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CONCAVITY PROPERTY OF MINIMAL L^2 INTEGRALS WITH LEBESGUE MEASURABLE GAIN

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Abstract. In this article, we present a concavity property of the minimal L^2 integrals related to multiplier ideal sheaves with Lebesgue measurable gain. As applications, we give necessary conditions for our concavity degenerating to linearity, characterizations for 1-dimensional case, and a characterization for the holding of the equality in optimal L^2 extension problem on open Riemann surfaces with weights may not be subharmonic.

Contents

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§1. Introduction

The multiplier ideal sheaves related to plurisubharmonic functions plays an important role in complex geometry and algebraic geometry (see, e.g., [3], [4], [6–8], [22],]23], [26–28], [30]). Recall the definition of the multiplier ideal sheaves as follows (see [4]).

The multiplier ideal sheaf $\mathcal{I}(\varphi)$ was defined as the sheaf of germs of holomorphic functions f such that $|f|^2e^{-\varphi}$ is locally integrable, where φ is a plurisubharmonic function on a complex manifold M.

The strong openness conjecture is $\mathcal{I}(\varphi)=\mathcal{I}_+(\varphi):=\cup_{\varepsilon>0}\mathcal{I}((1+\varepsilon)\varphi)$, which was posed by Demailly [3] and was proved by Guan–Zhou [19] (the dimension two case was proved by Jonsson–Mustata [21]). When $\mathcal{I}(\varphi)=\mathcal{O}$, this conjecture is called the openness conjecture, which was posed by Demailly–Kollár [7], and was proved by Berndtsson [1] (the dimension two case was proved by Favre–Jonsson [9]) by establishing an effectiveness result of the openness conjecture.

Stimulated by Berndtsson's effectiveness result, continuing the solution of the strong openness conjecture [19], Guan–Zhou [20] established a non-sharp effectiveness result of the strong openness conjecture. Recall that for the first time, Guan–Zhou [20] considered the minimal L^2 integral related to multiplier ideals on the sublevel set $\{\varphi<0\}$, that is, the pseudoconvex domain D.

In [14], by considering all the minimal L^2 integrals on the sublevels of the weights φ, Guan presented a sharp version of the effectiveness result of the strong openness conjecture, and obtained a concavity property of the minimal L^2 integrals without gain. In [13], Guan generalized the concavity property in [14] to minimal L^2 integrals with smooth gain.

In [15], Guan–Mi gave some applications of the concavity property in [13]: a necessary condition for the concavity degenerating to linearity, a characterization for 1-dimensional case, and a characterization for the holding of the equality in optimal L^2 extension problem on open Riemann surfaces with subharmonic weights. Recall that if the subharmonic weights degenerate to 0, the characterization for the holding of the equality in optimal L^2 extension problem on open Riemann surfaces is the solution of the equality part of the Suita conjecture in [18]; if the subharmonic weights degenerate to harmonic, the characterization for the holding of the equality in optimal L^2 extension problem on open Riemann surfaces is the solution of the equality part of the extended Suita conjecture in [18].

In the present article, we point out that the smooth gain of the general concavity property in [13] (see also [15]) can be replaced by Lebesgue measurable gain (Definition 1.1 and Theorem 1.3). As applications, we give necessary conditions for our concavity degenerating to linearity (§1.2.2), characterizations for 1-dimensional case (§1.2.3), and a characterization for the holding of the equality in optimal L^2 extension problem on open Riemann surfaces with weights may not be subharmonic (§1.2.4).

1.1 Concavity property of minimal L^2 integrals with Lebesgue measurable gain

Let M be a complex manifold. We call M that satisfies condition (a), if there exists a closed subset $X \subset M$ satisfying the following two statements:

- $(a1)$ X is locally negligible with respect to L^2 holomorphic functions; that is, for any local coordinate neighborhood $U \subset M$ and for any L^2 holomorphic function f on $U \backslash X$, there exists an L^2 holomorphic function $\tilde{f}$ on U such that $\tilde{f}|_{U \backslash X} = f$ with the same L^2 norm.
- $(a2)$ $M \backslash X$ is a Stein manifold.

Let M be an n-dimensional complex manifold satisfying condition (a), and let K_M be the canonical (holomorphic) line bundle on M. Let ψ be a plurisubharmonic function on M, and let φ be a Lebesgue measurable function on M, such that $\varphi + \psi$ is a plurisubharmonic function on M. Take $T = -\sup_M \psi$ (T maybe $-\infty$).

Definition 1.1. We call a positive measurable function c (so-called "gain") on $(T, +\infty)$ in class $\mathcal{P}_T$ if the following two statements hold:

1. $c(t)e^{-t}$ is decreasing with respect to t.
2. There is a closed subset E of M such that $E \subset \{z \in Z : \psi(z) = -\infty\}$ and for any compact subset $K \subseteq M \backslash E$, $e^{-\varphi}c(-\psi)$ has a positive lower bound on K, where Z is some analytic subset of M.

Remark 1.2. We recall a class $\mathcal{P}'_T$ of positive smooth functions in [13]. A positive smooth function c on $(T, +\infty)$ in class $\mathcal{P}'_T$ if the following three statements hold:

- $(1)'$ $\int_T^{+\infty} c(t)e^{-t}dt < +\infty$.
- $(2)'$ $c(t)e^{-t}$ is decreasing with respect to t.
- $(3)'$ For any compact subset $K \subseteq M$, $e^{-\varphi}c(-\psi)$ has a positive lower bound on K.

We compare these two classes of functions $\mathcal{P}_T$ and $\mathcal{P}'_T$. When $c \in \mathcal{P}_T$, c maybe non-smooth on $(T, +\infty)$ and $\int_T^{+\infty} c(t)e^{-t}dt$ maybe $+\infty$. When φ is continuous on M, condition $(3)'$ is equivalent to $\liminf_{t\to+\infty} c(t) > 0$. When φ is continuous on M and $\psi \in A(S)$ (see §1.2.1), the decreasing property of $c(t)e^{-t}$ implies that $c \in \mathcal{P}_T$ and $\liminf_{t\to+\infty} c(t)$ may be equal to 0.

Let Z_0 be a subset of $\{\psi = -\infty\}$ such that $Z_0 \cap Supp(\{\mathcal{O}/\mathcal{I}(\varphi+\psi)\}) \neq \emptyset$. Let $U \supseteq Z_0$ be an open subset of M, and let f be a holomorphic $(n, 0)$ form on U. Let $\mathcal{F} \supseteq \mathcal{I}(\varphi+\psi)|_U$ be a analytic subsheaf of $\mathcal{O}$ on U.

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Denote

$$\inf\left\{ \int_{\{\psi<-t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : (\tilde{f}-f) \in H^0(Z_0, (\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0}) \right.$$
$$\left. \& \, \tilde{f} \in H^0(\{\psi<-t\}, \mathcal{O}(K_M)) \right\},$$

by $G(t;\varphi,\psi,c)$ (so-called "minimal L^2 integrals related to multiplier ideal sheaves"), where $t \in [T,+\infty)$, c is a nonnegative function on $(T,+\infty)$, $|f|^2 := \sqrt{-1}^{n^2} f\wedge\bar{f}$ for any $(n,0)$ form f and $(\tilde{f}-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$ means $(\tilde{f}-f,z_0)\in(\mathcal{O}(K_M)\otimes\mathcal{F})_{z_0}$ for all $z_0\in Z_0$. If there is no holomorphic holomorphic $(n,0)$ form $\tilde{f}$ on $\{\psi<-t\}$ satisfying $(\tilde{f}-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$, we set $G(t;\varphi,\psi,c)=+\infty$. Without misunderstanding, we denote $G(t;\varphi,\psi,c)$ by $G(t)$, and when we focus on different φ, ψ, or c, we denote it by $G(t;\varphi)$, $G(t;\psi)$, or $G(t;c)$, respectively.

In the present article, we obtain the following concavity for $G(t)$.

THEOREM 1.3. *Let $c\in\mathcal{P}_T$. If there exists $t\in[T,+\infty)$ satisfying that $G(t)<+\infty$, then $G(h^{-1}(r))$ is concave with respect to $r\in(\int_{T_1}^{T}c(t)e^{-t}dt,\int_{T_1}^{+\infty}c(t)e^{-t}dt)$, $\lim_{t\to T+0}G(t)=G(T)$, and $\lim_{t\to+\infty}G(t)=0$, where $h(t)=\int_{T_1}^{t}c(t_1)e^{-t_1}dt_1$ and $T_1\in(T,+\infty)$.*

When $c(t)\in\mathcal{P}'_T$ and M is a Stein manifold, Theorem 1.3 is the concavity property in [13] (see also [15]).

Theorem 1.3 implies the following corollary.

COROLLARY 1.4. *If $\int_{T_1}^{+\infty}c(t)e^{-t}dt=+\infty$, where $c\in\mathcal{P}_T$, and $f\notin H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$, then $G(t)=+\infty$ for any $t\geq T$, that is, there is no holomorphic holomorphic $(n,0)$ form $\tilde{f}$ on $\{\psi<-t\}$ satisfying $(\tilde{f}-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$ and $\int_{\{\psi<-t\}}|\tilde{f}|^2e^{-\varphi}c(-\psi)<+\infty$.*

In the following, we give two corollaries of Theorem 1.3 when concavity degenerates to linearity.

COROLLARY 1.5. *Let $c\in\mathcal{P}_T$, and let $G(t)\in(0,+\infty)$ for some $t\geq T$, then $G(h^{-1}(r))$ is concave with respect to $r\in(\int_{T_1}^{T}c(t)e^{-t}dt,\int_{T_1}^{+\infty}c(t)e^{-t}dt]$ and the following three statements are equivalent:*

(1) $G(t)=\frac{G(T_1)}{\int_{T_1}^{+\infty}c(t_1)e^{-t_1}dt_1}\int_{t}^{+\infty}c(t_1)e^{-t_1}dt_1$ *holds for any $t\in[T,+\infty)$, that is, $G(\hat{h}^{-1}(r))$ is linear with respect to $r\in[0,\int_{T}^{+\infty}c(s)e^{-s}ds)$, where $\hat{h}(t)=\int_{t}^{+\infty}c(s)e^{-s}ds$.*

(2) *There exists $r_0\in(\int_{T_1}^{T}c(t)e^{-t}dt,\int_{T_1}^{+\infty}c(t)e^{-t}dt)$ such that*

$$\frac{G(h^{-1}(r_0))}{\int_{T_1}^{+\infty}c(t_1)e^{-t_1}dt_1-r_0}\leq\lim_{t\to T+0}\frac{G(t)}{\int_{t}^{+\infty}c(t_1)e^{-t_1}dt_1},$$

that is,

$$\frac{G(t_0)}{\int_{t_0}^{+\infty}c(t_1)e^{-t_1}dt_1}\leq\lim_{t\to T+0}\frac{G(t)}{\int_{t}^{+\infty}c(t_1)e^{-t_1}dt_1}$$

holds for some $t_0\in(T,+\infty)$.

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(3) $\lim_{r\to\int_{T_1}^{+\infty}c(t_1)e^{-t_1}dt_1-0}\frac{G(h^{-1}(r))}{\int_{T_1}^{+\infty}c(t)e^{-t}dt-r}\le\lim_{t\to T+0}\frac{G(t)}{\int_t^{+\infty}c(t_1)e^{-t_1}dt_1}$ *holds, that is,*

$$\lim_{t\to+\infty}\frac{G(t)}{\int_t^{+\infty}c(t_1)e^{-t_1}dt_1}\le\lim_{t\to T+0}\frac{G(t)}{\int_t^{+\infty}c(t_1)e^{-t_1}dt_1}$$

holds.

REMARK 1.6. Let $M=\Delta\subset\mathbb{C}$, and let $\psi=\psi+\varphi=2\log|z|$. Let $c(t)\equiv 1$, and let $\mathcal{F}=\mathcal{I}(\varphi+\psi)$. Let $f\equiv dz$ and $Z_0=o$ the origin of $\mathbb{C}$. It is clear that $G(\hat{h}^{-1}(r))=2\pi r$ is linear with respect to r, where $\hat{h}(t)=\int_t^{+\infty}c(l)e^{-l}dl$.

Let $c(t)$ be a nonnegative measurable function on $(T,+\infty)$. Set

$$\mathcal{H}^2(c,t)=\left\{\tilde{f}:\int_{\{\psi<-t\}}|\tilde{f}|^2e^{-\varphi}c(-\psi)<+\infty,(\tilde{f}-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})\right.$$
$$\left.\&\,\tilde{f}\in H^0(\{\psi<-t\},\mathcal{O}(K_M))\right\},$$

where $t\in[T,+\infty)$.

COROLLARY 1.7. *Let $c\in\mathcal{P}_T$, if $G(t)\in(0,+\infty)$ for some $t\ge T$ and $G(\hat{h}^{-1}(r))$ is linear with respect to $r\in[0,\int_T^{+\infty}c(s)e^{-s}ds)$, where $\hat{h}(t)=\int_t^{+\infty}c(l)e^{-l}dl$, then there is a unique holomorphic $(n,0)$ form F on M satisfying $(F-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$ and $G(t;c)=\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)$ for any $t\ge T$. Equality*

$$\int_{\{-t_1\le\psi<-t_2\}}|F|^2e^{-\varphi}a(-\psi)=\frac{G(T_1;c)}{\int_{T_1}^{+\infty}c(t)e^{-t}dt}\int_{t_2}^{t_1}a(t)e^{-t}dt \tag{1}$$

holds for any nonnegative measurable function a on $(T,+\infty)$, where $+\infty\ge t_1>t_2\ge T$ and $T_1\in(T,+\infty)$.

Furthermore, if $\mathcal{H}^2(\tilde{c},t_0)\subset\mathcal{H}^2(c,t_0)$ for some $t_0\ge T$, where $\tilde{c}$ is a nonnegative measurable function on $(T,+\infty)$, we have

$$G(t_0;\tilde{c})=\int_{\{\psi<-t_0\}}|F|^2e^{-\varphi}\tilde{c}(-\psi)=\frac{G(T_1;c)}{\int_{T_1}^{+\infty}c(s)e^{-s}ds}\int_{t_0}^{+\infty}\tilde{c}(s)e^{-s}ds. \tag{2}$$

When $c(t)\in\mathcal{P}'_T$ and M is a Stein manifold, Corollaries 1.5 and 1.7 can be referred to [15] (when $c\equiv 1$, M is a Stein manifold, φ is a smooth plurisubharmonic function on M and $\{\psi=-\infty\}$ is a closed subset of M, Xu–Zhou [32] also get the existence of F in Corollary 1.7 independently).

REMARK 1.8. Let $\tilde{c}\in\mathcal{P}_T$, if $\mathcal{H}^2(\tilde{c},t_1)\subset\mathcal{H}^2(c,t_1)$, then $\mathcal{H}^2(\tilde{c},t_2)\subset\mathcal{H}^2(c,t_2)$, where $t_1>t_2>T$. In the following, we give some sufficient conditions of $\mathcal{H}^2(\tilde{c},t_0)\subset\mathcal{H}^2(c,t_0)$ for $t_0>T$:

(1) $\tilde{c}\in\mathcal{P}_T$ and $\lim_{t\to+\infty}\frac{\tilde{c}(t)}{c(t)}>0$. Especially, $\tilde{c}\in\mathcal{P}_T$, c and $\tilde{c}$ are smooth on $(T,+\infty)$ and $\frac{d}{dt}(\log(\tilde{c}(t))\ge\frac{d}{dt}(\log c(t))$.
(2) $\tilde{c}\in\mathcal{P}_T$, $\mathcal{H}^2(c,t_0)\ne\emptyset$ and there exists $t>t_0$, such that $\{\psi<-t\}\subset\subset\{\psi<-t_0\}$, $\{z\in\overline{\{\psi<-t\}}:\mathcal{I}(\varphi+\psi)_z\ne\mathcal{O}_z\}\subset Z_0$ and $\mathcal{F}|_{\overline{\{\psi<-t\}}}=\mathcal{I}(\varphi+\psi)|_{\overline{\{\psi<-t\}}}$.

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The sufficiency of condition (1) is clear. For condition (2), assume that $\mathcal{H}^2(\tilde{c},t_0)\neq\emptyset$, then the following inequality gives the sufficiency of condition (2):

$$\begin{aligned}
&\int_{\{\psi<-t_0\}}|\tilde{F}|^2e^{-\varphi}c(-\psi)\\
&\leq 2\int_{\{\psi<-t\}}|\tilde{F}-F|^2e^{-\varphi}c(-\psi)+2\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)\\
&\quad+\int_{\{-t\leq\psi<-t_0\}}|\tilde{F}|^2e^{-\varphi}c(-\psi)\\
&\leq 2C\int_{\{\psi<-t\}}|\tilde{F}-F|^2e^{-\varphi-\psi}+2\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)\\
&\quad+\frac{\sup_{s\in(t_0,t]}c(s)}{\inf_{s\in(t_0,t]}\tilde{c}(s)}\int_{\{\psi<-t_0\}}|\tilde{F}|^2e^{-\varphi}\tilde{c}(-\psi)\\
&<+\infty,
\end{aligned}$$

where $\tilde{F}\in\mathcal{H}^2(\tilde{c},t_0)$ and $F\in\mathcal{H}^2(c,t_0)$.

1.2 Applications

In this section, we give some applications of our concavity property.

1.2.1. Applications in optimal L^2 extension theorem

Let M be an n-dimensional complex manifold, and let S be an analytic subset of M. Let dV_M be a continuous volume form on M.

We consider a class of plurisubharmonic functions Φ from M to $[-\infty,+\infty)$, such that:

(1) $S\subset\Phi^{-1}(-\infty)$, and $\Phi^{-1}(-\infty)$ is a closed subset of some analytic subset of M satisfying that Φ has locally lower bound on $M\backslash\Phi^{-1}(-\infty)$.

(2) If S is l-dimensional around a point $x\in S_{reg}$, there is a local coordinate $(z_1,\dots,z_n)$ on a neighborhood U of x such that $z_{l+1}=\cdots=z_n=0$ on $S\cap U$ and

$$\sup_{U-S}|\Phi(z)-(n-l)\log\sum_{l+1}^{n}|z_j|^2|<+\infty.$$

The set of such polar functions Φ will be denoted by $A(S)$. We call Φ is in class $A'(S)$, if the condition (2) is replaced by (2)$'$:

(2)$'$ if S is l-dimensional around a point $x\in S_{reg}$, there is a local coordinate $(z_1,\dots,z_n)$ on a neighborhood U of x such that $z_{l+1}=\cdots=z_n=0$ on $S\cap U$ and $\Phi(z)-(n-l)\log\sum_{l+1}^{n}|z_j|^2$ is continuous on U.

Let $\psi\in A(S)$. Following [24] (see also [18]), one can define a positive measure $dV_M[\psi]$ on S_{reg} as the minimum element of the partially ordered set of positive measures $d\mu$ satisfying

$$\int_{S_l}f\,d\mu\geq\limsup_{t\to+\infty}\frac{2(n-l)}{\sigma_{2n-2l-1}}\int_M\mathbb{I}_{\{-t-1<\psi<-t\}}fe^{-\psi}dV_M$$

for any nonnegative continuous function f with $supp f\subset\subset M$. Here, denote by σ_m, the volume of the unit sphere in $\mathbb{R}^{m+1}$. If $\psi\in A'(S)$, then $dV_M[\psi]|_{S_l}$ is a continuous volume form on S_l and $dV_M[\psi+h]|_{S_l} = e^{-h}dV_M[\psi]|_{S_l}$ (see [18]), where h is a continuous function on M.

Let us recall a class of complex manifolds (see [18]). Let M be a complex manifold with the volume form dV_M, and let S be an analytic subset of M. We say (M,S) satisfies condition (ab) if there exists a closed subset $X \subset M$ satisfying the following statements:

(a) X is locally negligible with respect to L^2 holomorphic functions.
(b) $M\backslash X$ is a Stein manifold which intersects with every component of S, such that $S_{sing} \subset X$.

We give the following L^2 extension theorem with an optimal estimate. When $c(t)$ is continuous, the theorem can be referred to [18].

THEOREM 1.9. *Let (M,S) satisfy condition (ab). Let $\psi \in A(S)$ satisfying $\psi < -T$ on M. Let φ be a continuous function on M, such that $\varphi+\psi$ is plurisubharmonic on M. Let $c(t)$ be a positive function on $(T,+\infty)$ such that $c(t)e^{-t}$ is decreasing and $\int_T^{+\infty} c(t)e^{-t}dt < +\infty$. Then for any holomorphic section f of $K_M|_S$ on S, such that*

$$\sum_{k=1}^{n} \frac{\pi^k}{k!} \int_{S_{n-k}} \frac{|f|^2}{dV_M} e^{-\varphi} dV_M[\psi] < +\infty,$$

there exists a holomorphic $(n,0)$ form F on M such that $F|_S = f$ and

$$\int_M |F|^2 e^{-\varphi} c(-\psi) \leq \left(\int_T^{+\infty} c(t)e^{-t}dt\right) \sum_{k=1}^{n} \frac{\pi^k}{k!} \int_{S_{n-k}} \frac{|f|^2}{dV_M} e^{-\varphi} dV_M[\psi].$$

By the definition of $dV_M[\psi]$, we know $\frac{|f|^2}{dV_M}dV_M[\psi]$ is independent of the choice of dV_M (see [18]).

Denote that $\|f\|_S := \sum_{k=1}^{n} \frac{\pi^k}{k!} \int_{S_{n-k}} \frac{|f|^2}{dV_M} e^{-\varphi} dV_M[\psi]$ and $\|F\|_M := \int_M |F|^2 e^{-\varphi} c(-\psi)$. Let $\mathcal{F}|_{Z_0} = \mathcal{I}(\psi)|_{S_{reg}}$ and choose the f in the definition of $G(t)$ by any holomorphic extension of the f in Theorem 1.9. Then $G(T) = \inf\{\|F\|_M : F$ is a holomorphic extension of f from S to $M\}$, and Theorem 1.9 tells us that

$$G(T) \leq \left(\int_T^{+\infty} c(t)e^{-t}dt\right) \|f\|_S \tag{3}$$

(when $G(T) < +\infty$, Lemma 2.6 shows that there exists a holomorphic extension F of f such that $G(T) = \|F\|_M$).

Using Corollary 1.7 and Theorem 1.9, we obtain a necessary condition of inequality (3) becomes an equality.

THEOREM 1.10. *Let (M,S) satisfy condition (ab). Let $\psi \in A(S)$, and let $\psi < -T$. Let φ be a continuous function on M, such that $\varphi+\psi$ is plurisubharmonic on M. Let $c(t)$ be a positive function on $(T,+\infty)$ such that $c(t)e^{-t}$ is decreasing and $\int_T^{+\infty} c(t)e^{-t}dt < +\infty$. Let f be a holomorphic section of $K_M|_S$ on S, such that*

$$\sum_{k=1}^{n} \frac{\pi^k}{k!} \int_{S_{n-k}} \frac{|f|^2}{dV_M} e^{-\varphi} dV_M[\psi] < +\infty.$$

If $G(T) = \left(\int_T^{+\infty} c(t)e^{-t}dt\right) \|f\|_S$, then $G(\hat{h}^{-1}(r))$ is linear with respect to r and there exists a unique holomorphic $(n,0)$ form F on M such that $F|_S = f$ and $G(T) = \|F\|_M$.

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For any $t \geq T$, there exists a unique holomorphic $(n,0)$ form F_t on $\{\psi < -t\}$ such that $F_t|_S = f$ and

$$\int_{\{\psi<-t\}} |F_t|^2 e^{-\varphi} c(-\psi) \leq \left(\int_t^{+\infty} c(l)e^{-l}dl\right) \sum_{k=1}^{n} \frac{\pi^k}{k!} \int_{S_{n-k}} \frac{|f|^2}{dV_M} e^{-\varphi} dV_M[\psi].$$

In fact, $F_t = F$ on $\{\psi < -t\}$.

If $\mathcal{H}^2(\tilde{c},t) \subset \mathcal{H}^2(c,t)$ for some $t \geq T$, where $\tilde{c}$ is a nonnegative measurable function on $(T,+\infty)$, then there exists a unique holomorphic $(n,0)$ form F_t on $\{\psi < -t\}$ such that $F_t|_S = f$ and

$$\int_{\{\psi<-t\}} |F_t|^2 e^{-\varphi} \tilde{c}(-\psi) \leq \left(\int_t^{+\infty} \tilde{c}(l)e^{-l}dl\right) \sum_{k=1}^{n} \frac{\pi^k}{k!} \int_{S_{n-k}} \frac{|f|^2}{dV_M} e^{-\varphi} dV_M[\psi].$$

In fact, $F_t = F$ on $\{\psi < -t\}$.

When $c(t) \in \mathcal{P}'_T$ and M is a Stein manifold, Theorem 1.10 was obtained by Guan–Mi in [15].

Using Theorem 1.9, we obtain the following optimal L^2 extension theorem.

COROLLARY 1.11. *Let M be an n-dimensional Stein manifold, and let S be an analytic subset of M. Let $\psi_1 \in A(S)$, and let ψ_2 be a plurisubharmonic function on M such that $\psi = \psi_1 + \psi_2 < -T$ on M. Let φ be a Lebesgue measurable function on M such that $\varphi + \psi_2$ is plurisubharmonic on M. Let $c(t)$ be a positive function on $(T,+\infty)$, such that $c(t)e^{-t}$ is decreasing, $\int_T^{+\infty} c(t)e^{-t}dt < +\infty$ and $e^{-\varphi}c(-\psi)$ has locally a positive lower bound on $M \backslash Z$, where Z is some analytic subset of M. For any holomorphic section f of $K_M|_{S_{reg}}$ on S_{reg} satisfying*

$$\sum_{k=1}^{n} \frac{\pi^k}{k!} \int_{S_{n-k}} \frac{|f|^2}{dV_M} e^{-\varphi-\psi_2} dV_M[\psi_1] < +\infty,$$

there exists a holomorphic $(n,0)$ form F on M such that $F|_{S_{reg}} = f$ and

$$\int_M |F|^2 e^{-\varphi} c(-\psi) \leq \left(\int_T^{+\infty} c(t)e^{-t}dt\right) \sum_{k=1}^{n} \frac{\pi^k}{k!} \int_{S_{n-k}} \frac{|f|^2}{dV_M} e^{-\varphi-\psi_2} dV_M[\psi_1].$$

Especially, when $c \equiv 1$ and $\psi_1 = 2\log|w|$, where w is a holomorphic function on M, such that dw does not vanish identically on any branch of $w^{-1}(0)$ and $S_{reg} = \{z \in M : w(z) = 0 \,\&\, dw(z) \neq 0\}$, Corollary 1.11 can be referred to [16] (see also [18]).

Denote that $\|f\|_S^* := \sum_{k=1}^{n} \frac{\pi^k}{k!} \int_{S_{n-k}} \frac{|f|^2}{dV_M} e^{-\varphi-\psi_2} dV_M[\psi_1]$. Let $\mathcal{F}|_{Z_0} = \mathcal{I}(\psi_1)|_{S_{reg}}$ and choose the f in the definition of $G(t)$ by any holomorphic extension of the f in Corollary 1.11. Then $G(T) = \inf\{\|F\|_M : F$ is a holomorphic extension of f from S to $M\}$, and Corollary 1.11 tells us that

$$G(T) \leq \left(\int_T^{+\infty} c(t)e^{-t}dt\right) \|f\|_S^*. \tag{4}$$

Similarly to Theorem 1.10, we give a necessary condition of inequality (4) becomes an equality.

COROLLARY 1.12. *Let M be an n-dimensional Stein manifold, and let S be an analytic subset of M. Let $\psi_1 \in A(S)$, and let ψ_2 be a plurisubharmonic function on M such that*

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$\psi=\psi_1+\psi_2<-T$ on M. Let φ be a Lebesgue measurable function on M such that $\varphi+\psi_2$ is plurisubharmonic on M. Let $c(t)\in\mathcal{P}_T$ such that $\int_T^{+\infty}c(t)e^{-t}dt<+\infty$. Let f be a holomorphic section of $K_M|_{S_{reg}}$ on S_{reg} satisfying

$$\sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}}\frac{|f|^2}{dV_M}e^{-\varphi-\psi_2}dV_M[\psi_1]<+\infty.$$

If $G(T)=\left(\int_T^{+\infty}c(t)e^{-t}dt\right)\|f\|_S^*$, then $G(\hat{h}^{-1}(r))$ is linear with respect to r and there exists a unique holomorphic $(n,0)$ form F on M such that $F|_S=f$ and $G(T)=\|F\|_M$.

For any $t\geq T$, there exists a unique holomorphic $(n,0)$ form F_t on $\{\psi<-t\}$ such that $F_t|_S=f$ and

$$\int_{\{\psi<-t\}}|F_t|^2e^{-\varphi}c(-\psi)\leq\left(\int_t^{+\infty}c(l)e^{-l}dl\right)\sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}}\frac{|f|^2}{dV_M}e^{-\varphi-\psi_2}dV_M[\psi_1].$$

In fact, $F_t=F$ on $\{\psi<-t\}$.

If $\mathcal{H}^2(\tilde{c},t)\subset\mathcal{H}^2(c,t)$ for some $t\geq T$, where $\tilde{c}$ is a nonnegative measurable function on $(T,+\infty)$, then there exists a unique holomorphic $(n,0)$ form F_t on $\{\psi<-t\}$ such that $F_t|_S=f$ and

$$\int_{\{\psi<-t\}}|F_t|^2e^{-\varphi}\tilde{c}(-\psi)\leq\left(\int_t^{+\infty}\tilde{c}(l)e^{-l}dl\right)\sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}}\frac{|f|^2}{dV_M}e^{-\varphi-\psi_2}dV_M[\psi_1].$$

In fact, $F_t=F$ on $\{\psi<-t\}$.

1.2.2. Necessary conditions of $G(\hat{h}^{-1}(r))$ is linear

In this section, we give some necessary conditions of $G(\hat{h}^{-1}(r))$ is linear.

THEOREM 1.13. *Let M be an n-dimensional complex manifold satisfying condition (a). Let $c\in\mathcal{P}_T$, and assume that there exists $t\geq T$ such that $G(t)\in(0,+\infty)$. If there exists a Lebesgue measurable function $\tilde{\varphi}\geq\varphi$ such that $\tilde{\varphi}+\psi$ is plurisubharmonic function on M and satisfies that:*

(1) $\tilde{\varphi}\neq\varphi$;
(2) $\lim_{t\to T+0}\sup_{\{\psi\geq-t\}}(\tilde{\varphi}-\varphi)=0$;
(3) *$\tilde{\varphi}-\varphi$ is bounded on M.*

Then $G(\hat{h}^{-1}(r))$ is not linear with respect to $r\in(0,\int_T^{+\infty}c(s)e^{-s}ds)$. Especially, if $\varphi+\psi$ is strictly plurisubharmonic at $z_1\in M$, $G(\hat{h}^{-1}(r))$ is not linear with respect to $r\in(0,\int_T^{+\infty}c(s)e^{-s}ds)$.

In the following, we give two necessary conditions for ψ when $G(\hat{h}^{-1}(r))$ is linear.

THEOREM 1.14. *Let M be an n-dimensional complex manifold satisfying condition (a). Let $c\in\mathcal{P}_T$, and assume that $G(T)\in(0,+\infty)$. If there exists a plurisubharmonic function $\tilde{\psi}\geq\psi$ on M satisfying that:*

(1) *$\tilde{\psi}<-T$ on M;*
(2) $\tilde{\psi}\neq\psi$;
(3) $\lim_{t\to+\infty}\sup_{\{\psi<-t\}}(\tilde{\psi}-\psi)=0$.

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Then $G(\hat{h}^{-1}(r))$ is not linear with respect to $r \in (0, \int_T^{+\infty} c(s)e^{-s}ds)$. Especially, if ψ is strictly plurisubharmonic at $z_1 \in M \backslash (\cap_t \overline{\{\psi < -t\}})$, $G(\hat{h}^{-1}(r))$ is not linear with respect to $r \in (0, \int_T^{+\infty} c(s)e^{-s}ds)$.

Let M be an n-dimensional Stein manifold, and let S be an analytic subset of M. Let ψ be a plurisubharmonic function on M, and let φ be a Lebesgue measurable function on M such that $\varphi + \psi$ is plurisubharmonic on M.

We call (φ, ψ) in class W if there exist two plurisubharmonic functions $\psi_1 \in A'(S)$ and ψ_2, such that $\varphi + \psi_2$ is plurisubharmonic function on M and $\psi = \psi_1 + \psi_2$.

The following theorem gives a necessary condition of $G(\hat{h}^{-1}(r))$ is linear when $(\varphi, \psi) \in W$.

THEOREM 1.15. *Let $c \in P_T$, and let $(\varphi, \psi) \in W$. Let $\mathcal{F}|_{Z_0} = \mathcal{I}(\psi_1)|_{S_{reg}}$. Assume that $G(T) \in (0, +\infty)$ and $\psi_2(z) > -\infty$ for almost every $z \in S_{reg}$. If $G(\hat{h}^{-1}(r))$ is linear with respect to $r \in (0, \int_T^{+\infty} c(s)e^{-s}ds)$, then we have*

$$\frac{G(T)}{\int_T^{+\infty} c(t)e^{-t}dt} = \sum_{k=1}^{n} \frac{\pi^k}{k!} \int_{S_{n-k}} \frac{|f|^2}{dV_M} e^{-\varphi - \psi_2} dV_M[\psi_1], \tag{5}$$

and there is no plurisubharmonic function $\tilde{\psi} \geq \psi$ on M satisfying that:

(1) $\tilde{\psi} < -T$;
(2) $\tilde{\psi} \neq \psi$;
(3) $(\varphi + \psi - \tilde{\psi}, \tilde{\psi}) \in W$.

1.2.3. Characterizations for the linearity of $G(\hat{h}(r))$ in 1-dimensional case

In this section, we consider the 1-dimensional case. Let $M = \Omega$ be an open Riemann surface admitted a nontrivial Green function G_Ω, we give characterizations of the linearity.

We recall some notations (see [18]). Let $p : \Delta \to \Omega$ be the universal covering from unit disk Δ to Ω. we call the holomorphic function f (resp. holomorphic (1,0) form F) on Δ a multiplicative function (resp. multiplicative differential (Prym differential)), if there is a character χ, which is the representation of the fundamental group of Ω, such that $g^* f = \chi(g) f$ (resp. $g^*(F) = \chi(g)F$), where $|\chi| = 1$ and g is an element of the fundamental group of Ω. Denote the set of such kinds of f (resp. F) by $\mathcal{O}^\chi(\Omega)$ (resp. $\Gamma^\chi(\Omega)$).

