we write $Q(X,Y)$ for the set of all edges of M with one end in X and one in Y. This set is the *cut* between X and Y. A cut with just k edges is a *k-cut*.

(1.1) *Each cut of M has at least two edges.*

Proof. Suppose $Q(X, Y)$ is a 0-cut. Define $U(X)$ as the union of the vertices of X and their incident edges, and let $U(Y)$ be defined analogously. By the connection of the 2-sphere M has a face K whose boundary meets both $U(X)$ and $U(Y)$. But then the boundary of K is not connected, contrary to the definition of a face.

Received August 22, 1961. Part of this work was done at the Combinatorial Symposium held by the RAND Corporation, July–August 1961.

https://doi.org/10.4153/CJM-1962-032-x Published online by Cambridge University Press

Next suppose $Q(X, Y)$ is a 1-cut, with edge A. Then A is incident with some face K. The boundary of K, a simple closed curve, must contain at least one other edge with one end in X and one in Y.

These contradictions establish the theorem.

(1.2) *Let $Q(X, Y)$ be a 2-cut or 3-cut of a planar map M. Let X' and X'' be complementary non-null subsets of X. Then there exists an edge C of M with one end in X' and the other in X''.*

Proof. Assume the theorem false. By (1.1) the cuts $Q(X', V(M) - X')$ and $Q(X'', V(M) - X'')$ have each at least two edges. Hence $Q(X, Y)$ is at least a 4-cut, contrary to hypothesis.

(1.3) *Suppose $Q(X, Y)$ and $Q(X_1, Y_1)$ are 2-cuts or 3-cuts of M with the same edges. Then the unordered pairs $\{X, Y\}$ and $\{X_1, Y_1\}$ are identical.*

Proof. We can adjust the notation so that $X \cap X_1$ is non-null. Assume $X \cap Y_1$ is also non-null. By (1.2) there is an edge A of M with one end in $X \cap X_1$ and one in $X \cap Y_1$. But then $A \in Q(X_1, Y_1)$ and $A \notin Q(X, Y)$, which is contrary to hypothesis. We deduce that $X \subseteq X_1$. A similar argument shows that $X_1 \subseteq X$. The theorem follows.

A planar map M is *trivalent* if each vertex v is incident with just three edges. This implies that v is incident with just three faces, and each pair of edges incident with v determines a unique face. A trivalent map is 2-*separable* if it has a 2-cut, and 3-*connected* otherwise.

A trivalent planar map is *rooted* if one vertex is distinguished as the *root* and its three incident edges are distinguished as the first, second, and third *major edges*. Two rooted trivalent planar maps M and N are *equivalent* if and only if there is an isomorphism of M onto N which preserves the root and first, second, and third major edges. Such an isomorphism is a *root-isomorphism* of M onto N.

(1.4) *Let $\{A, B\} = Q(X, Y)$ be a 2-cut of a trivalent map M. Then A and B have no common end.*

Proof. Suppose A and B have a common end x. Let C be the third edge incident with x and let y be its other end. We may suppose $x \in X$. Then also $y \in X$. But then C is the only edge of the cut $Q(X - \{x\}, Y \cup \{x\})$, which is contrary to (1.1).

(1.5) *Any root-isomorphism of a rooted map M onto itself is an identity.*

Proof. Let f be such a root-isomorphism. Let us call a face P of M *strictly invariant* if P itself and each of its incident edges and vertices is invariant under f. Clearly the three faces of M incident with the root are strictly invariant, and every face having a common edge with a strictly invariant face is strictly invariant. Hence every face of M is strictly invariant, by the connection of the 2-sphere, and the theorem follows.

2. Duality. Two planar maps M and M^* are *duals* if there is a 1–1 mapping f of $C(M)$ onto $C(M^*)$ with the following properties.

(i) *f maps $V(M)$ onto $F(M^*)$, $E(M)$ onto $E(M^*)$ and $F(M)$ onto $V(M^*)$.*

(ii) *Both f and f^{-1} preserve incidence relations.*

We call any such mapping f a *dual correspondence* between M and M^*.

(2.1) *Each planar map M has a dual planar map M^*.*

Proof. In each face K we select a point $p(K)$ and on each edge A we select a point $P(A)$. We join $p(K)$ to each point $P(A)$ in the boundary of K by an open arc $L(K, A)$ in K, arranging that the arcs $L(K, A)$ are disjoint. The point $p(K)$ and the arcs $L(K, A)$, for a given K, separate the remainder of K into as many simply connected domains as K has incident vertices. Any one of these domains can be specified by a symbol $D(K, v)$, where v is the single vertex of M on its boundary.

For each edge A of M we define $f(A)$ as the open arc formed by $P(A)$ and the two arcs $L(K, A)$. For each vertex v of M we define $f(v)$ as the union of v, the segments $vP(A)$ of the edges A of M incident with v, and the domains $D(K, v)$ derived from faces incident with v. We also write $f(K) = p(K)$. It is readily verified that the sets $f(K), f(A), f(v)$ are the cells of a planar map M^* and that f is a dual correspondence between M and M^*.

