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OPERATORS ON $C_0(L, X)$ WHOSE RANGE DOES NOT CONTAIN c_0

JARNO TALPONEN

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Abstract

This paper contains two results: (a) if $X \neq \{0\}$ is a Banach space and (L, τ) is a nonempty locally compact Hausdorff space without isolated points, then each linear operator $T : C_0(L, X) \to C_0(L, X)$ whose range does not contain an isomorphic copy of c_{00} satisfies the Daugavet equality $\|\mathbf{I} + T\| = 1 + \|T\|$; (b) if Γ is a nonempty set and X and Y are Banach spaces such that X is reflexive and Y does not contain c_0 isomorphically, then any continuous linear operator $T : c_0(\Gamma, X) \to Y$ is weakly compact.

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1. Introduction

In what follows X, Y and Z are real Banach spaces different from $\{0\}$, L is a nonempty locally compact Hausdorff space and $C_0(L, X)$ denotes the $\|\cdot\|_\infty$-normed Banach space of X-valued continuous functions on L vanishing at infinity. This work is related to the result due to Cembranos [2] that if K is an infinite compact and X is an infinite-dimensional Banach space, then the Banach space $C(K, X)$ of X-valued continuous functions on K contains a complemented copy of c_0. See also the related paper [1] where the Dieudonné property of $C(K, X)$ was studied.

Going in another direction, we study continuous linear operators of the type $(*)$ $T : C_0(L, X) \to Y$, where Y does not contain c_0 isomorphically. There is a structural disparity between spaces $C_0(L, X)$ and Y, since typically the former space contains copies of c_0 in abundance. This difference has a strong impact on the properties of T. Namely, it turns out that the range of T in Y is *small* in some sense.

If L does not contain isolated points, then an operator $T : C_0(L, X) \to C_0(L, X)$ of type $(*)$ satisfies the Daugavet type equality $\|\mathbf{I} + T\| = 1 + \|T\|$ (see Theorem 2.1). See [5] for a recent discussion on matters related to the Daugavet property.

If L is discrete, X is reflexive and c_0 is not contained in Y isomorphically, then an operator $T : C_0(L, X) \to Y$ is weakly compact (see Theorem 2.3).

Preliminaries. Here X and Y denote real Banach spaces. The closed unit ball and the unit sphere of X are denoted by $\mathbf{B}_X$ and $\mathbf{S}_X$, respectively. An identity mapping is denoted by $\mathbf{I}$. An operator $T\colon X \to Y$ is weakly compact if $\overline{T(\mathbf{B}_X)}$ is weakly compact. If $X \neq \{0\}$, then we say that X is *nontrivial*. For given sets $A \subset B$ the mapping $\chi_A\colon B \to \{0, 1\}$ is determined by $\chi_A(t) = 1$ if and only if $t \in A$. We refer to [3, 4, 6] for suitable background information including definitions and basic results.

2. Results

THEOREM 2.1. *Let X be a nontrivial Banach space and (L, τ) a nonempty locally compact Hausdorff space without isolated points. Then each linear operator $T\colon C_0(L, X) \to C_0(L, X)$ whose range does not contain an isomorphic copy of c_{00} satisfies the Daugavet equality*

$$\|\mathbf{I} + T\| = 1 + \|T\|.$$

Let us first make some preparations before giving the proof. It is easy to see that the range of T contains c_{00} isomorphically if and only if the closure of the range contains c_0.

The assumption that L does not contain isolated points cannot be removed. Indeed, if L is not a singleton, $t_0 \in L$ is an isolated point and X contains no isomorphic copy of c_0, then the linear operator

$$T\colon C_0(L, X) \to X; \quad F \mapsto -\chi_{\{t_0\}}(\cdot)F(\cdot)$$

is of type $(*)$ and satisfies $\|T\| = \|\mathbf{I} + T\| = 1$.

Theorem 2.1 holds analogously for $T\colon CB(L, X) \to CB(L, X)$, essentially with the same proof. Here $CB(L, X)$ is the $\|\cdot\|_\infty$-normed Banach space of X-valued bounded continuous functions on L.