For any harmonic function u on Ω, there exists a χ_u and a multiplicative function $f_u \in \mathcal{O}^{\chi_u}(\Omega)$, such that $|f_u| = p^*(e^u)$. If $u_1 - u_2 = \log|f|$, where u_1 and u_2 are harmonic function on Ω and f is holomorphic function on Ω, then $\chi_{u_1} = \chi_{u_2}$.

For the Green function $G_\Omega(z, z_0)$, one can also find a χ_{z_0} and a multiplicative function $f_{z_0} \in \mathcal{O}^{\chi_{z_0}}(\Omega)$, such that $|f_{z_0}| = p^*\left(e^{G_\Omega(z,z_0)}\right)$.

Let $M = \Omega$ be an open Riemann surface admitted a nontrivial Green function G_Ω. Let ψ be a subharmonic function on Ω satisfying $T = -\sup_\Omega \psi = 0$, and let φ be a Lebesgue measurable function on Ω, such that $\varphi + \psi$ is subharmonic on Ω. Let $Z_0 = z_0$ be a point in Ω.

Let w be a local coordinate on a neighborhood V_{z_0} of $z_0 \in \Omega$ satisfying $w(z_0) = 0$. Set $f = f_1(w)dw$ on V_{z_0}, where f is the holomorphic (1,0) form in the definition of $G(t)$ (see §1.1) and f_1 is a holomorphic function on V_{z_0}.

The following two theorems give characterizations of $G(\hat{h}^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(l)e^{-l}dl)$ for some kinds of (φ, ψ). Set $d^c = \frac{1}{2\pi i}(\partial - \bar{\partial})$.

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THEOREM 1.16. *Let $c \in \mathcal{P}_0$. Assume that $\varphi + a\psi$ is a subharmonic function on a neighborhood of z_0 for some $a \in [0,1)$, and $G(0) \in (0,+\infty)$. Then $G(\hat{h}^{-1}(r))$ is linear with respect to r if and only if the following statements hold:*

(1) $\varphi+\psi = 2\log|g| + 2G_\Omega(z,z_0) + 2u$, $ord_{z_0}(g) = ord_{z_0}(f_1)$, *and* $\mathcal{F}_{z_0} = \mathcal{I}(\varphi+\psi)_{z_0}$, *where g is a holomorphic function on Ω and u is a harmonic function on Ω.*
(2) $\psi = 2pG_\Omega(z,z_0)$ *on Ω for some $p > 0$.*
(3) $\chi_{-u} = \chi_{z_0}$, *where χ_{-u} and χ_{z_0} are the characters associated with the functions $-u$ and $G_\Omega(z,z_0)$, respectively.*

When $\psi = 2G_\Omega(z,z_0)$, $\mathcal{F}_{z_0} = \mathcal{I}(\varphi+\psi)_{z_0}$ and $c(t) \in \mathcal{P}_0'$, Theorem 1.16 can be referred to [15].

THEOREM 1.17. *Let $c \in \mathcal{P}_0$, and let $Z_0 = z_0$ be a point in Ω. Assume that $(\psi - 2pG_\Omega(z,z_0))(z_0) > -\infty$, where $p = \frac{1}{2}v(dd^c\psi, z_0)$, and $G(0) \in (0,+\infty)$. Then $G(\hat{h}^{-1}(r))$ is linear with respect to r if and only if the following statements hold:*

(1) $\varphi+\psi = 2\log|g| + 2G_\Omega(z,z_0) + 2u$, $ord_{z_0}(g) = ord_{z_0}(f_1)$, *and* $\mathcal{F}_{z_0} = \mathcal{I}(\varphi+\psi)_{z_0}$, *where g is a holomorphic function on Ω and u is a harmonic function on Ω.*
(2) $p > 0$ *and* $\psi = 2pG_\Omega(z,z_0)$ *on Ω.*
(3) $\chi_{-u} = \chi_{z_0}$, *where χ_{-u} and χ_{z_0} are the characters associated with the functions $-u$ and $G_\Omega(z,z_0)$, respectively.*

1.2.4. Characterizations for the holding of the equality in optimal L^2 extension problem on open Riemann surfaces with weights may not be subharmonic

Let $M = \Omega$ be an open Riemann surface admitted a nontrivial Green function G_Ω. Let ψ be a subharmonic function on Ω satisfying $T = -\sup_\Omega \psi = 0$, and let φ be a Lebesgue measurable function on Ω, such that $\varphi+\psi$ is subharmonic on Ω. Let $Z_0 = z_0$ be a point in Ω.

Let w be a local coordinate on a neighborhood V_{z_0} of $z_0 \in \Omega$ satisfying $w(z_0) = 0$. Let $f \equiv dw$ be a holomorphic $(1,0)$ form on V_{z_0}. Following the notations in Section 1.2.1. Now, we give characterizations for the holding of the equality in optimal L^2 extension problem on open Riemann surfaces with weights may not be subharmonic.

COROLLARY 1.18. *Let $M = \Omega$, $S = z_0$, and $T = 0$. Let $\varphi(z_0) > -\infty$. Assume that $\psi \in A(z_0)$, $e^{-\varphi-\psi}$ is not L^1 on any neighborhood of z_0 and $c(t) \in \mathcal{P}_0$ satisfying $\int_0^{+\infty} c(t)e^{-t}dt < +\infty$.*

Then there exists a holomorphic $(1,0)$ form F on Ω such that $F(z_0) = f(z_0)$ and

$$\int_\Omega |F|^2 e^{-\varphi} c(-\psi) \leq \left(\int_0^{+\infty} c(t)e^{-t}dt\right) \|f\|_{z_0}. \tag{6}$$

Moreover, equality $\left(\int_0^{+\infty} c(t)e^{-t}dt\right)\|f\|_{z_0} = \inf\{\|\tilde{F}\|_\Omega$: $\tilde{F}$ is a holomorphic extension of f from z_0 to $\Omega\}$ holds if and only if the following statements hold:

(1) $\varphi = 2\log|g| + 2u$, *where u is a harmonic function on Ω and g is a holomorphic function on Ω such that $g(z_0) \neq 0$.*
(2) $\psi = 2G_\Omega(z,z_0)$ *on Ω.*
(3) $\chi_{-u} = \chi_{z_0}$, *where χ_{-u} and χ_{z_0} are the characters associated with the functions $-u$ and $G_\Omega(z,z_0)$, respectively.*

When $\psi = 2G_\Omega(z,z_0)$ and $c(t) \in \mathcal{P}_0'$, Corollary 1.18 can be referred to [15].

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COROLLARY 1.19. *Let $M=\Omega$, $S=z_0$, and $T=0$. Let $(\varphi,\psi)\in W$, and let $\|f\|^*_{z_0}\in(0,+\infty)$. Let $c(t)\in\mathcal{P}_0$ such that $\int_0^{+\infty}c(t)e^{-t}dt<+\infty$. Then equality $\left(\int_0^{+\infty}c(t)e^{-t}dt\right)\|f\|^*_{z_0}=\inf\{\|F\|_\Omega : F$ is a holomorphic extension of f from z_0 to $\Omega\}$ holds if and only if the following statements hold:*

(1) *$\varphi=2\log|g|+2u$, where u is a harmonic function on Ω and g is a holomorphic function on Ω such that $g(z_0)\neq 0$.*
(2) *$\psi=2G_\Omega(z,z_0)$ on Ω.*
(3) *$\chi_{-u}=\chi_{z_0}$, where χ_{-u} and χ_{z_0} are the characters associated with the functions $-u$ and $G_\Omega(z,z_0)$, respectively.*

§2. Preparation

2.1 L^2 methods

We call a positive measurable function c on $(S,+\infty)$ in class $\tilde{\mathcal{P}}_S$ if $\int_S^s c(l)e^{-l}dl<+\infty$ for some $s>S$ and $c(t)e^{-t}$ is decreasing with respect to t. Note that $\mathcal{P}_T\subset\tilde{\mathcal{P}}_S$ when $S>T$.

In this section, we present the following lemma (proof can be referred to §7.1), whose various forms already appear in [14], [15], [17], [18] etc.:

LEMMA 2.1. *Let $B\in(0,+\infty)$ and $t_0\geq S$ be arbitrarily given. Let M be an n-dimensional Stein manifold. Let $\psi<-S$ be a plurisubharmonic function on M. Let φ be a plurisubharmonic function on M. Let F be a holomorphic $(n,0)$ form on $\{\psi<-t_0\}$, such that*

$$\int_{K\cap\{\psi<-t_0\}}|F|^2<+\infty \tag{7}$$

for any compact subset K of M, and

$$\int_M \frac{1}{B}\mathbb{I}_{\{-t_0-B<\psi<-t_0\}}|F|^2e^{-\varphi}\leq C<+\infty. \tag{8}$$

Then there exists a holomorphic $(n,0)$ form $\tilde{F}$ on M, such that

$$\int_M|\tilde{F}-(1-b_{t_0,B}(\psi))F|^2e^{-\varphi+v_{t_0,B}(\psi)}c(-v_{t_0,B}(\psi))\leq C\int_S^{t_0+B}c(t)e^{-t}dt, \tag{9}$$

where $b_{t_0,B}(t)=\int_{-\infty}^t\frac{1}{B}\mathbb{I}_{\{-t_0-B<s<-t_0\}}ds$, $v_{t_0,B}(t)=\int_{-t_0}^t b_{t_0,B}(s)ds-t_0$, and $c(t)\in\tilde{\mathcal{P}}_S$.

We give the proof of Lemma 2.1 in Section 7.1. It is clear that $\mathbb{I}_{(-t_0,+\infty)}(t)\leq b_{t_0,B}(t)\leq\mathbb{I}_{(-t_0-B,+\infty)}(t)$ and $\max\{t,-t_0-B\}\leq v_{t_0,B}(t)\leq\max\{t,-t_0\}$.

Lemma 2.1 implies the following lemma, which will be used in the proof of Theorem 1.3.

LEMMA 2.2. *Let M be an n-dimensional complex manifold satisfying condition (a), and let $c(t)\in\mathcal{P}_T$. Let $B\in(0,+\infty)$ and $t_0>t_1>T$ be arbitrarily given. Let $\psi<-T$ be a plurisubharmonic function on M. Let φ be a Lebesgue measurable function on M, such that $\varphi+\psi$ is plurisubharmonic on M. Let F be a holomorphic $(n,0)$ form on $\{\psi<-t_0\}$, such that*

$$\int_{\{\psi<-t_0\}}|F|^2e^{-\varphi}c(-\psi)<+\infty. \tag{10}$$

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Then there exists a holomorphic $(n,0)$ form $\tilde{F}$ on $\{\psi<-t_1\}$, such that

$$\int_{\{\psi<-t_1\}}|\tilde{F}-(1-b_{t_0,B}(\psi))F|^2e^{-\varphi-\psi+v_{t_0,B}(\psi)}c(-v_{t_0,B}(\psi))\leq C\int_{t_1}^{t_0+B}c(t)e^{-t}dt, \tag{11}$$

where $C=\int_M \frac{1}{B}\mathbb{I}_{\{-t_0-B<\psi<-t_0\}}|F|^2e^{-\varphi-\psi}<+\infty$, $b_{t_0,B}(t)=\int_{-\infty}^{t}\frac{1}{B}\mathbb{I}_{\{-t_0-B<s<-t_0\}}ds$, *and* $v_{t_0,B}(t)=\int_{-t_0}^{t}b_{t_0,B}(s)ds-t_0$.

Proof. As M is an n-dimensional complex manifold satisfying condition (a) and $c(t)\in\mathcal{P}_T$, there exist a closed subset $X\subset M$ and a closed subset $E\subset X\cap\{\psi=-\infty\}$ satisfying that X is locally negligible with respect to L^2 holomorphic functions, $M\backslash X$ is a Stein manifold, $e^{-\varphi}c(-\psi)$ has locally a positive lower bound on $M\backslash E$ and there exists an analytic subset Z of M such that $E\subset Z$.

Combining inequality (10) and $e^{-\varphi}c(-\psi)$ has locally a positive lower bound on $M\backslash E$, we obtain that

$$\int_{K\cap\{\psi<-t_0\}}|F|^2<+\infty$$

holds for any compact subset K of $M\backslash X$. Then Lemma 2.1 shows that there exists a holomorphic $(n,0)$ form $\tilde{F}_X$ on $\{\psi<-t_1\}\backslash X$, such that

$$\int_{\{\psi<-t_1\}\backslash X}|\tilde{F}_X-(1-b_{t_0,B}(\psi))F|^2e^{-\varphi-\psi+v_{t_0,B}(\psi)}c(-v_{t_0,B}(\psi))\leq C\int_{t_1}^{t_0+B}c(t)e^{-t}dt. \tag{12}$$

For any $z\in\{\psi<-t_1\}\cap(X\backslash E)$, there exists an open neighborhood V_z of z, such that $V_z\subset\subset\{\psi<-t_1\}\backslash E$. Note that $c(t)e^{-t}$ is decreasing on $(T,+\infty)$ and $v_{t_0,B}(\psi)\geq\psi$, then we have

$$\begin{aligned}&\int_{V_z\backslash X}|\tilde{F}_X-(1-b_{t_0,B}(\psi))F|^2e^{-\varphi}c(-\psi)\\ \leq&\int_{V_z\backslash X}|\tilde{F}_X-(1-b_{t_0,B}(\psi))F|^2e^{-\varphi-\psi+v_{t_0,B}(\psi)}c(-v_{t_0,B}(\psi))\\ <&+\infty.\end{aligned} \tag{13}$$

Note that there exists a positive number $C_1>0$ such that $e^{-\varphi}c(-\psi)>C_1$ on V_z and $\int_{V_z\backslash X}|(1-b_{t_0,B}(\psi))F|^2e^{-\varphi}c(-\psi)\leq\int_{\{\psi<-t_0\}}|F|^2e^{-\varphi}c(-\psi)<+\infty$, then we have

$$\begin{aligned}&\int_{V_z\backslash X}|\tilde{F}_X|^2\\ \leq&C_1\int_{V_z\backslash X}|\tilde{F}_X|^2e^{-\varphi}c(-\psi)\\ \leq&2C_1\left(\int_{V_z\backslash X}|(1-b_{t_0,B}(\psi))F|^2e^{-\varphi}c(-\psi)+\int_{V_z\backslash X}|\tilde{F}_X-(1-b_{t_0,B}(\psi))F|^2e^{-\varphi}c(-\psi)\right)\\ <&+\infty.\end{aligned} \tag{14}$$

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As X is locally negligible with respect to L^2 holomorphic functions, we can find a holomorphic extension $\tilde{F}_E$ of $\tilde{F}_X$ from $\{\psi < -t_1\}\backslash X$ to $\{\psi < -t_1\}\backslash E$ such that

$$\int_{\{\psi<-t_1\}\backslash E} |\tilde{F}_E - (1-b_{t_0,B}(\psi))F|^2 e^{-\varphi-\psi+v_{t_0,B}(\psi)} c(-v_{t_0,B}(\psi)) \le C\int_{t_1}^{t_0+B} c(t)e^{-t}dt. \quad (15)$$

Note that $E \subset \{\psi < -t_0\} \subset \{\psi < -t_1\}$, for any $z \in E$, there exists an open neighborhood U_z of z, such that $U_z \subset\subset \{\psi < -t_0\}$. As $\varphi + \psi$ is plurisubharmonic on M and $e^{v_{t_0,B}(\psi)}c(-v_{t_0,B}(\psi))$ has a positive lower bound on $\{\psi < -t_1\}$, then we have

$$\begin{aligned} &\int_{U_z\backslash E} |\tilde{F}_E - (1-b_{t_0,B}(\psi))F|^2 \\ \le & C_2\int_{\{\psi<-t_1\}\backslash E} |\tilde{F}_E - (1-b_{t_0,B}(\psi))F|^2 e^{-\varphi-\psi+v_{t_0,B}(\psi)} c(-v_{t_0,B}(\psi)) \\ < & +\infty, \end{aligned} \quad (16)$$

where C_2 is some positive number. As $U_z \subset\subset \{\psi < -t_0\}$, we have

$$\int_{U_z} |(1-b_{t_0,B}(\psi))F|^2 \le \int_{U_z} |F|^2 < +\infty. \quad (17)$$

Combining inequality (16) and (17), we obtain that $\int_{U_z\backslash E} |\tilde{F}_E|^2 < +\infty$.

As E is contained in some analytic subset of M, we can find a holomorphic extension $\tilde{F}$ of $\tilde{F}_E$ from $\{\psi < -t_1\}\backslash E$ to $\{\psi < -t_1\}$ such that

$$\int_{\{\psi<-t_1\}} |\tilde{F} - (1-b_{t_0,B}(\psi))F|^2 e^{-\varphi-\psi+v_{t_0,B}(\psi)} c(-v_{t_0,B}(\psi)) \le C\int_{t_1}^{t_0+B} c(t)e^{-t}dt. \quad (18)$$

This proves Lemma 2.2. ∎

2.2 Some properties of $G(t)$

We present some properties related to $G(t)$ in this section.

LEMMA 2.3 (See [12]). *Let N be a submodule of $\mathcal{O}^q_{\mathbb{C}^n,o}$, $1 \le q < +\infty$, and let $f_j \in \mathcal{O}_{\mathbb{C}^n}(U)^q$ be a sequence of $q-$tuples holomorphic in an open neighborhood U of the origin o. Assume that the f_j converge uniformly in U toward a $q-$tuples $f \in \mathcal{O}_{\mathbb{C}^n}(U)^q$, assume furthermore that all germs (f_j, o) belong to N. Then $(f, o) \in N$.*

The closedness of submodules will be used in the following discussion.

LEMMA 2.4. *Let M be a complex manifold. Let S be an analytic subset of M. Let $\{g_j\}_{j=1,2,\ldots}$ be a sequence of nonnegative Lebesgue measurable functions on M, which satisfies that g_j are almost everywhere convergent to g on M when $j \to +\infty$, where g is a nonnegative Lebesgue measurable function on M. Assume that for any compact subset K of $M\backslash S$, there exist $s_K \in (0,+\infty)$ and $C_K \in (0,+\infty)$ such that*

$$\int_K {g_j}^{-s_K} dV_M \le C_K$$

for any j, where dV_M is a continuous volume form on M.

Let $\{F_j\}_{j=1,2,\ldots}$ be a sequence of holomorphic $(n,0)$ form on M. Assume that there exists a positive constant C such that $\liminf_{j\to+\infty}\int_M |F_j|^2 g_j \le C$. Then there exists a subsequence

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$\{F_{j_l}\}_{l=1,2,\ldots}$, which satisfies that $\{F_{j_l}\}$ is uniformly convergent to a holomorphic $(n,0)$ form F on M on any compact subset of M when $l\to+\infty$, such that

$$\int_M |F|^2 g \le C.$$

Proof. As S is a analytic subset of M, by Local Parameterization Theorem (see [5]) and Maximum Principle, for any compact set $K\subset\subset M$, there exists $K_1\subset\subset M\backslash S$ satisfying

$$\sup_{z\in K}\frac{|F_j(z)|^2}{dV_M}\le C_1\sup_{z\in K_1}\frac{|F_j(z)|^2}{dV_M} \tag{19}$$

for any j, where C_1 is a constant depending on K but independent of j. Then there exists a compact set $K_2\subset\subset M\backslash S$ satisfying $K_1\subset K_2$ and

$$\begin{aligned}\left(\frac{|F_j(z)|^2}{dV_M}\right)^r &\le C_2\int_{K_2}\left(\frac{|F_j(z)|^2}{dV_M}\right)^r\\ &\le C_2\left(\int_{K_2}|F_j|^2 g_j\right)^r\left(\int_{K_2} g_j^{-\frac{r}{1-r}}\right)^{1-r}\end{aligned} \tag{20}$$

for any j and any $z\in K_1$, where $r\in(0,1)$ and C_2 is a constant. Let $\frac{r}{1-r}=s_{K_2}$, inequality (20) implies

$$\sup_{z\in K_1}\frac{|F_j(z)|^2}{dV_M}\le C_3\int_{K_2}|F_j|^2 g_j, \tag{21}$$

where C_3 is a constant. As $\liminf_{j\to+\infty}\int_M|F_j|^2g_j<C$, combining inequality (19), (21), and the diagonal method, we obtain a subsequence of $\{F_j\}$, denoted still by $\{F_j\}$, which is uniformly convergent to a holomorphic $(n,0)$ form F on M on any compact subset of M.

It follows from the Fatou's Lemma and $\lim_{j\to+\infty}\int_M|F_j|^2g_j\le C$ that

$$\begin{aligned}\int_M|F|^2g &= \int_M\lim_{j\to+\infty}|F_j|^2g_j\\ &\le \liminf_{j\to+\infty}\int_M|F_j|^2g_j\\ &\le C.\end{aligned}$$

Thus Lemma 2.4 holds. ∎

Let M be an n-dimensional complex manifold satisfying condition (a). Let ψ be a plurisubharmonic function on M, and let φ be a Lebesgue measurable function on M, such that $\varphi+\psi$ is a plurisubharmonic function on M. Let $c\in\mathcal{P}_T$. The following lemma is a characterization of $G(t)=0$ for any $t\ge T$, where $T=-\sup_M\psi$ and the meaning of $G(t)$ can be referred to Section 1.1.

LEMMA 2.5. *$f\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})\Leftrightarrow G(t)=0$.*

Proof. It is clear that $f\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})\Rightarrow G(t)=0$.

In the following part, we prove that $G(t)=0\Rightarrow f\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$. As $G(t)=0$, then there exists holomorphic $(n,0)$ forms $\{\tilde f_j\}_{j\in\mathbb{N}^+}$ on $\{\psi<-t\}$ such that $\lim_{j\to+\infty}\int_{\{\psi<-t\}}|\tilde f_j|^2e^{-\varphi}c(-\psi)=0$ and $(f_j-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$ for any j. As $e^{-\varphi}c(-\psi)$ has a positive lower bound on any compact subset of $M\backslash Z$, where Z is

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some analytic subset of M, it follows from Lemma 2.4 that there exists a subsequence of $\{\tilde{f}_j\}_{j\in\mathbb{N}^+}$ denoted by $\{\tilde{f}_{j_k}\}_{k\in\mathbb{N}^+}$ that compactly convergent to 0. It is clear that $\tilde{f}_{j_k}-f$ is compactly convergent to $0-f=-f$ on $U\cap\{\psi<-t\}$. It follows from Lemma 2.3 that $f\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$. This proves Lemma 2.5. ∎

The following lemma shows the existence and uniqueness of the holomorphic $(n,0)$ form related to $G(t)$.

LEMMA 2.6. *Assume that $G(t)<+\infty$ for some $t\in[T,+\infty)$. Then there exists a unique holomorphic $(n,0)$ form F_t on $\{\psi<-t\}$ satisfying $(F_t-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$ and $\int_{\{\psi<-t\}}|F_t|^2e^{-\varphi}c(-\psi)=G(t)$. Furthermore, for any holomorphic $(n,0)$ form $\hat{F}$ on $\{\psi<-t\}$ satisfying $(\hat{F}-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$ and $\int_{\{\psi<-t\}}|\hat{F}|^2e^{-\varphi}c(-\psi)<+\infty$, we have the following equality:*

$$\begin{aligned}&\int_{\{\psi<-t\}}|F_t|^2e^{-\varphi}c(-\psi)+\int_{\{\psi<-t\}}|\hat{F}-F_t|^2e^{-\varphi}c(-\psi)\\ =&\int_{\{\psi<-t\}}|\hat{F}|^2e^{-\varphi}c(-\psi).\end{aligned}\tag{22}$$

Proof. Firstly, we prove the existence of F_t. As $G(t)<+\infty$ then there exists holomorphic $(n,0)$ forms $\{f_j\}_{j\in\mathbb{N}^+}$ on $\{\psi<-t\}$ such that $\lim_{j\to+\infty}\int_{\{\psi<-t\}}|f_j|^2e^{-\varphi}c(-\psi)=G(t)$, and $(f_j-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$. As $e^{-\varphi}c(-\psi)$ has a positive lower bound on any compact subset of $M\backslash Z$, where Z is some analytic subset of M, it follows from Lemma 2.4 that there exists a subsequence of $\{f_j\}$ compact convergence to a holomorphic $(n,0)$ form F on $\{\psi<-t\}$ satisfying $\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)\le G(t)$. It follows from Lemma 2.3 that $(F-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$. Then we obtain the existence of $F_t(=F)$.

Secondly, we prove the uniqueness of F_t by contradiction: if not, there exist two different holomorphic $(n,0)$ forms f_1 and f_2 on on $\{\psi<-t\}$ satisfying $\int_{\{\psi<-t\}}|f_1|^2e^{-\varphi}c(-\psi)=\int_{\{\psi<-t\}}|f_2|^2=G(t)$, $(f_1-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$ and $(f_2-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$. Note that

$$\begin{aligned}&\int_{\{\psi<-t\}}\left|\frac{f_1+f_2}{2}\right|^2e^{-\varphi}c(-\psi)+\int_{\{\psi<-t\}}\left|\frac{f_1-f_2}{2}\right|^2e^{-\varphi}c(-\psi)\\ =&\frac{\int_{\{\psi<-t\}}|f_1|^2e^{-\varphi}c(-\psi)+\int_{\{\psi<-t\}}|f_2|^2e^{-\varphi}c(-\psi)}{2}=G(t),\end{aligned}\tag{23}$$

then we obtain that

$$\int_{\{\psi<-t\}}\left|\frac{f_1+f_2}{2}\right|^2e^{-\varphi}c(-\psi)<G(t),$$

and $(\frac{f_1+f_2}{2}-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$, which contradicts the definition of $G(t)$.

Finally, we prove equality (22). For any holomorphic h on $\{\psi<-t\}$ satisfying $\int_{\{\psi<-t\}}|h|^2e^{-\varphi}c(-\psi)<+\infty$ and $h\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$, it is clear that for any complex number α, $F_t+\alpha h$ satisfying $((F_t+\alpha h)-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$, and $\int_{\{\psi<-t\}}|F_t|^2e^{-\varphi}c(-\psi)\le\int_{\{\psi<-t\}}|F_t+\alpha h|^2e^{-\varphi}c(-\psi)<+\infty$. Note that

$$\int_{\{\psi<-t\}}|F_t+\alpha h|^2e^{-\varphi}c(-\psi)-\int_{\{\psi<-t\}}|F_t|^2e^{-\varphi}c(-\psi)\ge 0$$

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implies

$$\Re\int_{\{\psi<-t\}}F_t\bar{h}e^{-\varphi}c(-\psi)=0,$$

then

$$\int_{\{\psi<-t\}}|F_t+h|^2e^{-\varphi}c(-\psi)=\int_{\{\psi<-t\}}(|F_t|^2+|h|^2)e^{-\varphi}c(-\psi).$$

Choosing $h=\hat{F}-F_t$, we obtain equality (22). ∎

The following lemma shows the monotonicity and lower semicontinuity property of $G(t)$.

LEMMA 2.7. *$G(t)$ is decreasing with respect to $t\in[T,+\infty)$, such that $\lim_{t\to t_0+0}G(t)=G(t_0)$ for any $t_0\in[T,+\infty)$, and if $G(t)<+\infty$ for some $t\geq T$, then $\lim_{t\to+\infty}G(t)=0$. Especially $G(t)$ is lower semicontinuous on $[T,+\infty)$.*

Proof. By the definition of $G(t)$, it is clear that $G(t)$ is decreasing on $[T,+\infty)$. And it follows from the dominated convergence theorem that if $G(t)<+\infty$ for some $t\geq T$, then $\lim_{t\to+\infty}G(t)=0$. Then it suffices to prove $\lim_{t\to t_0+0}G(t)=G(t_0)$. We prove it by contradiction: if not, then $\lim_{t\to t_0+0}G(t)<G(t_0)$.

By Lemma 2.6, there exists a unique holomorphic $(n,0)$ form F_t on $\{\psi<-t\}$ satisfying $(F_t-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$ and $\int_{\{\psi<-t\}}|F_t|^2e^{-\varphi}c(-\psi)=G(t)$. Note that $G(t)$ is decreasing implies that $\int_{\{\psi<-t\}}|F_t|^2e^{-\varphi}c(-\psi)\leq\lim_{t\to t_0+0}G(t)$ for any $t>t_0$. If $\lim_{t\to t_0+0}G(t)=+\infty$, the equality $\lim_{t\to t_0+0}G(t)=G(t_0)$ is clear, thus it suffices to prove the case $\lim_{t\to t_0+0}G(t)<+\infty$. As $e^{-\varphi}c(-\psi)$ has a positive lower bound on any compact subset of $M\backslash Z$, where Z is some analytic subset of M, and $\int_{\{\psi<-t_1\}}|F_t|^2e^{-\varphi}c(-\psi)\leq\lim_{t\to t_0+0}G(t)<+\infty$ holds for any $t\in(t_0,t_1]$, where $t_1>t_0$ is a fixed number, it follows from Lemma 2.4 that there exists $\{F_{t_j}\}$ ($t_j\to t_0+0$, as $j\to+\infty$) uniformly convergent on any compact subset of $\{\psi<-t_1\}$. Using the diagonal method, we obtain a subsequence of $\{F_t\}$ (also denoted by $\{F_{t_j}\}$), which is convergent on any compact subset of $\{\psi<-t_0\}$.

Let $\hat{F}_{t_0}=\lim_{j\to+\infty}F_{t_j}$, which is a holomorphic $(n,0)$ form on $\{\psi<-t_0\}$. Then it follows from the decreasing property of $G(t)$ that

$$\int_K|\hat{F}_{t_0}|^2e^{-\varphi}c(-\psi)\leq\lim_{j\to+\infty}\int_K|F_{t_j}|^2e^{-\varphi}c(-\psi)\leq\lim_{j\to+\infty}G(t_j)\leq\lim_{t\to t_0+0}G(t)$$

for any compact set $K\subset\{\psi<-t_0\}$. It follows from Levi's Theorem that

$$\int_{\{\psi<-t_0\}}|\hat{F}_{t_0}|^2e^{-\varphi}c(-\psi)\leq\lim_{t\to t_0+0}G(t).$$

It follows from Lemma 2.3 that $\hat{F}_{t_0}\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$. Then we obtain that $G(t_0)\leq\int_{\{\psi<-t_0\}}|\hat{F}_{t_0}|^2e^{-\varphi}c(-\psi)\leq\lim_{t\to t_0+0}G(t)$, which contradicts $\lim_{t\to t_0+0}G(t)<G(t_0)$. ∎

We consider the derivatives of $G(t)$ in the following lemma.

LEMMA 2.8. *Assume that $G(t_1)<\infty$, where $t_1\in(T,+\infty)$, then for any $t_0>t_1$, we have*

$$\frac{G(t_1)-G(t_0)}{\int_{t_1}^{t_0}c(t)e^{-t}dt}\leq\liminf_{B\to0+0}\frac{G(t_0)-G(t_0+B)}{\int_{t_0}^{t_0+B}c(t)e^{-t}dt},$$

that is,

$$\frac{G(t_0)-G(t_1)}{\int_{T_1}^{t_0}c(t)e^{-t}dt-\int_{T_1}^{t_1}c(t)e^{-t}dt}\geq\limsup_{B\to 0+0}\frac{G(t_0+B)-G(t_0)}{\int_{T_1}^{t_0+B}c(t)e^{-t}dt-\int_{T_1}^{t_0+B}c(t)e^{-t}dt}.$$

Proof. It follows from Lemma 2.7 that $G(t)<+\infty$ for any $t\geq t_1$. By Lemma 2.6, there exists a holomorphic $(n,0)$ form F_{t_0} on $\{\psi<-t_0\}$, such that $(F_{t_0}-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$ and $\int_{\{\psi<-t_0\}}|F_{t_0}|^2e^{-\varphi}c(-\psi)=G(t_0)$.

It suffices to consider that $\liminf_{B\to 0+0}\frac{G(t_0)-G(t_0+B)}{\int_{t_0}^{t_0+B}c(t)e^{-t}dt}\in[0.+\infty)$ because of the decreasing property of $G(t)$. Then there exists $B_j\to 0+0$ $(j\to+\infty)$ such that

$$\lim_{j\to+\infty}\frac{G(t_0)-G(t_0+B_j)}{\int_{t_0}^{t_0+B_j}c(t)e^{-t}dt}=\liminf_{B\to 0+0}\frac{G(t_0)-G(t_0+B)}{\int_{t_0}^{t_0+B}c(t)e^{-t}dt}\tag{24}$$

and $\{\frac{G(t_0)-G(t_0+B_j)}{\int_{t_0}^{t_0+B_j}c(t)e^{-t}dt}\}_{j\in\mathbb{N}^+}$ is bounded. As $c(t)e^{-t}$ is decreasing and positive on $(T,+\infty)$, then

$$\begin{aligned}\lim_{j\to+\infty}\frac{G(t_0)-G(t_0+B_j)}{\int_{t_0}^{t_0+B_j}c(t)e^{-t}dt}&=\left(\lim_{j\to+\infty}\frac{G(t_0)-G(t_0+B_j)}{B_j}\right)\left(\frac{1}{\lim_{t\to t_0+0}c(t)e^{-t}}\right)\\&=\left(\lim_{j\to+\infty}\frac{G(t_0)-G(t_0+B_j)}{B_j}\right)\left(\frac{e^{t_0}}{\lim_{t\to t_0+0}c(t)}\right).\end{aligned}\tag{25}$$

Hence, $\left\{\frac{G(t_0)-G(t_0+B_j)}{B_j}\right\}_{j\in\mathbb{N}^+}$ is bounded with respect to j.