Let us define a *triangulation* as a planar map in which each face has just three incident edges and in which no two edges have the same pair of ends. The triangulations can be characterized as the dual planar maps of the trivalent planar maps without 2-cuts, that is the 3-connected trivalent planar maps.

We call a triangulation "*rooted*" if one face is distinguished as the *root* or *outside* and the three incident edges are distinguished as the *first*, *second*, and *third major edges*. Two rooted triangulations are *equivalent* if there is an isomorphism of one onto the other which preserves the root and each major edge. A rooted triangulation and a rooted 3-connected trivalent planar map are *duals* if the corresponding unrooted maps are duals, with a dual correspondence that relates root to root and kth major edge to kth major edge.

By (2.1) every rooted 3-connected trivalent planar map has a dual rooted triangulation, and conversely. It follows from the definitions that the duals of equivalent rooted 3-connected trivalent maps are equivalent rooted triangulations, and conversely.

Let q_n denote the number of inequivalent 3-connected trivalent rooted maps of $2n$ vertices. (The number of vertices of a trivalent map must be even.) By the above results q_n is also the number of inequivalent rooted triangulations of $2n$ faces. But rooted triangulations are studied in **(1)**, where they are called simply "triangulations." The number q_n is the number $\psi_{n-1,0}$ of **(1)** since a rooted triangulation with $2n$ faces, counting the outside, is one with $3n - 3$ internal edges (as defined in **(1)**). So by (4.10) of **(1)** we have

$$q_n = \psi_{n-1,0} = \frac{2\cdot(4n-3)!}{n!(3n-1)!}. \tag{2.2}$$

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3. Extensions. From now on the term "map" is to mean always a trivalent planar map.

Let M_1, M_2, and M be rooted maps, with roots r_1, r_2, and r respectively. Let A be an edge of M_1 incident with the vertices x_1 and y_1 and the faces P_1 and R_1. Let B be the first major edge of M_2 incident with the vertices r_2 and y_2 and the faces P_2 and R_2, these faces being incident with the second and third major edge respectively.

We say that M is an (x_1, P_1)-*extension* of M_1 by M_2 at A if the three maps are related as follows

(i) $C(M_1) - \{A, P_1, R_1\}$ *and* $C(M_2) - \{B, P_2, R_2\}$ *are disjoint subsets of* $C(M)$.

(ii) *The remaining cells of* M *are an edge* E_x *incident with* x_1 *and* r_2, *an edge* E_y *incident with* y_1 *and* y_2 *and two faces* P *and* R. *The face* P (R) *is incident only with* E_x, E_y, *and those other edges and vertices of* M *which are incident with* P_1 (R_1) *in* M_1 *or* P_2 (R_2) *in* M_2.

(iii) $r = r_1$ *and the* kth *major edge of* M *is the* kth *major edge of* M_1 *if this is not* A. *In the exceptional case the* kth *major edge of* M *is whichever of* E_x *and* E_y *is incident with* r.

When it seems unnecessary to specify x_1 and P_1 we shall speak of M simply as an *extension* of M_1 by M_2 at A.

(3.1) *Let* M_1 *and* M_2 *be rooted maps. Let* r_1, r_2, x_1, y_1, y_2, A, B, P_1, R_1, P_2, R_2 *be defined as above. Then we can construct an* (x_1, P_1)-*extension of* M_1 *at* A *by a rooted map* M_3 *equivalent to* M_2.

Proof. Choose a point O on A. Let Q be a circle on the sphere S_1 of M_1 with centre O and radius ϵ such that Q and its interior meet no edge or vertex of M_1 other than A. We can find a topological mapping ϕ of the sphere S_2 of M_2 onto S_1 with the following properties

(i) ϕ maps P_2 onto P_1.

(ii) $\phi(B)$ contains the whole of the boundary of P_1 outside Q. Moreover ϕ maps every edge and vertex of M_2, other than B, into the interior of Q.

(iii) x_1 lies between $\phi(r_2)$ and y_1 on $\phi(B)$ (see Fig. 1).

Let M_3 be the rooted map on S_1 determined by M_2 and the homeomorphism ϕ. On combining M_1 and M_3 we obtain the required (x_1, P_1)-extension of M_1 at A_1. We take E_x and E_y to be the disjoint open segments $x_1\phi(r_2)$ and $y_1\phi(y_2)$ respectively of $\phi(B)$, and we write $P = P_1 = \phi(P_2)$, $R = R_1 \cap \phi(R_2)$. This completes the proof of (3.1).

Suppose $A_1, A_2, \ldots, A_k$ are distinct edges of a rooted map N_0. Suppose N_1 is an extension of N_0 at A_1 by a rooted map M_1, N_2 is an extension of N_1 at A_2 by a rooted map M_2, and so on up to N_k. Then we call N_k a *multiple extension* of N_0 of order k.

