Canad. Math. Bull. Vol. **34** (4), 1991 pp. 499–513

BASIC DOUBLE SERIES, QUADRATIC TRANSFORMATIONS AND PRODUCTS OF BASIC SERIES

BASSAM NASSRALLAH

ABSTRACT. A basic double series is expressed in terms of two ${}_5\phi_4$ series which extends Bailey's transformation of an ${}_8\phi_7$ series into two ${}_4\phi_3$'s. From this formula we derive some quadratic transformations; one of them is a new q-analogue of a transformation due to Whipple. Product formulas as well as Gasper-Rahman's q-Clausen formula are also given as special cases.

1. **Introduction.** In an earlier version of [8], I felt the need for a formula that transforms a sum of two balanced ${}_5\phi_4$ series. It had to be an extension of Bailey's [4, 8.4(2)]

$$
\begin{aligned}
&{}_8W_7(A; D, E, F, G, H; A^2q^2/DEFGH)\\
&= \frac{(Aq, Aq/FG, Aq/FH, Aq/GH)_\infty}{(Aq/F, Aq/G, Aq/H, Aq/FGH)_\infty}\, {}_4\phi_3\left[\begin{matrix} Aq/DE, F, G, H \\ Aq/D, Aq/E, FGH/A \end{matrix}; q\right]\\
&+ \frac{(Aq, Aq/DE, F, G, H, A^2q^2/DFGH, A^2q^2/EFGH)_\infty}{(Aq/D, Aq/E, Aq/F, Aq/G, Aq/H, A^2q^2/DEFGH, FGH/Aq)_\infty}\\
&\cdot {}_4\phi_3\left[\begin{matrix} Aq/FG, Aq/FH, Aq/GH, A^2q^2/DEFGH \\ Aq^2/FGH, A^2q^2/DFGH, A^2q^2/EFGH \end{matrix}; q\right],
\end{aligned} \tag{1.1}
$$

where

$$
\begin{cases}
{}_{r+1}\phi_{r+j}\left[\begin{matrix} a_0, \dots, a_r \\ b_1, \dots, b_{r+j} \end{matrix}; q, z\right] = \sum_{k=0}^{\infty} \frac{(a_0, \dots, a_r)_k (-1)^{jk}}{(q, b_1, \dots, b_{r+j})_k} q^{j\binom{k}{2}} z^k,\\
{}_8W_7(a; b, c, d, e, f; a^2q^2/bcdef)\\
= {}_8\phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq/f \end{matrix}; q, a^2q^2/bcdef\right].
\end{cases} \tag{1.2}
$$

When ${}_{r+1}\phi_{r+j}$ is non-terminating, we assume $|q| < 1$ and it converges for all z when $j \neq 0$ and for $|z| < 1$ when $j = 0$. Throughout this article we use the following notation

$$(a, b)_\infty = (a)_\infty (b)_\infty, \qquad (a, b)_m = (a)_m (b)_m \tag{1.3}$$

This work was supported by NSERC (Canada) under grant 8398-01.
Received by the editors November 8, 1989.
AMS subject classification: 33A35, 33A99.

where

$$(1.4)\qquad \begin{cases} (a)_\infty \equiv (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), & \text{whenever the product converges,} \\ (a)_m \equiv (a;q)_m = \begin{cases} (a)_\infty/(aq^m)_\infty, & \text{for any complex number } m, \\ \prod_{n=0}^{m-1}(1-aq^n), & \text{when } m \text{ is a natural number.} \end{cases} \end{cases}$$

In (1.2) and (1.4), the q appearing immediately after the semicolons is called the *base.* It is usually suppressed in our notation unless different from q, in which case it is shown.

To find the required extension of (1.1), use was made of Askey-Wilson's q-analogue of the beta integral [3], namely

$$(1.5)\qquad \int_{-1}^{1} dx\, w(x;a,b,c,d) = \frac{2\pi(abcd)_\infty}{(q,ab,ac,ad,bc,bd,cd)_\infty},$$

$\max(|a|,|b|,|c|,|d|,|q|) < 1$, where

$$(1.6)\qquad \begin{cases} w(x;a,b,c,d) = \frac{h(x;1)h(x;-1)h(x;\sqrt{q})h(x;-\sqrt{q})}{\sqrt{1-x^2}\,h(x;a)h(x;b)h(x;c)h(x;d)}, \\ h(x;a) = \prod_{n=0}^{\infty}(1-2axq^n+a^2q^{2n}) = |(ae^{i\theta})_\infty|^2, \quad x=\cos\theta. \end{cases}$$

This is done in Section 2 where we derive

$$(1.7)\qquad \begin{aligned} &\sum_{k=0}^{\infty} \frac{(A, q\surd, -q\surd, D, E, F, B, C, G)_k (ACGq/a^2)_{2k}}{(q, \surd, -\surd, Aq/D, Aq/E, BCG/a^2, ACGq/Fa^2)_k} \\ &\cdot \frac{(A^2q^2/DEFa^2)^k}{(ACq/a^2, AGq/a^2)_k (Aq)_{2k}} \\ &\cdot {}_8W_7(ACGq^{2k}/a^2; Aq^{1+k}/B, Fq^k, Cq^k, Gq^k, CG/a^2; ABq/Fa^2) \\ = &\frac{(Aq/a^2, ACGq/a^2, ACq/Fa^2, AGq/Fa^2)_\infty}{(Aq/Fa^2, ACGq/Fa^2, ACq/a^2, AGq/a^2)_\infty} \\ &\cdot {}_5\phi_4\left[\begin{matrix} Aq/DE, F, B, C, G \\ Aq/D, Aq/E, Fa^2/A, BCG/a^2 \end{matrix}; q\right] \\ &+ \frac{(ACGq/a^2, Aq/DE, F, B, C, G, A^2q^2/DFa^2, A^2q^2/EFa^2)_\infty}{(Aq/D, Aq/E, BCG/a^2, ACGq/Fa^2, ACq/a^2, AGq/a^2)_\infty} \\ &\cdot \frac{(ABCGq/Fa^4)_\infty}{(Fa^2/Aq, ABq/Fa^2, A^2q^2/DEFa^2)_\infty} \\ &\cdot {}_5\phi_4\left[\begin{matrix} A^2q^2/DEFa^2, Aq/a^2, ABq/Fa^2, ACq/Fa^2, AGq/Fa^2 \\ A^2q^2/DFa^2, A^2q^2/EFa^2, Aq^2/Fa^2, ABCGq/Fa^4 \end{matrix}; q\right]. \end{aligned}$$