As $t\leq v_{t_0,B_j}(t)$, the decreasing property of $c(t)e^{-t}$ shows that

$$e^{-\psi+v_{t_0,B_j}(\psi)}c(-v_{t_0,B_j}(\psi))\geq c(-\psi).$$

Lemma 2.2 shows that for any B_j, there exists holomorphic $(n,0)$ form $\tilde{F}_j$ on $\{\psi<-t_1\}$, such that $(\tilde{F}_j-F_{t_0})\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))|_{Z_0})\subseteq H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$ and

$$\begin{aligned}&\int_{\{\psi<-t_1\}}|\tilde{F}_j-(1-b_{t_0,B_j}(\psi))F_{t_0}|^2e^{-\varphi}c(-\psi)\\&\leq\int_{\{\psi<-t_1\}}|\tilde{F}_j-(1-b_{t_0,B_j}(\psi))F_{t_0}|^2e^{-\varphi}e^{-\psi+v_{t_0,B_j}(\psi)}c(-v_{t_0,B_j}(\psi))\\&\leq\int_{t_1}^{t_0+B_j}c(t)e^{-t}dt\int_{\{\psi<-t_1\}}\frac{1}{B_j}(\mathbb{I}_{\{-t_0-B_j<\psi<-t_0\}})|F_{t_0}|^2e^{-\varphi-\psi}\\&\leq\frac{e^{t_0+B_j}\int_{t_1}^{t_0+B_j}c(t)e^{-t}dt}{\inf_{t\in(t_0,t_0+B_j)}c(t)}\int_{\{\psi<-t_1\}}\frac{1}{B_j}(\mathbb{I}_{\{-t_0-B_j<\psi<-t_0\}})|F_{t_0}|^2e^{-\varphi}c(-\psi)\\&\leq\frac{e^{t_0+B_j}\int_{t_1}^{t_0+B_j}c(t)e^{-t}dt}{\inf_{t\in(t_0,t_0+B_j)}c(t)}\times\left(\int_{\{\psi<-t_1\}}\frac{1}{B_j}\mathbb{I}_{\{\psi<-t_0\}}|F_{t_0}|^2e^{-\varphi}c(-\psi)\right.\\&\quad\left.-\int_{\{\psi<-t_1\}}\frac{1}{B_j}\mathbb{I}_{\{\psi<-t_0-B_j\}}|F_{t_0}|^2e^{-\varphi}c(-\psi)\right)\\&\leq\frac{e^{t_0+B_j}\int_{t_1}^{t_0+B_j}c(t)e^{-t}dt}{\inf_{t\in(t_0,t_0+B_j)}c(t)}\times\frac{G(t_0)-G(t_0+B_j)}{B_j}.\end{aligned}\tag{26}$$

Firstly, we will prove that $\int_{\{\psi<-t_1\}}|\tilde{F}_j|^2e^{-\varphi}c(-\psi)$ is bounded with respect to j.

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Note that

$$\begin{aligned}&\left(\int_{\{\psi<-t_1\}}|\tilde{F}_j-(1-b_{t_0,B_j}(\psi))F_{t_0}|^2e^{-\varphi}c(-\psi)\right)^{\frac{1}{2}}\\ \geq&\left(\int_{\{\psi<-t_1\}}|\tilde{F}_j|^2e^{-\varphi}c(-\psi)\right)^{\frac{1}{2}}-\left(\int_{\{\psi<-t_1\}}|(1-b_{t_0,B_j}(\psi))F_{t_0}|^2e^{-\varphi}c(-\psi)\right)^{\frac{1}{2}},\end{aligned}\tag{27}$$

then it follows from inequality (26) that

$$\begin{aligned}&\left(\int_{\{\psi<-t_1\}}|\tilde{F}_j|^2e^{-\varphi}c(-\psi)\right)^{\frac{1}{2}}\\ \leq&\left(\frac{e^{t_0+B_j}\int_{t_1}^{t_0+B_j}c(t)e^{-t}dt}{\inf_{t\in(t_0,t_0+B_j)}c(t)}\right)^{\frac{1}{2}}\left(\frac{G(t_0)-G(t_0+B_j)}{B_j}\right)^{\frac{1}{2}}\\ &+\left(\int_{\{\psi<-t_1\}}|(1-b_{t_0,B_j}(\psi))F_{t_0}|^2e^{-\varphi}c(-\psi)\right)^{\frac{1}{2}}.\end{aligned}\tag{28}$$

Since $\left\{\frac{G(t_0+B_j)-G(t_0)}{B_j}\right\}_{j\in\mathbb{N}^+}$ is bounded, $\lim_{j\to+\infty}\inf_{t\in(t_0,t_0+B_j)}c(t)\in(0,+\infty)$ and

$$\int_{\{\psi<-t_1\}}|(1-b_{t_0,B_j}(\psi))F_{t_0}|^2e^{-\varphi}c(-\psi)\leq\int_{\{\psi<-t_0\}}|F_{t_0}|^2e^{-\varphi}c(-\psi)<+\infty,$$

then $\int_{\{\psi<-t_1\}}|\tilde{F}_j|^2e^{-\varphi}c(-\psi)$ is bounded with respect to j.

Secondly, we will prove the main result.

It follows from $\int_{\{\psi<-t_1\}}|\tilde{F}_j|^2e^{-\varphi}c(-\psi)$ is bounded with respect to j and Lemma 2.4 that there exists a subsequence of $\{\tilde{F}_j\}$, denoted by $\{\tilde{F}_{j_k}\}_{k\in\mathbb{N}^+}$, which is uniformly convergent to a holomorphic $(n,0)$ form F_1 on $\{\psi<-t_1\}$ on any compact subset of $\{\psi<-t_1\}$ when $k\to+\infty$, such that

$$\int_{\{\psi<-t_1\}}|F_1|^2e^{-\varphi}c(-\psi)\leq\liminf_{j\to+\infty}\int_{\{\psi<-t_1\}}|\tilde{F}_j|^2e^{-\varphi}c(-\psi)<+\infty.$$

As $(\tilde{F}_j-F_{t_0})\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$ for any j, we have $(F_1-F_{t_0})\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$. Note that

$$\lim_{j\to+\infty}b_{t_0,B_j}(t)=\lim_{j\to+\infty}\int_{-\infty}^{t}\frac{1}{B_j}\mathbb{I}_{\{-t_0-B_j<s<-t_0\}}ds=\begin{cases}0, & \text{if}\quad x\in(-\infty,-t_0),\\ 1, & \text{if}\quad x\in[-t_0,+\infty),\end{cases}$$

and

$$\lim_{j\to+\infty}v_{t_0,B_j}(t)=\lim_{j\to+\infty}\int_{-t_0}^{t}b_{t_0,B_j}ds-t_0=\begin{cases}-t_0, & \text{if}\quad x\in(-\infty,-t_0),\\ t, & \text{if}\quad x\in[-t_0,+\infty).\end{cases}$$

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Following from equality (25), inequality (26), and the Fatou's Lemma, we have

$$\begin{aligned}
&\int_{\{\psi<-t_0\}}|F_1-F_{t_0}|^2e^{-\varphi-\psi-t_0}c(t_0)+\int_{\{-t_0\le\psi<-t_1\}}|F_1|^2e^{-\varphi}c(-\psi)\\
=&\int_{\{\psi<-t_1\}}\lim_{k\to+\infty}|\tilde{F}_{j_k}-(1-b_{t_0,B_{j_k}}(\psi))F_{t_0}|^2e^{-\varphi}e^{-\psi+v_{t_0,B_{j_k}}(\psi)}c(-v_{t_0,B_{j_k}}(\psi))\\
\le&\liminf_{k\to+\infty}\int_{\{\psi<-t_1\}}|\tilde{F}_{j_k}-(1-b_{t_0,B_{j_k}}(\psi))F_{t_0}|^2e^{-\varphi}e^{-\psi+v_{t_0,B_{j_k}}(\psi)}c(-v_{t_0,B_{j_k}}(\psi))\\
\le&\liminf_{k\to+\infty}\left(\frac{e^{t_0+B_{j_k}}\int_{t_1}^{t_0+B_{j_k}}c(t)e^{-t}dt}{\inf_{t\in(t_0,t_0+B_{j_k})}c(t)}\times\frac{G(t_0)-G(t_0+B_{j_k})}{B_{j_k}}\right)\\
=&\frac{e^{t_0}\int_{t_1}^{t_0}c(t)e^{-t}dt}{\lim_{t\to t_0+0}c(t)}\lim_{j\to+\infty}\frac{G(t_0)-G(t_0+B_j)}{B_j}\\
=&\int_{t_1}^{t_0}c(t)e^{-t}dt\lim_{j\to+\infty}\frac{G(t_0)-G(t_0+B_j)}{\int_{t_0}^{t_0+B_j}c(t)e^{-t}dt}.
\end{aligned}\tag{29}$$

As $e^{\psi}c(-\psi)\le e^{-t_0}c(t_0)$ on $\{\psi<-t_0\}$, it follows Lemma 2.6, equality (24) and inequality (29) that

$$\begin{aligned}
&\int_{t_1}^{t_0}c(t)e^{-t}dt\liminf_{B\to0+0}\frac{G(t_0)-G(t_0+B)}{\int_{t_0}^{t_0+B}c(t)e^{-t}dt}\\
=&\int_{t_1}^{t_0}c(t)e^{-t}dt\lim_{j\to+\infty}\frac{G(t_0)-G(t_0+B_j)}{\int_{t_0}^{t_0+B_j}c(t)e^{-t}dt}\\
\ge&\int_{\{\psi<-t_0\}}|F_1-F_{t_0}|e^{-\varphi-\psi-t_0}c(t_0)+\int_{\{-t_0\le\psi<-t_1\}}|F_1|^2e^{-\varphi}c(-\psi)\\
\ge&\int_{\{\psi<-t_0\}}|F_1-F_{t_0}|e^{-\varphi}c(-\psi)+\int_{\{-t_0\le\psi<-t_1\}}|F_1|^2e^{-\varphi}c(-\psi)\\
=&\int_{\{\psi<-t_1\}}|F_1|^2e^{-\varphi}c(-\psi)-\int_{\{\psi<-t_0\}}|F_{t_0}|^2e^{-\varphi}c(-\psi)\\
\ge&G(t_1)-G(t_0).
\end{aligned}\tag{30}$$

This proves Lemma 2.8. ∎

The following well-known property of concave functions will be used in the proof of Theorem 1.3.

LEMMA 2.9. *Let $a(r)$ be a lower semicontinuous function on (A,B) ($-\infty\le A<B\le+\infty$). Then $a(r)$ is concave if and only if*

$$\frac{a(r_2)-a(r_1)}{r_2-r_1}\ge\limsup_{r\to r_2+0}\frac{a(r)-a(r_2)}{r-r_2},\tag{31}$$

holds for any $A<r_1<r_2<B$.

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Proof. For the convenience of the reader, we recall the proof.

It suffices to prove that inequality (31) implies the concavity of $a(r)$. We prove by contradiction: if not, there exists $A<r_3<r_4<r_5<B$ such that

$$\frac{a(r_4)-a(r_3)}{r_4-r_3}<\frac{a(r_5)-a(r_3)}{r_5-r_3}<\frac{a(r_5)-a(r_4)}{r_5-r_4}. \tag{32}$$

Consider $\tilde{a}(r)=a(r)-a(r_5)-\frac{a(r_5)-a(r_3)}{r_5-r_3}(r-r_5)$ on (A,B). As $a(r)$ is lower semicontinuous on (A,B), then $\tilde{a}(r)$ is lower semicontinuous on (A,B). Note that $\tilde{a}(r_3)=\tilde{a}(r_5)=0$ and $\tilde{a}(r_4)<0$, then it follows from the lower semicontinuity of $\tilde{a}(r)$ that there exists $r_6\in(r_3,r_5)$ such that $\tilde{a}(r_6)=\inf_{r\in[r_3,r_5]}\tilde{a}(r)<0$. It clear that $\frac{\tilde{a}(r_6)-\tilde{a}(r_3)}{r_6-r_3}<0$ and $\limsup_{r\to r_6+0}\frac{\tilde{a}(r)-\tilde{a}(r_6)}{r-r_6}\geq 0$. Then we obtain that

$$\frac{a(r_6)-a(r_3)}{r_6-r_3}<\frac{a(r_5)-a(r_3)}{r_5-r_3}\leq\limsup_{r\to r_6+0}\frac{a(r)-a(r_6)}{r-r_6},$$

which contradict inequality (31). ∎

2.3 Some results used in the proofs of applications

In this section, we give some results which will be used in the proofs of applications in Section 1.2.

LEMMA 2.10. *If $c(t)$ is a positive measurable function on $(T,+\infty)$ such that $c(t)e^{-t}$ is decreasing on $(T,+\infty)$ and $\int_{T_1}^{+\infty}c(t)e^{-t}dt<+\infty$ for some $T_1>T$, then there exists a positive measurable function $\tilde{c}$ on $(T,+\infty)$, satisfying the following statements:*

(1) *$\tilde{c}\geq c$ on $(T,+\infty)$.*
(2) *$\tilde{c}(t)e^{-t}$ is strictly decreasing on $(T,+\infty)$ and $\tilde{c}$ is increasing on $(a,+\infty)$, where $a>T$ is a real number.*
(3) *$\int_{T_1}^{+\infty}\tilde{c}(t)e^{-t}dt<+\infty$.*

Moreover, if $\int_T^{+\infty}c(t)e^{-t}dt<+\infty$ and $c\in\mathcal{P}_T$, we can choose $\tilde{c}$ satisfying the above conditions, $\int_T^{+\infty}\tilde{c}(t)e^{-t}dt<+\infty$ and $\tilde{c}\in\mathcal{P}_T$.

Proof. Without loss of generality, we can assume that $T<0$. Let $a_n=c(n)e^{-n}$, where $n\in\mathbb{N}^+$. Take $b_1=a_1$, and we can define $b_n=\max\{\frac{b_{n-1}}{e},a_n\}$ for $n>1$, inductively. Since a_n is decreasing with respect to n, we have $b_n\geq b_{n+1}\geq\frac{b_n}{e}$ and $b_n\geq a_n$ for any $n\in\mathbb{N}^+$.

Let

$$\tilde{a}(t)=\begin{cases} eb_n(\frac{b_{n+1}}{b_n})^{t-n}, & \text{if} \quad t\in[n,n+1),\\ c(t)e^{-t+1}, & \text{if} \quad t\in(T,1].\end{cases}$$

It is clear that $\tilde{a}(t)\geq c(t)e^{-t}$, $\tilde{a}(t)$ is decreasing on $(T,+\infty)$ and continuous on $[1,+\infty)$. Let $\tilde{c}(t)=\tilde{a}(t)e^t$. When $t\in[n,n+1)$, as $eb_{n+1}\geq b_n$, we have $\tilde{c}(t)$ is increasing on $[n,n+1)$, which implies that $\tilde{c}(t)$ is increasing on $(1,+\infty)$.

As $\int_0^{+\infty}c(t)e^{-t}dt<+\infty$, then $\sum_{n=1}^{+\infty}a_n<+\infty$. In the following, we will prove $\int_0^{+\infty}\tilde{c}(t)e^{-t}<+\infty$. By the definition of $\tilde{c}(t)$, we have

$$\int_0^{+\infty}\tilde{c}(t)e^{-t}=\int_0^1\tilde{a}(t)dt+\sum_{n=1}^{+\infty}\int_n^{n+1}\tilde{a}(t)dt\leq c(0)e+e\sum_{n=1}^{+\infty}b_n. \tag{33}$$

Take $I = \{n_i : n_i$ is the ith positive integer such that $a_{n_i} = b_{n_i}\} \in \mathbb{N}^+$. Note that if $a_{n+1} \neq b_{n+1}$, then $b_{n+1} = \frac{b_n}{e}$, thus, we have

$$\begin{aligned}\sum_{n=1}^{+\infty} b_n &= \sum_{i=1} \sum_{j=0}^{n_{i+1}-n_i-1} b_{n_i+j} \\ &= \sum_{i=1} \sum_{j=0}^{n_{i+1}-n_i-1} b_{n_i} e^{-j} \\ &\leq \sum_{i=1} a_{n_i} \frac{e}{e-1} \\ &< +\infty,\end{aligned} \tag{34}$$

where if n_i is the largest integer such that $a_{n_i} = b_{n_i}$, take $n_{i+1} = +\infty$. Combining inequality (33) and (34), we obtain $\int_0^{+\infty} \tilde{c}(t)e^{-t}dt < +\infty$. By replacing $\tilde{c}(t)$ by $\tilde{c}(t)+1$, we have $\tilde{c} \geq c$, $\tilde{c}$ is increasing on $(1,+\infty)$, $\tilde{c}(t)e^{-t}$ is strictly decreasing on $(T,+\infty)$ and $\int_0^{+\infty} \tilde{c}(t)e^{-t}dt < +\infty$.

Moreover, if $\int_T^{+\infty} c(t)e^{-t}dt < +\infty$ and $c \in \mathcal{P}_T$, as $\tilde{c}(t) \geq c(t)$ on $(T,+\infty)$ and $\tilde{c}(t) = ec(t)+1$ on $(T,1)$, we have $\int_T^{+\infty} \tilde{c}(t)e^{-t}dt < +\infty$ and $\tilde{c} \in \mathcal{P}_T$. Thus, Lemma 2.10 holds. ∎

Let Ω be an open Riemann surface admitted a nontrivial Green function G_Ω. Let w be a local coordinate on a neighborhood V_{z_0} of $z_0 \in \Omega$ satisfying $w(z_0) = 0$.

LEMMA 2.11 (See [25], see also [31]). *$G_\Omega(z,z_0) = \sup_{v \in \Delta_0(z_0)} v(z)$, where $\Delta_0(z_0)$ is the set of negative subharmonic functions v on Ω satisfying that $v - \log|w|$ has a locally finite upper bound near z_0.*

LEMMA 2.12. *For any open neighborhood U of z_0, there exists $t > 0$ such that $\{G_\Omega(z,z_0) < -t\}$ is a relatively compact subset of U.*

Proof. Let w be a coordinate on a neighborhood $V_{z_0} \subset\subset U$ of z_0, such that $w(z_0) = 0$ and $G_\Omega(z,z_0) = \log|w(z)| + v(w(z))$, where v is a harmonic function on V_{z_0} and $\sup_{V_{z_0}} |v(w(z))| < +\infty$. Then there exists $t > 0$ such that $\{z \in V_{z_0} : \log|w(z)| + v(w(z)) < -t\} \subset\subset V_{z_0}$.

We claim that $\{z \in \Omega : G_\Omega(z,z_0) < -t\} \subset\subset V_{z_0}$, therefore Lemma 2.12 holds. In fact, set

$$\tilde{G}(z) = \begin{cases} G_\Omega(z,z_0), & \text{if } \ z \in V_{z_0}, \\ \max\{G_\Omega(z,z_0), -t\}, & \text{if } \ z \in \Omega \backslash V_{z_0}. \end{cases}$$

As $\{z \in V_{z_0} : \log|w(z)| + v(w(z)) < -t\} \subset\subset V_{z_0}$, we know $\tilde{G}(z)$ is subharmonic on Ω. Lemma 2.11 tells us $\tilde{G}(z) \leq G_\Omega(z,z_0)$, therefore $\{z \in \Omega : G_\Omega(z,z_0) < -t\} = \{z \in V_{z_1} : G_\Omega(z,z_0) < -t\} \subset\subset V_{z_0}$. ∎

LEMMA 2.13. *For any $z_0 \in \Omega$ and open subsets V_1 and U_1 of Ω satisfying $z_0 \in V_1 \subset\subset U_1 \subset\subset \Omega$, there exists a constant $N > 0$ such that*

$$G_\Omega(z,z_1) \geq N G_\Omega(z,z_0)$$

holds for any $(z,z_1) \in (\Omega \backslash U_1) \times V_1$.

Proof. As $V_1 \subset\subset U_1 \subset\subset \Omega$, fixed $z \in \Omega \backslash U_1$, $G_\Omega(z,z_1)$ is harmonic with respect to z_1 on a open neighborhood of $\overline{V_1}$. The Harnack inequality shows that there exists a constant $N > 0$

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such that

$$\sup_{z_1\in\overline{V_1}}(-G_\Omega(z,z_1))\le N\inf_{z_1\in\overline{V_1}}(-G_\Omega(z,z_1)) \tag{35}$$

holds of any $z\in\Omega\backslash U_1$. As $z_0\in V_1$, it follows from inequality (35) that

$$G_\Omega(z,z_1)\ge NG(z,z_0)$$

holds for any $(z,z_1)\in(\Omega\backslash U_1)\times V_1$. $\square$

The following lemma (proof can be referred to §7.2) will be used in the proof of Theorem 1.16.

LEMMA 2.14. *Let T be a closed positive $(1,1)$ current on Ω. For any open set $U\subset\subset\Omega$ satisfying $U\cap suppT\neq\emptyset$, there exists a subharmonic function $\Phi<0$ on Ω, which satisfies the following properties:*

(1) $i\partial\bar\partial\Phi\le T$ *and* $i\partial\bar\partial\Phi\not\equiv 0$;
(2) $\lim_{t\to 0+0}(\inf_{\{G_\Omega(z,z_0)\ge -t\}}\Phi(z))=0$;
(3) $supp(i\partial\bar\partial\Phi)\subset U$ *and* $\inf_{\Omega\backslash U}\Phi>-\infty$.

Now, we recall some notations. Let $c_\beta(z)$ be the logarithmic capacity which is locally defined by

$$c_\beta(z_0):=\exp\lim_{z\to z_0}(G_\Omega(z,z_0)-\log|w(z)|)$$

on Ω (see [25]). The weighted Bergman kernel $\kappa_{\Omega,\rho}$ with weight ρ of holomorphic $(1,0)$ form on Ω is defined by $\kappa_{\Omega,\rho}:=\sum_i e_i\otimes\bar e_i$, where $\{e_i\}_{i=1,2,\dots}$ are holomorphic $(1,0)$ forms on Ω and satisfy $\sqrt{-1}\int_\Omega\rho\frac{e_i}{\sqrt2}\wedge\frac{\bar e_j}{\sqrt2}=\delta_i^j$. Let $B_{\Omega,\rho}(z):=\frac{\kappa_{\Omega,\rho}(z)}{|dw|^2}$ on V_{z_0}.

THEOREM 2.15 [18]. *(A solution of the extended Suita Conjecture) Let u be a harmonic function on Ω. $c_\beta^2(z_0)\le\pi e^{-2u(z_0)}B_{\Omega,e^{-2u}}(z_0)$ holds, and the equality holds if and only if $\chi_{-u}=\chi_{z_0}$.*

§3. Proofs of Theorem 1.3 and Corollaries 1.4, 1.5, and 1.7

In this section, we prove Theorem 1.3 and Corollaries 1.4, 1.5, and 1.7.

3.1 Proof of Theorem 1.3

Firstly, we prove that if $G(t_0)<+\infty$ for some $t_0>T$, then $G(t_1)<+\infty$ for any $t_1\in(T,t_0)$. It follows from Lemma 2.6 that there exists a holomorphic $(n,0)$ form F_{t_0} on $\{\psi<-t_0\}$ satisfying $(F_{t_0}-f)\in H^0(Z_0,(\mathcal O(K_M)\otimes\mathcal F)|_{Z_0})$ and $\int_{\{\psi<-t_0\}}|F_{t_0}|^2e^{-\varphi}c(-\psi)=G(t_0)<+\infty$. Using Lemma 2.2, we get a holomorphic $(n,0)$ form $\tilde F$ on $\{\psi<-t_1\}$, such that

$$(\tilde F-F_{t_0})\in H^0(Z_0,(\mathcal O(K_M)\otimes\mathcal I(\varphi+\psi))|_{Z_0})\subset H^0(Z_0,(\mathcal O(K_M)\otimes\mathcal F)|_{Z_0})$$

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and

$$\begin{aligned}
&\int_{\{\psi<-t_1\}} |\tilde{F}-(1-b_{t_0,B}(\psi))F_{t_0}|^2 e^{-\varphi}c(-\psi) \\
\le &\int_{\{\psi<-t_1\}} |\tilde{F}-(1-b_{t_0,B}(\psi))F_{t_0}|^2 e^{-\varphi-\psi+v_{t_0,B}(\psi)}c(-v_{t_0,B}(\psi)) \\
\le &\left(\int_{t_1}^{t_0+B} c(t)e^{-t}dt\right)\int_{\{\psi<-t_1\}} \frac{1}{B}\mathbb{I}_{\{-t_0+B<\psi<-t_0\}}|F_{t_0}|^2 e^{-\varphi-\psi}.
\end{aligned} \tag{36}$$

Note that

$$\begin{aligned}
&\left(\int_{\{\psi<-t_1\}} |\tilde{F}|^2 e^{-\varphi}c(-\psi)\right)^{\frac{1}{2}} - \left(\int_{\{\psi<-t_1\}} |(1-b_{t_0,B}(\psi))F_{t_0}|^2 e^{-\varphi}c(-\psi)\right)^{\frac{1}{2}} \\
\le &\left(\int_{\{\psi<-t_1\}} |\tilde{F}-(1-b_{t_0,B}(\psi))F_{t_0}|^2 e^{-\varphi}c(-\psi)\right)^{\frac{1}{2}},
\end{aligned}$$

combining with inequality (36), we obtain

$$\begin{aligned}
&\left(\int_{\{\psi<-t_1\}} |\tilde{F}|^2 e^{-\varphi}c(-\psi)\right)^{\frac{1}{2}} \\
\le &\left(\left(\int_{t_1}^{t_0+B} c(t)e^{-t}dt\right)\int_{\{\psi<-t_1\}} \frac{1}{B}\mathbb{I}_{\{-t_0-B<\psi<-t_0\}}|F_{t_0}|^2 e^{-\varphi-\psi}\right)^{\frac{1}{2}} \\
&+\left(\int_{\{\psi<-t_1\}} |(1-b_{t_0,B}(\psi))F_{t_0}|^2 e^{-\varphi}c(-\psi)\right)^{\frac{1}{2}}.
\end{aligned} \tag{37}$$

As $b_{t_0,B}(\psi)=1$ on $\{\psi\ge t_0\}$, $0\le b_{t_0,B}(\psi)\le 1$, $\int_{\{\psi<-t_0\}}|F_{t_0}|^2e^{-\varphi}c(-\psi)<+\infty$, and $c(t)$ has a positive lower bound on any compact subset of $(T,+\infty)$, then

$$\left(\int_{\{\psi<-t_1\}} |(1-b_{t_0,B}(\psi))F_{t_0}|^2 e^{-\varphi}c(-\psi)\right)^{\frac{1}{2}} < +\infty$$

and

$$\begin{aligned}
&\left(\int_{t_1}^{t_0+B} c(t)e^{-t}dt\right)\int_{\{\psi<-t_1\}} \frac{1}{B}\mathbb{I}_{\{-t_0-B<\psi<-t_0\}}|F_{t_0}|^2 e^{-\varphi-\psi} \\
\le &\frac{e^{t_0+B}\int_{t_1}^{t_0+B} c(t)e^{-t}dt}{\inf_{t\in(t_0,t_0+B)}c(t)}\int_{\{\psi<-t_1\}} \frac{1}{B}\mathbb{I}_{\{-t_0-B<\psi<-t_0\}}|F_{t_0}|^2 e^{-\varphi}c(-\psi) \\
< &+\infty,
\end{aligned}$$

which implies that

$$\int_{\{\psi<-t_1\}} |\widetilde{F}|^2 e^{-\varphi}c(-\psi) < +\infty.$$

Then we obtain $G(t_1)\le \int_{\{\psi<-t_1\}}|\widetilde{F}|^2e^{-\varphi}c(-\psi)<+\infty$.

Now, assume that $G(t_0)<+\infty$ for some $t_0\geq T$ (otherwise it is clear that $G(t)\equiv+\infty$). As $G(h^{-1}(r))$ is lower semicontinuous (Lemma 2.7), then Lemmas 2.8 and 2.9 imply the concavity of $G(h^{-1}(r))$. It follows from Lemma 2.7 that $\lim_{t\to T+0}G(t)=G(T)$ and $\lim_{t\to+\infty}G(t)=0$, hence we prove Theorem 1.3.

3.2 Proof of Corollary 1.4

Note that if there exists a positive decreasing concave function $g(t)$ on $(a,b)\subset\mathbb{R}$ and $g(t)$ is not a constant function, then $b<+\infty$. We prove Corollary 1.4 by contradiction: if $G(t)<+\infty$ for some $t\geq T$, as $f\notin H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$, Lemma 2.5 shows that $G(t)\in(0,+\infty)$. Following from Theorem 1.3, we know $G(h^{-1}(r))$ is concave with respect to $r\in(\int_{T_1}^{T}c(t)e^{-t}dt,\int_{T_1}^{+\infty}c(t)e^{-t}dt)$ and $G(h^{-1}(r))$ is not a constant function, therefore we obtain $\int_{T_1}^{+\infty}c(t)e^{-t}dt<+\infty$, which contradicts to $\int_{T_1}^{+\infty}c(t)e^{-t}dt=+\infty$. Thus Corollary 1.4 holds.

3.3 Proof of Corollary 1.5

If $G(t)\in(0,+\infty)$ for some $t\geq T$, Corollary 1.4 and Lemma 2.5 show that $\int_{T_1}^{+\infty}c(t)e^{-t}dt<+\infty$. As $\lim_{t\to+\infty}G(t)=0$, then $G(h^{-1}(r))$ is concave on $(\int_{T_1}^{T}c(t)e^{-t}dt,\int_{T_1}^{+\infty}c(t)e^{-t}dt]$ by defining $G(+\infty)=0$. Then the concavity of $G(h^{-1}(r))$ implies that the three statements are equivalent.

3.4 Proof of Corollary 1.7

It follows from Corollary 1.5 that $G(t)=\frac{G(T_1)}{\int_{T_1}^{+\infty}c(s)e^{-s}ds}\int_t^{+\infty}c(s)e^{-s}ds$ for any $t\in[T,+\infty)$.

Firstly, we prove the existence and uniqueness of F.

Following the notations in Lemma 2.8, as $G(t)=\frac{G(T_1)}{\int_{T_1}^{+\infty}c(s)e^{-s}ds}\int_t^{+\infty}c(s)e^{-s}ds\in(0,+\infty)$ for any $t\in(T,+\infty)$, by choosing $t_1\in(T,+\infty)$ and $t_0>t_1$, we know that the inequality (30) must be equality, which implies that

$$\int_{\{\psi<-t_0\}}|F_1-F_{t_0}|^2e^{-\varphi}(e^{-\psi-t_0}c(t_0)-c(-\psi))=0, \tag{38}$$

where F_1 is a holomorphic $(n,0)$ form on $\{\psi<-t_1\}$ such that $(F_1-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$ and F_{t_0} is a holomorphic $(n,0)$ form on $\{\psi<-t_0\}$ such that $(F_{t_0}-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$. As $\int_{T_1}^{+\infty}c(t)e^{-t}<+\infty$ and $c(t)e^{-t}$ is decreasing, then there exists $t_2>t_0$ such that $c(t)e^{-t}<c(t_0)e^{-t_0}-\delta$ for any $t\geq t_2$, where δ is a positive constant. Then equality (38) implies that

$$\begin{aligned}
&\delta\int_{\{\psi<-t_2\}}|F_1-F_{t_0}|^2e^{-\varphi}e^{-\psi}\\
\leq&\int_{\{\psi<-t_2\}}|F_1-F_{t_0}|^2e^{-\varphi}(e^{-\psi-t_0}c(t_0)-c(-\psi))\\
\leq&\int_{\{\psi<-t_0\}}|F_1-F_{t_0}|^2e^{-\varphi}(e^{-\psi-t_0}c(t_0)-c(-\psi))\\
=&0.
\end{aligned}$$

It follows from $\varphi+\psi$ is plurisubharmonic function and F_1 and F_{t_0} are holomorphic $(n,0)$ forms that $F_1=F_{t_0}$ on $\{\psi<-t_0\}$. As $\int_{\{\psi<-t_0\}}|F_{t_0}|^2e^{-\varphi}c(-\psi)=G(t_0)$ and the inequality

(30) becomes equality, we have

$$\int_{\{\psi<-t_1\}}|F_1|^2e^{-\varphi}c(-\psi)=G(t_1).$$

Following from Lemma 2.6, there exists a unique holomorphic $(n,0)$ form F_t on $\{\psi<-t\}$ satisfying $(F_t-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$ and $\int_{\{\psi<-t\}}|F_t|^2e^{-\varphi}c(-\psi)=G(t)$ for any $t>T$. By discussion in the above, we know $F_t=F_{t'}$ on $\{\psi<-\max\{t,t'\}\}$ for any $t\in(T,+\infty)$ and $t'\in(T,+\infty)$. Hence, combining $\lim_{t\to T+0}G(t)=G(T)$, we obtain that there exists a unique holomorphic $(n,0)$ form F on M satisfying $(F-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$ and $\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)=G(t)$ for any $t\geq T$.

Secondly, we prove equality (1). As $a(t)$ is nonnegative measurable function on $(T,+\infty)$, then there exists a sequence of functions $\{\sum_{j=1}^{n_i}a_{ij}\mathbb{I}_{E_{ij}}\}_{i\in\mathbb{N}^+}$ ($n_i<+\infty$ for any $i\in\mathbb{N}^+$) satisfying $\sum_{j=1}^{n_i}a_{ij}\mathbb{I}_{E_{ij}}$ is increasing with respect to i and $\lim_{i\to+\infty}\sum_{j=1}^{n_i}a_{ij}\mathbb{I}_{E_{ij}}(t)=a(t)$ for any $t\in(T,+\infty)$, where E_{ij} is a Lebesgue measurable subset of $(T,+\infty)$ and $a_{ij}\geq 0$ is a constant. It follows from Levi's Theorem that it suffices to prove the case that $a(t)=\mathbb{I}_E(t)$, where $E\subset\subset(T,+\infty)$ is a Lebesgue measurable set.

Note that $G(t)=\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)=\frac{G(T_1)}{\int_{T_1}^{+\infty}c(s)e^{-s}ds}\int_t^{+\infty}c(s)e^{-s}ds$, then

$$\int_{\{-t_1\leq\psi<-t_2\}}|F|^2e^{-\varphi}c(-\psi)=\frac{G(T_1)}{\int_{T_1}^{+\infty}c(s)e^{-s}ds}\int_{t_2}^{t_1}c(s)e^{-s}ds \tag{39}$$

holds for any $T\leq t_2<t_1<+\infty$. It follows from the dominated convergence theorem and inequality (39) that

$$\int_{\{z\in M:-\psi(z)\in N\}}|F|^2e^{-\varphi}=0 \tag{40}$$

holds for any $N\subset\subset(T,+\infty)$ such that $\mu(N)=0$, where μ is Lebesgue measure.