Let the ends of A_i in N_0 be x_i and y_i. In the ith extension A_i is replaced

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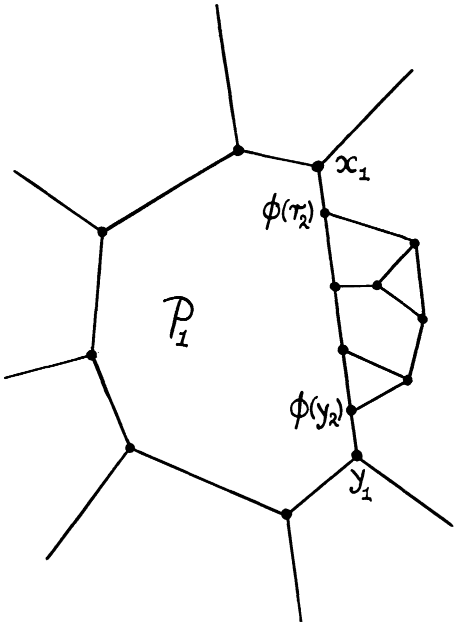

FIGURE 1.

by two distinct edges, B_i and C_i say, incident with x_i and y_i respectively. It follows from the definitions that

(3.2) *If* $1 \leqslant i \leqslant k$, *then* $\{B_i, C_i\}$ *is a* 2-*cut* $Q(V(N_k) - V(M_i), V(M_i))$ *of* N_k.

We call B_i and C_i the *representatives* of A_i in N_k. If A is any edge of N_0 which survives in N_k we say that A is its own representative in N_k.

Now consider a face P_0 of N_0. If P_0 is incident with A_1 then the first extension replaces it by a face P_1 incident with B_1 and C_1 and with precisely those edges of N_0, other than A_1, which are incident with P_0 in N_0. If P_0 is not incident with A_1 it becomes a face of N_i incident with the same edges as in N_0.

We then write $P_1 = P_0$. Similar considerations apply to P_1 in the second extension, and so on. We deduce

(3.3) *Let* P *be any face of* N_0. *Then there exists a face* P' *of* N_k *such that* P *is incident with* $A \in E(N_0)$ *if and only if* P' *is incident with the representatives of* A *in* N_k.

We can supplement this result as follows.

(3.4) P' *is uniquely determined when* P *is given. Moreover, the correspondence* $P \to P'$ *is a* $1 - 1$ *mapping of* $F(N_0)$ *onto the set* U *of all faces of* N_k *incident with vertices of* N_0.

Proof. If P and R are distinct faces of N_0 we cannot arrange that $P' = R'$.

For then P and R would have the same incident edges in N_0, by (3.3), which is impossible since N_0 is trivalent.

Suppose $P \in F(N_0)$ and that there are two distinct choices P_1' and P_2' for P'. Let A be any edge of N_0 incident with P in N_0, and let B be a representative of A in N_k. Let T be the second face of N_0 incident with A. Then T' must be either P_1' or P_2' since these are the two faces of N_k incident with B, and this is contrary to the preceding result. Hence P' is uniquely determined by P.

If $P \in F(N_0)$ it is trivial that $P' \in U$. Conversely, suppose $W \in U$. Then W is incident in N_k with a vertex $z \in V(N_0)$ and two edges B_1 and B_2 incident with z. These are representatives of distinct edges A_1 and A_2 of N_0 incident with z, by (1.4) and (3.2). There is a face P of N_0 incident with both A_1 and A_2. The face P' is incident with both B_1 and B_2 in N_k, and so it must coincide with N. The theorem folllows.

We call P' the *representative* of $P \in F(N_0)$ in N_k.

(3.5) *Let N be a rooted map and let V be a non-null proper subset of $V(N)$. Let it be given that N is a multiple extension of some rooted map M such that $V(M) = V$. Then the structure of M isu niquely determined by N and V. More precisely we can determine M to within a root-isomorphism reducing to an identity for the vertices.*

Proof. The vertices of M are given. Let B be any edge of N incident with a vertex x of V, and let its other end be u. If $u \in V$ then B is an edge of M, by (3.2).

Suppose $u \notin V$. Then B is a representative of some edge A of M incident with x. The second representative C of A has the following properties.

(i) *C has an end $y \in V$ and an end $v \notin V$.*

(ii) *$\{B, C\}$ is a 2-cut $Q(X, Y)$ of N such that $V \subseteq X$.*

(See (3.2).) We show that these properties determine C uniquely. For suppose there is a third edge C' of N with ends $y' \in V$ and $v' \notin V$, and suppose $\{B, C'\}$ is a 2-cut $Q(X', Y')$ of N such that $V \subseteq X'$. Then we have

$$B \in Q(X \cup X', Y \cap Y') \subseteq \{B, C, C'\}.$$

But $v \notin Y \cap Y'$ since $C \notin Q(X', Y')$, and similarly $v' \notin Y \cap Y'$. Hence $Q(X \cup X', Y \cap Y') = \{B\}$, which contradicts (1.1).

We see that the edges of M are determined through their representatives in N. The incidence relations between edges and vertices of M can thus be reconstructed. The faces of M can likewise be determined through their representatives, which are the members of the set corresponding to U in (3.4). The incidence relations between faces and edges of M are given by (3.3). The theorem follows.

