ABSTRACT DEFINITIONS FOR THE MATHIEU GROUPS M_{11} AND M_{12}.

W.O.J. Moser

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A list of known finite simple groups has been given by Dickson [3, 4]. With but five exceptions, all of them fall into infinite families. The five exceptional groups, discovered by Mathieu [8, 9], were further investigated by Jordan [7], Miller [10], de Séguier [11], Zassenhaus [13], and Witt [12]. In Witt's notation they are M_{11}, M_{12}, M_{22}, M_{23}, M_{24}. Generators for them may be seen in the book of Carmichael [1, pp. 151, 263, 288]; but only for the smallest of them, M_{11} of order 7920, has a set of defining relations been given.

Fryer [6] has shown that the permutations

$$A = (0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10),$$
$$B = (1\ 4\ 5\ 9\ 3)\ (2\ 8\ 10\ 7\ 6),$$
$$C = (2\ 10\ 8\ 6)\ (3\ 9\ 4\ 5)$$

generate a group M_{11}. In fact A and C suffice to generate the group since $B = A^2C^2A^8C^2A^2C^2$. Coxeter and Moser [2, pp. 98-100] have shown that the relations

$$(1) \qquad A^{11} = B^5 = C^4 = (A^4C^2)^3 = (BC^2)^2 = (ABC)^3 = E,$$

$$B^{-1}AB = A^4, \quad C^{-1}BC = B^2$$

(which are satisfied by the permutations) provide an abstract definition for M_{11}. The relation $(BC^2)^2 = E$ is in fact redundant, for the other relations imply

$$C^{-1}B^2C = B^4, \quad C^{-1}\cdot C^{-1}BC\cdot C = B^4, \ (BC^2)^2 = E.$$

The permutations A and $D = (10\ 7\ 2\ 6)\ (3\ 9\ 4\ 5)$ also generate a group M_{11} [1, p. 151]. A, D and $B = D^2A^2D^2A^6D^2A^2 = (1\ 4\ 5\ 9\ 3)\ (2\ 8\ 10\ 7\ 6)$ satisfy the relations.

$$(2) \qquad A^{11} = B^5 = C^4 = (AD^2)^3 = (A^{-1}DB)^3 = E,$$

$$B^{-1}AB = A^4, \quad D^{-1}BD = B^2.$$

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We now show that these relations also provide an abstract definition for M_{11}. It will suffice to show that the order of the abstract group G defined by (2) is at most 7920. Let H be the subgroup of G generated by the elements $S = A$, $V = B$, $T = D^2$. The relations (2) imply

$$S^{11} = V^5 = T^2 = (ST)^3 = (VT)^2 = E, \quad V^{-1}SV = S^4$$

which are Frasch's [5, p. 252; 2, p. 94 with $\alpha = 2$] relations for LF(2,11), the simple group of order 660. Hence H is precisely LF(2,11), We now enumerate the cosets of H in G by the Todd-Coxeter enumeration method [2, p. 12] defining the cosets by

$$1 = H,\ 2 = 1 \cdot C,\ 3 = 2 \cdot A,\ 4 = 3 \cdot A,\ \ldots,\ 12 = 11 \cdot A.$$

We have initially $1 \cdot A^4 = 1$, $1 \cdot B = 1$, $1 \cdot C^2 = 1$; the tables close up after the coset 12 has been inserted. Hence the order of G is 12.660 = 7920, and the abstract definition (2) for M_{11} is established. The tables are:

A	A	A	A	A	A	A	A	A	A	A	
1	1	1	1	1	1	1	1	1	1	1	1
2	3	4	5	6	7	8	9	10	11	12	2

B	B	B	B	B	
1	1	1	1	1	1
2	2	2	2	2	2
3	6	7	11	5	3
4	10	12	9	8	4

D	D	D	D	
1	2	1	2	1
5	12	5	12	5
3	8	11	4	3
6	10	7	9	6

B^{-1}	A	B		= A^4	
2	2	3	6	2	6
3	5	6	7	3	7
4	8	9	8	4	8
6	3	4	10	6	10
7	6	7	11	7	11
8	9	10	12	8	12
10	4	5	3	10	3
11	7	8	4	11	4

