A NOTE ON FOURIER TRANSFORMS AND IMBEDDING THEOREMS

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It is well known that Sobolev's Lemma on the continuity of functions possessing L^2 distributional derivatives of sufficiently high order is a simple consequence of elementary properties of the Fourier transform in L^2 (e.g. [1, p.174]). (In fact this statement remains true if 2 is replaced by p, $1 \leq p \leq 2$). In this note we show that imbedding theorems of the type $W^{m,p} \subset L^q$ can also be obtained using Fourier transforms and an elementary lemma which reduces the cases $p > 2$ to the case $p = 2$. The simplicity of this approach is obtained at the expense of a slight loss of generality in the imbedding theorem.

Let Ω be an open set in R_n. Let m be a positive integer and let p be real and satisfy $1 \leq p < \infty$. We denote by $W_o^{m,p}(\Omega)$ the closure of the set of infinitely differentiable functions with compact support in Ω with respect to the norm

$$||u||_{m,p} = \{ \sum_{|\alpha| \leq m} ||D^\alpha u||_{o,p}^p \}^{1/p}$$

where $||u||_{o,p}$ denotes the norm in $L^p = L^p(\Omega)$. As is customary

$$\alpha = (\alpha_1, \ldots, \alpha_n); \quad |\alpha| = \alpha_1 + \ldots + \alpha_n; \quad D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \ldots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n};$$

the α_i being non-negative integers. We prove the following

THEOREM (Sobolev): If $2n(n+2)^{-1} < p \leq nm^{-1}$ then

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$W_o^{m,p} \subset L^r$ with continuous imbedding for $p \leq r < np(n - mp)^{-1}$. ($p \leq r < \infty$ if $n = mp$).

The restriction $2n(n + 2)^{-1} < p$ occurs because the Fourier transform fails to be adequately defined in L^p for $p > 2$. This also results in loss of the endpoint $r = np(n - mp)^{-1}$. The conclusion for arbitrary m follows from that for the special case $m = 1$ since the mapping $u \rightarrow \frac{\partial u}{\partial x_j}$ is continuous from $W_o^{m,p}$ into $W_o^{m-1,p}$.

For the case $m = 1$, $2n(n + 2)^{-1} < p \leq 2$ the theorem can be proven as follows. For $u \in L^p$ let $\tilde{u}$ denote the function coinciding with u on Ω and equal to zero in $R_n - \Omega$. Let $\hat{u}$ be the Fourier transform of $\tilde{u}$, the transform variable being denoted by ξ. If $u \in W_o^{1,p}$ then $\tilde{u}, \widetilde{\frac{\partial u}{\partial x_j}} \in L^p(R_n)$ and so $\hat{u}, \xi_j \hat{u}, \in L^{p'}(R_n)$ where $p^{-1} + p'^{-1} = 1$. Thus $(1 + |\xi|)\hat{u} \in L^{p'}(R_n)$. Since $(1 + |\xi|)^{-1} \in L^q(R_n)$ for every $q > n$ it follows by Holder's inequality that $\hat{u} = (1 + |\xi|)^{-1}(1 + |\xi|)\hat{u} \in L^s(R_n)$ for every s satisfying $p' \geq s > s_o \equiv np'(n + p')^{-1}$. Since $2n(n + 2)^{-1} < p$ we have $s_o < 2$. Choosing s such that $s_o < s \leq 2$ we obtain $\hat{\hat{u}} \in L^{s'}(R_n)$ where $s^{-1} + s'^{-1} = 1$ and so by Fourier's inversion formula $u \in L^{s'}$ for $2 \leq s' < s'_o = np(n - p)^{-1}$. Since $L^p \cap L^{s'} \subset L^r$ whenever $p \leq r \leq s'$ it follows that $u \in L^r$ for $p \leq r < np(n - p)^{-1}$. The continuity of the imbedding in this case is an immediate consequence of the continuity of the Fourier transform as a mapping from L^p into $L^{p'}$.