4. The core of a rooted map.

(4.1) *Let M be a rooted map, with root r. Let $\{A, B\} = Q(X, Y)$ be a 2-cut of*

M, with $r \in X$. Then M is an extension, at an edge C, of a rooted map M_1 such that $V(M_1) = X$.

Proof. Let the ends of A be $a_X \in X$ and $a_Y \in Y$. Let those of B be $b_X \in X$ and $b_Y \in Y$ (Fig. 2). Choose a face P incident with A, and therefore B. If we remove A and B from the boundary of P we obtain two closed arcs. One

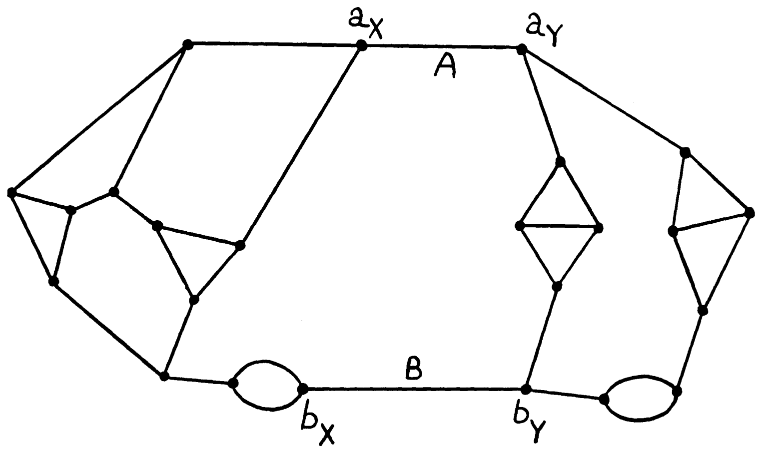

FIGURE 2.

joins a_X and b_X, passing through vertices of X only. The other joins a_Y and b_Y, passing through vertices of Y only. We denote these arcs by K_X and K_Y respectively. Similarly we obtain an arc L_X joining a_X and b_X, and an arc L_Y joining a_Y and b_Y, from the second face R incident with A and B.

We construct a rooted map M_1 as follows. We unite A, K_Y, and B to form a new edge C_1. We retain all the cells of M incident with vertices of X except A, B, and R, and we unite R with all the other cells incident with members of Y and not contained in C_1 to form a new face R_1. (This union of cells is a simply connected domain since its boundary is the simple closed curve $L_X \cup C_1$.) We give M_1 the same root and major edges of M, except that C_1 is to replace A or B if necessary.

A similar construction gives a rooted map M_2 whose cells are an edge $C_2 = A \cup B \cup K_X$, all the cells of M incident with vertices of Y except A, B, and R, and the union R_2 of all the cells of M incident with members of X and not in C_2. We take a_Y as the root of M_2 and C_2 as the first major edge. We choose the second major edge to be incident with P. The third one is then fixed.

We observe that M satisfies the definition of an (a_X, P)-extension of M_1 by M_2 at C_1.

In what follows we recognize each rooted map as a multiple extension of itself, of order zero.

(4.2) *Let M be a rooted map. Then M is a multiple extension of some 3-connected rooted map $\bar{M}$.*

Proof. If possible choose M so that the theorem fails and M has the least number of vertices consistent with this condition. By the convention just stated M is 2-separable. Choose a 2-cut $\{A, B\} = Q(X, Y)$ of M such that the root r of M is in X and such that Y has the greatest number of vertices consistent with this condition.

By (4.1) M is an extension of a rooted map M_1, where $V(M_1) = X$, at an edge C whose representatives in M are A and B. By the choice of M the rooted map M_1 is a multiple extension of some 3-connected rooted map $\bar{M}$.

C is not an edge of $\bar{M}$. For otherwise M would be a multiple extension of $\bar{M}$, contrary to its definition. We deduce that there is a 2-cut $Q(Z, T)$ of M_1, corresponding to one of the single extensions of $\bar{M}$, such that $r \in Z$ and at least one end of C is in T. If only one end of C is in T then C is a member of $Q(Z, T)$. It follows that the cut $Q(Z, T \cup Y)$ of M is a 2-cut having the same edges as $Q(Z, T)$ of M_1. But this is contrary to the choice of $Q(X, Y)$. The theorem follows.

We refer to $\bar{M}$ as the *core* of M. We proceed to show that the vertices and structure of this core are uniquely determined when M is given.

(4.3) *Let M' and M'' be two cores of a given rooted map M. Then there is a root-isomorphism of M' onto M'' which reduces to an identity for the vertices.*

Proof. By (3.5) it is sufficient to prove that $V(M') = V(M'')$.

Assume this equation false. Without loss of generality we may assume that $V(M')$ has a vertex v not in $V(M'')$. Since M is a multiple extension of M'' there is a 2-cut $\{A, B\} = Q(X, Y)$ of M such that $v \in Y$ and $V(M'') \subseteq X$. Since the root r of M is common to $V(M')$ and $V(M'')$ there exists a cut $J = Q(X \cap V(M'), \; Y \cap V(M'))$ of M'.