The radical signs in the above equation are over the top *l. h. s.* term, A in this case. Clearly if we let $a^2 = CG$ and replace C by H, we get (1.1).

To see some of the applications of (1.7) one should be able to manipulate the ${}_8W_7$ series on the *l.h.s.* In Sections 3 and 4 we consider some of the cases when this ${}_8W_7$ series is summable. This leads to some nice quadratic transformation formulas as well as other summation formulas.

In Sections 5 and 6 we look into the cases when the ${}_8W_7$ series is transformable into ${}_2\phi_1$ series. This will lead to product formulas among which is Gasper-Rahman's q-analogue of Clausen's formula [6].

Before going on to the next section, we should mention that if we go from the *r. h. s.* to the *l. h. s.* in (1.1) after interchanging the roles of the ${}_4\phi_3$ series on the *r. h. s.* we obtain

$$(1.8)\quad \begin{aligned} &{}_8W_7(A;D,E,F,G,H;A^2q^2/DEFGH)\\ &\quad=\frac{(Aq,Aq/GH,A^2q^2/DEFG,A^2q^2/DEFH)_\infty}{(Aq/G,Aq/H,A^2q^2/DEF,A^2q^2/DEFGH)_\infty}\\ &\qquad\cdot{}_8W_7(A^2q/DEF;Aq/EF,Aq/DF,Aq/DE,G,H;Aq/GH),\end{aligned}$$

which is a limiting case of [11, (3.4.2.4)]. This transformation will be very useful in the coming sections.

2. **Derivation of (1.7).** Equation (1.7), once known, can be proved easily using (1.1): transform the ${}_8W_7$ series on the *l. h. s.* into two ${}_4\phi_3$ series, interchange the summations and use the summation formulas for ${}_6\phi_5$ series [11, (IV. 7) and (IV.9)]. But to be fair to the original method of obtaining this formula, we present the following.

Let $G = ae^{i\theta}$, $H = ae^{-i\theta}$ in (1.1), rearrange the terms, multiply by $dx\, w(x;a,b,c,d)$ and integrate from -1 to 1 to get

$$(2.1)\quad \begin{aligned} &\frac{(Aq/F,Aq/Fa^2)_\infty}{(Aq,Aq/a^2)_\infty}\sum_{k=0}^{\infty}\frac{(A,q\sqrt{A},-q\sqrt{A},D,E,F)_k}{(q,\sqrt{A},-\sqrt{A},Aq/D,Aq/E,Aq/F)_k}\\ &\quad\cdot\left(A^2q^2/DEFa^2\right)^k\int_{-1}^{1}dx\,w(x;aq^k,b,c,d)\left|\frac{Aq^{1+k}e^{i\theta}/a)_\infty}{(Aqe^{i\theta}/Fa)_\infty}\right|^2\\ &=\sum_{k=0}^{\infty}\frac{(Aq/DE,F)_k}{(q,Aq/D,Aq/E,Fa^2/A)_k}q^k\int_{-1}^{1}dx\,w(x;aq^k,b,c,d)\\ &\quad+\frac{(Aq/DE,F,A^2q^2/DFa^2,A^2q^2/EFa^2,Aq/Fa^2)_\infty}{(Aq/D,Aq/E,Aq/a^2,A^2q^2/DEFa^2,Fa^2/Aq)_\infty}\\ &\quad\cdot\sum_{k=0}^{\infty}\frac{(Aq/a^2,A^2q^2/DEFa^2)_k}{(q,Aq^2/Fa^2,A^2q^2/DFa^2,A^2q^2/EFa^2)_k}q^k\\ &\quad\cdot\int_{-1}^{1}dx\,w(x;Aq^{1+k}/Fa,b,c,d).\end{aligned}$$

By (1.5), we can easily evaluate the integrals on the *r. h. s.* of (2.1). Equally easy is the evaluation of the integral on the *l. h. s.* when we use our integral representation formula for an ${}_8W_7$ series [9]. The latter gives

$$(2.2)\quad \begin{aligned} &\int_{-1}^{1}dx\,w(x;aq^k,b,c,d)\left|\frac{(Aq^{1+k}e^{i\theta}/a)_\infty}{(Aqe^{i\theta}/Fa)_\infty}\right|^2\\ &=\frac{2\pi(abcdq^k,Acdq^{1+k}/F,Aq^{1+2k},Acq^{1+k}/a,Adq^{1+k}/a)_\infty}{(q,abq^k,acq^k,adq^k,Aq^{1+k}/F,bc,bd,cd,Acq/Fa,Adq/Fa,Acdq^{1+k})_\infty}\\ &\quad\cdot{}_8W_7(Acdq^{2k};Aq^{1+k}/ab,Fq^k,acq^k,adq^k,cd;Abq/Fa).\end{aligned}$$

https://doi.org/10.4153/CMB-1991-080-7 Published online by Cambridge University Press

At this point, evaluate the integrals on the *r. h. s.* of (2.1) by (1.5), substitute (2.2) for the integral on the *l. h. s.*, simplify using $(aq^m)_\infty = (a)_\infty/(a)_m$ and set $ab = B$, $ac = C$, $ad = G$. This will lead directly to (1.7).