As $c(t)e^{-t}$ is decreasing on $(T,+\infty)$, there are at most countable points denoted by $\{s_j\}_{j\in\mathbb{N}^+}$ such that $c(t)$ is not continuous at s_j. Then there is a decreasing sequence open sets $\{U_k\}$, such that $\{s_j\}_{j\in\mathbb{N}^+}\subset U_k\subset(T,+\infty)$ for any j, and $\lim_{k\to+\infty}\mu(U_k)=0$. Choosing any closed interval $[t_2',t_1']\subset(T,+\infty)$. Then we have

$$\begin{aligned}&\int_{\{-t_1'\leq\psi<-t_2'\}}|F|^2e^{-\varphi}\\ =&\int_{\{z\in M:-\psi(z)\in(t_2',t_1']\setminus U_k\}}|F|^2e^{-\varphi}+\int_{\{z\in M:-\psi(z)\in[t_2',t_1']\cap U_k\}}|F|^2e^{-\varphi}\\ =&\lim_{n\to+\infty}\sum_{i=0}^{n-1}\int_{\{z\in M:-\psi(z)\in I_{n,i}\setminus U_k\}}|F|^2e^{-\varphi}+\int_{\{z\in M:-\psi(z)\in[t_2',t_1']\cap U_k\}}|F|^2e^{-\varphi},\end{aligned} \tag{41}$$

where $I_{n,i}=(t_1'-(i+1)\alpha_n,t_1'-i\alpha_n]$ and $\alpha_n=\frac{t_1'-t_2'}{n}$. Note that

$$\begin{aligned}&\lim_{n\to+\infty}\sum_{i=0}^{n-1}\int_{\{z\in M:-\psi(z)\in I_{n,i}\setminus U_k\}}|F|^2e^{-\varphi}\\ \leq&\limsup_{n\to+\infty}\sum_{i=0}^{n-1}\frac{1}{\inf_{I_{n,i}\setminus U_k}c(t)}\int_{\{z\in M:-\psi(z)\in I_{n,i}\setminus U_k\}}|F|^2e^{-\varphi}c(-\psi).\end{aligned} \tag{42}$$

It follows from equality (39) that inequality (42) becomes

$$\begin{aligned}&\lim_{n\to+\infty}\sum_{i=0}^{n-1}\int_{\{z\in M:-\psi(z)\in I_{n,i}\backslash U_k\}}|F|^2e^{-\varphi}\\ \le&\frac{G(T_1)}{\int_{T_1}^{+\infty}c(s)e^{-s}ds}\limsup_{n\to+\infty}\sum_{i=0}^{n-1}\frac{1}{\inf_{I_{n,i}\backslash U_k}c(t)}\int_{I_{n,i}\backslash U_k}c(s)e^{-s}ds.\end{aligned}\tag{43}$$

It is clear that $c(t)$ is uniformly continuous and has a positive lower bound and upper bound on $[t_2',t_1']\backslash U_k$. Then we have

$$\begin{aligned}&\limsup_{n\to+\infty}\sum_{i=0}^{n-1}\frac{1}{\inf_{I_{n,i}\backslash U_k}c(t)}\int_{I_{n,i}\backslash U_k}c(s)e^{-s}ds\\ \le&\limsup_{n\to+\infty}\sum_{i=0}^{n-1}\frac{\sup_{I_{n,i}\backslash U_k}c(t)}{\inf_{I_{n,i}\backslash U_k}c(t)}\int_{I_{n,i}\backslash U_k}e^{-s}ds\\ =&\int_{(t_2',t_1']\backslash U_k}e^{-s}ds.\end{aligned}\tag{44}$$

Combining inequality (41), (43), and (44), we have

$$\begin{aligned}&\int_{\{-t_1'\le\psi<-t_2'\}}|F|^2e^{-\varphi}\\ =&\int_{\{z\in M:-\psi(z)\in(t_2',t_1']\backslash U_k\}}|F|^2e^{-\varphi}+\int_{\{z\in M:-\psi(z)\in[t_2',t_1']\cap U_k\}}|F|^2e^{-\varphi}\\ \le&\frac{G(T_1)}{\int_{T_1}^{+\infty}c(s)e^{-s}ds}\int_{(t_2',t_1']\backslash U_k}e^{-s}ds+\int_{\{z\in M:-\psi(z)\in[t_2',t_1']\cap U_k\}}|F|^2e^{-\varphi}.\end{aligned}\tag{45}$$

Let $k\to+\infty$, following from equality (40) and inequality (45), we obtain that

$$\int_{\{-t_1'\le\psi<-t_2'\}}|F|^2e^{-\varphi}\le\frac{G(T_1)}{\int_{T_1}^{+\infty}c(s)e^{-s}ds}\int_{t_2'}^{t_1'}e^{-s}ds.\tag{46}$$

Following from a similar discussion, we obtain

$$\int_{\{-t_1'\le\psi<-t_2'\}}|F|^2e^{-\varphi}\ge\frac{G(T_1)}{\int_{T_1}^{+\infty}c(s)e^{-s}ds}\int_{t_2'}^{t_1'}e^{-s}ds,$$

then combining inequality (46), we know that

$$\int_{\{-t_1'\le\psi<-t_2'\}}|F|^2e^{-\varphi}=\frac{G(T_1)}{\int_{T_1}^{+\infty}c(s)e^{-s}ds}\int_{t_2'}^{t_1'}e^{-s}ds.\tag{47}$$

Then it is clear that for any open set $U\subset(T,+\infty)$ and compact set $V\subset(T,+\infty)$

$$\int_{\{z\in M:-\psi(z)\in U\}}|F|^2e^{-\varphi}=\frac{G(T_1)}{\int_{T_1}^{+\infty}c(s)e^{-s}ds}\int_U e^{-s}ds$$

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and

$$\int_{\{z\in M:-\psi(z)\in V\}}|F|^2e^{-\varphi}=\frac{G(T_1)}{\int_{T_1}^{+\infty}c(s)e^{-s}ds}\int_V e^{-s}ds.$$

As $E\subset\subset(T,+\infty)$, then $E\cap(t_2,t_1]$ is Lebesgue measurable subset of $(T+\frac{1}{n},n)$ for some large n, where $T\le t_2<t_1\le+\infty$. Then there exist a sequence of compact sets $\{V_j\}$ and a sequence of open sets $\{V'_j\}$ satisfying $V_1\subset\cdots\subset V_j\subset V_{j+1}\subset\cdots\subset E\cap(t_2,t_1]\subset\cdots\subset V'_{j+1}\subset V'_j\subset\ldots\subset V'_1\subset\subset(T,+\infty)$ and $\lim_{j\to+\infty}\mu(V'_j-V_j)=0$, where μ is Lebesgue measure. Then we have

$$\begin{aligned}\int_{\{-t_1\le\psi<-t_2\}}|F|^2e^{-\varphi}\mathbb{I}_E(-\psi)=&\int_{\{z\in M:-\psi(z)\in E\cap(t_2,t_1]\}}|F|^2e^{-\varphi}\\ \le&\liminf_{j\to+\infty}\int_{\{z\in M:-\psi(z)\in V'_j\}}|F|^2e^{-\varphi}\\ \le&\liminf_{j\to+\infty}\frac{G(T_1)}{\int_{T_1}^{+\infty}c(s)e^{-s}ds}\int_{V'_j}e^{-s}\\ =&\frac{G(T_1)}{\int_{T_1}^{+\infty}c(s)e^{-s}ds}\int_{E\cap(t_2,t_1]}e^{-s}ds\\ =&\frac{G(T_1)}{\int_{T_1}^{+\infty}c(s)e^{-s}ds}\int_{t_2}^{t_1}e^{-s}\mathbb{I}_E(s)ds\end{aligned}$$

and

$$\begin{aligned}\int_{\{-t_1\le\psi<-t_2\}}|F|^2e^{-\varphi}\mathbb{I}_E(-\psi)\ge&\liminf_{j\to+\infty}\int_{\{z\in M:-\psi(z)\in V_j\}}|F|^2e^{-\varphi}\\ \ge&\liminf_{j\to+\infty}\frac{G(T_1)}{\int_{T_1}^{+\infty}c(s)e^{-s}ds}\int_{V_j}e^{-s}\\ =&\frac{G(T_1)}{\int_{T_1}^{+\infty}c(s)e^{-s}ds}\int_{t_2}^{t_1}e^{-s}\mathbb{I}_E(s)ds,\end{aligned}$$

which implies that $\int_{\{-t_1\le\psi<-t_2\}}|F|^2e^{-\varphi}\mathbb{I}_E(-\psi)=\frac{G(T_1)}{\int_{T_1}^{+\infty}c(s)e^{-s}ds}\int_{t_2}^{t_1}e^{-s}\mathbb{I}_E(s)ds$. Hence, we obtain that equality (1) holds.

Finally, we prove equality (2).

By the definition of $G(t_0;\tilde{c})$, we have $G(t_0;\tilde{c})\le\int_{\{\psi<-t_0\}}|F|^2e^{-\varphi}\tilde{c}(-\psi)$, then we only consider the case $G(t_0;\tilde{c})<+\infty$.

By the definition of $G(t_0;\tilde{c})$, we can choose a holomorphic $(n,0)$ form $F_{t_0,\tilde{c}}$ on $\{\psi<-t_0\}$ satisfying $(F_{t_0,\tilde{c}}-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$ and $\int_{\{\psi<-t_0\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi}\tilde{c}(-\psi)<+\infty$. As $\mathcal{H}^2(\tilde{c},t_0)\subset\mathcal{H}^2(c,t_0)$, we have $\int_{\{\psi<-t_0\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi}c(-\psi)<+\infty$. By using Lemma 2.6, we obtain that

$$\begin{aligned}\int_{\{\psi<-t\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi}c(-\psi)=&\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)\\ &+\int_{\{\psi<-t\}}|F_{t_0,\tilde{c}}-F|^2e^{-\varphi}c(-\psi)\end{aligned}$$

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for any $t \geq t_0$, then

$$\begin{aligned}\int_{\{-t_3\leq\psi<-t_4\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi}c(-\psi)=&\int_{\{-t_3\leq\psi<-t_4\}}|F|^2e^{-\varphi}c(-\psi)\\&+\int_{\{-t_3\leq\psi<-t_4\}}|F_{t_0,\tilde{c}}-F|^2e^{-\varphi}c(-\psi)\end{aligned}\tag{48}$$

holds for any $t_3 > t_4 \geq t_0$. It follows from the dominant convergence theorem, equality (48), equality (40), and $c(t) > 0$ for any $t > T$, that

$$\int_{\{z\in M:-\psi(z)=t\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi}=\int_{\{z\in M:-\psi(z)=t\}}|F_{t_0,\tilde{c}}-F|^2e^{-\varphi}\tag{49}$$

holds for any $t > t_0$.

Choosing any closed interval $[t'_4, t'_3] \subset (t_0, +\infty) \subset (T, +\infty)$. Note that $c(t)$ is uniformly continuous and have positive lower bound and upper bound on $[t'_4, t'_3]\backslash U_k$, where $\{U_k\}_k$ is a decreasing sequence of open subsets of $(T, +\infty)$, such that c is continuous on $(T, +\infty)\backslash U_k$ and $\lim_{k\to+\infty}\mu(U_k) = 0$. Take $N = \cap_{k=1}^{+\infty} U_k$. Note that

$$\begin{aligned}&\int_{\{-t'_3\leq\psi<-t'_4\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi}\\&=\lim_{n\to+\infty}\sum_{i=0}^{n-1}\int_{\{z\in M:-\psi(z)\in I_{n,i}\backslash U_k\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi}+\int_{\{z\in M:-\psi(z)\in(t'_4,t'_3]\cap U_k\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi}\\&\leq\limsup_{n\to+\infty}\sum_{i=0}^{n-1}\frac{1}{\inf_{I_{n,i}\backslash U_k}c(t)}\int_{\{z\in M:-\psi(z)\in I_{n,i}\backslash U_k\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi}c(-\psi)\\&\quad+\int_{\{z\in M:-\psi(z)\in(t'_4,t'_3]\cap U_k\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi},\end{aligned}\tag{50}$$

where $I_{n,i} = (t'_3 - (i+1)\alpha_n, t'_3 - i\alpha_n]$ and $\alpha_n = \frac{t'_3 - t'_4}{n}$. It following from equality (48), (49), (40), and the dominated theorem that

$$\begin{aligned}&\int_{\{z\in M:-\psi(z)\in I_{n,i}\backslash U_k\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi}c(-\psi)\\&=\int_{\{z\in M:-\psi(z)\in I_{n,i}\backslash U_k)\}}|F|^2e^{-\varphi}c(-\psi)+\int_{\{z\in M:-\psi(z)\in I_{n,i}\backslash U_k)\}}|F_{t_0,\tilde{c}}-F|^2e^{-\varphi}c(-\psi).\end{aligned}\tag{51}$$

As $c(t)$ is uniformly continuous and have positive lower bound and upper bound on $[t'_4, t'_3]\backslash U_k$, combining equality (51), we have

$$\begin{aligned}&\limsup_{n\to+\infty}\sum_{i=0}^{n-1}\frac{1}{\inf_{I_{n,i}\backslash U_k}c(t)}\int_{\{z\in M:-\psi(z)\in I_{n,i}\backslash U_k\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi}c(-\psi)\\&=\limsup_{n\to+\infty}\sum_{i=0}^{n-1}\frac{1}{\inf_{I_{n,i}\backslash U_k}c(t)}(\int_{\{z\in M:-\psi(z)\in I_{n,i}\backslash U_k)\}}|F|^2e^{-\varphi}c(-\psi)\\&+\int_{\{z\in M:-\psi(z)\in I_{n,i}\backslash U_k)\}}|F_{t_0,\tilde{c}}-F|^2e^{-\varphi}c(-\psi))\end{aligned}$$

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$$\begin{aligned}&\leq \limsup_{n\to+\infty}\sum_{i=0}^{n-1}\frac{\sup_{I_{n,i}\setminus U_k}c(t)}{\inf_{I_{n,i}\setminus U_k}c(t)}\Big(\int_{\{z\in M:-\psi(z)\in I_{n,i}\setminus U_k\}}|F|^2e^{-\varphi}\\&+\int_{\{z\in M:-\psi(z)\in I_{n,i}\setminus U_k\}}|F_{t_0,\tilde{c}}-F|^2e^{-\varphi}\Big)\\&=\int_{\{z\in M:-\psi(z)\in(t_4',t_3']\setminus U_k\}}|F|^2e^{-\varphi}+\int_{\{z\in M:-\psi(z)\in(t_4',t_3']\setminus U_k\}}|F_{t_0,\tilde{c}}-F|^2e^{-\varphi}.\end{aligned}\tag{52}$$

It follows from inequality (50) and (52), we obtain that

$$\begin{aligned}&\int_{\{-t_3'\leq\psi<-t_4'\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi}\\&\leq\int_{\{z\in M:-\psi(z)\in(t_4',t_3']\setminus U_k\}}|F|^2e^{-\varphi}+\int_{\{z\in M:-\psi(z)\in(t_4',t_3']\setminus U_k\}}|F_{t_0,\tilde{c}}-F|^2e^{-\varphi}\\&+\int_{\{z\in M:-\psi(z)\in(t_4',t_3']\cap U_k\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi}.\end{aligned}\tag{53}$$

It follows from $F_{t_0,\tilde{c}}\in\mathcal{H}^2(c,t_0)$ that $\int_{\{-t_3'\leq\psi<-t_4'\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi}<+\infty$. Let $k\to+\infty$, following from equality (40), inequality (53), and the dominated theorem, we have

$$\begin{aligned}\int_{\{-t_3'\leq\psi<-t_4'\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi}\leq&\int_{\{-t_3'\leq\psi<-t_4'\}}|F|^2e^{-\varphi}\\&+\int_{\{z\in M:-\psi(z)\in(t_4',t_3']\setminus N\}}|F_{t_0,\tilde{c}}-F|^2e^{-\varphi}\\&+\int_{\{z\in M:-\psi(z)\in(t_4',t_3']\cap N\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi}.\end{aligned}\tag{54}$$

Following from a similar discussion, we can obtain that

$$\begin{aligned}\int_{\{-t_3'\leq\psi<-t_4'\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi}\geq&\int_{\{-t_3'\leq\psi<-t_4'\}}|F|^2e^{-\varphi}\\&+\int_{\{z\in M:-\psi(z)\in(t_4',t_3']\setminus N\}}|F_{t_0,\tilde{c}}-F|^2e^{-\varphi}\\&+\int_{\{z\in M:-\psi(z)\in(t_4',t_3']\cap N\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi},\end{aligned}$$

then combining inequality (54), we have

$$\begin{aligned}\int_{\{-t_3'\leq\psi<-t_4'\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi}=&\int_{\{-t_3'\leq\psi<-t_4'\}}|F|^2e^{-\varphi}\\&+\int_{\{z\in M:-\psi(z)\in(t_4',t_3']\setminus N\}}|F_{t_0,\tilde{c}}-F|^2e^{-\varphi}\\&+\int_{\{z\in M:-\psi(z)\in(t_4',t_3']\cap N\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi}.\end{aligned}\tag{55}$$

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Using equality (40), (49), (55), and Levi's Theorem, we have

$$\begin{aligned}\int_{\{z\in M:-\psi(z)\in U\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi}=&\int_{\{z\in M:-\psi(z)\in U\}}|F|^2e^{-\varphi}\\&+\int_{\{z\in M:-\psi(z)\in U\backslash N\}}|F_{t_0,\tilde{c}}-F|^2e^{-\varphi}\\&+\int_{\{z\in M:-\psi(z)\in U\cap N\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi}\end{aligned}\tag{56}$$

holds for any open set $U\subset\subset(t_0,+\infty)$, and

$$\begin{aligned}\int_{\{z\in M:-\psi(z)\in V\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi}=&\int_{\{z\in M:-\psi(z)\in V\}}|F|^2e^{-\varphi}\\&+\int_{\{z\in M:-\psi(z)\in V\backslash N\}}|F_{t_0,\tilde{c}}-F|^2e^{-\varphi}\\&+\int_{\{z\in M:-\psi(z)\in V\cap N\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi}\end{aligned}\tag{57}$$

holds for any compact set $V\subset(t_0,+\infty)$. For any measurable set $E\subset\subset(t_0,+\infty)$, there exists a sequence of compact sets $\{V_l\}$, such that $V_l\subset V_{l+1}\subset E$ for any l and $\lim_{l\to}\mu(V_l)=\mu(E)$, hence

$$\begin{aligned}\int_{\{\psi<-t_0\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi}\mathbb{I}_E(-\psi)\geq&\lim_{l\to+\infty}\int_{\{\psi<-t_0\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi}\mathbb{I}_{V_j}(-\psi)\\\geq&\lim_{j\to+\infty}\int_{\{\psi<-t_0\}}|F|^2e^{-\varphi}\mathbb{I}_{V_j}(-\psi)\\=&\int_{\{\psi<-t_0\}}|F|^2e^{-\varphi}\mathbb{I}_E(-\psi).\end{aligned}\tag{58}$$

It is clear that for any $t>t_0$, there exists a sequence of functions $\{\sum_{j=1}^{n_i}a_{ij}\mathbb{I}_{E_{ij}}\}_{i=1}^{+\infty}$ defined on $(t,+\infty)$, satisfying $E_{ij}\subset\subset(t,+\infty)$, $\sum_{j=1}^{n_{i+1}}a_{i+1j}\mathbb{I}_{E_{i+1j}}(s)\geq\sum_{j=1}^{n_i}a_{ij}\mathbb{I}_{E_{ij}}(s)$, and $\lim_{i\to+\infty}\sum_{j=1}^{n_i}a_{ij}\mathbb{I}_{E_{ij}}(s)=\tilde{c}(s)$ for any $s>t$. Combining Levi's Theorem and inequality (58), we have

$$\int_{\{\psi<-t_0\}}|F_{t_0,\tilde{c}}|^2e^{-\varphi}\tilde{c}(-\psi)\geq\int_{\{\psi<-t_0\}}|F|^2e^{-\varphi}\tilde{c}(-\psi).\tag{59}$$

By the definition of $G(t_0,\tilde{c})$, we have $G(t_0,\tilde{c})=\int_{\{\psi<-t_0\}}|F|^2e^{-\varphi}\tilde{c}(-\psi)$. Then equality (2) holds.

§4. Proofs of Theorems 1.9 and 1.10, and Corollaries 1.11 and 1.12

In this section, we prove Theorems 1.9 and 1.10, and Corollaries 1.11 and 1.12.

4.1 Proof of Theorem 1.9

The following remark shows that it suffices to consider Theorem 1.9 for the case $c(t)$ has a positive lower bound and upper bound on $(t',+\infty)$ for any $t'>T$.

Remark 4.1. Take c_j is a positive measurable function on $(T,+\infty)$, such that $c_j(t)=c(t)$ when $t<T+j$, $c_j(t)=\min\{c(T+j),\frac{1}{j}\}$ when $t\geq T+j$. It is clear that $c_j(t)e^{-t}$ is

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decreasing with respect to t, and $\int_T^{+\infty} c_j(t)e^{-t} < +\infty$. As

$$\lim_{j\to+\infty}\int_{T+j}^{+\infty} c_n(t)e^{-t} = 0,$$

we have

$$\lim_{j\to+\infty}\int_T^{+\infty} c_j(t)e^{-t} = \int_T^{+\infty} c(t)e^{-t}.$$

If Theorem 1.9 holds in this case, then there exists a holomorphic $(n,0)$ form F_j on M such that $F_j|_S = f$ and

$$\int_M |F_j|^2 e^{-\varphi} c_j(-\psi) \le \left(\int_T^{+\infty} c_j(t)e^{-t}dt\right)\sum_{k=1}^n \frac{\pi^k}{k!}\int_{S_{n-k}} |f|^2 e^{-\varphi} dV_M[\psi].$$

Note that ψ has locally lower bound on $M\backslash\psi^{-1}(-\infty)$ and $\psi^{-1}(-\infty)$ is a closed subset of some analytic subset of M, it follows from Lemma 2.4 that there exists a subsequence of $\{F_j\}$, denoted still by $\{F_j\}$, which is uniformly convergent to a holomorphic $(n,0)$ form F on any compact subset of M and

$$\begin{aligned}\int_M |F|^2 e^{-\varphi} c(-\psi) &\le \lim_{j\to+\infty}\left(\int_T^{+\infty} c_j(t)e^{-t}dt\right)\sum_{k=1}^n \frac{\pi^k}{k!}\int_{S_{n-k}} |f|^2 e^{-\varphi} dV_M[\psi] \\ &= \left(\int_T^{+\infty} c(t)e^{-t}dt\right)\sum_{k=1}^n \frac{\pi^k}{k!}\int_{S_{n-k}} |f|^2 e^{-\varphi} dV_M[\psi].\end{aligned}$$

Since $F_j|_S = f$ for any j, we have $F|_S = f$.

By the definition of condition (ab), $\liminf_{t\to+\infty} c(t) > 0$, it suffices to prove the case that M is Stein manifold and $S_{reg} = S$. Without loss of generality, we can assume that $supp(\mathcal{O}_M/\mathcal{I}(\psi)) = S_{reg}$ (if $supp(\mathcal{O}_M/\mathcal{I}(\psi)) \ne S_{reg}$, there exists a analytic subset X of M such that (M,X) satisfies condition (ab) and $supp(\mathcal{O}_M/\mathcal{I}(\psi))\backslash S_{reg} \in X$).

Since M is Stein, we can find a sequence of Stein manifolds $\{D_m\}_{m=1}^{+\infty}$ satisfying $D_m \subset\subset D_{m+1}$ for any m and $\cup_{m=1}^{+\infty} D_m = M$, and there is a holomorphic $(n,0)$ form $\tilde{F}$ on M such that $\tilde{F}|_S = f$.

Note that $\int_{D_m} |\tilde{F}|^2 < +\infty$ for any m and

$$\int_{D_m} \mathbb{I}_{\{-t_0-1<\psi<-t_0\}} |\tilde{F}|^2 e^{-\varphi-\psi} < +\infty$$

for any m and $t_0 > T$. Using Lemma 2.1, for any D_m and $t_0 > T$, there exists a holomorphic $(n,0)$ form F_{m,t_0} on D_m, such that

$$\begin{aligned}&\int_{D_m} |F_{m,t_0} - (1-b_{t_0,1}(\psi))\tilde{F}|^2 e^{-\varphi-\psi+v_{t_0,1}(\psi)} c(-v_{t_0,1}(\psi)) \\ \le&\left(\int_T^{t_0+1} c(t)e^{-t}dt\right)\int_{D_m} \mathbb{I}_{\{-t_0-1<\psi<-t_0\}} |\tilde{F}|^2 e^{-\varphi-\psi},\end{aligned} \tag{60}$$

where $b_{t_0,1}(t) = \int_{-\infty}^t \mathbb{I}_{\{-t_0-1<s<-t_0\}} ds$, $v_{t_0,1}(t) = \int_{-t_0}^t b_{t_0,1}(s)ds - t_0$. Note that $e^{-\psi}$ is not locally integrable along S, and $b_{t_0,1}(t) = 0$ when $-t$ is large enough, then $(F_{m,t_0} - (1 - b_{t_0,1}(\psi))\tilde{F})|_{D_m\cap S} = 0$, and therefore $F_{m,t_0}|_{D_m\cap S} = f$.

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Note that $v_{t_0,1}(\psi) \geq \psi$ and $c(t)e^{-t}$ is decreasing, then the inequality (60) becomes

$$\begin{aligned}&\int_{D_m} |F_{m,t_0} - (1-b_{t_0,1}(\psi))\tilde{F}|^2 e^{-\varphi} c(-\psi)\\ \leq &\left(\int_T^{t_0+1} c(t)e^{-t}dt\right)\int_{D_m} \mathbb{I}_{\{-t_0-1<\psi<-t_0\}}|\tilde{F}|^2 e^{-\varphi-\psi}.\end{aligned} \tag{61}$$

As $\sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}}\frac{|f|^2}{dV_M}e^{-\varphi}dV_M[\psi] < +\infty$, by definition of $dV_M[\psi]$ and $supp(\mathcal{O}_M/\mathcal{I}(\psi)) = S_{reg}$, we have

$$\begin{aligned}&\limsup_{t_0\to+\infty}\left(\int_T^{t_0+1} c(t)e^{-t}dt\right)\int_{D_m} \mathbb{I}_{\{-t_0-1<\psi<-t_0\}}|\tilde{F}|^2 e^{-\varphi-\psi}\\ \leq &\left(\int_T^{+\infty} c(t)e^{-t}dt\right)\sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}\cap D_m}\frac{|f|^2}{dV_M}e^{-\varphi}dV_M[\psi]\\ < &+\infty.\end{aligned} \tag{62}$$

Note that $e^{-\varphi}c(-\psi)$ has a positive lower bound on D_m, then it follows from inequality (61) and (62) that $\sup_{t_0}\int_{D_m}|F_{m,t_0} - (1-b_{t_0,1}(\psi))\tilde{F}|^2 < +\infty$.

Combining with

$$\sup_{t_0}\int_{D_m}|(1-b_{t_0,1}(\psi))\tilde{F}|^2 \leq \sup_{t_0}\int_{D_m}\mathbb{I}_{\{\psi<-t_0\}}|\tilde{F}|^2 < +\infty, \tag{63}$$

one can obtain that $\sup_{t_0}\int_{D_m}|F_{m,t_0}|^2 < +\infty$, which implies that there exists a subsequence of $\{F_{m,t_0}\}_{t_0\to+\infty}$ (also denoted by $\{F_{m,t_0}\}_{t_0\to+\infty}$) compactly convergent to a holomorphic $(n,0)$ form on D_m denoted by F_m. Then it follows from inequality (61), inequality (62), and the Fatou's Lemma that

$$\begin{aligned}\int_{D_m}|F_m|^2 e^{-\varphi}c(-\psi) &= \int_{D_m}\liminf_{t_0\to+\infty}|F_{m,t_0} - (1-b_{t_0,1}(\psi))\tilde{F}|^2 e^{-\varphi}c(-\psi)\\ &\leq \liminf_{t_0\to+\infty}\int_{D_m}|F_{m,t_0} - (1-b_{t_0,1}(\psi))\tilde{F}|^2 e^{-\varphi}c(-\psi)\\ &\leq \limsup_{t_0\to+\infty}\left(\int_T^{t_0+1} c(t)e^{-t}dt\right)\int_{D_m} \mathbb{I}_{\{-t_0-1<\psi<-t_0\}}|\tilde{F}|^2 e^{-\varphi-\psi}\\ &\leq \left(\int_T^{+\infty} c(t)e^{-t}dt\right)\sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}\cap D_m}\frac{|f|^2}{dV_M}e^{-\varphi}dV_M[\psi]\\ &\leq \left(\int_T^{+\infty} c(t)e^{-t}dt\right)\sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}}\frac{|f|^2}{dV_M}e^{-\varphi}dV_M[\psi],\end{aligned} \tag{64}$$

and $F_m|_{D_m\cap S} = f$. Inequality (64) implies that

$$\int_{D_m}|F_{m'}|^2 e^{-\varphi}c(-\psi) \leq \left(\int_T^{+\infty} c(t)e^{-t}dt\right)\sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}}\frac{|f|^2}{dV_M}e^{-\varphi}dV_M[\psi]$$

holds for any $m' \geq m$. As $e^{-\varphi}c(-\psi)$ has a positive lower bound on any D_m, by the diagonal method, we obtain a subsequence of $\{F_m\}$, denoted also by $\{F_m\}$, which is uniformly convergent to a holomorphic $(n,0)$ form F on M on any compact subset of M satisfying

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that $F|_S = f$ and

$$\int_M |F|^2 e^{-\varphi} c(-\psi) \leq \left(\int_T^{+\infty} c(t)e^{-t}dt\right) \sum_{k=1}^{n} \frac{\pi^k}{k!} \int_{S_{n-k}} \frac{|f|^2}{dV_M} e^{-\varphi} dV_M[\psi].$$

Thus Theorem 1.9 holds.

4.2 Proof of Theorem 1.10

If $\sum_{k=1}^{n} \frac{\pi^k}{k!} \int_{S_{n-k}} \frac{|f|^2}{dV_M} e^{-\varphi} dV_M[\psi] = 0$, it is clear that $F \equiv 0$ satisfying all requirements in Theorem 1.10. In the following part, we consider the case $\sum_{k=1}^{n} \frac{\pi^k}{k!} \int_{S_{n-k}} \frac{|f|^2}{dV_M} e^{-\varphi} dV_M[\psi] \in (0,+\infty)$.

Using Theorem 1.9, for any $t > T$, there exists a holomorphic $(n,0)$ form F_t on $\{\psi < -t\}$ such that $F_t|_S = f$ and

$$\int_{\{\psi<-t\}} |F_t|^2 e^{-\varphi} c(-\psi) \leq \left(\int_t^{+\infty} c(l)e^{-l}dl\right) \sum_{k=1}^{n} \frac{\pi^k}{k!} \int_{S_{n-k}} \frac{|f|^2}{dV_M} e^{-\varphi} dV_M[\psi].$$

Then we have inequality

$$\frac{G(t)}{\int_t^{+\infty} c(l)e^{-l}dl} \leq \frac{G(T)}{\int_T^{+\infty} c(t)e^{-t}dt} \tag{65}$$

holds for any $t > T$. As (M,S) satisfies condition (ab), and $\psi \in A(S)$, Theorem 1.3 tells us $G(\hat{h}^{-1}(r))$ is concave with respect to r. Combining inequality (65) and Corollary 1.5, we obtain that $G(\hat{h}^{-1}(r))$ is linear with respect to r. Note that $\frac{G(T)}{\int_T^{+\infty} c(t)e^{-t}dt} = \|f\|_S$, Corollary 1.7 shows that the rest results of Theorem 1.10 hold.

4.3 Proof of Corollary 1.11

In this section, we prove Corollary 1.11 by using Theorem 1.9.

Since M is Stein, we can find a sequence of Stein manifolds $\{D_l\}_{l=1}^{+\infty}$ satisfying $D_l \subset\subset D_{l+1}$ for any l and $\cup_{l=1}^{+\infty} D_l = M$. Since ψ_2 and $\psi_2 + \varphi$ are plurisubharmonic functions on M, there exist smooth plurisubharmonic functions Ψ_m and $\Phi_{m'}$, which are decreasingly convergent to ψ_2 and $\psi_2 + \varphi$, respectively.

Fixed D_l, we can choose large enough m such that $\Psi_m + \psi_1 < -T$ on D_l. Note that $dV_M[\Psi_m + \psi_1] = e^{-\Psi_m} dV_M[\psi_1]$ and

$$\sum_{k=1}^{n} \frac{\pi^k}{k!} \int_{S_{n-k}} \frac{|f|^2}{dV_M} e^{-\Phi_{m'}} dV_M[\psi_1] \leq \sum_{k=1}^{n} \frac{\pi^k}{k!} \int_{S_{n-k}} \frac{|f|^2}{dV_M} e^{-\varphi-\psi_2} dV_M[\psi_1] < +\infty.$$

Using Theorem 1.9, for any D_l, there exists a holomorphic $(n,0)$ form $F_{l,m'}$ on D_l, satisfying $F_{l,m'}|_S = f$ and

$$\int_{D_l} |F_{l,m'}|^2 e^{-\Phi_{m'}+\Psi_m} c(-\Psi_m - \psi_1) \leq \left(\int_T^{+\infty} c(t)e^{-t}dt\right) \sum_{k=1}^{n} \frac{\pi^k}{k!} \int_{S_{n-k}} \frac{|f|^2}{dV_M} e^{-\Phi_{m'}} dV_M[\psi_1]. \tag{66}$$

As $e^{-\Phi_{m'}+\Psi_m} c(-\Psi_m - \psi_1)$ has locally uniformly positive lower bound for any m' on $D_m \backslash Z$, where Z is some analytic subset of M, it follows from Lemma 2.4 that there exists a subsequence of $\{F_{l,m'}\}_{m'\to+\infty}$, also denoted by $\{F_{l,m'}\}_{m'\to+\infty}$, which satisfies that $\{F_{l,m'}\}_{m'\to+\infty}$ is uniformly convergent to a holomorphic $(n,0)$ form F_l on any compact

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subset of D_l. Following from inequality (66), Fatou's Lemma and $c(t)e^{-t}$ is decreasing, we have

$$\begin{aligned}\int_{D_l}|F_l|^2e^{-\varphi}c(-\psi_2-\psi_1) &\leq \int_{D_l}|F_l|^2e^{-\varphi-\psi_2+\Psi_m}c(-\Psi_m-\psi_1)\\ &= \int_{D_l}\lim_{m'\to+\infty}|F_l|^2e^{-\Phi_{m'}+\Psi_m}c(-\Psi_m-\psi_1)\\ &\leq \liminf_{m'\to+\infty}\int_{D_l}|F_l|^2e^{-\Phi_{m'}+\Psi_m}c(-\Psi_m-\psi_1)\\ &\leq \left(\int_T^{+\infty}c(t)e^{-t}dt\right)\sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}}\frac{|f|^2}{dV_M}e^{-\varphi-\psi_2}dV_M[\psi_1].\end{aligned} \tag{67}$$

Note that $e^{-\varphi}c(-\psi_2-\psi_1)$ has locally a positive lower bound on $M\backslash Z$, where Z is some analytic subset of M, by using Lemma 2.4 and the diagonal method, we obtain that there exists a subsequence of $\{F_l\}$, also denoted by $\{F_l\}$, which satisfies that $\{F_l\}$ is uniformly convergent to a holomorphic $(n,0)$ form F on M on any compact subset of M. Following from inequality (67) and Fatou's Lemma, we have

$$\begin{aligned}\int_{M}|F|^2e^{-\varphi}c(-\psi_2-\psi_1) &= \int_{M}\lim_{l\to+\infty}\mathbb{I}_{D_l}|F_l|^2e^{-\varphi}c(-\psi_2-\psi_1)\\ &\leq \liminf_{l\to+\infty}\int_{D_l}|F_l|^2e^{-\varphi}c(-\psi_2-\psi_1)\\ &\leq \left(\int_T^{+\infty}c(t)e^{-t}dt\right)\sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}}\frac{|f|^2}{dV_M}e^{-\varphi-\psi_2}dV_M[\psi_1].\end{aligned} \tag{68}$$

This proves Corollary 1.11.