In the process of constructing M from M' we may suppose that the successive extensions are made at edges $C_1, C_2, \ldots, C_k$ of M', by the rooted maps $M_1, M_2, \ldots, M_k$ respectively.

Suppose $C \in J$. If $C \in E(M)$ then C is either A or B. In the remaining case we have $C = C_j$, where $1 \leqslant j \leqslant k$. Then either C has A or B as a representative in M or $V(M_j)$ meets both X and Y. In the latter alternative either A or B has both ends in $V(M_j)$, by (1.2) and (3.2). Since the k sets $V(M_i)$ are disjoint it follows from these observations that J is at most a 2-cut of M'. But M' is 3-connected. This contradiction establishes the theorem.

5. A census of rooted maps.

We write p_n for the number of inequivalent rooted maps of $2n$ vertices. We introduce the generating functions

$$p(x) = \sum_{n=1}^{\infty} p_n x^n,$$

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$$q(x) = \sum_{n=1}^{\infty} q_n x^n,$$

where q_n is defined as in § 2.

By (2.2) the function $q(x)$ is the one denoted by $xg(x)$ in **(1)**. By equations (4.8) and (4.9) of that paper there is a function $\theta(x)$ satisfying

$$x = \theta(x)\{1 - \theta(x)\}^3, \tag{5.1}$$

$$q(x) = \theta(x)\{1 - 2\theta(x)\}. \tag{5.2}$$

(5.3) *Let M be a rooted map of $2m$ vertices. Then the number of inequivalent multiple extensions of M having $2n$ vertices is the coefficient of x^n in*

$$x^m\{1 + p(x)\}^{3m}$$

Proof. M has just $3m$ edges. Let them be enumerated as $A_1, A_2, \ldots, A_{3m}$. Suppose a multiple extension M' of M is formed by extending M first at A_i by a rooted map M_i and then at A_j by a rooted map M_j. It follows from the definition of an extension that M' can be regarded equally well as the result of extending M first at A_j by M_j and then at A_i by M_i.

Hence if N is any multiple extension of M we can suppose that the successive extensions of M at edges $A_{k(1)}$, $A_{k(2)}$, etc. are made in the order of these edges in the sequence $(A_1, A_2, \ldots, A_{3m})$.

For each edge A_i of M we pick out an incident vertex x_i and an incident face P_i. Suppose one of the extensions used in the construction of N is the extension of a rooted map N_j by M_j at A_j. Then by adjusting the root and major edges of M_j we can arrange that this operation is an (x_i, P_{ij})-extension, where P_{ij} is the representative of P_i in N_j. In what follows we adopt this convention.

Consider the construction of a multiple extension N of M. A given edge A_i can either be left unaltered or used in an extension by a rooted map M_i of (say) $2s$ vertices. In the latter alternative the extension can be made in just p_s essentially different ways. Hence the number of ways of forming a multiple extension N with $2n$ vertices is the stated coefficient.

To complete the proof we observe that no two of these ways give equivalent multiple extensions. For equivalent multiple extensions of M must use the same set of edges, by (3.5) and (1.5), and the extending rooted map M_i for an edge A_i is uniquely determined, to within an equivalence, by the structure of N.

By (4.2) the rooted maps are the multiple extensions of the 3-connected rooted maps. So by (4.3) and (5.3) p_n is equal to the coefficient of x^n in

$$\sum_{m=1}^{\infty} q_m x^m \{1 + p(x)\}^{3m}.$$

Thus $p(x)$ satisfies the functional equation

$$p(x) = q(x\{1 + p(x)\}^3). \tag{5.4}$$

To solve this we write

$$\phi = \theta(x\{1 + p(x)\}^3).$$

We then have

$$x\{1 + p(x)\}^3 = \phi(1 - \phi)^3, \qquad \text{by}(5.1).$$

(5.5) $$p(x) = \phi(1 - 2\phi), \qquad \text{by (5.2) and (5.4)}.$$

Hence

$$x(1 + \phi - 2\phi^2)^3 = \phi(1 - \phi)^3,$$
$$x(1 - \phi)^3(1 + 2\phi)^3 = \phi(1 - \phi)^3,$$

(5.6) $$\phi = x(1 + 2\phi)^3.$$

Applying Lagrange's theorem to (5.5) and (5.6) we obtain

$$\begin{aligned} p(x) &= \sum_{n=1}^{\infty} \frac{x^n}{n!}\left[\left(\frac{d}{d\phi}\right)^{n-1}\{(1 + 2\phi)^{3n}(1 - 4\phi)\}\right]_{\phi=0} \\ &= \sum_{n=1}^{\infty} \frac{x^n}{n!}\left[\left(\frac{d}{d\phi}\right)^{n-1}\{3(1 + 2\phi)^{3n} - 2(1 + 2\phi)^{3n+1}\}\right]_{\phi=0} \\ &= \sum_{n=1}^{\infty} \frac{x^n}{n!}\left[\frac{3\cdot(3n)!}{(2n + 1)!}2^{n-1}(1 + 2\phi)^{2n+1} \right. \\ &\qquad \left. - \frac{2\cdot(3n + 1)!}{(2n + 2)!}2^{n-1}(1 + 2\phi)^{2n+2}\right]_{\phi=0} \\ &= \sum_{n=1}^{\infty} \frac{x^n}{n!}\cdot\frac{2^{n-1}(3n)!}{(2n + 2)!}\{3(2n + 2) - 2(3n + 1)\}, \end{aligned}$$

(5.7) $$p(x) = \sum_{n=1}^{\infty} \frac{2^{n+1}(3n)!}{n!(2n + 2)!}x^n.$$

An application of Stirling's theorem gives the following asymptotic formula.