The validity of the theorem in the case $m = 1, 2 < p \leq n$ is a consequence of the

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LEMMA. Let $p > 2$. If $u \in W_o^{1,p} \cap L^q$ for all q such that $p \leq q < q_o$ then $u \in L^r$ for all r such that $p \leq r < r_o = 2n(n-2)^{-1}[1 + (p-2)q_o/2p]$. Moreover, if $||u||_{o,q} \leq$ const. $||u||_{1,p}$ then $||u||_{o,r} \leq$ const. $||u||_{1,p}$.

Proof. If $np(n-2)^{-1} \leq r_1 < r_o$ then $r_1 = 2n(n-2)^{-1}[1+(p-2)q/2p]$ where $p \leq q < q_o$. Let $v = |u|^s$ where $s = 1 + (p-2)q/2p > 1$ so that $\frac{\partial v}{\partial x_j} = s|u|^{s-1} \operatorname{sgn} u \frac{\partial u}{\partial x_j} \in L^2$ by Holder's inequality. Also $p/s \leq 2 \leq q/s$ so that $v \in L^{p/s} \cap L^{q/s} \subset L^2$. Thus $v \in W_o^{1,2} \subset L^t$ for $2 \leq t < 2n(n-2)^{-1}$ by the previous case. Therefore $u \in L^r$ for any $r = st$ satisfying $2s \leq r < r_1$. But $2s = p$ if $q = p$ and r_1 can be made as close as desired to r_o. Hence $u \in L^r$ for any r such that $p \leq r < r_o$.

Now if $C_1, \ldots C_5$ denote various constants we have by the previous case that

$$\begin{aligned}||u||_{o,r} &= ||v||_{o,t}^{1/s} \\ &\leq C_1 ||v||_{1,2}^{1/s} \\ &\leq C_2 [||v||_{o,2} + \sum_{j=1}^{n} ||\frac{\partial v}{\partial x_j}||_{o,2}]^{1/s}\end{aligned}$$

But $v \in L^{p/s} \cap L^{q/s}$ and $\alpha p/s + (1-\alpha) q/s = 2$ where $0 \leq \alpha \leq 1$. Thus by Holder's inequality and since $||u||_{o,q} \leq$ const. $||u||_{1,p}$ we have

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$$||v||_{o,2} \leq ||v||_{o,p/s}^{\alpha p/2s} ||v||_{o,q/s}^{(1-\alpha)q/2s} = ||u||_{o,p}^{\alpha p/2} ||u||_{o,q}^{(1-\alpha)q/2}$$

$$\leq C_3 ||u||_{1,p}^{s}$$

Also by Holder's inequality

$$||\frac{\partial v}{\partial x_j}||_{o,2} \leq ||u||_{o,q}^{s-1} ||\frac{\partial u}{\partial x_j}||_{o,p} \leq C_4 ||u||_{1,p}^{s}$$

so that $||u||_{o,r} \leq C_5 ||u||_{1,p}$ completing the proof.

REMARK. If $p > 2$ then $W^{1,p} \subset L^r$ for $p \leq r < np(n-2)^{-1}$ and $||u||_{o,r} \leq$ const. $||u||_{1,p}$ where the constant is independent of u. The proof is the same as for the lemma taking $q = p$.

The proof of Sobolev's theorem for $m = 1$, $p > 2$ can now be completed. Let $r_o = p$, $r_k = 2n(n-2)^{-1}[1 + (p-2)r_{k-1}/2p]$. Successive applications of the lemma show that $W_o^{1,p} \subset L^r$ for $p \leq r < r_k$, $k = 1, 2, 3, \ldots$ with continuous imbedding. Clearly $r_k \to np(n-p)^{-1}$ as $k \to \infty$ ($r_k \to \infty$ if $n = p$) whence the theorem.

REMARK. The lemma can be modified to yield a proof of the imbedding theorem for the case $p > p_o \geq 1$ when the theorem has already been established for $p = p_o$.

REFERENCE

1. K. Yosida, Functional analysis, Academic Press, New York, 1965.

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