For a linear operator $T\colon C_0(L, X) \to Y$ we denote

$$\mathrm{osc}_T(A) = \sup\{\|TF\|: F \in \mathbf{B}_{C_0(L,X)}, L \setminus A \subset F^{-1}(0)\} \quad \text{for } A \subset L.$$

LEMMA 2.2. *Let $T\colon C_0(L, X) \to Y$ be a linear operator, where Y does not contain c_0 isomorphically. Suppose that $(V_n)_{n\in\mathbb{N}}$ is a sequence of pair-wise disjoint nonempty open subsets of L. Then $\mathrm{osc}_T(V_n) \to 0$ as $n \to 0$.*

PROOF. By passing to a subsequence it suffices, without loss of generality, to show that $\inf_{n\in\mathbb{N}} \mathrm{osc}_T(V_n) = 0$. Indeed, assume to the contrary that there is some $d > 0$ such that $\mathrm{osc}_T(V_n) \geq d$ for all $n \in \mathbb{N}$. This means that one can find a sequence

$$(F_n)_{n\in\mathbb{N}} \subset \left(\frac{1}{d} + 1\right)\mathbf{B}_{C_0(L,X)}$$

such that F_n is supported in V_n and $\|T(F_n)\| = 1$ for $n \in \mathbb{N}$. Note that for each finite subset $I \subset \mathbb{N}$ it holds that

$$\sum_{i\in I} F_i \in \left(\frac{1}{d} + 1\right)\mathbf{B}_{C_0(L,X)}$$

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as V_n are pair-wise disjoint. Since T is linear and continuous, we obtain that

$$\sup_{\epsilon, I} \left\| \sum_{i \in I} T(\epsilon_i F_i) \right\| \leq \left(\frac{1}{d} + 1 \right) \|T\|,$$

where the supremum is taken over all signs $\epsilon : \mathbb{N} \to \{-1, 1\}$ and finite subsets $I \subset \mathbb{N}$.

Recall the well-known result due to Bessaga and Pelczynski (see for example [4, page 202]) that in a Banach space Y a sequence $(y_n) \subset S_{\mathrm{Y}}$ is equivalent to the standard unit vector basis of c_0 if and only if

$$\sup_{\epsilon, I} \left\| \sum_{i \in I} \epsilon_i y_i \right\| < \infty$$

(supremum taken as above). By placing $y_i = T(F_i)$ we obtain that the range of T contains c_{00} isomorphically, which contradicts the assumptions. Hence, $\inf_{n \in \mathbb{N}} \mathrm{osc}_T(V_n) = 0$. □

Proof of Theorem 2.1. Recall that as (L, τ) is a locally compact Hausdorff space it is completely regular, that is, for each closed set $C \subset L$ and $t \in L \setminus C$ there is a continuous map $s : L \to \mathbb{R}$ such that $s(C) = \{0\}$ and $s(t) = 1$.

Suppose that there are no isolated points in (L, τ). Let $T : C_0(L, \mathrm{X}) \to C_0(L, \mathrm{X})$ be a linear operator. If the operator norm of T is 0 or ∞, then the Daugavet equation holds trivially, so that we may concentrate on the case $\|T\| = C \in (0, \infty)$. Let $k \in \mathbb{N}$. Fix $F \in \mathbf{S}_{C_0(L,\mathrm{X})}$ such that $G = TF$ satisfies $\|G\| > C - (1/k)$. Consider the open subspace $U = \{t \in L : \|G(t)\| > C - (1/k)\}$ of L.

We can pick a sequence $(V_n)_{n \in \mathbb{N}} \subset U$ of pair-wise disjoint open subsets as follows. Clearly $\overline{U}$ is also a locally compact (even compact) Hausdorff space which does not contain isolated points. Hence, U itself is not a singleton, and we may take two points $t_0, t_1 \in U$, $t_0 \neq t_1$. Since U is a Hausdorff space, there are disjoint open neighbourhoods $U_0, U_1 \subset U$ of t_0 and t_1, respectively. By repeating the same reasoning, pick $t_{10}, t_{11} \in U_1$, $t_{10} \neq t_{11}$ and disjoint open neighbourhoods U_{10}, $U_{11} \subset U_1$ of t_{10} and t_{11}, respectively. Similarly, pick $t_{110}, t_{111} \in U_{11}$, $t_{110} \neq t_{111}$ and the corresponding disjoint open neighbourhoods U_{110}, $U_{111} \subset U_{11}$. Proceeding in this manner yields a sequence of pair-wise disjoint open subsets by letting $V_n = U_s$, $s \in \{1\}^n \times \{0\}$ for $n \in \mathbb{N}$.