3. **Special cases of (1.7); quadratic transformations.** In this section we look into the cases when the ${}_8W_7$ series on the *l. h. s.* of (1.7) is summable. Two summation formulas will be used. They are

$$(3.1)\qquad {}_8W_7(ax^2; a, x, -x, x\sqrt{q}, -x\sqrt{q}; aq) = \frac{(ax^2q, a^2q)_\infty}{(aq, a^2x^2q)_\infty},\ |aq| < 1,$$

and

$$(3.2)\qquad {}_8W_7(ax^2/q; a, x, -x, x\sqrt{q}, -x\sqrt{q}; a/q) = \frac{(ax^2, a^2/q)_\infty}{(a^2x^2/q, a/q)_\infty},\ |a/q| < 1.$$

The proofs of both (3.1) and (3.2) are similar; they are done with the help of (1.1). For the sake of completion we present the proof of (3.1).

PROOF OF (3.1). By (1.1), the *l. h. s.* of (3.1) is equal to

$$\begin{aligned}
&\frac{(ax^2q, x\sqrt{q}, -x\sqrt{q}, -a)_\infty}{(x^2q, ax\sqrt{q}, -ax\sqrt{q}, -1)_\infty}\,{}_4\phi_3\left[\begin{matrix} x\sqrt{q}, -x\sqrt{q}, a, -aq \\ axq, -axq, -q \end{matrix}; q\right] \\
&\quad + \frac{(ax^2q, -aq, x\sqrt{q}, -x\sqrt{q}, a)_\infty}{(x^2q, ax\sqrt{q}, -ax\sqrt{q}, aq, -1)_\infty}\,{}_4\phi_3\left[\begin{matrix} x\sqrt{q}, -x\sqrt{q}, -a, aq \\ axq, -axq, -q \end{matrix}; q\right] \\
(3.3)\qquad &= \frac{(ax^2q, x\sqrt{q}, -x\sqrt{q}, -aq)_\infty}{(x^2q, ax\sqrt{q}, -ax\sqrt{q}, -1)_\infty}\left\{\sum_{k=0}^{\infty} \frac{(x^2q, a^2; q^2)_k}{(q^2, a^2x^2q^2; q^2)_k}(1+aq^k)q^k \right. \\
&\quad \left. + \sum_{k=0}^{\infty} \frac{(x^2q, a^2; q^2)_k}{(q^2, a^2x^2q^2; q^2)_k}(1-aq^k)q^k \right\} \\
&= \frac{(ax^2q, x\sqrt{q}, -x\sqrt{q}, -aq)_\infty}{(x^2q, ax\sqrt{q}, -ax\sqrt{q}, -q)_\infty}\,{}_2\phi_1\left[\begin{matrix} x^2q, a^2 \\ a^2x^2q^2 \end{matrix}; q^2, q\right].
\end{aligned}$$

The ${}_2\phi_1$ series on the *r. h. s.* of the last equality in (3.3) is summable by q-Gauss's formula [11, (IV. 2)] and is equal to

$$(3.4)\qquad \frac{(a^2q, x^2q^2; q^2)_\infty}{(a^2x^2q^2, q; q^2)_\infty}.$$

Equation (3.1) is obtained from (3.3) and (3.4) with the use of

$$(3.5)\qquad (a; q^2)_n = (\sqrt{a}, -\sqrt{a})_n, \qquad [11, (II.\,16)],$$

$$(3.6)\qquad (a)_{2n} = (a, aq; q^2)_n, \qquad [11, (II.\,17)],$$

and the fact that

$$(3.7)\qquad (\sqrt{q}, -\sqrt{q}, -q)_\infty = 1.$$

Now let's go back to (1.7) and set $A = \alpha q$, $a = \sqrt{\alpha q/x}$, $B = -q\sqrt{\alpha q}$, $C = -G = q\sqrt{\alpha}$, $F = \sqrt{\alpha q}$. The *l. h. s.* becomes

$$(3.8)\quad \begin{aligned} &\sum_{k=0}^{\infty} \frac{(\alpha q, q\surd, -q\surd, D, E, \sqrt{\alpha q}, -q\sqrt{\alpha q}, q\sqrt{\alpha}, -q\sqrt{\alpha})_k (-\alpha x q^3)_{2k}}{(q, \surd, -\surd, \alpha q^2/D, \alpha q^2/E, xq^2\sqrt{\alpha q}, -xq^2\sqrt{\alpha q}, xq^2\sqrt{\alpha}, -xq^2\sqrt{\alpha})_k} \\ &\cdot \frac{(xq^2\sqrt{\alpha q}/DE)^k}{(\alpha q^2)_{2k}} \\ &\cdot {}_8W_7(-\alpha x q^{2+2k}; q^k\sqrt{\alpha q}, -q^k\sqrt{\alpha q}, q^{1+k}\sqrt{\alpha}, -q^{1+k}\sqrt{\alpha}, -xq; -xq^2) \\ &= \sum_{k=0}^{\infty} \frac{(\alpha q, -q\sqrt{\alpha q}, D, E)_k (-\alpha x q^3)_{2k}}{(q, -\sqrt{\alpha q}, \alpha q^2/D, \alpha q^2/E)_k (\alpha x^2 q^4)_{2k}} (xq^2\sqrt{\alpha q}/DE)^k \\ &\cdot \frac{(-\alpha x q^{3+2k}, x^2 q^3)_\infty}{(-xq^2, \alpha x^2 q^{4+2k})_\infty}, \end{aligned}$$

by (3.1), (3.5) and (3.6).