(5.8) $$p_n \sim \frac{1}{4}\sqrt{\frac{3}{\pi}}\, n^{-5/2}\left(\frac{27}{2}\right)^n$$

From (5.7) we have $p_1 = 1$, $p_2 = 4$ and $p_3 = 24$. The corresponding rooted maps are shown in Figure 3. Rooted maps differing only by a permutation of major edges are represented by a single diagram, the number of corresponding rooted maps being written underneath. Other numerical values for p_n are given in Table I.

6. Cross-connections. Let J be a simple closed curve on a 2-sphere and let D be one of its residual domains. Let $2n$ distinct points $P_1, P_2, \ldots, P_{2n}$ be chosen on J. By a *cross-connection* of these points in D we understand a set of n non-intersecting open arcs in D joining the $2n$ points P_i in pairs. Two such cross-connections are *equivalent* if one can be converted into the

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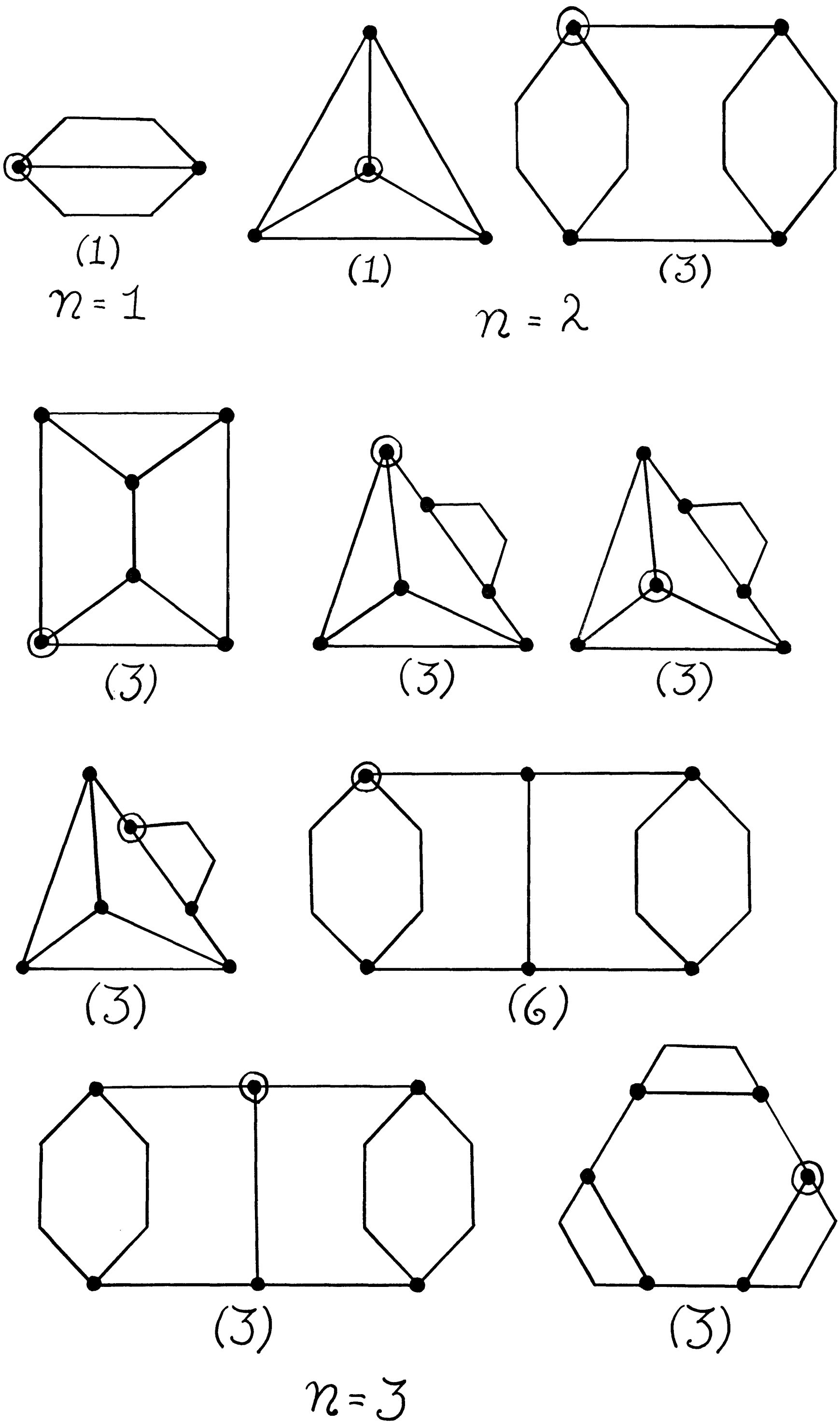

FIGURE 3.

other by a topological mapping of $J \cup D$ onto itself which leaves each point of J invariant. Clearly the number of inequivalent cross-connections of the given points in D is a function γ_n of n only. We write $\gamma_0 = 1$.