Since $c_0 \not\subset \overline{T(C_0(L, \mathrm{X}))}$ we obtain, by using Lemma 2.2, that $\mathrm{osc}_T(V_n) \to 0$ as $n \to \infty$. Fix $n \in \mathbb{N}$ such that $\mathrm{osc}_T(V_n) < (1/k)$. Let $u_0 \in V_n$. By using the complete regularity of L one can find a continuous map $s : L \to [0, 1]$ such that $s(L \setminus V_n) = \{0\}$ and $s(u_0) = 1$. Observe that the mappings $s(\cdot)F(\cdot)$ and $(s(\cdot)/\max(1, \|G(\cdot)\|_{\mathrm{X}}))G(\cdot)$ are elements of $\mathbf{B}_{C_0(L,\mathrm{X})}$. Hence,

$$\|T(s(\cdot)F(\cdot))\|_{C_0(L,\mathrm{X})} \leq \frac{1}{k} \quad \text{and} \quad \left\| T\left(\frac{s(\cdot)}{\max(1, \|G(\cdot)\|_{\mathrm{X}})} G(\cdot) \right) \right\|_{C_0(L,\mathrm{X})} \leq \frac{1}{k} \quad (2.1)$$

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by the definition of $\mathrm{osc}_T(V_n)$. Note that

$$\left\|(1-s(t))F(t)+\frac{s(t)}{\max(1,\ \|G(t)\|_{\mathrm{X}})}G(t)\right\|_{\mathrm{X}} \leq (1-s(t))\|F(t)\|_{\mathrm{X}}+s(t)\leq 1$$

for all $t\in L$. Hence,

$$E(\cdot)\doteq(1-s(\cdot))F(\cdot)+\frac{s(\cdot)}{\max(1,\ \|G(\cdot)\|_{\mathrm{X}})}G(\cdot)$$

defines an element of $\mathbf{B}_{C_0(L,\mathrm{X})}$. Note that E is a kind of interpolation of F and G.

Observe that

$$\|G-T(E)\|\leq\frac{2}{k}$$

according to (2.1), and that

$$\begin{aligned}\|E+G\|_{C_0(L,\mathrm{X})}\geq\|(E+G)u_0\|_{\mathrm{X}}&=\left\|\frac{s(u_0)}{\|G(u_0)\|_{\mathrm{X}}}G(u_0)+G(u_0)\right\|_{\mathrm{X}}\\&=1+\|G(u_0)\|_{\mathrm{X}}>1+C-\frac{1}{k}.\end{aligned}$$

Thus,

$$\|\mathbf{I}+T\|\geq\|E+T(E)\|\geq\|E+G\|-\|G-T(E)\|>1+C-\frac{3}{k}$$

and by letting $k\to\infty$ we obtain that $\|\mathbf{I}+T\|\geq 1+C=1+\|T\|$. By the triangle inequality $\|\mathbf{I}+T\|\leq 1+\|T\|$ and we have the claim. □

Let us recall a few classical results due to James which are applied here frequently: a closed convex subset $C\subset\mathrm{X}$ is weakly compact if and only if each $f\in\mathrm{X}^*$ attains its supremum over C, and X is reflexive if and only if $\mathbf{B}_{\mathrm{X}}$ is weakly compact (see, for example, [4, Chapter 3]).

THEOREM 2.3. *Let Γ be a nonempty set and X, Y be Banach spaces such that X is reflexive and Y does not contain c_0 isomorphically. Then any continuous linear operator $T: c_0(\Gamma,\mathrm{X})\to\mathrm{Y}$ is weakly compact.*

The above result holds similarly for $\ell^\infty(\Gamma,\mathrm{X})$ in place of $c_0(\Gamma,\mathrm{X})$, essentially with the same proof. Note that the operators $\mathbf{I}: c_0(\mathbb{N},\mathrm{X})\to c_0(\mathbb{N},\mathrm{X})$ and $T: c_0(\mathbb{N},\mathrm{Y})\to\mathrm{Y};\ (y_n)\mapsto(y_1,0,0,\ldots)$ are not weakly compact for any nontrivial X and nonreflexive Y according to the James characterization of reflexivity. Hence, neither of the assumptions about the reflexivity or the noncontainment of c_0 can be removed.

PROOF OF THEOREM 2.3. Let $T: c_0(\Gamma,\mathrm{X})\to\mathrm{Y}$ be a continuous linear operator such that Y does not contain c_0. One may write $c_0(\Gamma,\mathrm{X})=C_0(\Gamma,\mathrm{X})$ isometrically where Γ on the right-hand side is interpreted as a discrete topological space.