Simplifying the *r. h. s.* of (3.8), equating it to the *r. h. s.* of (1.7) with the special values of the parameters and then rearranging the terms yield

$$(3.9)\quad \begin{aligned} &{}_4\phi_3\left[\begin{matrix} \alpha q, -q\sqrt{\alpha q}, D, E, \\ -\sqrt{\alpha q}, \alpha q^2/D, \alpha q^2/E \end{matrix}; xq^2\sqrt{\alpha q}/DE\right] \\ &= \frac{(1-xq)(xq^2\sqrt{\alpha q})}{(x\sqrt{q/\alpha})_\infty}\, {}_5\phi_4\left[\begin{matrix} \sqrt{\alpha q}, -q\sqrt{\alpha q}, q\sqrt{\alpha}, -q\sqrt{\alpha}, \alpha q^2/DE \\ \alpha q^2/D, \alpha q^2/E, \sqrt{\alpha q}/x, xq^2\sqrt{\alpha q} \end{matrix}; q\right] \\ &+ \frac{(\alpha q^2/DE, xq^2\sqrt{\alpha q}/D, xq^2\sqrt{\alpha q}/E, \alpha q)_\infty}{(\alpha q^2/D, \alpha q^2/E, xq^2\sqrt{\alpha q}/DE, \sqrt{\alpha/q}/x)_\infty (1+\sqrt{\alpha q})} \\ &\cdot {}_5\phi_4\left[\begin{matrix} xq, -xq^2, xq^{3/2}, -xq^{3/2}, xq^2\sqrt{\alpha q}/DE \\ xq^2\sqrt{\alpha q}/D, xq^2\sqrt{\alpha q}/E, xq^2/\sqrt{\alpha q}, x^2q^3 \end{matrix}; q\right]. \end{aligned}$$

This is another q-analogue of Whipple's quadratic transformation [12]

$$(3.10)\quad \begin{aligned} &{}_3F_2\left[\begin{matrix} a, b, c \\ 1+a-b, 1+a-c \end{matrix}; x\right] \\ &= (1-x)^{-a}\, {}_3F_2\left[\begin{matrix} 1+a-b-c, \frac{a}{2}, \frac{a+1}{2} \\ 1+a-b, 1+a-c \end{matrix}; -\frac{4x}{(1-x)^2}\right]. \end{aligned}$$

The other one is Gasper-Rahman's extension of Carlitz's quadratic transformation [5], namely

$$(3.11)\quad \begin{aligned} &{}_3\phi_2\left[\begin{matrix} \alpha, D, E \\ \alpha q/D, \alpha q/E \end{matrix}; \alpha x q/DE\right] \\ &= \frac{(\alpha x)_\infty}{(x)_\infty}\, {}_5\phi_4\left[\begin{matrix} \sqrt{\alpha}, -\sqrt{\alpha}, \sqrt{\alpha q}, -\sqrt{\alpha q}, \alpha q/DE \\ \alpha q/D, \alpha q/E, \alpha x, q/x \end{matrix}; q\right] \\ &+ \frac{(\alpha, \alpha x q/D, \alpha x q/E, \alpha q/DE)_\infty}{(\alpha q/D, \alpha q/E, \alpha x q/DE, x^{-1})_\infty} \\ &\cdot {}_5\phi_4\left[\begin{matrix} x\sqrt{\alpha}, -x\sqrt{\alpha}, x\sqrt{\alpha q}, -x\sqrt{\alpha q}, \alpha x q/DE \\ \alpha x q/D, \alpha x q/E, xq, \alpha x^2 \end{matrix}; q\right]. \end{aligned}$$

The ${}_3F_2$ function used in (3.10) is the ordinary hypergeometric function, see [4,11]. It's obtained from (1.2) by letting $q \longrightarrow 1^-$ after a typical parameter a is replaced by q^a.

The quadratic transformation (3.11) is easily obtained from (1.7) as follows. First we use (1.8) to transform the ${}_8W_7$ series on the $l.h.s.$ of (1.7). This results in

(3.12)

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(A, q\surd, -q\surd, D, E, F, C, G, A^2CGq^2/Fa^4)_k}{(q, \surd, -\surd, Aq/D, Aq/E, ACGq/Fa^2, ACq/a^2)_k} \\
&\cdot \frac{(A^2q^2/DEFa^2)^k}{(AGq/a^2, a^2q^2/Fa^2)_k} \\
&\cdot {}_8W_7(A^2CGq^{1+k}/Fa^4; Aq^{1+k}/B, AGq/Fa^2, ACq/Fa^2, Aq/a^2, CG/a^2; Bq^k) \\
&= \frac{(Aq/a^2, BCG/a^2, Aq, A^2CGq^2/Fa^4, ABq/Fa^2, ACq/Fa^2)_\infty}{(Aq/Fa^2, ACGq/Fa^2, ACq/a^2, AGq/a^2, B, A^2q^2/Fa^2)_\infty} \\
&\quad \cdot \frac{(AGq/Fa^2)_\infty}{(ABCGq/Fa^4)_\infty} {}_5\phi_4\left[\begin{matrix} Aq/DE, F, B, C, G \\ Aq/D, Aq/E, Fa^2/A, BCG/a^2 \end{matrix}; q\right] \\
&\quad + \frac{(Aq/DE, F, C, G, A^2q^2/DFa^2, A^2q^2/EFa^2)_\infty}{(Aq/D, Aq/E, ACGq/Fa^2, ACq/a^2, AGq/a^2, Fa^2/Aq)_\infty} \\
&\quad \cdot \frac{(Aq, A^2CGq^2/Fa^4)_\infty}{(A^2q^2/DEFa^2, A^2q^2/Fa^2)_\infty} \\
&\quad \cdot {}_5\phi_4\left[\begin{matrix} A^2q^2/DEFa^2, Aq/a^2, ABq/Fa^2, ACq/Fa^2, AGq/Fa^2 \\ A^2q^2/DFa^2, A^2q^2/EFa^2, Aq^2/Fa^2, ABCGq/Fa^4 \end{matrix}; q\right].
\end{aligned}$$