4.4 Proof of Corollary 1.12

If $\|f\|_S^*=0$, it is clear that $F\equiv 0$ satisfying all requirements in Corollary 1.12. In the following part, we consider the case $\|f\|_S^*\in(0,+\infty)$.

Using Corollary 1.11, for any $t>T$, there exists a holomorphic $(n,0)$ form F_t on $\{\psi<-t\}$ such that $F_t|_S=f$ and

$$\int_{\{\psi<-t\}}|F_t|^2e^{-\varphi}c(-\psi)\leq\left(\int_t^{+\infty}c(l)e^{-l}dl\right)\|f\|_S^*.$$

Then we have inequality

$$\frac{G(t)}{\int_t^{+\infty}c(l)e^{-l}dl}\leq\frac{G(T)}{\int_T^{+\infty}c(t)e^{-t}dt} \tag{69}$$

holds for any $t>T$. Theorem 1.3 tells us $G(\hat{h}^{-1}(r))$ is concave with respect to r. Combining inequality (69) and Corollary 1.5, we obtain that $G(\hat{h}^{-1}(r))$ is linear with respect to r. Note that $\frac{G(T)}{\int_T^{+\infty}c(t)e^{-t}dt}=\|f\|_M^*$, Corollary 1.7 shows that the rest results of Theorem 1.12 hold.

§5. Proofs of Theorems 1.13–1.15

In this section, we prove Theorems 1.13–1.15.

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5.1 Proof of Theorem 1.13

We prove the theorem by comparing $G(t;\varphi)$ and $G(t;\tilde{\varphi})$. Let us assume that $G(\hat{h}^{-1}(r);\varphi)$ is linear with respect to r to get a contradiction.

As $G(\hat{h}^{-1}(r);\varphi)$ is linear with respect to r, it follows from Corollary 1.7 that there exists a holomorphic $(n,0)$ form F on M such that $(F-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$ and $\forall t\geq T$ equality

$$G(t;\varphi)=\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)$$

holds. As $\tilde{\varphi}+\psi$ is plurisubharmonic and $\tilde{\varphi}-\varphi$ is bounded on M, it follows from Theorem 1.3 that $G(\hat{h}^{-1}(r);\tilde{\varphi})$ is concave with respect to r.

As $\tilde{\varphi}+\psi\geq\varphi+\psi$, $\tilde{\varphi}+\psi\neq\varphi+\psi$ and both of them are plurisubharmonic functions on M, then there exists a subset U of M such that $e^{-\tilde{\varphi}}<e^{-\varphi}$ on a subset U and $\mu(U)>0$, where μ is Lebesgue measure on M. As $F\not\equiv 0$, inequality

$$\frac{G(T_0;\tilde{\varphi})}{\int_{T_0}^{+\infty}c(s)e^{-s}ds}\leq\frac{\int_{\{\psi<-T_0\}}|F|^2e^{-\tilde{\varphi}}c(-\psi)}{\int_{T_0}^{+\infty}c(s)e^{-s}ds}<\frac{G(T_0;\varphi)}{\int_{T_0}^{+\infty}c(s)e^{-s}ds}\tag{70}$$

holds for some $T_0>T$. For $t>T$, there exists a holomorphic $(n,0)$ form F_t on $\{\psi<-t\}$ such that $(F_t-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$ and

$$G(t;\tilde{\varphi})=\int_{\{\psi<-t\}}|F_t|^2e^{-\tilde{\varphi}}c(-\psi)<+\infty.$$

As $\tilde{\varphi}-\varphi$ is bounded on M, we have $\int_{\{\psi<-t\}}|F_t|^2e^{-\varphi}c(-\psi)<+\infty$. It follows from Lemma 2.6 that

$$\begin{aligned}G(t_1;\tilde{\varphi})-G(t_2;\tilde{\varphi})&\geq\int_{\{-t_2\leq\psi<-t_1\}}|F_{t_1}|^2e^{-\tilde{\varphi}}c(-\psi)\\&\geq\left(\inf_{\{-t_2\leq\psi\}}e^{\varphi-\tilde{\varphi}}\right)\int_{\{-t_2\leq\psi<-t_1\}}|F_{t_1}|^2e^{-\varphi}c(-\psi)\\&\geq\left(\inf_{\{-t_2\leq\psi\}}e^{\varphi-\tilde{\varphi}}\right)\int_{\{-t_2\leq\psi<-t_1\}}|F|^2e^{-\varphi}c(-\psi)\end{aligned}\tag{71}$$

holds for $T<t_1<t_2<+\infty$. As $\lim_{t\to T+0}\sup_{z\in\{\psi\geq-t\}}((\tilde{\varphi}-\varphi)(z))=0$, it follows from inequality (70) and (71) that

$$\begin{aligned}\liminf_{t_2\to T+0}\frac{G(t_1;\tilde{\varphi})-G(t_2;\tilde{\varphi})}{\int_{t_1}^{t_2}c(s)e^{-s}ds}&\geq\liminf_{t_2\to T+0}\left(\inf_{z\in\{-t_2\leq\psi\}}e^{\varphi-\tilde{\varphi}}\right)\frac{\int_{\{-t_2\leq\psi<-t_1\}}|F|^2e^{-\varphi}c(-\psi)}{\int_{t_1}^{t_2}c(s)e^{-s}ds}\\&=\frac{G(T_0;\varphi)}{\int_{T_0}^{+\infty}c(s)e^{-s}ds}\\&>\frac{G(T_0;\tilde{\varphi})}{\int_{T_0}^{+\infty}c(s)e^{-s}ds},\end{aligned}$$

which contradicts the concavity of $G(\hat{h}^{-1}(r);\tilde{\varphi})$. Thus the assumption does not hold, that is, $G(\hat{h}^{-1}(r);\varphi)$ is not linear with respect to r.

Especially, if $\varphi+\psi$ is strictly plurisubharmonic at $z_1\in M$, we can construct a $\tilde{\varphi}\geq\varphi$ satisfying the three statements in Theorem 1.13, which implies $G(\hat{h}^{-1}(r);\varphi)$ is not linear with respect to r. In fact, there is a small open neighborhood (U,w) of z_1 and

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$w=(w_1,\ldots,w_n)$ is the local coordinate on U such that $i\partial\bar{\partial}(\varphi+\psi)>\epsilon\omega$ for some $\epsilon>0$, where $\omega=i\sum_{j=1}^n dw_j\wedge d\bar{w}_j$ on U. Let ρ be a smooth nonnegative function on M satisfying $\rho\not\equiv 0$ and $supp\rho\subset\subset U$. It is clear that there exists a positive number δ such that

$$i\partial\bar{\partial}(\varphi+\psi+\delta\rho)>0$$

holds on U. Let $\tilde{\varphi}=\varphi+\delta\rho$, it is clear that $\tilde{\varphi}$ satisfies the three statements in Theorem 1.13. Thus we complete the proof of Theorem 1.13.

5.2 Proof of Theorem 1.14

Let $\tilde{\varphi}=\varphi+\psi-\tilde{\psi}$, then $\tilde{\varphi}+\tilde{\psi}=\varphi+\psi$ is a plurisubharmonic function on M. We prove the theorem by comparing $G(t;\varphi,\psi)$ and $G(t;\tilde{\varphi},\tilde{\psi})$. Let us assume that $G(\hat{h}^{-1}(r);\varphi,\psi)$ is linear with respect to r to get a contradiction.

Since $G(T;\varphi,\psi)\in(0,+\infty)$, $G(\hat{h}^{-1}(r))$ is linear and Corollary 1.7, we have $\int_T^{+\infty}c(t)e^{-t}dt<+\infty$. As $G(\hat{h}^{-1}(r);\varphi,\psi)$ is linear with respect to r, it follows from Corollary 1.7, Remark 1.8, and Lemma 2.10 that we can assume $c(t)e^{-t}$ is strictly decreasing on $(T,+\infty)$ and $c(t)$ is increasing on $(a,+\infty)$ for some $a>T$.

Using Corollary 1.7, there exists a holomorphic $(n,0)$ form F on M, such that $(F-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$ and $\forall t\geq T$ equality

$$G(t;\varphi,\psi)=\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)$$

holds.

Since $\lim_{t\to+\infty}\sup_{\{\psi<-t\}}(\tilde{\psi}-\psi)=0$, we have $Z_0\subset\{\psi=-\infty\}=\{\tilde{\psi}=-\infty\}$. As $c(t)e^{-t}$ is decreasing and $\tilde{\psi}\geq\psi$, we have $e^{-\varphi}c(-\psi)=e^{-\varphi-\psi}e^{\psi}c(-\psi)\leq e^{-\tilde{\varphi}-\tilde{\psi}}e^{\tilde{\psi}}c(-\tilde{\psi})=e^{-\tilde{\varphi}}c(-\tilde{\psi})$. It follows from Theorem 1.3 that $G(\hat{h}^{-1}(r);\tilde{\varphi},\tilde{\psi})$ is concave with respect to r.

We claim that

$$\lim_{t\to T+0}\frac{G(t;\tilde{\varphi},\tilde{\psi})}{\int_t^{+\infty}c(s)e^{-s}ds}>\frac{G(T;\varphi,\psi)}{\int_T^{+\infty}c(s)e^{-s}ds}. \tag{72}$$

In fact, we just need to prove the inequality for the case $G(T;\tilde{\varphi},\tilde{\psi})<+\infty$. It follows from Lemma 2.6 that there exists a holomorphic $(n,0)$ form F_T on M such that $(F_T-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$ and

$$G(T;\tilde{\varphi},\tilde{\psi})=\int_M|F_T|^2e^{-\tilde{\varphi}}c(-\tilde{\psi})\in(0,+\infty),$$

where $G(T;\tilde{\varphi},\tilde{\psi})>0$ follows from $G(T;\varphi,\psi)>0$. As $\tilde{\psi}\geq\psi$, $\tilde{\psi}\neq\psi$ and both of them are plurisubharmonic functions on M, then there exists a subset U of M such that $\tilde{\psi}>\psi$ on a subset U and $\mu(U)>0$, where μ is Lebesgue measure on M. As $F_T\not\equiv 0$ and $c(t)e^{-t}$ is strictly decreasing on $(T,+\infty)$, we have

$$\begin{aligned}\frac{G(T;\tilde{\varphi},\tilde{\psi})}{\int_T^{+\infty}c(s)e^{-s}ds}&=\frac{\int_M|F_T|^2e^{-\tilde{\varphi}}c(-\tilde{\psi})}{\int_T^{+\infty}c(s)e^{-s}ds}\\&>\frac{\int_M|F_T|^2e^{-\varphi}c(-\psi)}{\int_T^{+\infty}c(s)e^{-s}ds}\\&\geq\frac{G(T;\varphi,\psi)}{\int_T^{+\infty}c(s)e^{-s}ds}.\end{aligned}$$

Then the claim holds.

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As $c(t)$ is increasing on $(a,+\infty)$ and $\lim_{t\to+\infty}\sup_{\{\psi<-t\}}(\tilde{\psi}-\psi)=0$, we obtain that

$$\begin{aligned}\lim_{t\to+\infty}\frac{G(t;\tilde{\varphi},\tilde{\psi})}{\int_t^{+\infty}c(s)e^{-s}ds}&\leq\lim_{t\to+\infty}\frac{\int_{\{\tilde{\psi}<-t\}}|F|^2e^{-\tilde{\varphi}}c(-\tilde{\psi})}{\int_t^{+\infty}c(s)e^{-s}ds}\\&\leq\lim_{t\to+\infty}\frac{\int_{\{\psi<-t\}}|F|^2e^{-\varphi-\psi}e^{\tilde{\psi}}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds}\\&\leq\lim_{t\to+\infty}\left(\sup_{\{\psi<-t\}}e^{\tilde{\psi}-\psi}\right)\frac{\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds}\\&=\frac{\int_{\{\psi<-T\}}|F|^2e^{-\varphi}c(-\psi)}{\int_T^{+\infty}c(s)e^{-s}ds}.\end{aligned}\tag{73}$$

Combining inequality (72) and (73), we have

$$\lim_{t\to+\infty}\frac{G(t;\tilde{\varphi},\tilde{\psi})}{\int_t^{+\infty}c(s)e^{-s}ds}<\lim_{t\to T+0}\frac{G(t;\tilde{\varphi},\tilde{\psi})}{\int_t^{+\infty}c(s)e^{-s}ds},$$

which contradicts the concavity of $G(\hat{h}^{-1}(r);\tilde{\varphi},\tilde{\psi})$. Thus the assumption does not hold, that is, $G(\hat{h}^{-1}(r);\varphi,\psi)$ is not linear with respect to r.

Especially, if ψ is strictly plurisubharmonic at $z_1\in M\backslash(\cap_t\overline{\{\psi<-t\}})$, we can construct a $\tilde{\psi}\geq\psi$ satisfying the three statements in Theorem 1.14, which implies $G(\hat{h}^{-1}(r);\varphi,\psi)$ is not linear with respect to r. In fact, there is a small open neighborhood (U,w) of z_1 and $w=(w_1,\dots,w_n)$ is the local coordinate on U such that $U\cap(\cap_t\overline{\{\psi<-t\}})=\emptyset$ and $i\partial\bar{\partial}\psi>\epsilon\omega$ for some $\epsilon>0$, where $\omega=i\sum_{j=1}^n dw_j\wedge d\bar{w}_j$ on U. Let ρ be a smooth nonnegative function on M satisfying $\rho\not\equiv0$ and $supp\rho\subset\subset U$. It is clear that there exists a positive number δ such that

$$i\partial\bar{\partial}(\psi+\delta\rho)>0$$

holds on U and $\psi+\delta\rho<-T$ on M. Let $\tilde{\psi}=\psi+\delta\rho$, it is clear that $\tilde{\psi}$ satisfies the three statements in Theorem 1.14. Thus we complete the proof of Theorem 1.14.

5.3 A limiting property of $G(t)$

The following proposition gives a limiting property of $G(t)$, which will be used in the proof of Theorem 1.15 and Corollary 1.18.

PROPOSITION 5.1. *Let M be an n-dimensional Stein manifold, and let S be an analytic subset of M. Let $c\in P_T$, and let $(\varphi,\psi)\in W$. Let $\mathcal{F}|_{Z_0}=\mathcal{I}(\psi_1)|_{S_{reg}}$. Assume that $G(T)\in(0,+\infty)$ and $\psi_2(z)>-\infty$ for almost every $z\in S_{reg}$.*

Assume that $c(t)$ is increasing on $(a,+\infty)$ for some $a>T$. Then we have

$$\lim_{t\to+\infty}\frac{G(t)}{\int_t^{+\infty}c(l)e^{-l}dl}=\sum_{k=1}^n\frac{\pi^k}{k!}\int_{S_{n-k}}\frac{|f|^2}{dV_M}e^{-\varphi-\psi_2}dV_M[\psi_1].\tag{74}$$

Proof. $\lim_{t\to+\infty}\frac{G(t)}{\int_t^{+\infty}c(l)e^{-l}dl}\le\sum_{k=1}^n\frac{\pi^k}{k!}\int_{S_{n-k}}\frac{|f|^2}{dV_M}e^{-\varphi-\psi_2}dV_M[\psi_1]$ can be obtained by using Corollary 1.11. Thus, we just need to prove that

$$\lim_{t\to+\infty}\frac{G(t)}{\int_t^{+\infty}c(l)e^{-l}dl}\ge\sum_{k=1}^n\frac{\pi^k}{k!}\int_{S_{n-k}}\frac{|f|^2}{dV_M}e^{-\varphi-\psi_2}dV_M[\psi_1].$$

For any $t\ge T$, there exists a holomorphic $(n,0)$ form F_t on $\{\psi<-t\}$, such that $F_t|_S=f$ and $\int_{\{\psi<-t\}}|F_t|^2e^{-\varphi}c(-\psi)=G(t)$.

Let $\{U^\alpha\}_{\alpha\in\mathbb{N}}$ be a coordinate patches of $M\backslash S_{sing}$, biholomorphic to polydisks, and admit the following property: if $U^\alpha\cap S_{reg}\ne\emptyset$, and we denote the corresponding coordinates by $(z^\alpha,w^\alpha)\in\Delta^l\times\Delta^{n-l}$, where $z^\alpha=(z_1^\alpha,\dots,z_l^\alpha)$ and $w^\alpha=(w_1^\alpha,\dots,w_{n-l}^\alpha)$ for some $l\in\{0,1,2\dots,n-1\}$, then $U^\alpha\cap S=U^\alpha\cap S_l=\{w^\alpha=0\}$. Let $\{v^\alpha\}$ be a partition of unity subordinate to $\{U^\alpha\}$.

As $\varphi+\psi_2$ is plurisubharmonic, then there exist smooth plurisubharmonic functions Φ_n on M decreasingly convergent to $\varphi+\psi_2$. Thus, we have

$$\int_{\{\psi<-t\}}v^\alpha|F_t|^2e^{-\varphi}c(-\psi)\ge\int_{\{\psi<-t\}}v^\alpha|F_t|^2e^{-\Phi_n+\psi_2}c(-\psi) \tag{75}$$

for any $n\in\mathbb{N}$.

Firstly, we consider $\int_{\{\psi<-t\}}v^\alpha|F_t|^2e^{-\Phi_n+\psi_2}c(-\psi)$, where $U^\alpha\cap S_l\ne\emptyset$.

Note that $\psi=\psi_1+\psi_2$ and $\psi_1\in A'(S)$, then for small enough $s>0$, $\psi_1=(n-l)\log(|w^\alpha|^2)+h_1$ on $\Delta^l\times\{|w^\alpha|<s\}$ and h_1 is continuous on $\Delta^l\times\{|w^\alpha|<s\}$. For any $\epsilon>0$, there exists $s>0$ such that $v^\alpha(z^\alpha,w^\alpha)\ge\max\{v^\alpha(z^\alpha,0)-\epsilon,0\}$, $\Phi_n(z^\alpha,w^\alpha)\le\Phi_n(z^\alpha,0)+\epsilon$, and $h_1(z^\alpha,w^\alpha)\le h_1(z^\alpha,0)+\epsilon$ on $\Delta^l\times\{|w^\alpha|<s\}$. Let $\psi_s(z^\alpha)=\sup_{|w^\alpha|<s}\psi_2(z^\alpha,w^\alpha)$. As $\psi_2(z)>-\infty$ for almost every $z\in S_{reg}$, we know $\psi_s(z^\alpha)>-\infty$ for almost every $z^\alpha\in\Delta^l$. Let $v_\epsilon^\alpha:=\max\{v^\alpha(z^\alpha,0)-\epsilon,0\}$. As $c(t)$ is increasing for $t>a$, then we have

$$\begin{aligned}&\int_{\{\psi<-t\}}v^\alpha|F_t|^2e^{-\Phi_n+\psi_2}c(-\psi)\\ \ge&\int_{\{\psi_2+h_1+(n-l)\log(|w^\alpha|^2)<-t\}\cap\{|w^\alpha|<s\}}v_\epsilon^\alpha|F_t|^2e^{-\Phi_n(z^\alpha,0)-\epsilon+\psi_2}c(-\psi)\\ \ge&\int_{\{\psi_s+h_1(z^\alpha,0)+\epsilon+(n-l)\log(|w^\alpha|^2)<-t\}\cap\{|w^\alpha|<s\}}v_\epsilon^\alpha|F_t|^2e^{-\Phi_n(z^\alpha,0)-\epsilon+\psi_2}\\ &\times c(-\psi_s-h_1(z^\alpha,0)-\epsilon-(n-l)\log(|w^\alpha|^2))\end{aligned} \tag{76}$$

for $t>a$.

Without loss of generality, assume that $dV_M=(\wedge_{k=1}^l idz_k^\alpha\wedge d\bar z_k^\alpha)\wedge(\wedge_{k=1}^{n-l}idw_k^\alpha\wedge d\bar w_k^\alpha)$, $dV_\alpha=\wedge_{k=1}^l idz_k^\alpha\wedge d\bar z_k^\alpha$ on U^α, and $dV'_\alpha=\wedge_{k=1}^{n-l}idw_k^\alpha\wedge d\bar w_k^\alpha$. Let $h_2(z^\alpha):=\psi_s(z^\alpha)+h_1(z^\alpha,0)+\epsilon$. As $\frac{|F_t|^2}{dV_M}e^{\psi_2}$ is plurisubharmonic on $\Delta^l\times\{|w^\alpha|<s\}$, then we obtain that inequality

$$\begin{aligned}&\int_{\{h_2+(n-l)\log(|w^\alpha|^2)<-t\}\cap\{|w^\alpha|<s\}}\frac{|F_t|^2}{dV_M}e^{\psi_2}c(-h_2-(n-l)\log(|w^\alpha|^2))dV'_\alpha\\ \ge&\frac{|f(z^\alpha,0)|^2}{dV_M}e^{\psi_2(z^\alpha,0)}\int_{\{h_2+(n-l)\log(|w^\alpha|^2)<-t\}\cap\{|w^\alpha|<s\}}c(-h_2-(n-l)\log(|w^\alpha|^2))dV'_\alpha\\ =&2^{n-l}\frac{\sigma_{2n-2l-1}}{2(n-l)}\frac{|f(z^\alpha,0)|^2}{dV_M}e^{\psi_2(z^\alpha,0)}\int_{\max\{t,-h_2(z^\alpha)-2(n-l)\log(s)\}}^{+\infty}c(l)e^{-l}dl\end{aligned} \tag{77}$$

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holds for any $z^\alpha \in \Delta^l$. It follows from inequality (76) and (77) that

$$\begin{aligned}
&\int_{\{\psi<-t\}} v^\alpha |F_t|^2 e^{-\Phi_n+\psi_2} c(-\psi) \\
\geq &\int_{\Delta^l} v_\epsilon^\alpha e^{-\Phi_n(z^\alpha,0)-\epsilon} \\
&\times \int_{\{h_2+(n-l)\log(|w^\alpha|^2)<-t\}\cap\{|w^\alpha|<s\}} \frac{|F_t|^2}{dV_M} e^{\psi_2} c(-h_2-(n-l)\log(|w^\alpha|^2)) dV_M \\
\geq &2^{n-l}\frac{\sigma_{2n-2l-1}}{2(n-l)} \int_{\Delta^l} v_\epsilon^\alpha e^{-\Phi_n(z^\alpha,0)-\epsilon} \frac{|f(z^\alpha,0)|^2}{dV_M} e^{\psi_2(z^\alpha,0)} e^{-h_2} \\
&\times \left(\int_{\max\{t,-h_2(z^\alpha)-2(n-l)\log(s)\}}^{+\infty} c(l)e^{-l} dl\right) dV_\alpha
\end{aligned} \tag{78}$$

for $t>a$.

Next, we prove that

$$\liminf_{t\to+\infty} \frac{\int_{\{\psi<-t\}} v^\alpha |F_t|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(l)e^{-l}dl} \geq \frac{\pi^{n-l}}{(n-l)!} \int_{S_l} v^\alpha e^{-\varphi-\psi_2} \frac{|f(z^\alpha,0)|^2}{dV_M} dV_M[\psi_1]. \tag{79}$$

It follows from $\psi_s(z^\alpha)>-\infty$ for almost every $z^\alpha \in \Delta^l$ that $h_2(z^\alpha)>-\infty$ for almost every $z^\alpha \in \Delta^l$. Thus, we have

$$\liminf_{t\to+\infty} \frac{\int_{\max\{t,-h_2(z^\alpha)-2(n-l)\log(s)\}}^{+\infty} c(l)e^{-l}dl}{\int_t^{+\infty} c(l)e^{-l}dl} = 1 \tag{80}$$

for almost every $z^\alpha \in \Delta^l$. Combining inequality (78), equality (80), and Fatou's Lemma, we have

$$\begin{aligned}
&\liminf_{t\to+\infty} \frac{\int_{\{\psi<-t\}} v^\alpha |F_t|^2 e^{-\Phi_n+\psi_2} c(-\psi)}{\int_t^{+\infty} c(l)e^{-l}dl} \\
\geq &2^{n-l}\frac{\sigma_{2n-2l-1}}{2(n-l)} \int_{\Delta^l} v_\epsilon^\alpha e^{-\Phi_n(z^\alpha,0)-\epsilon} \frac{|f(z^\alpha,0)|^2}{dV_M} e^{\psi_2(z^\alpha,0)} e^{-h_2} \\
&\times \liminf_{t\to+\infty} \frac{\int_{\max\{t,-h_2(z^\alpha)-2(n-l)\log(s)\}}^{+\infty} c(l)e^{-l}dl}{\int_t^{+\infty} c(l)e^{-l}dl} dV_\alpha \\
= &2^{n-l}\frac{\sigma_{2n-2l-1}}{2(n-l)} \int_{\Delta^l} v_\epsilon^\alpha e^{-\Phi_n(z^\alpha,0)-\epsilon} \frac{|f(z^\alpha,0)|^2}{dV_M} e^{\psi_2(z^\alpha,0)} e^{-h_2} dV_\alpha.
\end{aligned} \tag{81}$$

As $dV_M = dV_\alpha \wedge (\wedge_{k=1}^{n-l} i dw_k^\alpha \wedge d\bar{w}_k^\alpha)$ and $\psi_1 = (n-l)\log(|w^\alpha|^2)+h_1$, by definition of $dV_M[\psi_1]$, we have $dV_M[\psi_1] = 2^{n-l}e^{-h_1}dV_\alpha$ on $\Delta^l \subset S_l$. Then inequality (81) becomes

$$\begin{aligned}
&\liminf_{t\to+\infty} \frac{\int_{\{\psi<-t\}} v^\alpha |F_t|^2 e^{-\Phi_n+\psi_2} c(-\psi)}{\int_t^{+\infty} c(l)e^{-l}dl} \\
\geq &\frac{\sigma_{2n-2l-1}}{2(n-l)} \int_{S_l} v_\epsilon^\alpha e^{-\Phi_n(z^\alpha,0)-\epsilon} \frac{|f(z^\alpha,0)|^2}{dV_M} e^{\psi_2(z^\alpha,0)-\psi_s(z^\alpha)-\epsilon} dV_M[\psi_1].
\end{aligned} \tag{82}$$

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When $s \to 0$, $\psi_s(z^\alpha)$ is decreasing to $\psi_2(z^\alpha,0)$ for any $z^\alpha \in \Delta^l$. As $\psi_2(z^\alpha,0) > -\infty$ for almost every $z^\alpha \in \Delta^l$, let $s \to 0$ and $\epsilon \to 0$, then inequality (82) implies that

$$\begin{aligned}&\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}} v^\alpha |F_t|^2 e^{-\Phi_n+\psi_2} c(-\psi)}{\int_t^{+\infty} c(l)e^{-l}dl}\\ \geq &\frac{\pi^{n-l}}{(n-l)!}\int_{S_l} v^\alpha e^{-\Phi_n(z^\alpha,0)}\frac{|f(z^\alpha,0)|^2}{dV_M}dV_M[\psi_1].\end{aligned}\tag{83}$$

Note that Φ_n decreasing to $\varphi+\psi_2$, then inequality (83) implies that inequality (79) holds.

Following from inequality (79) and the concavity of $G(t)$, we have

$$\lim_{t\to+\infty}\frac{G(t)}{\int_t^{+\infty}c(l)e^{-l}dl} \geq \sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}}\frac{|f|^2}{dV_M}e^{-\varphi-\psi_2}dV_M[\psi_1].$$

Thus, Proposition 5.1 holds. ∎

5.4 Proof of Theorem 1.15

Assume that $G(\hat{h}^{-1}(r))$ is linear with respect to r. As $G(T) \in (0,+\infty)$, we have $\int_T^{+\infty} c(t)e^{-t}dt < +\infty$. It follows from Corollary 1.7, Remark 1.8, and Lemma 2.10 that we can assume $c(t)e^{-t}$ is strictly decreasing on $(T,+\infty)$ and $c(t)$ is increasing on $(a,+\infty)$ for some $a > T$. Thus, Proposition 5.1 shows that equality (5) holds.

In the following part, assume that there exists $\tilde{\psi}$ satisfying the three statements in Theorem 1.15 to get a contradiction. We prove it by comparing $G(t;\varphi,\psi)$ and $G(t;\tilde{\varphi},\tilde{\psi})$, where $\tilde{\varphi} = \varphi+\psi-\tilde{\psi}$. It follows from Proposition 5.1 and the linearity of $G(\hat{h}^{-1}(r);\varphi,\psi)$ that $\sum_{k=1}^{n}\int_{S_{n-k}}\frac{\pi^k}{k!}\frac{|F|^2}{dV_M}e^{-\varphi-\psi_2}dV_M[\psi_1] < +\infty$ and equality

$$\frac{G(t;\varphi,\psi)}{\int_t^{+\infty}c(l)e^{-l}dl} = \sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}}\frac{|F|^2}{dV_M}e^{-\varphi-\psi_2}dV_M[\psi_1]\tag{84}$$

holds for any $t \geq T$.

As $(\tilde{\varphi},\tilde{\psi}) \in W$, there exist plurisubharmonic functions $\tilde{\psi}_1$ and $\tilde{\psi}_2$ such that $\tilde{\psi} = \tilde{\psi}_1+\tilde{\psi}_2$, $\tilde{\psi}_1 \in A'(S)$ and $\tilde{\varphi}+\tilde{\psi}_2$ is plurisubharmonic on M. $dV_M[\psi_1] = e^{-\psi_1+\tilde{\psi}_1}dV_M[\tilde{\psi}_1]$ implies that

$$\sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}}\frac{|F|^2}{dV_M}e^{-\tilde{\varphi}-\tilde{\psi}_2}dV_M[\tilde{\psi}_1] = \sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}}\frac{|F|^2}{dV_M}e^{-\varphi-\psi_2}dV_M[\psi_1] < +\infty.$$

It follows from Corollary 1.11 that

$$\begin{aligned}\frac{G(T;\tilde{\varphi},\tilde{\psi})}{\int_T^{+\infty}c(l)e^{-l}dl} &\leq \sum_{k=1}^{n}\int_{S_{n-k}}\frac{\pi^k}{k!}\frac{|F|^2}{dV_M}e^{-\tilde{\varphi}-\tilde{\psi}_2}dV_M[\tilde{\psi}_1]\\ &= \sum_{k=1}^{n}\int_{S_{n-k}}\frac{\pi^k}{k!}\frac{|F|^2}{dV_M}e^{-\varphi-\psi_2}dV_M[\psi_1].\end{aligned}\tag{85}$$

Since $\tilde{\psi} \geq \psi$, $\tilde{\psi} \neq \psi$, there exists a subset U of M such that $\mu(U) > 0$ and $\tilde{\psi} > \psi$ on U, where μ is Lebesgue measure on M. As $c(t)e^{-t}$ is strictly decreasing on $(T,+\infty)$, we have

$G(T;\tilde{\varphi},\tilde{\psi})>G(T;\varphi,\psi)$. Then inequality (85) implies that

$$\sum_{k=1}^{n}\int_{S_{n-k}}\frac{\pi^k}{k!}\frac{|F|^2}{dV_M}e^{-\varphi-\psi_2}dV_M[\psi_1]\geq\frac{G(T;\tilde{\varphi},\tilde{\psi})}{\int_T^{+\infty}c(l)e^{-l}dl}>\frac{G(T;\varphi,\psi)}{\int_T^{+\infty}c(l)e^{-l}dl},$$

which contradicts equality (84). Thus Theorem 1.15 holds.

§6. Proofs of Theorem 1.16, Theorem 1.17, Corollary 1.18 and Corollary 1.19

In this section, we prove Theorems 1.16 and 1.17, and Corollaries 1.18 and 1.19.

6.1 A necessary condition of linearity

The following proposition give a necessary condition of $G(\hat{h}^{-1}(r))$ is linear, and will be used in the proof of Theorem 1.16.

PROPOSITION 6.1. *Let Ω be an open Riemann surface. Let $c\in\mathcal{P}_0$, and assume that there exists $t\geq 0$ such that $G(t)\in(0,+\infty)$. If $G(\hat{h}^{-1}(r))$ is linear with respect to r, then there is no Lebesgue measurable function $\tilde{\varphi}\geq\varphi$ such that $\tilde{\varphi}+\psi$ is subharmonic function on M and satisfies:*

(1) *$\tilde{\varphi}\neq\varphi$ and $\mathcal{I}(\tilde{\varphi}+\psi)=\mathcal{I}(\varphi+\psi)$;*
(2) *$\lim_{t\to 0+0}\sup_{\{\psi\geq -t\}}(\tilde{\varphi}-\varphi)=0$;*
(3) *there exists an open subset $U\subset\subset\Omega$ such that $\sup_{\Omega\setminus U}(\tilde{\varphi}-\varphi)<+\infty$, $e^{-\tilde{\varphi}}c(-\psi)$ has a positive lower bound on U and $\int_U|F_1-F_2|^2e^{-\varphi}c(-\psi)<+\infty$ for any $F_1\in\mathcal{H}^2(c,\tilde{\varphi},t)$ and $F_2\in\mathcal{H}^2(c,\varphi,t)$, where $U\subset\subset\{\psi<-t\}$.*

Proof. We prove the lemma by comparing $G(t;\varphi)$ and $G(t;\tilde{\varphi})$. In the following, let us assume that there exists a Lebesgue measurable function $\tilde{\varphi}$ satisfying these properties in Proposition 6.1 to get a contradiction.