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We claim that the sum

$$\sum_{\gamma\in\Gamma} \mathrm{osc}_T(\{\gamma\})$$

is defined and finite. Indeed, otherwise one can extract pair-wise disjoint (open) subsets $\Gamma_n \subset \Gamma$, $n \in \mathbb{N}$, such that

$$\sum_{\gamma\in\Gamma_n} \mathrm{osc}_T(\{\gamma\}) \geq 1$$

for $n \in \mathbb{N}$. However, since $c_0 \not\subset \mathrm{Y}$, Lemma 2.2 yields that this case does not occur. Thus there exists a sequence $(\gamma_n)_{n\in\mathbb{N}} \subset \Gamma$ such that $\mathrm{osc}_T(\Gamma \setminus \{\gamma_n\}_{n\leq k}) \to 0$ as $k \to \infty$.

In order to verify the statement of the theorem we must show that $\overline{T(\mathbf{B}_{c_0(\Gamma,\mathrm{X})})}$ is weakly compact. In doing this we apply the James characterization of weakly compact sets. Fix $f \in \mathrm{Y}^*$. It suffices to show that f attains its supremum over $\overline{T(\mathbf{B}_{c_0(\Gamma,\mathrm{X})})}$. Observe that $f \circ T$ defines an element of $C_0(\Gamma, \mathrm{X})^*$. For each $k \in \mathbb{N}$ define a contractive linear projection

$$P_k : c_0(\Gamma, \mathrm{X}) \to c_0(\Gamma, \mathrm{X}) \quad \text{by } P_k f(\cdot) = \chi_{\{\gamma_n\}_{n\leq k}}(\cdot) f(\cdot).$$

Put $g_k = f \circ T \circ P_k$ and $\mathrm{Z}_k = P_k(c_0(\Gamma, \mathrm{X}))$ for $k \in \mathbb{N}$. Note that g_k restricted to Z_k satisfies $g_k|_{\mathrm{Z}_k} \in \mathrm{Z}_k^*$, where $\mathrm{Z}_k^* = \ell^1(\{\gamma_n\}_{n\leq k}, \mathrm{X}^*)$ isometrically for $k \in \mathbb{N}$. Clearly $g_{k+l}|_{\mathrm{Z}_k} = g_k|_{\mathrm{Z}_k}$ for $k, l \in \mathbb{N}$. Hence, there is a sequence $(x_n^*)_{n\in\mathbb{N}} \subset \mathrm{X}^*$ such that

$$g_k\bigg(\sum_{n=1}^{k} \chi_{\{\gamma_n\}}(\cdot) y_n\bigg) = \sum_{n=1}^{k} x_n^*(y_n) \tag{2.2}$$

for $\sum_{n=1}^{k} \chi_{\{\gamma_n\}} y_n \in \mathrm{Z}_k$, $k \in \mathbb{N}$.

Observe that since X is reflexive, according to the James characterization of reflexivity there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbf{S}_{\mathrm{X}}$ such that $x_n^*(x_n) = \|x_n^*\|$ for $n \in \mathbb{N}$. It follows that

$$\sum_{n=1}^{k} x_n^*(x_n) = \|g_k\| \quad \text{for } k \in \mathbb{N}. \tag{2.3}$$

Now, since $\mathrm{osc}_T(\Gamma \setminus \{\gamma_n\}_{n\leq k}) \to 0$ as $k \to \infty$, we obtain that

$$\lim_{k\to\infty} \|T - T \circ P_k\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|g_k - f \circ T\| = 0. \tag{2.4}$$

By putting these observations together and using the continuity of T we obtain that the sequence

$$\bigg(T\bigg(\sum_{n=1}^{k} \chi_{\{\gamma_n\}}(\cdot) x_n\bigg)\bigg)_{k\in\mathbb{N}} \subset \mathrm{Y}$$

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is Cauchy. On the other hand,

$$\|f \circ T\| \geq f \circ T\left(\sum_{n=1}^{k} \chi_{\{\gamma_n\}}(\cdot)x_n\right) \geq \|g_k\| - \|g_k - f \circ T\| \to \|f \circ T\| \quad \text{as } k \to \infty$$

by using (2.2), (2.3) and (2.4). We conclude that

$$y \doteq \lim_{k\to\infty} T\left(\sum_{n=1}^{k} \chi_{\{\gamma_n\}}(\cdot)x_n\right) \in \overline{T(\mathbf{B}_{c_0(\Gamma,\mathrm{X})})}$$

satisfies

$$f(y) = \|f \circ T\| = \sup_{z\in T(B_{c_0(\Gamma,\mathrm{X})})} f(z),$$

which completes the proof. □

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JARNO TALPONEN, Department of Mathematics and Statistics,
University of Helsinki, Box 68, Gustaf Hällströminkatu 2b,
FI-00014 Helsinki, Finland
e-mail: talponen@cc.helsinki.fi

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