Now let $A = \alpha$, $a^2 = q\sqrt{\alpha}/x$, $B = -\sqrt{\alpha}$, $C = -G = \sqrt{\alpha q}$ and $F = \sqrt{\alpha}$. The ${}_8W_7$ series on the $l.h.s.$ of (3.12) becomes summable by (3.2), and upon simplifying and rearranging the terms we obtain (3.11).

Other quadratic transformations can be obtained by taking other special values of the parameters. For example, if $A = \alpha$, $a = \alpha q/x$, $B = -\sqrt{\alpha q}$, $C = -G = q\sqrt{\alpha}$, $F = \sqrt{\alpha q}$, then (1.7) and (3.2) give Gasper-Rahman's transformation of a very well-poised ${}_5\phi_4$ series into two balanced ${}_5\phi_4$ series [5, (3.5)].

A special case of (3.9), when $xq = 1$ and $\alpha q = A$, is

$$\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} A, -q\sqrt{A}, D, E \\ -\sqrt{A}, Aq/D, Aq/E \end{matrix}; q\sqrt{A}/DE\right] \\
&= \frac{(Aq, q\sqrt{A}/D, q\sqrt{A}/E, Aq/DE)_\infty}{(Aq/D, Aq/E, q\sqrt{A}, q\sqrt{A}/DE)_\infty},
\end{aligned} \tag{3.13}$$

provided $|q\sqrt{A}/DE| < 1$, which is the q-analogue of Dixon's summation formula [11, (IV. 6)]. We will be using this formula in the next section.

4. **More special cases of (1.7).** In this section we look into the special cases when the ${}_8W_7$ series on the $l.h.s.$ of (1.7) reduces to a ${}_4\phi_3$ series of the form summable by (3.13). To do this, we consider (3.12) instead and set $A = C^2/q$, $a = -C^{3/2}/(\alpha Fq)^{1/4}$,

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$B = C^2/\alpha$, $C = -G$. The ${}_8W_7$ series on the *l. h. s.* becomes

$$
\begin{aligned}
&{}_8W_7(-\alpha q^k; \alpha q^k, -\sqrt{\alpha q/F}, \sqrt{\alpha q/F}, \sqrt{\alpha F q}/C, -\sqrt{\alpha F q}/C; C^2 q^k/\alpha) \\
&= {}_4\phi_3\left[\begin{matrix}\alpha^2 q^{2k}, -q^2\sqrt{\cdot}, \alpha q/F, \alpha F q/C^2 \\ -\sqrt{\cdot}, \alpha F q^{1+2k}, \alpha C^2 q^{1+2k}/F\end{matrix}; q^2, C^2 q^k/\alpha\right] \\
&= \frac{(\alpha^2 q^{2+2k}, F q^{1+k}, C^2 q^{1+k}/F, C^2 q^{2k}; q^2)_\infty}{(\alpha F q^{1+2k}, \alpha C^2 q^{1+2k}/F, \alpha q^{2+k}, C^2 q^k/\alpha; q^2)_\infty},
\end{aligned}
\tag{4.1}
$$

by (3.13). Using (4.1) in (3.12), after substituting the above special values for the parameters, results in

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(C^2/q, q\sqrt{\cdot}, -q\sqrt{\cdot}, D, E, F)_k (F q^{1+k}, C^2 q^{1+k}/F; q^2)_\infty}{(q, \sqrt{\cdot}, -\sqrt{\cdot}, C^2/D, C^2/E, \alpha q)_k (\alpha q^{2+k}, C^2 q^k/\alpha; q^2)_\infty} \cdot (C\sqrt{\alpha q/F}/DE)^k \\
&= \frac{(\sqrt{\alpha F q}/C, -C\sqrt{Fq/\alpha}, C^2, C\sqrt{q/\alpha F}, -\sqrt{\alpha q/F}, \sqrt{\alpha q/F})_\infty}{(\sqrt{\alpha q/F}/C, C^2/\alpha, -q, C, -C, \alpha q)_\infty} \\
&\quad \cdot {}_5\phi_4\left[\begin{matrix}C^2/DE, F, C^2/\alpha, C, -C \\ C^2/D, C^2/E, C\sqrt{Fq/\alpha}, -C\sqrt{Fq/\alpha}\end{matrix}; q\right] \\
&\quad + \frac{(C^2/DE, C\sqrt{\alpha q/F}/D, C\sqrt{\alpha q/F}/E, F, C^2)_\infty}{(C^2/D, C^2/E, C\sqrt{\alpha q/F}/DE, C\sqrt{F/\alpha q}, \alpha q)_\infty} \\
&\quad \cdot {}_5\phi_4\left[\begin{matrix}C\sqrt{\alpha q/F}/DE, C\sqrt{q/\alpha F}, \sqrt{\alpha F q}/C, \sqrt{\alpha q/F}, -\sqrt{\alpha q/F} \\ C\sqrt{\alpha q/F}/D, C\sqrt{\alpha q/F}/E, q\sqrt{\alpha q/F}/C, -q\end{matrix}; q\right].
\end{aligned}
\tag{4.2}
$$

Of course if $\alpha = C^2/Fq$, (4.2) gives the sum of the ${}_6\phi_5$ series mentioned at the beginning of Section 2.