As $G(\hat{h}^{-1}(r);\varphi)$ is linear with respect to r, it follows from Corollary 1.7 that there exists a holomorphic (1,0) form F on Ω such that $(F-f,z_0)\in(\mathcal{O}(K_\Omega)\otimes\mathcal{F})_{z_0}$ and $\forall t\geq 0$ equality

$$G(t;\varphi)=\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)$$

holds. As $\tilde{\varphi}+\psi$ is subharmonic and there exists a subset $U\subset\subset\Omega$ such that $\sup_{\Omega\setminus U}(\tilde{\varphi}-\varphi)<+\infty$, $e^{-\tilde{\varphi}}c(-\psi)$ has a positive lower bound on U and $\mathcal{I}(\tilde{\varphi}+\psi)=\mathcal{I}(\varphi+\psi)$, it follows from Theorem 1.3 that $G(\hat{h}^{-1}(r);\tilde{\varphi})$ is concave with respect to r.

As $\tilde{\varphi}+\psi\geq\varphi+\psi$, $\tilde{\varphi}+\psi\neq\varphi+\psi$ and both of them are subharmonic functions on Ω, then there exists a subset V of Ω such that $e^{-\tilde{\varphi}}<e^{-\varphi}$ on a subset V and $\mu(V)>0$, where μ is Lebesgue measure on Ω. As $F\not\equiv 0$, inequality

$$\frac{G(T_0;\tilde{\varphi})}{\int_{T_0}^{+\infty}c(s)e^{-s}ds}\leq\frac{\int_{\{\psi<-T_0\}}|F|^2e^{-\tilde{\varphi}}c(-\psi)}{\int_{T_0}^{+\infty}c(s)e^{-s}ds}<\frac{G(T_0;\varphi)}{\int_{T_0}^{+\infty}c(s)e^{-s}ds} \tag{86}$$

holds for some $T_0>0$. For $t>0$, there exists a holomorphic (1,0) form F_t on $\{\psi<-t\}$ such that $(F_t-f)_{z_0}\in(\mathcal{O}(K_\Omega)\otimes\mathcal{F})_{z_0}$ and

$$G(t;\tilde{\varphi})=\int_{\{\psi<-t\}}|F_t|^2e^{-\tilde{\varphi}}c(-\psi)<+\infty.$$

As there exists a subset $U \subset\subset \Omega$ such that $\sup_{\Omega\backslash U}(\tilde{\varphi}-\varphi)<+\infty$, we get that

$$\begin{aligned}\int_{\{\psi<-t\}}|F_t|^2e^{-\varphi}c(-\psi)=&\int_{\{\psi<-t\}\cap U}|F_t|^2e^{-\varphi}c(-\psi)+\int_{\{\psi<-t\}\backslash U}|F_t|^2e^{-\varphi}c(-\psi)\\ \leq&2\int_{\{\psi<-t\}\cap U}|F|^2e^{-\varphi}c(-\psi)+2\int_{\{\psi<-t\}\cap U}|F_t-F|^2e^{-\varphi}c(-\psi)\\ &+e^{\sup_{\Omega\backslash U}(\tilde{\varphi}-\varphi)}\int_{\{\psi<-t\}\backslash U}|F_t|^2e^{-\tilde{\varphi}}c(-\psi)\\ <&+\infty\end{aligned}\tag{87}$$

holds for small enough $t>0$. It follows from Lemma 2.6 that

$$\begin{aligned}G(t_1;\tilde{\varphi})-G(t_2;\tilde{\varphi})\geq&\int_{\{-t_2\leq\psi<-t_1\}}|F_{t_1}|^2e^{-\tilde{\varphi}}c(-\psi)\\ \geq&\left(\inf_{z\in\{-t_2\leq\psi\}}e^{\varphi-\tilde{\varphi}}\right)\int_{\{-t_2\leq\psi<-t_1\}}|F_{t_1}|^2e^{-\varphi}c(-\psi)\\ \geq&\left(\inf_{z\in\{-t_2\leq\psi\}}e^{\varphi-\tilde{\varphi}}\right)\int_{\{-t_2\leq\psi<-t_1\}}|F|^2e^{-\varphi}c(-\psi)\end{aligned}\tag{88}$$

holds for small enough t_1 and t_2 such that $0<t_1<t_2<+\infty$. As $\lim_{t\to0+0}\sup_{\{\psi\geq-t\}}(\tilde{\varphi}-\varphi)=0$, it follows from inequality (86) and (88) that

$$\begin{aligned}\liminf_{t_2\to0+0}\frac{G(t_1;\tilde{\varphi})-G(t_2;\tilde{\varphi})}{\int_{t_1}^{t_2}c(s)e^{-s}ds}\geq&\liminf_{t_2\to0+0}\left(\inf_{z\in\{-t_2\leq\psi\}}e^{\varphi-\tilde{\varphi}}\right)\frac{\int_{\{-t_2\leq\psi<-t_1\}}|F|^2e^{-\varphi}c(-\psi)}{\int_{t_1}^{t_2}c(s)e^{-s}ds}\\ =&\frac{G(T_0;\varphi)}{\int_{T_0}^{+\infty}c(s)e^{-s}ds}\\ >&\frac{G(T_0;\tilde{\varphi})}{\int_{T_0}^{+\infty}c(s)e^{-s}ds},\end{aligned}$$

which contradicts the concavity of $G(\hat{h}^{-1}(r);\tilde{\varphi})$. Thus the assumption does not hold, and we complete the proof of Proposition 6.1. □

6.2 Proof of Theorem 1.16

Firstly, we prove the sufficiency by using Theorem 2.15. The following remark shows that it suffices to prove the sufficiency for the case $\psi=2G_\Omega(z,z_0)$.

REMARK 6.2. Let $\tilde{\varphi}=\varphi+a\psi$, $\tilde{c}(t)=c(\frac{t}{1-a})e^{-\frac{at}{1-a}}$ and $\tilde{\psi}=(1-a)\psi$ for some $a\in(-\infty,1)$. It is clear that $e^{-\tilde{\varphi}}\tilde{c}(-\tilde{\psi})=e^{-\varphi}c(-\psi)$, $(1-a)\int_t^{+\infty}c(l)e^{-l}dl=\int_{(1-a)t}^{+\infty}\tilde{c}(l)e^{-l}dl$ and $G(t;\varphi,\psi,c)=G((1-a)t;\tilde{\varphi},\tilde{\psi},\tilde{c})$.

Let $\tilde{c}\equiv1$ on $(0,+\infty)$. Set $\hat{f}=\frac{f}{g}$, $\hat{\varphi}=\varphi-2\log|g|=2u$, and $\hat{\mathcal{F}}_{z_0}=\mathcal{I}(\hat{\varphi}+\psi)_{z_0}=\mathcal{I}(2G_\Omega(z,z_0))_{z_0}$. Denote

$$\inf\{\int_{\{\psi<-t\}}|\tilde{f}|^2e^{-\hat{\varphi}}:(\tilde{f}-\hat{f})_{z_0}\in(\mathcal{O}(K_\Omega)\otimes\hat{\mathcal{F}})_{z_0}\,\&\,\tilde{f}\in H^0(\{\psi<-t\},\mathcal{O}(K_\Omega))\}$$

by $\hat{G}(t;\tilde{c})$. Without loss of generality, we can assume that $\hat{f}(z_0) = dw$, where w is a local coordinate on a neighborhood V_{z_0} of z_0 satisfying $w(z_0) = 0$. By definition of $G(t;\tilde{c})$ and $B_{\Omega,e^{-2u}}(z_0)$, it is clear that $G(t;\tilde{c}) = \hat{G}(t;\tilde{c})$ and $\hat{G}(0;\tilde{c}) = \frac{2}{B_{\Omega,e^{-2u}}(z_0)} = \inf\{\int_\Omega |\tilde{f}|^2 e^{-2u} : \tilde{f}$ is a holomorphic extension of $\hat{f}$ from z_0 to $\Omega\}$. Theorem 2.15 shows that $G(0;\tilde{c}) = \hat{G}(0;\tilde{c}) = 2\pi\frac{e^{-2u(z_0)}}{c_\beta^2(z_0)}$. Note that $\|\hat{f}\|_{z_0} = \pi\int_{z_0}\frac{|\hat{f}|^2}{dV_M}e^{-\hat{\varphi}}dV_\Omega[2G_\Omega(z,z_0)] = 2\pi\frac{e^{-2u(z_0)}}{c_\beta^2(z_0)}$, therefore Theorem 1.10 tells us that $G(-\log r;\tilde{c})$ and $\hat{G}(-\log r;\tilde{c})$ is linear with respect to r.

As $\psi = 2G_\Omega(z,z_0)$, Lemma 2.12 shows that, for any $t_0 \geq 0$, there exists $t > t_0$ such that $\{G_\Omega(z,z_0) < -t\}$ is a relatively compact subset of Ω and g has no zero point in $\{G_\Omega(z,z_0) < -t\}\backslash\{z_0\}$. Combining Corollary 1.7, Remark 1.8, and $G(-\log r;\tilde{c})$ is linear with respect to r, we obtain that $G(\hat{h}^{-1}(r))$ is linear with respect to r, where $\hat{h}(t) = \int_t^{+\infty} c(l)e^{-l}dl$.

In the following part, we prove the necessity in three steps.

By Remark 6.2, without loss of generality, we can assume that φ is subharmonic near z_0. As $\varphi+\psi$ is a subharmonic function on Ω, it follows from Weierstrass Theorem on open Riemann surfaces (see [11]) and Siu's Decomposition Theorem that

$$\varphi+\psi = 2\log|g| + 2G_\Omega(z,z_0) + 2u, \tag{89}$$

where g is a holomorphic function on Ω, and u is a subharmonic function on Ω such that $v(dd^c u, z) \in [0,1)$ for any $z \in \Omega$.

Step 1: $\mathcal{F}_{z_0} = \mathcal{I}(\varphi+\psi)_{z_0}$, $ord_{z_0}(g) = ord_{z_0}(f_1)$ and $v(dd^c\psi, z_0) > 0$.

As $\mathcal{I}(\varphi+\psi)_{z_0} = \mathcal{I}(2\log|g| + 2G_\Omega(z,z_0))_{z_0} \subset \mathcal{F}_{z_0}$ and $G(0) \neq 0$, we have $ord_{z_0}(g)+1 > ord_{z_0}(f_1)$. Corollary 1.7 tells us there exists a holomorphic (1,0) form on Ω such that $(F - f, z_0) \in (\mathcal{O}(K_\Omega)\otimes\mathcal{F})_{z_0}$ and $G(t) = \int_{\{\psi<-t\}}|F|^2 e^{-\varphi}c(-\psi)$ for $t \geq 0$. Denote that $\tilde{c}(t) = \max\{c(t), e^{rt}\}$ on $(0,+\infty)$, where $r \in (0,1)$. Set $F = \tilde{F}dw$ on V_{z_0}, and it follows from Corollary 1.7 and Remark 1.8 that $|\tilde{F}|^2 e^{-\varphi-r\psi}$ is locally integrable near z_0 for any $r \in (0,1)$, which implies that $ord_{z_0}(\tilde{F}) \geq ord_{z_0}(g)$.

We prove $\mathcal{F}_{z_0} = \mathcal{I}(\varphi+\psi)_{z_0}$ by contradiction: if not, then $\mathcal{F}_{z_0} \subsetneqq \mathcal{I}(2\log|g| + 2G_\Omega(z,z_0))_{z_0}$. Since $ord_{z_0}(\tilde{F}) \geq ord_{z_0}(g)$, we have $(\tilde{F}, z_0) \in \mathcal{F}_{z_0}$, which contradicts to $G(0) \neq 0$. Thus $\mathcal{F}_{z_0} = \mathcal{I}(\varphi+\psi)_{z_0}$.

As $ord_{z_0}(\tilde{F}) \geq ord_{z_0}(g)$, $ord_{z_0}(g)+1 > ord_{z_0}(f_1)$ and $(\tilde{F} - f_1, z_0) \in \mathcal{I}(2\log|g| + 2G_\Omega(z,z_0))_{z_0}$, we have $ord_{z_0}(g) = ord_{z_0}(f_1)$.

We prove $v(dd^c\psi, z_0) > 0$ by contradiction: if not, $v(dd^c\psi, z_0) = 0$ shows that $\mathcal{I}(\varphi+\psi)_{z_0} = \mathcal{I}(\varphi)_{z_0}$. Without loss of generality, we can assume that $c(t) > 1$ for large enough t, then $|\tilde{F}|^2 e^{-\varphi}$ is locally integrable near z_0, which contradicts to $(\tilde{F}, z_0) \notin \mathcal{F}_{z_0}$. Thus $v(dd^c\psi, z_0) > 0$.

Step 2: $\psi = 2pG_\Omega(z,z_0)$ for some $p > 0$.

As ψ is subharmonic function on Ω, it follows from Siu's Decomposition Theorem that $\psi = 2pG_\Omega(z,z_0) + \psi_1$ such that $v(dd^c\psi_1, z_0) = 0$.

Firstly, we prove ψ_1 is harmonic near z_0 by contradiction : if not, there exists a closed positive (1,1) current $T \not\equiv 0$, such that $suppT \subset\subset V_{z_0}$, $T \leq \frac{1}{2}i\partial\bar{\partial}\psi_1$ on V_{z_0}, where V_{z_0} is an open neighborhood of z_0, satisfying that g has not zero point on $\overline{V_{z_0}}\backslash\{z_0\}$, φ is subharmonic on a neighborhood of $\overline{V_{z_0}}$ and $V_{z_0} \subset\subset \Omega$. Note that $\{z \in \overline{V_{z_0}} : \mathcal{I}(\varphi+\psi)_z \neq \mathcal{O}_z\} = \{z_0\}$.

Using Lemma 2.14, there exists a subharmonic function $\Phi < 0$ on Ω, which satisfies the following properties: $i\partial\bar{\partial}\Phi \leq T$ and $i\partial\bar{\partial}\Phi \not\equiv 0$; $\lim_{t\to 0+0}(\inf_{\{G_\Omega(z,z_0)\geq -t\}}\Phi(z)) = 0$;

$supp(i\partial\bar{\partial}\Phi) \subset V_{z_0}$ and $\inf_{\Omega\backslash V_{z_0}} \Phi > -\infty$. It following from Lemma 2.11, $v(dd^c\psi, z_0) > 0$ and $\psi < 0$ on Ω, that $\lim_{t\to 0+0}(\inf_{\{\psi\geq -t\}} \Phi(z)) = 0$.

Set $\tilde{\varphi} = \varphi - \Phi$, then $\tilde{\varphi} + \psi = \varphi + 2pG_\Omega(z,z_0) + \psi_1 - \Phi$ on V_{z_0}, where $\psi_1 - \Phi$ is subharmonic on V_{z_0}. It is clear that $\tilde{\varphi} \geq \varphi$ and $\tilde{\varphi} \neq \varphi$. $suppT \subset\subset V_{z_0}$ and $i\partial\bar{\partial}\Phi \leq T \leq i\partial\bar{\partial}\psi_1$ on V_{z_0} show that $\tilde{\varphi} + \psi$ is subharmonic on Ω, $\mathcal{I}(\tilde{\varphi}+\psi) = \mathcal{I}(\varphi+\psi) = \mathcal{I}(2\log|g| + 2G_\Omega(z,z_0))$.

Without loss of generality, we can assume that $c(t) > e^{\frac{t}{2}}$ for any $t > 0$. $T \leq \frac{1}{2}i\partial\bar{\partial}\psi_1$ on V_{z_0} and $i\partial\bar{\partial}\Phi \subset\subset V_{z_0}$ show that $\frac{1}{2}\psi - \Phi$ is subharmonic on Ω, which implies that $e^{-\tilde{\varphi}}c(-\psi) \geq e^{-\varphi}e^{\Phi - \frac{1}{2}\psi}$ has a positive lower bound on V_{z_0}. Notice that $\inf_{\Omega\backslash V_{z_0}}(\varphi - \tilde{\varphi}) = \inf_{\Omega\backslash V_{z_0}} \Phi > -\infty$ and $\int_{V_{z_0}} |F_1 - F_2|^2 e^{-\varphi}c(-\psi) \leq C\int_{V_{z_0}} |F_1 - F_2|^2 e^{-\varphi-\psi} < +\infty$ for any $F_1 \in \mathcal{H}^2(c,\tilde{\varphi},t)$ and $F_2 \in \mathcal{H}^2(c,\varphi,t)$, where $V_{z_0} \subset\subset \{\psi < -t\}$, then $\tilde{\varphi}$ satisfies the conditions in Proposition 6.1, which contradicts to the result of Proposition 6.1. Thus ψ_1 is harmonic near z_0.

Then, we prove $\psi = 2pG_\Omega(z,z_0)$. Using Remark 6.2, it suffices to consider the case $p = 1$, where $p = \frac{1}{2}v(dd^c\psi, z_0)$. By Siu's Decomposition Theorem and Lemma 2.11, there exists a subharmonic function $\psi_2 \leq 0$ on Ω such that $\psi = 2G_\Omega(z,z_0) + \psi_2$. Note that $\psi_2(z_0) > -\infty$.

As Ω is an open Riemann surface, there exists a holomorphic function f_2 on Ω, such that $ord_{z_0}(f_2) = ord_{z_0}(f_1)$ and $\{z \in \Omega : f_2(z) = 0\} = \{z_0\}$. Set $\tilde{f} = \frac{f}{f_2}$, $\tilde{\varphi} = \varphi - 2\log|f_2|$, and $\tilde{\mathcal{F}}_{z_0} = \mathcal{I}(\tilde{\varphi}+\psi)_{z_0} = \mathcal{I}(2G_\Omega(z,z_0))_{z_0}$. Denote

$$\inf\left\{ \int_{\{\psi<-t\}} |F|^2 e^{-\tilde{\varphi}}c(-\psi) : (F - \tilde{f})_{z_0} \in (\mathcal{O}(K_\Omega)\otimes\tilde{\mathcal{F}})_{z_0} \right.$$
$$\left. \& F \in H^0(\{\psi < -t\}, \mathcal{O}(K_\Omega)) \right\}$$

by $\tilde{G}(t)$. By the definition of $G(t)$ and $\tilde{G}(t)$, we know $G(t) = \tilde{G}(t)$ for any $t \geq 0$, therefore $\tilde{G}(\hat{h}^{-1}(r))$ is linear with respect to r. Note that $(\tilde{\varphi},\psi) \in W$, $(\tilde{\varphi}+\psi-2G_\Omega(z,z_0), 2G_\Omega(z,z_0)) \in W$, $\psi_2(z_0) > -\infty$ and $\psi_2 \leq 0$, then Theorem 1.15 shows that $\psi = 2G_\Omega(z,z_0)$.

Step 3. u is harmonic on Ω and $\chi_{-u} = \chi_{z_0}$.

Without loss of generality, we can assume that $\psi = 2G_\Omega(z,z_0)$. Lemma 2.12 shows that, for any $t_0 \geq 0$, there exists $t > t_0$ such that $\{G_\Omega(z,z_0) < -t\}$ is a relatively compact subset of Ω and g has no zero point in $\{G_\Omega(z,z_0) < -t\}\backslash\{z_0\}$. Combining Corollary 1.7, Remark 1.8, and $G(\hat{h}^{-1}(r);c)$ is linear with respect to r, we obtain that $G(-\log r;\tilde{c} \equiv 1)$ is linear with respect to r and $G(0;\tilde{c}) \in (0,+\infty)$.

Now, we assume that u is not harmonic to get a contradiction. There exists a closed positive $(1,1)$ current $T \not\equiv 0$, such that $suppT \subset\subset \Omega$ and $T \leq i\partial\bar{\partial}u$. There exists an open subset $U \subset\subset \Omega$, such that $suppT \subset U$.

Using Lemma 2.14, there exists a subharmonic function $\Phi < 0$ on Ω, which satisfies the following properties: $i\partial\bar{\partial}\Phi \leq T$ and $i\partial\bar{\partial}\Phi \not\equiv 0$; $\lim_{t\to 0+0}(\inf_{\{G_\Omega(z,z_0)\geq -t\}} \Phi(z)) = 0$; $supp(i\partial\bar{\partial}\Phi) \subset U$ and $\inf_{\Omega\backslash U} \Phi > -\infty$.

Set $\tilde{\varphi} = \varphi - \Phi$, then $\tilde{\varphi} = 2\log|g| + 2u - \Phi$ is subharmonic on Ω. It is clear that $\tilde{\varphi} \geq \varphi$, $\tilde{\varphi} \neq \varphi$ and $\tilde{\varphi} + \psi$ is subharmonic on Ω, $\mathcal{I}(\tilde{\varphi}+\psi) = \mathcal{I}(\varphi+\psi) = \mathcal{I}(2\log|g| + 2G_\Omega(z,z_0))$.

As $\tilde{\varphi}$ is subharmonic on Ω, we have $e^{-\tilde{\varphi}}$ has a positive lower bound on U. Note that $\mathcal{I}(\varphi) = \mathcal{I}(\tilde{\varphi})$, then

$$\int_U |F_1 - F_2|^2 e^{-\varphi} \leq 2\int_U |F_1|^2 e^{-\varphi} + 2\int_U |F_2|^2 e^{-\varphi} < +\infty$$

for any $F_1 \in \mathcal{H}^2(\tilde{c},\tilde{\varphi},t)$ and $F_2 \in \mathcal{H}^2(\tilde{c},\varphi,t)$, where $U \subset\subset \{\psi < -t\}$ and $\tilde{c} \equiv 1$. Since $\inf_{\Omega\setminus U}(\varphi - \tilde{\varphi}) = \inf_{\Omega\setminus U}\Phi > -\infty$, then $\tilde{\varphi}$ satisfies the conditions in Proposition 6.1, which contradicts to the result of Proposition 6.1. Thus, u is harmonic on Ω.

Finally, we prove $\chi_{-u} = \chi_{z_0}$ by using Theorem 2.15.

Recall some notations in the proof of sufficiency. Set $\hat{f} = \frac{f}{g}$, $\hat{\varphi} = \varphi - 2\log|g| = 2u$, and $\hat{\mathcal{F}}_{z_0} = \mathcal{I}(\hat{\varphi}+\psi)_{z_0} = \mathcal{I}(2G_\Omega(z,z_0))_{z_0}$. Denote

$$\inf\left\{\int_{\{\psi<-t\}}|\tilde{f}|^2e^{-\hat{\varphi}} : (\tilde{f}-\hat{f})_{z_0} \in (\mathcal{O}(K_\Omega)\otimes\hat{\mathcal{F}})_{z_0} \right.$$
$$\left. \&\, \tilde{f} \in H^0(\{\psi<-t\},\mathcal{O}(K_\Omega))\right\}$$

by $\hat{G}(t;\tilde{c})$. Without loss of generality, we can assume that $\hat{f}(z_0) = dw$, where w is a local coordinate on a neighborhood V_{z_0} of z_0 satisfying $w(z_0) = 0$. By definition of $G(t;\tilde{c})$ and $B_{\Omega,e^{-2u}}(z_0)$, it is clear that $G(-\log r;\tilde{c}) = \hat{G}(-\log r;\tilde{c})$ is linear with respect to r and $\hat{G}(0;\tilde{c}) = \frac{2}{B_{\Omega,e^{-2u}}(z_0)} = \inf\{\int_\Omega|\tilde{f}|^2e^{-2u} : \tilde{f}$ is a holomorphic extension of $\hat{f}$ from z_0 to $\Omega\}$.

Note that $\|\hat{f}\|_{z_0} = 2\pi\frac{e^{-2u(z_0)}}{c_\beta^2(z_0)}$, then Theorem 1.15 shows that

$$\hat{G}(0,\tilde{c}) = 2\pi\frac{e^{-2u(z_0)}}{c_\beta^2(z_0)},$$

that is, $c_\beta^2(z_0) = \pi e^{-2u(z_0)}B_{\Omega,e^{-2u}}(z_0)$. Therefore, Theorem 2.15 shows that $\chi_{-u} = \chi_{z_0}$.

Thus, Theorem 1.16 holds.

6.3 Proof of Theorem 1.17

Theorem 1.16 implies the sufficiency. Thus, we just need to prove the necessity.

As $\varphi+\psi$ is a subharmonic function on Ω, it follows from Weierstrass Theorem on open Riemann surfaces (see [11]) and Siu's Decomposition Theorem that

$$\varphi+\psi = 2\log|g| + 2G_\Omega(z,z_0) + 2u, \tag{90}$$

where g is a holomorphic function on Ω, and u is a subharmonic function on Ω such that $v(dd^cu,z) \in [0,1)$ for any $z \in \Omega$.

As $\mathcal{I}(\varphi+\psi)_{z_0} = \mathcal{I}(2\log|g| + 2G_\Omega(z,z_0))_{z_0} \subset \mathcal{F}_{z_0}$ and $G(0) \neq 0$, we have $ord_{z_0}(g)+1 > ord_{z_0}(f_1)$. Corollary 1.7 tells us there exists a holomorphic (1,0) form on Ω such that $(F-f,z_0) \in (\mathcal{O}(K_\Omega)\otimes\mathcal{F})_{z_0}$ and $G(t) = \int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)$ for $t \geq 0$. Let $\tilde{c}(t) = \max\{c(t),e^{rt}\}$ defined on $(0,+\infty)$, where $r \in (0,1)$. Set $F = \tilde{F}dw$ on V_{z_0}, and it follows from Corollary 1.7 and Remark 1.8 that $|\tilde{F}|^2e^{-\varphi-r\psi}$ is locally integrable near z_0 for any $r \in (0,1)$. Note that

$$\int_U|\tilde{F}|^2e^{-\frac{\varphi+\psi}{p}} \leq \left(\int_U|\tilde{F}|^{2p}e^{-\varphi-\psi+ps\psi}\right)^{\frac{1}{p}}\left(\int_U e^{-qs\psi}\right)^{\frac{1}{q}}$$

holds for any $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, U is a small open neighborhood of z_0, and $s \in (0,1)$. For any $p \in (1,+\infty)$, we can choose small enough U and small enough $s \in (0,1)$ such that $\int_U|\tilde{F}|^{2p}e^{-\varphi-\psi+ps\psi} < +\infty$ and $\int_U e^{-qs\psi} < +\infty$, which implies that $(\tilde{F},z_0) \in \mathcal{I}(\frac{\varphi+\psi}{p})_{z_0} \subset \mathcal{I}(\frac{2\log|g|+2G_\Omega(z,z_0)}{p})_{z_0}$. Therefore, we have $ord_{z_0}(\tilde{F}) \geq ord_{z_0}(g)$.

We prove $\mathcal{F}_{z_0} = \mathcal{I}(\varphi+\psi)_{z_0}$ by contradiction: if not, then $\mathcal{F}_{z_0} \subsetneqq \mathcal{I}(2\log|g| + 2G_\Omega(z,z_0))_{z_0}$. Since $ord_{z_0}(\tilde{F}) \geq ord_{z_0}(g)$, we have $(\tilde{F}, z_0) \in \mathcal{F}_{z_0}$, which contradicts to $G(0) \neq 0$. Thus $\mathcal{F}_{z_0} = \mathcal{I}(\varphi+\psi)_{z_0}$.

As $ord_{z_0}(\tilde{F}) \geq ord_{z_0}(g)$, $ord_{z_0}(g) + 1 > ord_{z_0}(f_1)$ and $(\tilde{F} - f_1, z_0) \in \mathcal{I}(2\log|g| + 2G_\Omega(z,z_0))_{z_0}$, we have $ord_{z_0}(g) = ord_{z_0}(f_1)$.

We prove $v(dd^c\psi, z_0) > 0$ by contradiction: if not, as $|\tilde{F}|^2 e^{-\varphi - r\psi}$ is locally integrable near z_0 for any $r \in (0,1)$ and $ord_{z_0}(g) = ord_{z_0}(\tilde{F})$, we have $e^{-2G_\Omega(z,z_0)+(1-r)\psi}$ is locally integrable near z_0. Therefore, there exists $s > 0$ such that

$$\int_{\Delta_s} \frac{e^{(1-r)\psi}}{|w|^2} < +\infty,$$

where w is a local coordinate near z_0 such that $w(z_0) = 0$. As $e^{(1-r)\psi}$ is subharmonic, we have

$$2\pi e^{(1-r)\psi(z_0)} \int_0^s \frac{1}{t} dt = \int_{\Delta_s} \frac{e^{(1-r)\psi}}{|w|^2} < +\infty,$$

which contradicts to $\psi(z_0) > -\infty$. Thus $v(dd^c\psi, z_0) > 0$ holds.

Using Remark 6.2, it suffices to consider the case $p = 1$, where $p = \frac{1}{2} v(dd^c\psi, z_0)$. By Siu's Decomposition Theorem and Lemma 2.11, there exists a subharmonic function $\psi_2 \leq 0$ on Ω such that $\psi = 2G_\Omega(z,z_0) + \psi_2$. Following the assumption in Theorem 1.17, we know $\psi_2(z_0) > -\infty$.

As Ω is an open Riemann surface, there exists a holomorphic function f_2 on Ω, such that $ord_{z_0}(f_2) = ord_{z_0}(f_1)$ and $\{z \in \Omega : f_2 = 0\} = \{z_0\}$. Set $\tilde{f} = \frac{f}{f_2}$, $\tilde{\varphi} = \varphi - 2\log|f_2|$, and $\tilde{\mathcal{F}}_{z_0} = \mathcal{I}(\tilde{\varphi} + \psi)_{z_0} = \mathcal{I}(2G_\Omega(z,z_0))_{z_0}$. Denote

$$\inf\left\{ \int_{\{\psi < -t\}} |F|^2 e^{-\tilde{\varphi}} c(-\psi) : (F - \tilde{f})_{z_0} \in (\mathcal{O}(K_\Omega) \otimes \tilde{\mathcal{F}})_{z_0} \right.$$
$$\left. \& \, F \in H^0(\{\psi < -t\}, \mathcal{O}(K_\Omega)) \right\}$$

by $\tilde{G}(t)$. By the definition of $G(t)$ and $\tilde{G}(t)$, we know $G(t) = \tilde{G}(t)$ for any $t \geq 0$, therefore $\tilde{G}(\tilde{h}^{-1}(r))$ is linear with respect to r. Note that $(\tilde{\varphi}, \psi) \in W$, $(\tilde{\varphi} + \psi - 2G_\Omega(z,z_0), 2G_\Omega(z,z_0)) \in W$, $\psi_2(z_0) > -\infty$ and $\psi_2 \leq 0$, then Theorem 1.15 shows that $\psi = 2G_\Omega(z,z_0)$.

As $\varphi + \psi$ is subharmonic on Ω and $\psi = 2G_\Omega(z,z_0)$, we have φ is subharmonic on Ω. Then Theorem 1.16 implies that u is harmonic on Ω and $\chi_{-u} = \chi_{z_0}$.

Thus, Theorem 1.17 holds.

6.4 Proof of Corollary 1.18

The following remark shows that it suffices to prove the existence of holomorphic extension satisfying inequality (6) for the case $c(t)$ has a positive lower bound and upper bound on $(t', +\infty)$ for any $t' > 0$.

REMARK 6.3. Take c_j is a positive measurable function on $(0, +\infty)$, such that $c_j(t) = c(t)$ when $t < j$, $c_j(t) = \min\{c(j), \frac{1}{j}\}$ when $t \geq j$. It is clear that $c_j(t)e^{-t}$ is decreasing with respect to t, and $\int_0^{+\infty} c_j(t)e^{-t} < +\infty$. As

$$\lim_{j\to+\infty}\int_j^{+\infty} c_j(t)e^{-t} = 0,$$

we have

$$\lim_{j\to+\infty}\int_0^{+\infty} c_j(t)e^{-t} = \int_0^{+\infty} c(t)e^{-t}.$$

If the existence of holomorphic extension satisfying inequality (6) holds in this case, then there exists a holomorphic (1,0) form F_j on Ω such that $F_j(z_0) = f(z_0)$ and

$$\int_\Omega |F_j|^2 e^{-\varphi} c_j(-\psi) \le \left(\int_0^{+\infty} c_j(t)e^{-t}dt\right) \|f\|_{z_0}.$$

Note that ψ has locally lower bound on $\Omega\backslash\psi^{-1}(-\infty)$ and $\psi^{-1}(-\infty)$ is a closed subset of an analytic subset Z of Ω. For any compact subset K of $\Omega\backslash Z$, there exists $s_K > 0$ such that $\int_K e^{-s_K\psi} dV_\Omega < +\infty$, where dV_Ω is a continuous volume form on Ω. Then we have

$$\int_K \left(\frac{e^\varphi}{c_j(-\psi)}\right)^{s_K} dV_\Omega = \int_K \left(\frac{e^{\varphi+\psi}}{c_j(-\psi)}\right)^{s_K} e^{-s_K\psi} dV_\Omega \le C\int_K e^{-s_K\psi} dV_\Omega < +\infty,$$

where C is a constant independent of j. It follows from Lemma 2.4 ($g_j = e^{-\varphi}c_j(-\psi)$) that there exists a subsequence of $\{F_j\}$, denoted still by $\{F_j\}$, which is uniformly convergent to a holomorphic (1,0) form F on any compact subset of Ω and

$$\begin{aligned}\int_\Omega |F|^2 e^{-\varphi} c(-\psi) &\le \lim_{j\to+\infty}\left(\int_0^{+\infty} c_j(t)e^{-t}dt\right)\|f\|_{z_0}\\ &= \left(\int_0^{+\infty} c(t)e^{-t}dt\right)\|f\|_{z_0}.\end{aligned}$$

Since $F_j(z_0) = f(z_0)$ for any j, we have $F(z_0) = f(z_0)$.

As $\psi \in A(z_0)$ and $e^{-\varphi-\psi}$ is not L^1 on any neighborhood of z_0, it follows from Siu's Decomposition Theorem and the following lemma that $\psi(z) - 2G_\Omega(z,z_0)$ and $\varphi(z)+\psi(z)-2G_\Omega(z,z_0)$ is subharmonic on Ω with respect to z. Denote that $\psi_2(z) = \psi(z) - 2G_\Omega(z,z_0)$.