Let us suppose that the enumeration of the points P_i corresponds to their cyclic order on J.

To construct a cross-connection we may begin by joining P_1 to any other point P_i by an arc L_1 in D. For a given i this can be done in essentially only one way. The points P_1 and P_i separate J into two arcs J_1 and J_2. We distinguish J_1 by postulating that it contains the arc K in J which joins P_1

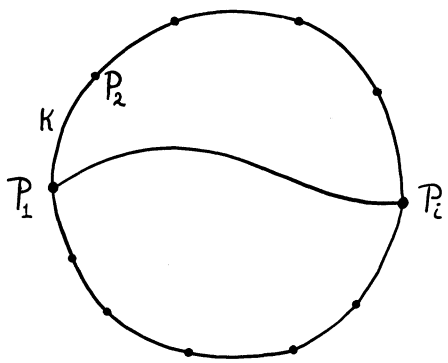

FIGURE 4.

and P_2 without passing through any other point P_k (or a specified arc K of this kind if $n = 1$). The arc L_1 separates D into two simply connected domains D_1 and D_2 with boundaries $L_1 \cup J_1$ and $L_1 \cup J_2$ respectively. The construction must be completed by combining a cross-connection of the $i - 2$ points $P_2, \ldots, P_{i-1}$ in D with a cross-connection of the points $P_{i+1}, \ldots, P_{2n}$ in D_2.

We observe that i must be an even number, $2j + 2$ say, and that γ_n satisfies the recursion formula

$$\gamma_n = \sum_{j=0}^{n-1} \gamma_j \gamma_{n-j-1}. \tag{6.1}$$

We introduce the generating function

$$\gamma(x) = \sum_{n=0}^{\infty} \gamma_n x^{n+1}.$$

In terms of this function (6.1) becomes

$$\begin{aligned} \gamma(x) &= x + (\gamma(x))^2, \\ \gamma(x) &= \tfrac{1}{2}(1 - (1 - 4x)^{\frac{1}{2}}). \end{aligned} \tag{6.2}$$

Hence, by the binomial theorem,

$$\gamma(x) = \sum_{n=0}^{\infty} \frac{(2n)!}{n!(n+1)!} x^{n+1}. \tag{6.3}$$

7. Hamiltonian polygons. A *polygon* in a map M is a simple closed curve made up of edges and vertices of M. It is a *Hamiltonian* polygon if it includes all the vertices.

A *Hamiltonian* rooted map is a rooted map M in which one Hamiltonian polygon passing through the first and second major edges is distinguished as the *principal* polygon. Two Hamiltonian rooted maps are *equivalent* if and only if there is an isomorphism of one onto the other which preserves the principal polygon as well as the root and each major edge.

We denote the number of inequivalent Hamiltonian rooted maps of $2n$ vertices by u_n.

Consider the construction of a Hamiltonian rooted map M of $2n$ vertices. We may represent the principal polygon by a great circle J. We can fix a positive direction on J, proceeding from the root r along the first major edge, and we can distinguish each vertex of M by its position in the cyclic order of vertices on J. Of the residual domains of J one can be distinguished as containing the third major edge. We denote this residual domain by D_1 and the other by D_2.

With this much given we must complete the construction of M as follows. First we partition $V(M)$ into two sets W_1 and W_2, each with an even number of vertices, such that $r \in W_1$. If W_2 is to have $2j$ vertices this can be done in

$$\binom{2n-1}{2j}$$

ways. Then we make a cross-connection between the members of W_1 in D_1 and a cross-connection between the members of W_2 in D_2. Our construction is then complete. The above process can be carried through in

$$\sum_{j=0}^{n-1} \binom{2n-1}{2j} \gamma_j \gamma_{n-j}$$

essentially different ways. Hence, by (6.3),

$$\begin{aligned} u_n &= \sum_{j=0}^{n-1} \frac{(2n-1)!}{(2j)!(2n-2j-1)!} \cdot \frac{(2j)!}{j!(j+1)!} \cdot \frac{(2n-2j)!}{(n-j)!(n-j+1)!} \\ &= \sum_{j=0}^{n-1} \frac{2\cdot(2n-1)!}{j!(j+1)!(n-j-1)!(n-j+1)!} \\ &= \frac{2\cdot(2n-1)!}{(n-1)!(n+2)!} \sum_{j=0}^{n-1} \binom{n-1}{j}\binom{n+2}{j+1} \end{aligned}$$