If we let $E = 1$ in (4.2) and replace F, C^2/α, C by a, b, c, respectively, we obtain

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix}a, b, c, -c \\ c^2, \sqrt{abq}, -\sqrt{abq}\end{matrix}; q\right] \\
&\quad + \frac{(\sqrt{q/ab}, a, b, c, -c, -q, c^2\sqrt{q/ab})_\infty}{(\sqrt{aq/b}, \sqrt{bq/a}, -\sqrt{abq}, \sqrt{ab/q}, -c\sqrt{q/ab}, c\sqrt{q/ab}, c^2)_\infty} \\
&\quad \cdot {}_4\phi_3\left[\begin{matrix}\sqrt{aq/b}, \sqrt{bq/a}, c\sqrt{q/ab}, -c\sqrt{q/ab} \\ c^2\sqrt{q/ab}, q\sqrt{q/ab}, -q\end{matrix}; q\right] \\
&= \frac{(\sqrt{q/ab}, \sqrt{abq})_\infty (aq, bq, c^2 q/a, c^2 q/b; q^2)_\infty}{(\sqrt{aq/b}, \sqrt{bq/a})_\infty (q, abq, c^2 q, c^2 q/ab; q^2)_\infty}.
\end{aligned}
\tag{4.3}
$$

On the other hand, if we replace $\sqrt{aq/b}$, $c\sqrt{q/ab}$, $q\sqrt{q/ab}$ by a, c, e, respectively, and rearrange the terms, then (4.3) becomes

$$
\begin{aligned}
(4.4)\qquad &{}_4\phi_3\left[\begin{matrix} a, q/a, c, -c \\ c^2q/e, e, -q \end{matrix}; q\right] \\
&+ \frac{(a, q/a, -q^2/e, q/e, c, -c, c^2q^2/e^2)_\infty}{(e/q, aq/e, q^2/ae, cq/e, -cq/e, -q, c^2q/e)_\infty} \\
&\cdot {}_4\phi_3\left[\begin{matrix} aq/e, q^2/ae, cq/e, -cq/e \\ c^2q^2/e^2, q^2/e, -q^2/e \end{matrix}; q\right] \\
&= \frac{(q/e)_\infty(ac^2q/e, c^2q^2/ae; q^2)_\infty}{(c^2q/e)_\infty(aq/e, q^2/ae; q^2)_\infty}.
\end{aligned}
$$

Equations (4.3) and (4.4) are the non-terminating extensions of Andrews' q-analogues of the Watson and Whipple summations respectively [1, Theorem 1 and Theorem 2]. By (1.1), they can be seen to be the same as Gasper-Rahman's [5, (3.21) and (3.22)].

The fact that either of (4.3) or (4.4) furnishes an extension of Andrews q-analogues of Watson's and Whipple's summation formulas is reflected in the following. If in (4.3), for example, we replace a, b, c by q^a, q^b, q^c, respectively, and let $q \longrightarrow 1^-$, we obtain

$$
\begin{aligned}
(4.5)\qquad &{}_3F_2\left[\begin{matrix} a, b, c \\ \frac{1+a+b}{2}, 2c \end{matrix}; 1\right] \\
&+ \Gamma\left[\begin{matrix} \frac{1+a-b}{2}, \frac{1+b-a}{2}, \frac{a+b-1}{2}, \frac{1-a-b}{2} + c, 2c \\ \frac{1-a-b}{2}, a, b, c, \frac{1-a-b}{2} + 2c \end{matrix}\right] \\
&\cdot {}_3F_2\left[\begin{matrix} \frac{1+a-b}{2}, \frac{1+b-a}{2}, \frac{1-a-b}{2} + c \\ \frac{3-a-b}{2}, \frac{1-a-b}{2} + 2c \end{matrix}; 1\right] \\
&= \Gamma\left[\begin{matrix} \frac{1+a-b}{2}, \frac{1+b-a}{2}, c + \frac{1}{2}, \frac{1-a-b}{2} + c, \frac{1}{2} \\ \frac{1-a-b}{2}, \frac{a+1}{2}, \frac{b+1}{2}, \frac{1-a}{2} + c, \frac{1-b}{2} + c \end{matrix}\right],
\end{aligned}
$$

which can be proved directly using the above mentioned Watson's and Whipple's summations [11, p. 245 (III. 23) and (III.24)].

In the process of taking the limit $q \longrightarrow 1^-$, we have used Jackson's q-Gamma function

$$
(4.6)\qquad \Gamma_q(x) = \frac{(q)_\infty}{(q^x)_\infty}(1-q)^{1-x}, \quad \lim_{q\to 1^-} \Gamma_q(x) = \Gamma(x),
$$

see [2] for example.