LEMMA 6.4 [29]. *Let u is a subharmonic function on Ω. If $v(dd^cu,z_0) < 1$, then e^{-u} is L^1 on a neighborhood of z_0.*

As Ω is a Stein manifold and $\varphi+\psi_2$ is subharmonic on Ω, there exist smooth subharmonic functions Φ_l on Ω, which are decreasingly convergent to $\varphi+\psi_2$. We can find a sequence of open Riemann surfaces $\{D_m\}_{m=1}^{+\infty}$ satisfying $z_0 \in D_m \subset\subset D_{m+1}$ for any m and $\cup_{m=1}^{+\infty} D_m = \Omega$, and there is a holomorphic $(n,0)$ form $\tilde{F}$ on Ω such that $\tilde{F}(z_0) = f(z_0)$.

Note that $\int_{D_m} |\tilde{F}|^2 < +\infty$ for any m and

$$\int_{D_m} \mathbb{I}_{\{-t_0-1<\psi<-t_0\}} |\tilde{F}|^2 e^{-\Phi_l - 2G_\Omega(\cdot,z_0)} \le e^{t_0+1}\int_{D_m} |\tilde{F}|^2 e^{-\Phi_l+\psi_2} < +\infty$$

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for any m, l and $t_0>T$. Using Lemma 2.1 ($\varphi\sim\Phi_l+2G_\Omega(\cdot,z_0)$), for any D_m, $l\in\mathbb{N}^+$, and $t_0>T$, there exists a holomorphic $(1,0)$ form F_{l,m,t_0} on D_m, such that

$$\begin{aligned}&\int_{D_m}|F_{l,m,t_0}-(1-b_{t_0,1}(\psi))\tilde{F}|^2e^{-\Phi_l-2G_\Omega(\cdot,z_0)+v_{t_0,1}(\psi)}c(-v_{t_0,1}(\psi))\\ &\leq\left(\int_0^{t_0+1}c(t)e^{-t}dt\right)\int_{D_m}\mathbb{I}_{\{-t_0-1<\psi<-t_0\}}|\tilde{F}|^2e^{-\Phi_l-2G_\Omega(\cdot,z_0)},\end{aligned}\tag{91}$$

where $b_{t_0,1}(t)=\int_{-\infty}^t\mathbb{I}_{\{-t_0-1<s<-t_0\}}ds$, $v_{t_0,1}(t)=\int_{-t_0}^t b_{t_0,1}(s)ds-t_0$. Note that $e^{-2G_\Omega(\cdot,z_0)}$ is not L^1 on any neighborhood of z_0, and $b_{t_0,1}(t)=0$ when $-t$ is large enough, then $(F_{l,m,t_0}-(1-b_{t_0,1}(\psi))\tilde{F})(z_0)=0$, and therefore $F_{l,m,t_0}(z_0)=f(z_0)$.

Note that $v_{t_0,1}(\psi)\geq\psi$ and $c(t)e^{-t}$ is decreasing, then the inequality (91) becomes

$$\begin{aligned}&\int_{D_m}|F_{l,m,t_0}-(1-b_{t_0,1}(\psi))\tilde{F}|^2e^{-\Phi_l+\psi_2}c(-\psi)\\ &\leq\left(\int_0^{t_0+1}c(t)e^{-t}dt\right)\int_{D_m}\mathbb{I}_{\{-t_0-1<\psi<-t_0\}}|\tilde{F}|^2e^{-\Phi_l-2G_\Omega(\cdot,z_0)}.\end{aligned}\tag{92}$$

There exist smooth subharmonic functions Ψ_k on Ω, which are decreasingly convergent to ψ_2. By definition of $dV_\Omega[\psi]$, we have

$$\begin{aligned}&\limsup_{t_0\to+\infty}\int_{D_m}\mathbb{I}_{\{-t_0-1<\psi<-t_0\}}|\tilde{F}|^2e^{-\Phi_l+\Psi_k-\psi}\\ \leq&\pi\int_{z_0}\frac{|f|^2}{dV_\Omega}e^{-\Phi_l+\Psi_k}dV_\Omega[\psi]\\ <&+\infty.\end{aligned}\tag{93}$$

Combining inequality (92) and (93), let $t_0\to+\infty$, we have

$$\begin{aligned}&\limsup_{t_0\to+\infty}\int_{D_m}|F_{l,m,t_0}-(1-b_{t_0,1}(\psi))\tilde{F}|^2e^{-\Phi_l+\psi_2}c(-\psi)\\ \leq&\limsup_{t_0\to+\infty}\left(\int_0^{t_0+1}c(t)e^{-t}dt\right)\int_{D_m}\mathbb{I}_{\{-t_0-1<\psi<-t_0\}}|\tilde{F}|^2e^{-\Phi_l+\Psi_k-\psi}\\ \leq&\pi\left(\int_0^{+\infty}c(t)e^{-t}dt\right)\int_{z_0}\frac{|f|^2}{dV_\Omega}e^{-\Phi_l+\Psi_k}dV_\Omega[\psi].\end{aligned}\tag{94}$$

Let $k\to+\infty$, inequality (94) implies that

$$\begin{aligned}&\limsup_{t_0\to+\infty}\int_{D_m}|F_{l,m,t_0}-(1-b_{t_0,1}(\psi))\tilde{F}|^2e^{-\Phi_l+\psi_2}c(-\psi)\\ \leq&\pi\left(\int_0^{+\infty}c(t)e^{-t}dt\right)\int_{z_0}\frac{|f|^2}{dV_\Omega}e^{-\Phi_l+\psi_2}dV_\Omega[\psi].\end{aligned}\tag{95}$$

Note that

$$\limsup_{t_0\to+\infty}\int_{D_m}|(1-b_{t_0,1}(\psi))\tilde{F}|^2e^{-\Phi_l+\psi_2}c(-\psi)<+\infty,$$

then we have

$$\limsup_{t_0\to+\infty}\int_{D_m}|F_{l,m,t_0}|^2e^{-\Phi_l+\psi_2}c(-\psi)<+\infty.$$

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Using Lemma 2.4, we obtain that there exists a subsequence of $\{F_{l,m,t_0}\}_{t_0\to+\infty}$ (also denoted by $\{F_{l,m,t_0}\}_{t_0\to+\infty}$) compactly convergent to a holomorphic $(1,0)$ form on D_m denoted by $F_{l,m}$. Then it follows from inequality (95) and Fatou's Lemma that

$$\begin{aligned}\int_{D_m}|F_{l,m}|^2e^{-\Phi_l+\psi_2}c(-\psi)=&\int_{D_m}\liminf_{t_0\to+\infty}|F_{l,m,t_0}-(1-b_{t_0,1}(\psi))\tilde{F}|^2e^{-\Phi_l+\psi_2}c(-\psi)\\ \le&\liminf_{t_0\to+\infty}\int_{D_m}|F_{l,m,t_0}-(1-b_{t_0,1}(\psi))\tilde{F}|^2e^{-\Phi_l+\psi_2}c(-\psi)\\ \le&\pi\left(\int_0^{+\infty}c(t)e^{-t}dt\right)\int_{z_0}\frac{|f|^2}{dV_\Omega}e^{-\Phi_l+\psi_2}dV_\Omega[\psi].\end{aligned}\tag{96}$$

As $\|f\|_{z_0}=\pi\int_{z_0}\frac{|f|^2}{dV_\Omega}e^{-\varphi}dV_\Omega[\psi]<+\infty$ and Φ_l are decreasingly convergent to $\varphi+\psi_2$, we have

$$\lim_{l\to+\infty}\pi\int_{z_0}\frac{|f|^2}{dV_\Omega}e^{-\Phi_l+\psi_2}dV_\Omega[\psi]=\|f\|_{z_0}<+\infty.\tag{97}$$

It follows from inequality (96) and (97) that

$$\limsup_{l\to+\infty}\int_{D_m}|F_{l,m}|^2e^{-\Phi_l+\psi_2}c(-\psi)\le\left(\int_0^{+\infty}c(t)e^{-t}dt\right)\|f\|_{z_0}<+\infty.\tag{98}$$

Using Lemma 2.4 ($g_l=e^{-\Phi_l+\psi_2}c(-\psi)$), we obtain that there exists a subsequence of $\{F_{l,m}\}_{l\to+\infty}$ (also denoted by $\{F_{l,m}\}_{l\to+\infty}$) compactly convergent to a holomorphic $(1,0)$ form on D_m denoted by F_m and

$$\int_{D_m}|F_m|^2e^{-\varphi}c(-\psi)\le\left(\int_0^{+\infty}c(t)e^{-t}dt\right)\|f\|_{z_0}.\tag{99}$$

Inequality (99) implies that

$$\int_{D_m}|F_{m'}|^2e^{-\varphi}c(-\psi)\le\pi\left(\int_0^{+\infty}c(t)e^{-t}dt\right)\|f\|_{z_0}$$

holds for any $m'\ge m$. Note that $\varphi+\psi$ and ψ are subharmonic on Ω and $\varphi=(\varphi+\psi)-\psi$. Using Lemma 2.4, the diagonal method and Levi's Theorem, we obtain a subsequence of $\{F_m\}$, denoted also by $\{F_m\}$, which is uniformly convergent to a holomorphic $(1,0)$ form F on Ω satisfying that $F(z_0)=f(z_0)$ and

$$\int_\Omega|F|^2e^{-\varphi}c(-\psi)\le\left(\int_0^{+\infty}c(t)e^{-t}dt\right)\|f\|_{z_0}.$$

Thus, the existence of holomorphic extension satisfying inequality (6) holds.

In the following part, we prove the characterization for $\left(\int_0^{+\infty}c(t)e^{-t}dt\right)\|f\|_{z_0}=\inf\{\|\tilde{F}\|_\Omega:\tilde{F}$ is a holomorphic extension of f from z_0 to $\Omega\}$.

Firstly, we prove the necessity. If $\|f\|_{z_0}=0$, then $F\equiv 0$, which contradicts to $F(z_0)=f(z_0)\neq 0$. Thus, we only consider the case $\|f\|_{z_0}\in(0,+\infty)$.

As $\{\psi<-t\}$ is an open Riemann surface. Note that $dV_\Omega[\psi+t]=e^{-t}dV_\Omega[\psi]$. By the above discussion ($\psi\sim\psi+t$, $c(\cdot)\sim c(\cdot+t)$ and $\Omega\sim\{\psi<-t\}$), for any $t>0$, there exists a holomorphic $(n,0)$ form F_t on $\{\psi<-t\}$ such that $F_t(z_0)=f(z_0)$ and

$$\int_{\{\psi<-t\}}|F_t|^2e^{-\varphi}c(-\psi)\le\left(\int_t^{+\infty}c(l)e^{-l}dl\right)\|f\|_{z_0}.$$

Let $\mathcal{F}|_{Z_0}=\mathcal{I}(\psi)_{z_0}$, by the definition of $G(t)$, we obtain that inequality

$$\frac{G(t)}{\int_t^{+\infty}c(l)e^{-l}dl}\le\frac{G(0)}{\int_0^{+\infty}c(t)e^{-t}dt} \tag{100}$$

holds for any $t\ge 0$. Theorem 1.3 tells us $G(\hat{h}^{-1}(r))$ is concave with respect to r. Combining inequality (100) and Corollary 1.5, we obtain that $G(\hat{h}^{-1}(r))$ is linear with respect to r. As $\psi-2G_\Omega(z,z_0)$ is bounded near z_0 and $G(0)=(\int_0^{+\infty}c(t)e^{-t}dt)\|f\|_{z_0}\in(0,+\infty)$, Theorem 1.17 shows that statements (1)–(3) hold.

Now, we prove the sufficiency. Let $\mathcal{F}|_{Z_0}=\mathcal{I}(2G_\Omega(z,z_0))_{z_0}$, then Theorem 1.16 shows that $G(\hat{h}^{-1}(r))$ is linear with respect to r. It follows from Lemma 2.10 and Corollary 1.7 that there exists $\tilde{c}\in\mathcal{P}_0$ such that $\tilde{c}(t)$ is increasing on $(a,+\infty)$ for some $a>0$, and $G(\hat{h}_{\tilde{c}}^{-1}(r);\tilde{c})$ is linear with respect to r, where $\hat{h}_{\tilde{c}}(t)=\int_t^{+\infty}\tilde{c}(l)e^{-l}dl$. Using Proposition 5.1, we have $G(0;\tilde{c})=\|f\|_{z_0}(\int_0^{+\infty}\tilde{c}(l)e^{-l}dl)$. Following from Corollary 1.7 and Remark 1.8, we obtain that $G(0;c)=\|f\|_{z_0}(\int_0^{+\infty}c(l)e^{-l}dl)$, which implies that $\|f\|_{z_0}(\int_0^{+\infty}c(l)e^{-l}dl)=\{\|\tilde{F}\|_\Omega:\tilde{F}$ is a holomorphic extension of f from z_0 to $\Omega\}$.

Thus, Corollary 1.18 holds.

6.5 Proof of Corollary 1.19

Note that $2G_\Omega(z,z_0)\in A'(z_0)$ and $\psi_1\in A'(z_0)$, we have

$$\|f\|_{z_0}^*=\pi\int_{z_0}\frac{|f|^2}{dV_\Omega}e^{-\varphi-\psi_2}dV_\Omega[\psi_1]=\pi\int_{z_0}\frac{|f|^2}{dV_\Omega}e^{-\varphi-\psi+2G_\Omega(z,z_0)}dV_\Omega[2G_\Omega(z,z_0)]. \tag{101}$$

Corollary 1.18 implies the sufficiency. Thus, it suffices to prove the necessity.

Let $\mathcal{F}|_{Z_0}=\mathcal{I}(\psi_1)_{z_0}$. It follows from Lemma 2.6 that there exists a unique holomorphic extension from z_0 to Ω, such that $\|F\|_\Omega\le\|f\|_{z_0}^*(\int_0^{+\infty}c(l)e^{-l}dl)$. Using Corollary 1.12, we know that $G(\hat{h}^{-1}(r))$ is linear with respect to r, therefore

$$\frac{G(t)}{\int_t^{+\infty}c(l)e^{-l}dl}=\|f\|_{z_0}^* \tag{102}$$

holds for any $t\ge 0$.

Let $\tilde{\psi}=2G_\Omega(z,z_0)$. Lemma 2.11 tells us $\psi-\tilde{\psi}\le 0$ on Ω. Let $\tilde{\varphi}=\varphi+\psi-\tilde{\psi}$, then we compare $G(t;\varphi,\psi)$ and $G(t;\tilde{\varphi},\tilde{\psi})$ to prove $\psi-\tilde{\psi}\equiv 0$. As $\|f\|_{z_0}^*<+\infty$ and $e^{-\tilde{\varphi}}c(-\tilde{\psi})=e^{-\varphi-\psi}e^{\tilde{\psi}}c(-\tilde{\psi})\ge e^{-\varphi}c(-\psi)$, it follows from Corollary 1.11 and equality (101) that $G(0;\tilde{\varphi},\tilde{\psi})\le\|f\|_{z_0}^*\left(\int_0^{+\infty}c(t)e^{-t}dt\right)$. Without loss of generality, we can assume that $c(t)e^{-t}$ is strictly decreasing on $(0,+\infty)$. We prove $\psi-\tilde{\psi}\equiv 0$ by contradiction: if not, $c(t)e^{-t}$ is strictly decreasing on $(0,+\infty)$ implies that $G(0;\tilde{\varphi},\tilde{\psi})>G(0;\varphi,\psi)$, which contradicts to $G(0;\tilde{\varphi},\tilde{\psi})\le\|f\|_{z_0}^*\left(\int_0^{+\infty}c(t)e^{-t}dt\right)=G(0;\varphi,\psi)$. Thus, we have $\psi=2G_\Omega(z,z_0)$. Combining the linearity of $G(\hat{h}(r);\varphi,\psi)$, $G(0;\varphi,\psi)=\|f\|_{z_0}^*\left(\int_0^{+\infty}c(t)e^{-t}dt\right)\in(0,+\infty)$ and Theorem 1.16, we obtain that the other two statements in Corollary 1.19 hold.

Thus, Corollary 1.19 holds.

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§7. Appendix

7.1 Proof of Lemma 2.1

In this section, we prove Lemma 2.1.

It follows from Lemma 7.4 that there exist smooth strongly plurisubharmonic functions ψ_m and φ_m on M decreasingly convergent to ψ and φ, respectively.

The following remark shows that it suffices to consider Lemma 2.1 for the case that M is a relatively compact open Stein submanifold of a Stein manifold, and F is a holomorphic $(n,0)$ form on $\{\psi < -t_0\}$ such that $\int_{\{\psi<-t_0\}} |F|^2 < +\infty$, which implies that $\sup_m \sup_M \psi_m < -T$ and $\sup_m \sup_M \varphi_m < +\infty$ on M.

Remark 7.1. It is well known that there exist open Stein submanifolds $D_1 \subset\subset \cdots \subset\subset D_j \subset\subset D_{j+1} \subset\subset \cdots$ such that $\cup_{j=1}^{+\infty} D_j = M$.

If inequality (9) holds on any D_j and inequality (8) holds on M, then for any $B > 0$, we obtain a sequence of holomorphic $(n,0)$ forms $\tilde{F}_j$ on D_j such that

$$\begin{aligned}&\int_{D_j} |\tilde{F}_j - (1-b_{t_0,B}(\psi))F|^2 e^{-\varphi+v_{t_0,B}(\psi)} c(-v_{t_0,B}(\psi)) \\ &\le \int_T^{t_0+B} c(t)e^{-t}dt \int_{D_j} \frac{1}{B}\mathbb{I}_{\{-t_0-B<\psi<-t_0\}} |F|^2 e^{-\varphi} \le C \int_T^{t_0+B} c(t)e^{-t}dt\end{aligned} \tag{103}$$

is bounded with respect to j. Note that for any given j, $e^{-\varphi+v_{t_0,B}(\psi)} c(-v_{t_0,B}(\psi))$ has a positive lower bound, then it follows that for any any given j, $\int_{D_j} |\tilde{F}_{j'} - (1-b_{t_0,B}(\psi))F|^2$ is bounded with respect to $j' \ge j$. Combining with

$$\int_{D_j} |(1-b_{t_0,B}(\psi))F|^2 \le \int_{D_j \cap \{\psi<-t_0\}} |F|^2 < +\infty \tag{104}$$

and inequality (9), one can obtain that $\int_{D_j} |\tilde{F}_{j'}|^2$ is bounded with respect to $j' \ge j$.

By the diagonal method, there exists a subsequence $F_{j''}$ uniformly convergent on any D_j to a holomorphic $(n,0)$ form on M denoted by $\tilde{F}$. Then it follows from inequality (103) and Fatou's Lemma that

$$\int_{D_j} |\tilde{F} - (1-b_{t_0,B}(\psi))F|^2 e^{-(\varphi - v_{t_0,B}(\psi))} c(-v_{t_0,B}(\psi)) \le C \int_T^{t_0+B} c(t)e^{-t}dt,$$

then one can obtain Lemma 2.1 when j goes to $+\infty$.

Next, we recall some lemmas on L^2 estimates for some $\bar{\partial}$ equations.

Lemma 7.2 (See [2], [4]). *Let X be a complete Kähler manifold equipped with a (non necessarily complete) Kähler metric ω, and let E be a Hermitian vector bundle over X. Assume that there are smooth and bounded functions η, $g > 0$ on X such that the (Hermitian) curvature operator*

$$\boldsymbol{B} := [\eta\sqrt{-1}\Theta_E - \sqrt{-1}\partial\bar{\partial}\eta - \sqrt{-1}g\partial\eta \wedge \bar{\partial}\eta, \Lambda_\omega]$$

is positive definite everywhere on $\Lambda^{n,q}T_X^* \otimes E$, for some $q \ge 1$. Then for every form $\lambda \in L^2(X, \Lambda^{n,q}T_X^* \otimes E)$ such that $D''\lambda = 0$ and $\int_X \langle \boldsymbol{B}^{-1}\lambda, \lambda\rangle dV_M < \infty$, there exists $u \in L^2(X, \Lambda^{n,q-1}T_X^* \otimes E)$ such that $D''u = \lambda$ and

$$\int_X (\eta + g^{-1})^{-1} |u|^2 dV_M \le \int_X \langle \boldsymbol{B}^{-1}\lambda, \lambda\rangle dV_M.$$

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LEMMA 7.3 (See [18]). *Let X and E be as in the above lemma and θ be a continuous* $(1,0)$ *form on X. Then we have*

$$[\sqrt{-1}\theta\wedge\bar{\theta},\Lambda_\omega]\alpha=\bar{\theta}\wedge(\alpha\llcorner(\bar{\theta})^\sharp),$$

for any $(n,1)$ *form α with value in E. Moreover, for any positive* $(1,1)$ *form β, we have* $[\beta,\Lambda_\omega]$ *is semipositive.*

The following lemma belongs to Fornaess and Narasimhan on approximation property of plurisubharmonic functions of Stein manifolds.

LEMMA 7.4 [10]. *Let X be a Stein manifold and $\varphi\in PSH(X)$. Then there exists a sequence $\{\varphi_n\}_{n=1,2,\ldots}$ of smooth strongly plurisubharmonic functions such that $\varphi_n\downarrow\varphi$.*

For the sake of completeness, let us recall some steps in the proof in [14] (see also [17], [18], [20]) with some slight modifications in order to prove Lemma 2.1.

It follows from Remark 7.1 that it suffices to consider that M is a Stein manifold, and F is holomorphic $(n,0)$ form on $\{\psi<-t_0\}$ and

$$\int_{\{\psi<-t_0\}}|F|^2<+\infty, \tag{105}$$

and there exist smooth plurisubharmonic functions ψ_m and φ_m on M decreasingly convergent to ψ and φ, respectively, satisfying $\sup_m\sup_M\psi_m<-T$ and $\sup_m\sup_M\varphi_m<+\infty$.

Step 1: Construct some functions.

Let $\varepsilon\in(0,\frac{1}{8}B)$. Let $\{v_\varepsilon\}_{\varepsilon\in(0,\frac{1}{8}B)}$ be a family of smooth increasing convex functions on $\mathbb{R}$, which are continuous functions on $\mathbb{R}\cup\{-\infty\}$, such that:

(1) $v_\varepsilon(t)=t$ for $t\geq -t_0-\varepsilon$, $v_\varepsilon(t)=constant$ for $t<-t_0-B+\varepsilon$ and are pointwise convergent to $v_{t_0,B}(t)$.
(2) $v''_\varepsilon(t)$ are pointwise convergent to $\frac{1}{B}\mathbb{I}_{(-t_0-B,-t_0)}$, when $\varepsilon\to 0$, and $0\leq v''_\varepsilon(t)\leq \frac{2}{B}\mathbb{I}_{(-t_0-B+\varepsilon,-t_0-\varepsilon)}$ for any $t\in\mathbb{R}$.
(3) $v'_\varepsilon(t)$ are pointwise convergent to $b_{t_0,B}(t)$ which is a continuous function on $\mathbb{R}$, when $\varepsilon\to 0$, and $0\leq v'_\varepsilon(t)\leq 1$ for any $t\in\mathbb{R}$.

One can construct the family $\{v_\varepsilon\}_{\varepsilon\in(0,\frac{1}{8}B)}$ by the setting

$$\begin{aligned}v_\varepsilon(t):=&\int_{-\infty}^{t}\left(\int_{-\infty}^{t_1}\left(\frac{1}{B-4\varepsilon}\mathbb{I}_{(-t_0-B+2\varepsilon,-t_0-2\varepsilon)}*\rho_{\frac{1}{4}\varepsilon}\right)(s)ds\right)dt_1\\&-\int_{-\infty}^{-t_0}\left(\int_{-\infty}^{t_1}\left(\frac{1}{B-4\varepsilon}\mathbb{I}_{(-t_0-B+2\varepsilon,-t_0-2\varepsilon)}*\rho_{\frac{1}{4}\varepsilon}\right)(s)ds\right)dt_1-t_0,\end{aligned} \tag{106}$$

where $\rho_{\frac{1}{4}\varepsilon}$ is the kernel of convolution satisfying $supp(\rho_{\frac{1}{4}\varepsilon})\subset(-\frac{1}{4}\varepsilon,\frac{1}{4}\varepsilon)$. Then it follows that

$$v''_\varepsilon(t)=\frac{1}{B-4\varepsilon}\mathbb{I}_{(-t_0-B+2\varepsilon,-t_0-2\varepsilon)}*\rho_{\frac{1}{4}\varepsilon}(t),$$

and

$$v'_\varepsilon(t)=\int_{-\infty}^{t}\left(\frac{1}{B-4\varepsilon}\mathbb{I}_{(-t_0-B+2\varepsilon,-t_0-2\varepsilon)}*\rho_{\frac{1}{4}\varepsilon}\right)(s)ds.$$

Let $\eta = s(-v_\varepsilon(\psi_m))$ and $\phi = u(-v_\varepsilon(\psi_m))$, where $s \in C^\infty((S,+\infty))$ satisfies $s > 0$ and $s' > 0$, and $u \in C^\infty((S,+\infty))$, such that $u''s - s'' > 0$, and $s' - u's = 1$. It follows from $\sup_m \sup_M \psi_m < -S$ and $\max\{t, -t_0 - B\} \le v_\epsilon(t) \le \max\{t, t_0\}$ that $\phi = u(-v_\varepsilon(\psi_m))$ are uniformly bounded on M with respect to m and ε, and $u(-v_\varepsilon(\psi))$ are uniformly bounded on M with respect to ε. Let $\Phi = \phi + \varphi_{m'}$, and let $\tilde{h} = e^{-\Phi}$.

Let $f(x) = 2\mathbb{I}_{(-\frac{1}{2},\frac{1}{2})} * \rho(x)$ be a smooth function on $\mathbb{R}$, where ρ is is the kernel of convolution satisfying $supp(\rho) \subset (-\frac{1}{3}, \frac{1}{3})$ and $\rho \ge 0$.

Let $g_l(x) = \begin{cases} lf(lx), & \text{if} \quad x \le 0, \\ lf(l^2x), & \text{if} \quad x > 0, \end{cases}$ then $\{g_l\}_{l \in \mathbb{N}^+}$ be a family of smooth functions on $\mathbb{R}$ satisfying that:

(1) $supp(g_l) \subset [-\frac{1}{l}, \frac{1}{l}]$, $g_l(x) \ge 0$ for any $x \in \mathbb{R}$.
(2) $\int_{-\frac{1}{l}}^0 g_l(x)dx = 1$, $\int_0^{\frac{1}{l}} g_l(x)dx \le \frac{1}{l}$ for any $l \in \mathbb{N}^+$.

Set $c_l(t) = e^t \int_{\mathbb{R}} h(e^y(t-S)+S)g_l(y)dy$, where $h(t) = e^{-t}c(t)$ and $c \in \tilde{\mathcal{P}}_S$. It is easy to get

$$c_l(t) - c(t) \ge e^t \int_{-\frac{1}{l}}^0 (h(e^y(t-S)+S) - h(t))g_l(y)dy \ge 0.$$

Set $\tilde{h}(t) = h(e^t + S)$ and $\tilde{g}_l(t) = g_l(-t)$, then $c_l(t) = e^t \tilde{h} * \tilde{g}_l(\ln(t-S)) \in C^\infty(S,+\infty)$. Because $h(t)$ is decreasing with respect to t, so is $c_l(t)e^{-t}$. And

$$\begin{aligned}\int_S^s c_l(t)e^{-t}dt &= \int_S^s \int_{\mathbb{R}} h(e^y(t-S)+S)g_l(y)dydt \\ &= \int_{\mathbb{R}} e^{-y}g_l(y) \int_S^{e^y(s-S)+S} h(t)dtdy \\ &\le \int_{\mathbb{R}} e^{-y}g_l(y)dy \int_S^{e(s-S)+S} h(t)dt \\ &< +\infty,\end{aligned}$$

then $c_l(t) \in \tilde{\mathcal{P}}_S$ for any $l \in \mathbb{N}^+$.

As $h(t)$ is decreasing with respect to t, then set $h^-(t) = \lim_{s\to t-0} h(s) \ge h(t)$ and $c^-(t) = \lim_{s\to t-0} c(s) \ge c(t)$, then we claim that $\lim_{l\to+\infty} c_l(t) = c^-(t)$. In fact, we have

$$\begin{aligned}|c_l(t) - c^-(t)| \le& e^t \int_{-\frac{1}{l}}^0 |h(e^y(t-S)+S) - h^-(t)|g_l(y)dy \\ &+ e^t \int_0^{\frac{1}{l}} h(e^y(t-S)+S)g_l(y)dy.\end{aligned} \tag{107}$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ and $|h(t-\delta) - h^-(t)| < \varepsilon$. Then $\exists N > 0$, $\forall l > N$, such that $e^y(t-S)+S > t-\delta$ for all $y \in [-\frac{1}{l}, 0)$ and $\frac{1}{l} < \varepsilon$. It following from (107) that

$$|c_l(t) - c^-(t)| \le \varepsilon e^t + \varepsilon h(t)e^t,$$

hence, $\lim_{l\to+\infty} c_l(t) = c^-(t)$ for any $t > S$.

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Step 2: Solving $\bar{\partial}-$equation with smooth polar function and smooth weight.

Now, let $\alpha \in \Lambda_x^{n,1}T_M^*$, for any $x \in M$. Using inequality $s>0$ and the fact that φ_m is plurisubharmonic on M, we get

$$\begin{aligned}\langle \mathbf{B}\alpha,\alpha\rangle_{\tilde{h}} =&\langle[\eta\sqrt{-1}\Theta_{\tilde{h}}-\sqrt{-1}\partial\bar{\partial}\eta-\sqrt{-1}g\partial\eta\wedge\bar{\partial}\eta,\Lambda_\omega]\alpha,\alpha\rangle_{\tilde{h}}\\ \geq&\langle[\eta\sqrt{-1}\partial\bar{\partial}\phi-\sqrt{-1}\partial\bar{\partial}\eta-\sqrt{-1}g\partial\eta\wedge\bar{\partial}\eta,\Lambda_\omega]\alpha,\alpha\rangle_{\tilde{h}},\end{aligned} \tag{108}$$

where $g>0$ is a smooth and bounded function on M. We need the following calculations to determine g:

$$\partial\bar{\partial}\eta=-s'(-v_\varepsilon(\psi_m))\partial\bar{\partial}(v_\varepsilon(\psi_m))+s''(-v_\varepsilon(\psi_m))\partial v_\varepsilon(\psi_m)\wedge\bar{\partial}v_\varepsilon(\psi_m), \tag{109}$$

and

$$\partial\bar{\partial}\phi=-u'(-v_\varepsilon(\psi_m))\partial\bar{\partial}v_\varepsilon(\psi_m)+u''(-v_\varepsilon(\psi_m))\partial v_\varepsilon(\psi_m)\wedge\bar{\partial}v_\varepsilon(\psi_m). \tag{110}$$

Then we have

$$\begin{aligned}&-\partial\bar{\partial}\eta+\eta\partial\bar{\partial}\phi-g(\partial\eta)\wedge\bar{\partial}\eta\\ =&(s'-su')\partial\bar{\partial}v_\varepsilon(\psi_m)+((u''s-s'')-gs'^2)\partial(-v_\varepsilon(\psi_m))\bar{\partial}(-v_\varepsilon(\psi_m))\\ =&(s'-su')(v'_\varepsilon(\psi_m)\partial\bar{\partial}\psi_m+v''_\varepsilon(\psi_m)\partial(\psi_m)\wedge\bar{\partial}(\psi_m))\\ &+((u''s-s'')-gs'^2)\partial(-v_\varepsilon(\psi_m))\wedge\bar{\partial}(-v_\varepsilon(\psi_m)).\end{aligned} \tag{111}$$

We omit composite item $-v_\varepsilon(\psi_m)$ after $s'-su'$ and $(u''s-s'')-gs'^2$ in the above equalities. Let $g=\frac{u''s-s''}{s'^2}(-v_\varepsilon(\psi_m))$. It follows that $\eta+g^{-1}=(s+\frac{s'^2}{u''s-s''})(-v_\varepsilon(\psi_m))$.

As $v'_\varepsilon\geq 0$ and $s'-su'=1$, using Lemma 7.3, equality (111) and inequality (108), we obtain

$$\begin{aligned}\langle \mathbf{B}\alpha,\alpha\rangle_{\tilde{h}} =&\langle[\eta\sqrt{-1}\Theta_{\tilde{h}}-\sqrt{-1}\partial\bar{\partial}\eta-\sqrt{-1}g\partial\eta\wedge\bar{\partial}\eta,\Lambda_\omega]\alpha,\alpha\rangle_{\tilde{h}}\\ \geq&\langle[(v''_\varepsilon\circ\psi_m)\sqrt{-1}\partial\psi_m\wedge\bar{\partial}\psi_m,\Lambda_\omega]\alpha,\alpha\rangle_{\tilde{h}}\\ =&\langle(v''_\varepsilon\circ\psi_m)\bar{\partial}\psi_m\wedge(\alpha\llcorner(\bar{\partial}\psi_m)^\sharp),\alpha\rangle_{\tilde{h}}.\end{aligned} \tag{112}$$

Using the definition of contraction, Cauchy–Schwarz inequality and the inequality (112), we have

$$\begin{aligned}|\langle(v''_\varepsilon\circ\psi_m)\bar{\partial}\psi_m\wedge\gamma,\tilde{\alpha}\rangle_{\tilde{h}}|^2 =&|\langle(v''_\varepsilon\circ\psi_m)\gamma,\tilde{\alpha}\llcorner(\bar{\partial}\psi_m)^\sharp\rangle_{\tilde{h}}|^2\\ \leq&\langle(v''_\varepsilon\circ\psi_m)\gamma,\gamma\rangle_{\tilde{h}}(v''_\varepsilon\circ\psi_m)|\tilde{\alpha}\llcorner(\bar{\partial}\psi_m)^\sharp|^2_{\tilde{h}}\\ =&\langle(v''_\varepsilon\circ\psi_m)\gamma,\gamma\rangle_{\tilde{h}}\langle(v''_\varepsilon\circ\psi_m)\bar{\partial}\psi_m\wedge(\tilde{\alpha}\llcorner(\bar{\partial}\psi_m)^\sharp),\alpha\rangle_{\tilde{h}}\\ \leq&\langle(v''_\varepsilon\circ\psi_m)\gamma,\gamma\rangle_{\tilde{h}}\langle\mathbf{B}\tilde{\alpha},\tilde{\alpha}\rangle_{\tilde{h}},\end{aligned} \tag{113}$$

for any $(n,0)$ form γ.