Since

$$\binom{n-1}{j}$$

is the coefficient of x^j in $(1+x)^{n-1}$ and

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$$\binom{n+2}{j+1}$$

is the coefficient of x^{n-j+1} in $(1+x)^{n+2}$ it follows that

$$\sum_{j=0}^{n-1}\binom{n-1}{j}\binom{n+2}{j+1}$$

is the coefficient of x^{n+1} in $(1+x)^{2n+1}$, which is

$$\frac{(2n+1)!}{n!(n+1)!}.$$

Hence

$$u_n = \frac{1}{2}\frac{(2n)!(2n+2)!}{n!((n+1)!)^2(n+2)!}. \tag{7.1}$$

Applying Stirling's theorem to this we obtain

$$u_n \sim \frac{2}{\pi} n^{-3}(16)^n \tag{7.2}$$

The average number of Hamiltonian polygons, passing through the first and second major edges, in a rooted map of $2n$ vertices is the fraction u_n/p_n. By (5.7) and (7.1) we have

$$\frac{u_n}{p_n} = \frac{(2n)!((2n+2)!)^2}{2^{n+2}((n+1)!)^2(n+2)!(3n)!}, \tag{7.3}$$

and by (5.8) and (7.2)

$$\frac{u_n}{p_n} \sim \frac{8}{\sqrt{(3\pi)}} n^{-\frac{1}{2}}\left(\frac{32}{27}\right)^n. \tag{7.4}$$

If we remove the restriction to Hamiltonian polygons through the first and second major edges the above average must be multiplied by 3.

We observe that if it is assumed that almost all maps with a specified Hamiltonian polygon are without non-trivial automorphisms then the average number of Hamiltonian polygons in an unrooted map with $2n$ vertices must be asymptotically,

$$8\sqrt{\frac{3}{\pi}}\, n^{-\frac{1}{2}}\left(\frac{32}{27}\right)^n$$

In Table I we give the values of p_n, u_n and u_n/p_n, the first two accurately and the last to three decimal places, for $1 \leqslant n \leqslant 11$.

8. 3-connected Hamiltonian rooted maps. Let M be a rooted map, and H a Hamiltonian polygon of M, passing through the first and second major edges. We write $\{M, H\}$ for the Hamiltonian rooted map formed from M by taking H as the principal polygon. We observe that the edges of any 2-cut in M belong to H.

TABLE I

n	p_n	u_n	u_n/p_n
1	1	1	1.000
2	4	5	1.250
3	24	35	1.458
4	176	294	1.670
5	1456	2772	1.904
6	13056	28314	2.169
7	124032	306735	2.473
8	1230592	3476330	2.825
9	12629760	40831076	3.233
10	133186560	493684828	3.707
11	1436098560	6114096716	4.257

Let $\bar{M}$ be the core of M. The edges of $\bar{M}$ having representatives in H evidently define a Hamiltonian polygon of $\bar{M}$. We denote this by $\bar{H}$ and call the Hamiltonian rooted map $\{\bar{M}, \bar{H}\}$ the *core* of $\{M, H\}$.

To construct a Hamiltonian rooted map $\{M, H\}$ with a given core $\{\bar{M}, \bar{H}\}$ we must form M as a multiple extension of $\bar{M}$, operating only at the edges of H. If M has $2m$ vertices there are just $2m$ such edges. Suppose one of the single extensions is by a rooted map M_i at an edge A_i of M. As in (5.3) we may suppose it specified as an (x_i, P_{ij})-extension. We must make a corresponding change in the principal polygon by replacing its edge A_i by the new representatives of A_i and the edges, other than A_i, of any Hamiltonian polygon H_i of M_i which passes through the first major edge A_i. The polygon H_i may pass through either the second or the third major edge of M_i. Hence if M_i is to have $2j$ vertices the complete extension can be made in just $2u_j$ essentially different ways.

We write v_n for the number of inequivalent 3-connected Hamiltonian rooted maps of $2n$ vertices. We write also

$$u(x) = \sum_{n=1}^{\infty} u_n x^n,$$

$$v(x) = \sum_{n=1}^{\infty} v_n x^n.$$

(Thus $1 - 2x - 4xu(x)$ is the hypergeometric function $F(-\frac{1}{2}, \frac{1}{2}, 2; 16x)$.)

From the above observations we deduce that the number of inequivalent Hamiltonian rooted maps of $2n$ vertices having a given core of $2m$ vertices is the coefficient of x^n in

$$x^m\{1 + 2u(x)\}^{2m}.$$

By (4.3) this gives us the functional equation

$$u(x) = v(x\{1 + 2u(x)\}^2). \tag{8.1}$$

Some values of v_n calculated from this equation are given in Table II. They have been verified by actual counts of Hamiltonian polygons as far as $v_6 = 518$. The numbers q_n are from **(1)**.

TABLE II

n	q_n	v_n	v_n/q_n
1	1	1	1.000
2	1	1	1.000
3	3	3	1.000
4	13	14	1.077
5	68	80	1.176
6	399	518	1.298
7	2530	3647	1.442
8	16965	27274	1.608
9	118668	213480	1.799
10	857956	1731652	2.018

Reference

1. W. T. Tutte, *A census of planar triangulations*, Can. J. Math., *14* (1962), 21–38.

University of Toronto