5. **Products of basic series.** As seen in the previous two sections, the ${}_8W_7$ series on the $l.h.s.$ of (1.7) can be handled in many ways. In this section we will consider some of the cases when it is transformable into a ${}_2\phi_1$ series. Our means in doing so is Gasper-Rahman's quadratic transformation [5], namely

$$
\begin{aligned}
(5.1)\qquad &{}_8W_7(ax/b; x, \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}; xq/b^2) \\
&= \frac{(axq/b, x^2q/b^2)_\infty}{(xq/b, ax^2q/b^2)_\infty}\, {}_2\phi_1\left[\begin{matrix} a, \ b \\ aq/b \end{matrix}; xq/b^2\right].
\end{aligned}
$$

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Let $A = \alpha/\beta$, $a^2 = -\alpha q/x$, $B = q\sqrt{\alpha}/\beta$, $C = -G = \sqrt{\alpha q}$ and $F = -\sqrt{\alpha}$ in (1.7). It becomes

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(\alpha/\beta, q\sqrt{\ }, -q\sqrt{\ }, D, E, -\sqrt{\alpha}, q\sqrt{\alpha}/\beta)_k}{(q, \sqrt{\ }, -\sqrt{\ }, \alpha q/\beta D, \alpha q/\beta E)_k} \\
&\cdot \frac{(\sqrt{\alpha q}, -\sqrt{\alpha q})_k (\alpha x q/\beta)_{2k} (xq\sqrt{\alpha}/DE\beta^2)^k}{(xq\sqrt{\alpha}/\beta, -xq\sqrt{\alpha}/\beta, x\sqrt{\alpha q}/\beta, -x\sqrt{\alpha q}/\beta)_k (\alpha q/\beta)_{2k}} \\
&\cdot {}_8W_7(\alpha x q^{2k}/\beta; \sqrt{\alpha}q^k, -\sqrt{\alpha}q^k, \sqrt{\alpha q}q^k, -\sqrt{\alpha q}q^k, x; xq/\beta^2) \\
&= \frac{(-x/\beta, \alpha x q/\beta, x\sqrt{q}/\beta, -x\sqrt{q}/\beta)_\infty}{(x/\beta\sqrt{\alpha}, -xq\sqrt{\alpha}/\beta, x\sqrt{\alpha q}/\beta, -x\sqrt{\alpha q}/\beta)_\infty} \\
&\qquad \cdot {}_5\phi_4\left[\begin{matrix} \alpha q/\beta DE, -\sqrt{\alpha}, \sqrt{\alpha q}, -\sqrt{\alpha q}, q\sqrt{\alpha}/\beta \\ \alpha q/\beta D, \alpha q/\beta E, \beta q\sqrt{\alpha}/x, xq\sqrt{\alpha}/\beta \end{matrix}; q\right] \\
&\qquad + \frac{(\alpha x q/\beta, \alpha q/\beta DE, -\sqrt{\alpha}, \sqrt{\alpha q}, -\sqrt{\alpha q}, q\sqrt{\alpha}/\beta)_\infty}{(\alpha q/\beta D, \alpha q/\beta E, xq\sqrt{\alpha}/\beta, -xq\sqrt{\alpha}/\beta, x\sqrt{\alpha q}/\beta)_\infty} \\
&\qquad \cdot \frac{(xq\sqrt{\alpha}/D\beta^2, xq\sqrt{\alpha}/E\beta^2, x^2q/\beta^2)_\infty}{(-x\sqrt{\alpha q}/\beta, \beta\sqrt{\alpha}/x, xq/\beta^2, xq\sqrt{\alpha}/DE\beta^2)_\infty} \\
&\qquad \cdot {}_5\phi_4\left[\begin{matrix} xq\sqrt{\alpha}/DE\beta^2, -x/\beta, xq/\beta^2, x\sqrt{q}/\beta, -x\sqrt{q}/\beta \\ xq\sqrt{\alpha}/D\beta^2, xq\sqrt{\alpha}/E\beta^2, xq/\beta\sqrt{\alpha}, x^2q/\beta^2 \end{matrix}; q\right].
\end{aligned} \tag{5.2}$$

By (5.1), the ${}_8W_7$ series in the above equation is equal to

$$\frac{(\alpha x q^{1+2k}/\beta, x^2q/\beta^2)_\infty}{(xq/\beta, \alpha x^2 q^{1+2k}/\beta^2)_\infty} {}_2\phi_1\left[\begin{matrix} \alpha q^{2k}, \beta \\ \alpha q^{1+2k}/\beta \end{matrix}; xq/\beta^2\right], \tag{5.3}$$

and thus the *l. h. s.* becomes

$$\begin{aligned}
&\frac{(\alpha x q/\beta, x^2q/\beta^2)_\infty}{(xq/\beta, \alpha x^2 q/\beta^2)_\infty} \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{(\alpha/\beta, q\sqrt{\ }, -q\sqrt{\ }, D, E, q\sqrt{\alpha}/\beta)_k}{(q, \sqrt{\ }, -\sqrt{\ }, \alpha q/\beta D, \alpha q/\beta E, \sqrt{\alpha})_k} \\
&\qquad \cdot \frac{(\alpha)_{2k+\ell}(\beta)_\ell}{(\alpha q/\beta)_{2k+\ell}(q)_\ell} (xq\sqrt{\alpha}/DE\beta^2)^k (xq/\beta^2)^\ell.
\end{aligned} \tag{5.4}$$

It is easy to see that

$$(\beta)_\ell = \frac{(\beta q^{-k})_{k+\ell}}{(\beta q^{-k})_k} = \frac{(\beta q^{-k})_{k+\ell}}{(q/\beta)_k(-\beta)^k} q^{\frac{k}{2}(k+1)}. \tag{5.5}$$