It follows from $s>0$ and $\varphi_{m'}$ is strongly plurisubharmonic that $\mathbf{B}$ is positive definite everywhere on $\Lambda^{n,1}T_M^*$. As F is holomorphic on $\{\psi<-t_0\}$ and $Supp(v''_\varepsilon(\psi_m))\subset\{\psi<-t_0\}$, then $\lambda:=\bar{\partial}[(1-v'_\varepsilon(\psi_m))F]$ is well defined and smooth on M.

Taking $\gamma=F$, and $\tilde{\alpha}=\mathbf{B}^{-1}\lambda$, note that $\tilde{h}=e^{-\Phi}$, using inequality (113), we have

$$\langle\mathbf{B}^{-1}\lambda,\lambda\rangle_{\tilde{h}}\leq v''_{t_0,\varepsilon}(\psi_m)|\tilde{F}|^2e^{-\Phi}.$$

Then it follows that

$$\int_M\langle\mathbf{B}^{-1}\lambda,\lambda\rangle_{\tilde{h}}\leq\int_M v''_{t_0,\varepsilon}(\psi_m)|\tilde{F}|^2e^{-\Phi}.$$

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Assume that we can choose η and ϕ such that $e^{v_\varepsilon\circ\psi_m}e^{\phi}c_l(-v_\varepsilon\circ\psi_m)=(\eta+g^{-1})^{-1}$. Using Lemma 7.2, we have locally L^1 function $u_{m,m',\varepsilon,l}$ on M such that $\bar\partial u_{m,m',\varepsilon,l}=\lambda$, and

$$\begin{aligned}&\int_M |u_{m,m',\varepsilon,l}|^2 e^{v_\varepsilon(\psi_m)-\varphi_{m'}}c_l(-v_\varepsilon\circ\psi_m)\\ =&\int_M |u_{m,m',\varepsilon,l}|^2(\eta+g^{-1})^{-1}e^{-\Phi}\\ \le&\int_M \langle \mathbf{B}^{-1}\lambda,\lambda\rangle_{\tilde h}\\ \le&\int_M v''_\varepsilon(\psi_m)|F|^2e^{-\Phi}\\ =&\int_M v''_\varepsilon(\psi_m)|F|^2e^{-\phi-\varphi_{m'}}.\end{aligned}\tag{114}$$

Let $F_{m,m',\varepsilon,l}:=-u_{m,m',\varepsilon,l}+(1-v'_\varepsilon(\psi_m))F$. Then inequality (114) becomes

$$\begin{aligned}&\int_M |F_{m,m',\varepsilon,l}-(1-v'_\varepsilon(\psi_m))F|^2e^{v_\varepsilon(\psi_m)-\varphi_{m'}}c_l(-v_\varepsilon\circ\psi_m)\\ \le&\int_M (v''_\varepsilon(\psi_m))|F|^2e^{-\phi-\varphi_{m'}}.\end{aligned}\tag{115}$$

Step 3: Singular polar function and smooth weight.

As $\sup_{m,\varepsilon}\sup_M|\phi|=\sup_{m,\varepsilon}|u(-v_\varepsilon(\psi_m))|<+\infty$ and $\sup_M\varphi_{m'}<+\infty$, note that

$$v''_\varepsilon(\psi_m)|F|^2e^{-\phi-\varphi_{m'}}\le \frac{2}{B}\mathbb{I}_{\{\psi<-t_0\}}|F|^2\sup_{m,\varepsilon}e^{-\phi-\varphi_{m'}}$$

on M, then it follows from inequality (105) and the dominated convergence theorem that

$$\lim_{m\to+\infty}\int_M v''_\varepsilon(\psi_m)|F|^2e^{-\phi-\varphi_{m'}}=\int_M v''_\varepsilon(\psi)|F|^2e^{-u(-v_\varepsilon(\psi))-\varphi_{m'}}.\tag{116}$$

Note that $\inf_m\inf_M e^{v_\varepsilon(\psi_m)-\varphi_{m'}}c_l(-v_\varepsilon\circ\psi_m)>0$, then it follows from inequality (115) and (116) that $\sup_m\int_M|F_{m,m',\varepsilon,l}-(1-v'_\varepsilon(\psi_m))F|^2<+\infty$. Note that

$$|(1-v'_\varepsilon(\psi_m))F|\le|\mathbb{I}_{\{\psi<-t_0\}}F|,\tag{117}$$

then it follows from inequality (105) that $\sup_m\int_M|F_{m,m',\varepsilon,l}|^2<+\infty$, which implies that there exists a subsequence of $\{F_{m,m',\varepsilon,l}\}_m$ (also denoted by $\{F_{m,m',\varepsilon,l}\}_m$) compactly convergent to a holomorphic $F_{m',\varepsilon,l}$ on M.

Note that $e^{v_\varepsilon(\psi_m)-\varphi_{m'}}c_l(-v_\varepsilon\circ\psi_m)$ are uniformly bounded on M with respect to m, then it follows from $|F_{m,m',\varepsilon,l}-(1-v'_\varepsilon(\psi_m))F|^2\le 2(|F_{m,m',\varepsilon,l}|^2+|(1-v'_\varepsilon(\psi_m))F|^2)\le 2(|F_{m,m',\varepsilon,l}|^2+|\mathbb{I}_{\{\psi<-t_0\}}F^2|)$ and the dominated convergence theorem that

$$\begin{aligned}&\lim_{m\to+\infty}\int_K|F_{m,m',\varepsilon,l}-(1-v'_\varepsilon(\psi_m))F|^2e^{v_\varepsilon(\psi_m)-\varphi_{m'}}c_l(-v_\varepsilon\circ\psi_m)\\ &=\int_K|F_{m',\varepsilon,l}-(1-v'_\varepsilon(\psi))F|^2e^{v_\varepsilon(\psi)-\varphi_{m'}}c_l(-v_\varepsilon\circ\psi)\end{aligned}\tag{118}$$

holds for any compact subset K on M. Combining with inequality (115) and (116), one can obtain that

$$\begin{aligned}&\int_K |F_{m',\varepsilon,l}-(1-v'_\varepsilon(\psi))F|^2 e^{v_\varepsilon(\psi)-\varphi_{m'}}c_l(-v_\varepsilon\circ\psi)\\ &\leq \int_M v''_\varepsilon(\psi)|F|^2 e^{-u(-v_\varepsilon(\psi))-\varphi_{m'}},\end{aligned}\tag{119}$$

which implies

$$\begin{aligned}&\int_M |F_{m',\varepsilon,l}-(1-v'_\varepsilon(\psi))F|^2 e^{v_\varepsilon(\psi)-\varphi_{m'}}c_l(-v_\varepsilon\circ\psi)\\ &\leq \int_M v''_\varepsilon(\psi)|F|^2 e^{-u(-v_\varepsilon(\psi))-\varphi_{m'}}.\end{aligned}\tag{120}$$

Step 4: Nonsmooth cut-off function.

Note that $\sup_\varepsilon\sup_M e^{-u(-v_\varepsilon(\psi))-\varphi_{m'}}<+\infty$, and

$$v''_\varepsilon(\psi)|F|^2 e^{-u(-v_\varepsilon(\psi))-\varphi_{m'}} \leq \frac{2}{B}\mathbb{I}_{\{-t_0-B<\psi<-t_0\}}|F|^2\sup_\varepsilon\sup_M e^{-u(-v_\varepsilon(\psi))-\varphi_{m'}},$$

then it follows from inequality (105) and the dominated convergence theorem that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_M v''_\varepsilon(\psi)|F|^2 e^{-u(-v_\varepsilon(\psi))-\varphi_{m'}}\\ =&\int_M \frac{1}{B}\mathbb{I}_{\{-t_0-B<\psi<-t_0\}}|F|^2 e^{-u(-v_{t_0,B}(\psi))-\varphi_{m'}}\\ \leq&(\sup_M e^{-u(-v_{t_0,B}(\psi))})\int_M \frac{1}{B}\mathbb{I}_{\{-t_0-B<\psi<-t_0\}}|F|^2 e^{-\varphi_{m'}}<+\infty.\end{aligned}\tag{121}$$

Note that $\inf_\varepsilon\inf_M e^{v_\varepsilon(\psi)-\varphi_{m'}}c_l(-v_\varepsilon\circ\psi)>0$, then it follows from inequality (120) and (121) that $\sup_\varepsilon\int_M|F_{m',\varepsilon,l}-(1-v'_\varepsilon(\psi))F|^2<+\infty$. Combining with

$$\sup_\varepsilon\int_M |(1-v'_\varepsilon(\psi))F|^2 \leq \int_M \mathbb{I}_{\{\psi<-t_0\}}|F|^2<+\infty,\tag{122}$$

one can obtain that $\sup_\varepsilon\int_M|F_{m',\varepsilon,l}|^2<+\infty$, which implies that there exists a subsequence of $\{F_{m',\varepsilon,l}\}_{\varepsilon\to 0}$ (also denoted by $\{F_{m',\varepsilon,l}\}_{\varepsilon\to 0}$) compactly convergent to a holomorphic $(n,0)$ form on M denoted by $F_{m',l}$. Then it follows from inequality (120), inequality (121), and Fatou's Lemma that

$$\begin{aligned}&\int_M |F_{m',l}-(1-b_{t_0,B}(\psi))F|^2 e^{v_{t_0,B}(\psi)-\varphi_{m'}}c_l(-v\circ\psi)\\ =&\int_M \liminf_{\varepsilon\to 0}|F_{m',\varepsilon,l}-(1-v'_\varepsilon(\psi))F|^2 e^{v_\varepsilon(\psi)-\varphi_{m'}}c_l(-v_\varepsilon\circ\psi)\\ \leq&\liminf_{\varepsilon\to 0}\int_M |F_{m',\varepsilon,l}-(1-v'_\varepsilon(\psi))F|^2 e^{v_\varepsilon(\psi)-\varphi_{m'}}c_l(-v_\varepsilon\circ\psi)\\ \leq&\liminf_{\varepsilon\to 0}\int_M v''_\varepsilon(\psi)|F|^2 e^{-u(-v_\varepsilon(\psi))-\varphi_{m'}}\\ \leq&(\sup_M e^{-u(-v_{t_0,B}(\psi))})\int_M \frac{1}{B}\mathbb{I}_{\{-t_0-B<\psi<-t_0\}}|F|^2 e^{-\varphi_{m'}}.\end{aligned}\tag{123}$$

Step 5: Singular weight.

Note that

$$\int_M \frac{1}{B}\mathbb{I}_{\{-t_0-B<\psi<-t_0\}}|F|^2e^{-\varphi_{m'}} \leq \int_M \frac{1}{B}\mathbb{I}_{\{-t_0-B<\psi<-t_0\}}|F|^2e^{-\varphi} < +\infty, \tag{124}$$

and $\sup_M e^{-u(-v_{t_0,B}(\psi))} < +\infty$, then it from (123) that

$$\sup_{m'}\int_M |F_{m',l}-(1-b(\psi))F|^2e^{v(\psi)-\varphi_{m'}}c_l(-v\circ\psi) < +\infty.$$

Combining with $\inf_{m'}\inf_M e^{v(\psi)-\varphi_{m'}}c_l(-v(\psi)) > 0$, one can obtain that

$$\sup_{m'}\int_M |F_{m',l}-(1-b(\psi))F|^2 < +\infty.$$

Note that

$$\int_M |(1-b(\psi))F|^2 \leq \int_M |\mathbb{I}_{\{\psi<-t_0\}}F|^2 < +\infty. \tag{125}$$

Then $\sup_{m'}\int_M |F_{m',l}|^2 < +\infty$, which implies that there exists a compactly convergent subsequence of $\{F_{m',l}\}_{m'}$ (also denoted by $\{F_{m',l}\}_{m'}$), which converges to a holomorphic $(n,0)$ form F_l on M. Then it follows from inequality (123), inequality (124), and Fatou's Lemma that

$$\begin{aligned}
&\int_M |F_l-(1-b_{t_0,B}(\psi))F|^2e^{v_{t_0,B}(\psi)-\varphi}c_l(-v_{t_0,B}\circ\psi)\\
=&\int_M \liminf_{m'\to+\infty}|F_{m',l}-(1-b_{t_0,B}(\psi))F|^2e^{v_{t_0,B}(\psi)-\varphi_{m'}}c_l(-v_{t_0,B}\circ\psi)\\
\leq&\liminf_{m'\to+\infty}\int_M |F_{m',l}-(1-b_{t_0,B}(\psi))F|^2e^{v_{t_0,B}(\psi)-\varphi_{m'}}c_l(-v_{t_0,B}\circ\psi)\\
\leq&\liminf_{m'\to+\infty}(\sup_M e^{-u(-v_{t_0,B}(\psi))})\int_M \frac{1}{B}\mathbb{I}_{\{-t_0-B<\psi<-t_0\}}|F|^2e^{-\varphi_{m'}}\\
\leq&(\sup_M e^{-u(-v_{t_0,B}(\psi))})\int_M \frac{1}{B}\mathbb{I}_{\{-t_0-B<\psi<-t_0\}}|F|^2e^{-\varphi}.
\end{aligned} \tag{126}$$

Step 6: ODE system.

we need to find η and ϕ such that $(\eta+g^{-1}) = e^{-\psi_m}e^{-\phi}\frac{1}{c_l(-v_\varepsilon(\psi_m))}$ on M and $s'-u's=1$. As $\eta = s(-v_\varepsilon(\psi_m))$ and $\phi = u(-v_\varepsilon(\psi_m))$, we have $(\eta+g^{-1})e^{v_\varepsilon(\psi_m)}e^{\phi} = (s+\frac{s'^2}{u''s-s''})e^{-t}e^u \circ (-v_\varepsilon(\psi_m))$.

Summarizing the above discussion about s and u, we are naturally led to a system of ODEs (see [16–18], [20]):

$$\begin{aligned}
&1).\ \left(s+\frac{s'^2}{u''s-s''}\right)e^{u-t} = \frac{1}{c_l(t)},\\
&2).\ s'-su' = 1,
\end{aligned} \tag{127}$$

where $t\in(T,+\infty)$.

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It is not hard to solve the ODE system (127) and get $u(t)=-\log(\int_S^t c_l(t_1)e^{-t_1}dt_1)$ and $s(t)=\frac{\int_S^t(\int_S^{t_2}c_l(t_1)e^{-t_1}dt_1)dt_2}{\int_S^t c_l(t_1)e^{-t_1}dt_1}$ (see [18]). It follows that $s\in C^\infty((S,+\infty))$ satisfies $s>0$ and $s'>0$, $u\in C^\infty((S,+\infty))$ satisfies $u''s-s''>0$.

As $u(t)=-\log(\int_S^t c_l(t_1)e^{-t_1}dt_1)$ is decreasing with respect to t, then it follows from $-S\geq v(t)\geq\max\{t,-t_0-B_0\}\geq -t_0-B_0$ for any $t\leq 0$ that

$$\sup_M e^{-u(-v(\psi))}\leq \sup_{t\in(S,t_0+B]} e^{-u(t)}=\int_S^{t_0+B}c_l(t_1)e^{-t_1}dt_1, \tag{128}$$

then it follows from inequality (8) and inequality (126) that

$$\int_M |F_l-(1-b_{t_0,B}(\psi))F|^2e^{v_{t_0,B}(\psi)-\varphi}c_l(-v_{t_0,B}(\psi))\leq C\int_S^{t_0+B}c_l(t_1)e^{-t_1}dt_1. \tag{129}$$

Step 7: Nonsmooth function c.

By the construction of c_l in Step 1, we have

$$\begin{aligned}
&\int_S^{t_0+B}c_l(t_1)e^{-t_1}dt_1\\
=&\int_S^{t_0+B}\int_{\mathbb{R}}h((t_1-S)e^y+S)g_l(y)dydt_1\\
=&\int_{\mathbb{R}}e^{-y}g_l(y)\int_S^{(t_0+B-S)e^y+S}h(s)dsdy\\
=&\int_{\mathbb{R}}e^{-y}g_l(y)dy\int_S^{t_0+B}h(s)ds+\int_{\mathbb{R}}e^{-y}g_l(y)\int_{t_0+B}^{(t_0+B-S)e^y+S}h(s)dsdy.
\end{aligned}\tag{130}$$

As

$$\begin{aligned}
&\lim_{l\to+\infty}\left|\int_{\mathbb{R}}e^{-y}g_l(y)dy-1\right|\\
\leq&\lim_{l\to+\infty}\left|\int_{-\frac{1}{l}}^0(e^{-y}-1)g_l(y)dy\right|+\lim_{l\to+\infty}\left|\int_0^{\frac{1}{l}}e^{-y}g_l(y)dy\right|\\
=&0
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_{\mathbb{R}}e^{-y}g_l(y)\int_{t_0+B}^{(t_0+B-S)e^y+S}h(s)dsdy\right|\\
\leq& e^{\frac{1}{l}}\left(1+\frac{1}{l}\right)h((t_0+B-S)e^{-1}+S)(t_0+B-S)(e^{\frac{1}{l}}-e^{-\frac{1}{l}}),
\end{aligned}$$

then it follows from inequality (130) that

$$\lim_{l\to+\infty}\int_S^{t_0+B}c_l(t_1)e^{-t_1}dt_1=\int_S^{t_0+B}c(t_1)e^{-t_1}dt_1. \tag{131}$$

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Combining with $\inf_l \inf_M e^{v_{t_0,B}(\psi)-\varphi} c_l(-v(\psi)) \geq \inf_M e^{v_{t_0,B}(\psi)-\varphi} c(-v(\psi)) > 0$, we obtain that

$$\sup_l \int_M |F_l - (1-b_{t_0,B}(\psi))F|^2 < +\infty.$$

Note that

$$\int_M |(1-b_{t_0,B}(\psi))F|^2 \leq \int_M |\mathbb{I}_{\{\psi<-t_0\}}F|^2 < +\infty, \tag{132}$$

then $\sup_l \int_M |F_l|^2 < +\infty$, which implies that there exists a compactly convergent subsequence of $\{F_l\}$ (also denoted by $\{F_l\}$), which converges to a holomorphic $(n,0)$ form $\tilde{F}$ on M. Then it follows from inequality (129) and the Fatou's Lemma that

$$\begin{aligned}
&\int_M |\tilde{F} - (1-b_{t_0,B}(\psi))F|^2 e^{v_{t_0,B}(\psi)-\varphi} c(-v_{t_0,B}(\psi)) \\
\leq &\int_M |\tilde{F} - (1-b_{t_0,B}(\psi))F|^2 e^{v_{t_0,B}(\psi)-\varphi} c^-(-v_{t_0,B}(\psi)) \\
= &\int_M \liminf_{l\to+\infty} |F_l - (1-b_{t_0,B}(\psi))F|^2 e^{v_{t_0,B}(\psi)-\varphi} c_l(-v_{t_0,B}(\psi)) \\
\leq &\liminf_{l\to+\infty} \int_M |F_l - (1-b_{t_0,B}(\psi))F|^2 e^{v_{t_0,B}(\psi)-\varphi} c_l(-v_{t_0,B}(\psi)) \\
\leq &C \liminf_{l\to+\infty} \int_S^{t_0+B} c_l(t_1)e^{-t_1} dt_1 \\
= &C \int_S^{t_0+B} c(t_1)e^{-t_1} dt_1.
\end{aligned}$$

Thus, we prove Lemma 2.1.

7.2 Proof of Lemma 2.14

The proof is from [15] with a few minor modifications.

Choose $p \in suppT \cap U$. By Lemma 2.12, there exist a real number $t > 0$ and a coordinate (V, w), such that $w(p) = 0$, $w(V) \cong B(0,1)$ and $V \subset\subset \{G_\Omega(z,p) < -t\} \subset\subset U$. There exists a cut-off function θ on Ω, such that $\theta \equiv 1$ on $w^{-1}(B(0,\frac{1}{4}))$ and $supp\theta \subset\subset w^{-1}(B(0,\frac{1}{2}))$.

Let $\tilde{T} = \theta T$, then $\tilde{T}$ is a closed positive $(1,1)$ current on Ω with $supp\tilde{T} \subset\subset w^{-1}(B(0,\frac{1}{2}))$ and $\tilde{T} \not\equiv 0$. Now, we prove that exists a subharmonic function $\Phi < 0$ on Ω, which satisfies the following properties: $i\partial\bar{\partial}\Phi = \tilde{T}$; $\lim_{t\to 0+0}(\inf_{\{G_\Omega(z,z_0)\geq -t\}} \Phi(z)) = 0$; $\inf_{\Omega\setminus U} \Phi > -\infty$. Then Φ satisfies the requirements in Lemma 2.14.

Step 1: Construct Φ.

Let $\rho \in C^\infty(\mathbb{C})$ be a function with $supp\rho \subset B(0,\frac{1}{2})$ and $\rho(z)$ depends only on $|z|$, $\rho \geq 0$ and $\int_{\mathbb{C}} \rho(z) d\lambda_z = 1$. Let $\rho_n(z) = n\rho(nz)$, ρ_n is a family of smoothing kernels.

As $w(V) \cong B(0,1)$, without misunderstanding we see (V, z_1) and $(B(0,1), w)$ the same. As $supp\tilde{T} \subset\subset w^{-1}(B(0,\frac{1}{2}))$ and $supp\rho \subset B(0,\frac{1}{2})$, denote that $T_n = \tilde{T} * \rho_n$ be the convolution of $\tilde{T}$. In fact, for any test function $h \in C_c^\infty(\Omega)$, $((h \circ w^{-1}) * \rho_n)(w)$ ($h * \rho_n(w)$ for short) is well defined on $w^{-1}(B(0,\frac{1}{2}))$, and $\langle T_n(z_1), h(z_1)\rangle =< \tilde{T}(w), h * \rho_n(w) >$. Then T_n is a smooth closed positive $(1,1)$ current on Ω with $suppT_n \subset\subset w^{-1}(B(0,\frac{1}{2}+\frac{1}{2n}))$.

Let $u_n(z) = \langle T_n(z_1), \frac{1}{\pi} G_\Omega(z,z_1)\rangle$. $G_\Omega(z,z_1)$ is locally integrable with respect $z_1 \in \Omega$ for any fixed $z \in \Omega$ implies that $u_n(z) > -\infty$ for any $z \in \Omega$. For fixed z and fixed n, we will prove $\langle T_n(z_1), \frac{1}{\pi}G_\Omega(z,z_1)\rangle = \langle \tilde{T}(w), (\frac{1}{\pi}G_\Omega(z,\cdot) * \rho_n)(w)\rangle$. For fixed z, $G_\Omega(z,z_1)$ is a subharmonic function on Ω. There exists a sequence of smooth subharmonic functions $G_m(z_1)$ decreasingly converge to $G_\Omega(z,z_1)$ with respect to m. As $G_m(z_1)$ is smooth, we have

$$\langle T_n(z_1), \frac{1}{\pi}G_m(z_1)\rangle = < \tilde{T}(w), \frac{1}{\pi}G_m * \rho_n(w) > . \tag{133}$$

As $\tilde{T}$ and T_n are closed positive $(1,1)$ current on Ω with $suppT_n \subset\subset V$ and $supp\tilde{T} \subset\subset V$, and $G_m(z_1)$ decreasingly converge to $G_\Omega(z,z_1) < 0$ with respect to m on Ω, it follows from Levi's Theorem and equality (133) that

$$\begin{aligned}\langle T_n(z_1), \frac{1}{\pi}G_\Omega(z,z_1)\rangle &= \lim_{m\to+\infty}\langle T_n(z_1), \frac{1}{\pi}G_m(z_!)\rangle \\ &= \lim_{m\to+\infty} < \tilde{T}(w), \frac{1}{\pi}G_m * \rho_n(w) > \\ &= < \tilde{T}(w), \frac{1}{\pi}G_\Omega(z,\cdot) * \rho_n(w) > .\end{aligned}$$

Fixed $z \in \Omega$, as $\frac{1}{\pi}G_\Omega(z,z_1)$ is subharmonic, then $\frac{1}{\pi}G_\Omega(z,\cdot) * \rho_n$ is decreasingly convergent to $\frac{1}{\pi}G_\Omega(z,z_1)$ with respect to n. Note that $\tilde{T}$ is a positive $(1,1)$ current on Ω, then $u_n(z)$ is decreasing with respect to n. Let $\Phi(z) = \lim_{n\to+\infty} u_n(z)$. $G_\Omega(z,z_1) < 0$ on $\Omega\times\Omega$ shows that $u_n(z) < 0$ and $\Phi(z) < 0$ on Ω.

Step 2: $i\partial\bar{\partial}\Phi = \tilde{T}$.

Firstly, we show that both $\{u_n\}$ and Φ is L^1_{loc} function on Ω. As $u_n \le 0$ on Ω and u_n is decreasingly convergent to Φ with respect to n on Ω, it suffices to prove that, for any $q \in \Omega$, there exists an open subset $K \subset\subset \Omega$, such that $q \in K$ and $\int_K |u_n| dV_\Omega \le C$, where dV_Ω is some continuous volume form and C is a constant which independent of n.

It is clear that there exists a compact subset D of V such that $supp\tilde{T} \subset D$ and $suppT_n \subset D$ for any n. When $q \notin V$, where exists a coordinate w_1 on a neighborhood V' of q, such that $w_1(q) = 0$, $V' \subset\subset \Omega$, $w_1(V') \cong B(3,1)$, and $\overline{V'} \cap D = \emptyset$. Note that for any $(z,z_1) \in V'\times D$, $G(z,z_1) < 0$ on $\Omega\times\Omega$, $G(z,z_1)$ is harmonic with respect to z or z_1 when fixed another one and $\int_{z_1\in V}|G(q,z_1)| < +\infty$. Without loss of generality, we see (V',z) and $(B(3,1),w_1)$ the same and assume that $dV_\Omega = d\lambda_z$ on V', where $d\lambda_z$ is the Lebesgue measure on $\mathbb{C}$. Then we have

$$\begin{aligned}\int_{V'} |u_n| dV_\Omega &= \frac{1}{\pi}\int_{z\in V'}\int_{z_1\in V} |G_\Omega(z,z_1)| T_n(z_1) d\lambda_z \\ &= \frac{1}{\pi}\int_{z_1\in V}\int_{z\in V'} |G_\Omega(z,z_1)| d\lambda_z T_n(z_1) \\ &= \frac{1}{\pi}\int_{z_1\in V} \pi |G_\Omega(q,z_1)| T_n(z_1) \\ &\le \|T_n\| \sup_{z_1\in V} |G_\Omega(q,z_1)| \\ &= \|\tilde{T}\| \sup_{z_1\in V} |G_\Omega(q,z_1)|.\end{aligned} \tag{134}$$

When $q \in V$, $G_\Omega(w,\tilde{w}) = \log|w-\tilde{w}| + v(w,\tilde{w})$ on $V\times V$, where $v(w,\tilde{w})$ is harmonic with respect to w or $\tilde{w}$ when fixed another one. Without loss of generality, we see (V,z) and $(B(0,1),w)$ the same and assume that $dV_\Omega = d\lambda_w$ on V, where $d\lambda_w$ is the Lebesgue measure

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on $\mathbb{C}$. Then we have

$$\begin{aligned}\int_V u_n dV_\Omega &= \frac{1}{\pi}\int_{w\in V}\int_{\tilde{w}\in V} G_\Omega(w,\tilde{w})T_n(\tilde{w})d\lambda_w \\ &= \frac{1}{\pi}\int_{\tilde{w}\in V}\int_{w\in V} G_\Omega(w,\tilde{w})d\lambda_w T_n(\tilde{w}) \\ &= \frac{1}{\pi}\int_{\tilde{w}\in V}\int_{w\in V}\log|w-\tilde{w}|d\lambda_w T_n(\tilde{w}) + \frac{1}{\pi}\int_{\tilde{w}\in V}\int_{w\in V} v(w,\tilde{w})d\lambda_w T_n(\tilde{w}).\end{aligned} \tag{135}$$

Note that

$$\int_{w\in V}\log|w-\tilde{w}|d\lambda_w \geq -\int_{w\in B(0,2)}|\log|w||d\lambda_w > -\infty$$

holds for any $\tilde{w}\in V$,

$$\int_{w\in V} v(w,\tilde{w})d\lambda_w = \pi v(q,\tilde{w})$$

holds for any $\tilde{w}\in V$ and $\inf_{\tilde{w}\in V} v(q,\tilde{w}) > -\infty$, then equality (135) implies that there exists a constant $N>0$ such that

$$\int_V u_n dV_\Omega \geq N\|T_n\|. \tag{136}$$

By the definition of T_n, we know $\|T_n\| = \|\tilde{T}\| < +\infty$. As $u_n \leq 0$, combining inequality (134) and (136), we obtain that any $q\in\Omega$ there exists an open subset $K\subset\subset\Omega$, such that $q\in K$ and $\int_K |u_n|dV_\Omega \leq C$, where dV_Ω is some continuous volume form and C is a constant which independent of n. Hence, we know $\{u_n\}\in L^1_{loc}(\Omega)$ and $\Phi\in L^1_{loc}(\Omega)$.

Now, we consider $i\partial\bar{\partial}\Phi$. Let $g\in C_c^\infty(X)$ be a test function. It follows from $\Phi\in L^1_{loc}(\Omega)$ and the dominated convergence theorem that

$$\begin{aligned}\langle i\partial\bar{\partial}\Phi, g\rangle &= \langle \Phi(z), i\partial\bar{\partial}g(z)\rangle \\ &= \lim_{n\to+\infty}\langle u_n(z), i\partial\bar{\partial}g(z)\rangle.\end{aligned} \tag{137}$$

As $u_n(z) = \langle T_n(z_1), \frac{1}{\pi}G_\Omega(z,z_1)\rangle$, using Fubini's Theorem, equality (137) becomes

$$\begin{aligned}\langle i\partial\bar{\partial}\Phi, g\rangle &= \lim_{n\to+\infty}\langle\langle T_n(z_1), \frac{1}{\pi}G_\Omega(z,z_1)\rangle, i\partial\bar{\partial}g(z)\rangle \\ &= \lim_{n\to+\infty}\langle T_n(z_1), \langle\frac{1}{\pi}G_\Omega(z,z_1), i\partial\bar{\partial}g(z)\rangle\rangle.\end{aligned} \tag{138}$$

Since T_n is positive $(1,1)$ current on Ω, T_n converge weakly to $\tilde{T}$ and $\frac{i}{\pi}\partial_z\bar{\partial}_z G_\Omega(z,z_1) = \delta_{z_1}$, it follows from equality (138) that

$$\begin{aligned}\langle i\partial\bar{\partial}\Phi, g\rangle &= \lim_{n\to+\infty}\langle T_n(z_1), \langle\frac{1}{\pi}G_\Omega(z,z_1), i\partial\bar{\partial}g(z)\rangle\rangle \\ &= \lim_{n\to+\infty}\langle T_n(z_1), g(z_1)\rangle \\ &= \langle\tilde{T}, g\rangle,\end{aligned} \tag{139}$$

which implies that $i\partial\bar{\partial}\Phi = \tilde{T}$.

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Step 3: $\lim_{t\to 0+0}(\inf_{\{G_\Omega(z,z_0)\geq -t\}}\Phi(z))=0$ *and* $\inf_{\Omega\setminus U}\Phi>-\infty$.

Let $W\subset\subset\Omega$ be an open set of Ω which satisfies $\overline{V}\cup\{z_0\}\subset W$ and $\overline{W}\cap\{-t\leq G_\Omega(z,z_0)\}=\emptyset$, where t is a small enough positive number. Then for every fixed $z\in\{-t\leq G_\Omega(z,z_0)\}$, $G_\Omega(z,z_1)$ is harmonic function on a neighborhood of $\overline{W}$ with respect to z_1. By the Harnack inequality of harmonic function, there exists a $M>0$ such that

$$\sup_{z_1\in\overline{W}}(-G_\Omega(z,z_1))\leq M\inf_{z_1\in\overline{W}}(-G_\Omega(z,z_1)).$$

As $z\in\{-t\leq G_\Omega(z,z_0)\}$, we have

$$Mt>-MG_\Omega(z,z_0)\geq M\inf_{z_1\in\overline{W}}(-G_\Omega(z,z_1)\geq\sup_{z_1\in\overline{W}}(-G_\Omega(z,z_1)\geq 0,$$

which means that $\lim_{t\to 0+0}(\inf_{\{G_\Omega(z,z_0)\geq -t\}\times\overline{W}}G_\Omega(z,z_1))=0$.

Note that $0\geq u_n(z)=\langle T_n(z_1),\frac{1}{\pi}G_\Omega(z,z_1)\rangle\geq\frac{1}{\pi}\inf_{\{G_\Omega(z,z_0)\geq -t\}\times\overline{W}}G_\Omega(z,z_1)\|T_n\|$ holds for any n and $z\in\{-t\leq G_\Omega(z,z_0)\}$, as $\|T_n\|=\|\tilde{T}\|<+\infty$ and u_n is decreasingly convergent to Φ, then we have

$$\lim_{t\to 0+0}(\inf_{\{G_\Omega(z,z_0)\geq -t\}}\Phi(z))\geq\lim_{t\to 0+0}\frac{1}{\pi}\inf_{\{G_\Omega(z,z_0)\geq -t\}\times\overline{W}}G_\Omega(z,z_1)\|\tilde{T}\|=0.$$

Next, we prove $\inf_{\Omega\setminus U}\Phi>-\infty$. Note that $p\in V\subset\subset\{G_\Omega(z,p)<-t\}\subset\subset U\subset\subset\Omega$, it follows from Lemma 2.13 that there exists a constant $N>0$, such that

$$G_\Omega(z,z_1)\geq NG_\Omega(z,p)\geq -Nt \tag{140}$$

holds for any $(z,z_1)\in(\Omega\setminus U,V)$. As $u_n(z)=\langle T_n(z_1),\frac{1}{\pi}G_\Omega(z,z_1)\rangle$ and $suppT_n\subset\subset V$ for any n, then we have $u_n(z)\geq -\frac{Nt}{\pi}\|T_n\|$ hold on $z\in\Omega\setminus U$. Note that $\|T_n\|=\|\tilde{T}\|$ and u_n is decreasingly convergent to Φ, then we have $\inf_{\Omega\setminus U}\Phi>-\infty$.

Thus, Lemma 2.14 holds.

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