D^{-1}	B	D		= B^2	
1	2	2	1	1	1
3	4	10	7	3	7
5	12	9	6	5	6
6	9	8	11	6	11
8	3	6	10	8	10
11	8	4	3	11	3
12	5	3	8	12	8

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A^{-1}	D	B	A^{-1}	D	B	A^{-1}	D	B	
1	1	2	2	12	5	3	2	1	1
5	4	3	6	5	12	9	8	11	5

We now proceed to the group M_{12} of order 95040 = 7920.12 generated by the permutations

$$A,\ D,\quad U = (0\ \ \infty)(1\ \ 10)(2\ \ 5)(3\ \ 7)(4\ \ 8)(6\ \ 9)$$

[1, p. 151] . Using the redundant generator B, we observe that the permutations A, B, D, U satisfy the relations

(3) $$A^{11} = B^5 = D^4 = (AD^2)^3 = (A^{-1}DB)^3 = (UA)^3 = E,$$

$$B^{-1}AB = A^4,\quad D^{-1}BD = B^2,\quad B = UD^{-1}UD,$$

$$UA^2D^{-1}A^4U = A^{-1}D^2A^3D^2A^4DA^5. \qquad *)$$

In order to establish (3) as an abstract definition of M_{12}, it will suffice to show that the order of the abstract group M defined by (3) is at most 95040. Let N be the subgroup of M generated by the elements A, B, D. Relations (3) imply relations (2); hence N is M_{11} of order 7920. We now enumerate the cosets of N in M, defining the cosets by

$$\infty = N,\quad 0 = \infty\cdot U,\quad 1 = 0\cdot A,\quad 2 = 1\cdot A,\ \ldots\ ,\ 10 = 9\cdot A.$$

We have initially $\infty\cdot A = \infty$, $\infty\cdot B = \infty$, $\infty\cdot D = \infty$; the tables close up after the coset 10 has been inserted. Hence the order of M is 7920.12 and the abstract definition (3) is established. The tables are:

A	A	A	A	A	A	A	A	A	A	A	
∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
0	1	2	3	4	5	6	7	8	9	10	0

B	B	B	B	B	
∞	∞	∞	∞	∞	∞
0	0	0	0	0	0
1	4	5	9	3	1
2	8	10	7	6	2

C	C	C	C	
∞	∞	∞	∞	∞
0	0	0	0	0
1	1	1	1	1
2	6	10	7	2
4	5	3	9	4
8	8	8	8	8

U	U	
∞	0	∞
1	10	1
2	5	2
3	7	3
4	8	4
6	9	6

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B^{-1}	A	B	=	A^4	
0	0	1	4	0	4
1	3	4	5	1	5
2	6	7	6	2	6
3	9	10	7	3	7
4	1	2	8	4	8
5	4	5	9	5	9
8	2	3	1	8	1
9	5	6	2	9	2

C^{-1}	B	C	=	B^2	
1	1	4	5	1	5
2	7	6	10	2	10
3	5	9	4	3	4
5	4	5	3	5	3
6	2	8	8	6	8
8	8	10	7	8	7
10	6	2	6	10	6

U	A	U	A	U	A	
∞	0	1	10	0	∞	∞

U	A^2	C^{-1}	A^4	U	=	A^{-1}	C^2	A^3	C^2	A^4	C^5
∞	0	2	7	0	∞	∞	∞	∞	∞	∞	∞	∞

U	C^{-1}	U	C	=	B	
∞	0	0	∞	∞	∞	∞
0	∞	∞	0	0	0	0
3	7	10	1	1	3	1

B	U	B	U	
1	4	8	10	1
2	8	4	5	2
6	2	5	9	6
7	6	9	3	7

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University of Saskatchewan

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