Substituting this into (5.4), replacing $k+\ell$ by j and simplifying give

$$\frac{(\alpha x q/\beta, x^2q/\beta^2)_\infty}{(xq/\beta, \alpha x^2 q/\beta^2)_\infty} \sum_{j=0}^{\infty} A_j \frac{(\alpha, \beta)_j}{(q, \alpha q/\beta)_j} (xq/\beta^2)^j, \tag{5.6}$$

where A_j is equal to

$$\begin{aligned}
&{}_8W_7(\alpha/\beta; D, E, q\sqrt{\alpha}/\beta, \alpha q^j, q^{-j}; q\sqrt{\alpha}/\beta DE) \\
&= \frac{(\alpha q/\beta, \sqrt{\alpha}/D)_j}{(\alpha q/\beta D, \sqrt{\alpha})_j} {}_4\phi_3\left[\begin{matrix} q^{-j}, D, q\sqrt{\alpha}/\beta, q^{1-j}/\beta E \\ \alpha q/\beta E, q^{1-j}/\beta, Dq^{1-j}/\sqrt{\alpha} \end{matrix}; q\right],
\end{aligned} \tag{5.7}$$

by Watson's transformation [4, 8.5 (2)], the terminating case of (1.1). Next, we substitute (5.7) into (5.6) and use

$$(5.8)\qquad (q^{1-N}/a)_n = \frac{(a)_N q^{\frac{n}{2}(n+1)}}{(a)_{N-n}(-a)^n q^{Nn}}, \quad [11, \text{ p. } 241 \text{ (II.9)}],$$

to get

$$(5.9)\qquad \begin{aligned} &\frac{(\alpha xq/\beta, x^2q/\beta^2)_\infty}{(xq/\beta, \alpha x^2q/\beta^2)_\infty} \sum_{j=0}^{\infty}\sum_{i=0}^{j} \frac{(\alpha,\beta E)_j(\beta,\sqrt{\alpha}/D)_{j-i}}{(\alpha q/\beta D,\sqrt{\alpha})_j(q,\beta E)_{j-i}} \\ &\quad\cdot \frac{(D, q\sqrt{\alpha}/\beta)_i}{(q,\alpha q/\beta E)_i}(xq/\beta^2)^j(\sqrt{\alpha}/DE)^i \\ &= \frac{(\alpha xq/\beta, x^2q/\beta^2)_\infty}{(xq/\beta, \alpha x^2q/\beta^2)_\infty} \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\alpha,\beta E)_{k+i}(\beta,\sqrt{\alpha}/D)_k}{(\alpha q/\beta D,\sqrt{\alpha})_{k+i}(q,\beta E)_k} \\ &\quad\cdot \frac{(D, q\sqrt{\alpha}/\beta)_i}{(q,\alpha q/\beta E)_i}(xq/\beta^2)^k(xq\sqrt{\alpha}/DE\beta^2)^i. \end{aligned}$$

Finally, we equate this to the $r.h.s.$ of (5.2) to obtain

(5.10)

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{(\alpha,\beta,\sqrt{\alpha}/D)_k}{(q,\alpha q/\beta D,\sqrt{\alpha})_k}(xq/\beta^2)^k \cdot {}_4\phi_3\left[\begin{matrix}\alpha q^k, \beta Eq^k, D, q\sqrt{\alpha}/\beta \\ \alpha q^{1+k}/\beta D, \sqrt{\alpha}q^k, \alpha q/\beta E\end{matrix}; xq\sqrt{\alpha}/DE\beta^2\right] \\ &= \frac{(xq\sqrt{\alpha}/\beta, -x/\beta)_\infty}{(x/\beta\sqrt{\alpha}, -xq/\beta)_\infty}\, {}_5\phi_4\left[\begin{matrix}\alpha q/\beta DE, -\sqrt{\alpha}, \sqrt{\alpha q}, -\sqrt{\alpha q}, q\sqrt{\alpha}/\beta \\ \alpha q/\beta D, \alpha q/\beta E, \beta q\sqrt{\alpha}/x, xq\sqrt{\alpha}/\beta\end{matrix}; q\right] \\ &\quad+ \frac{(\alpha q/\beta DE, xq\sqrt{\alpha}/D\beta^2, xq\sqrt{\alpha}/E\beta^2, \alpha, xq/\beta, q\sqrt{\alpha}/\beta)_\infty}{(xq\sqrt{\alpha}/DE\beta^2, \alpha q/\beta D, \alpha q/\beta E, \sqrt{\alpha}, \beta\sqrt{\alpha}/x, xq/\beta^2)_\infty} \\ &\quad\cdot {}_5\phi_4\left[\begin{matrix}xq\sqrt{\alpha}/DE\beta^2, -x/\beta, xq/\beta^2, x\sqrt{q}/\beta, -x\sqrt{q}/\beta \\ xq\sqrt{\alpha}/D\beta^2, xq\sqrt{\alpha}/E\beta^2, xq/\beta\sqrt{\alpha}, x^2q/\beta^2\end{matrix}; q\right]. \end{aligned}$$

An interesting special case of (5.10) is when $D = q/\beta$, $E = \sqrt{\alpha}/\beta$. In this case the ${}_4\phi_3$ on the $l.h.s.$ reduces to

$$(5.11)\qquad {}_2\phi_1\left[\begin{matrix}q\sqrt{\alpha}/\beta, q/\beta \\ q\sqrt{\alpha}\end{matrix}; x\right] = \frac{(xq/\beta^2)_\infty}{(x)_\infty}\, {}_2\phi_1\left[\begin{matrix}\beta\sqrt{\alpha}, \beta \\ q\sqrt{\alpha}\end{matrix}; xq/\beta^2\right],$$

by Heine's transformation formula [4, 8.4 (2)]

$$(5.12)\qquad {}_2\phi_1\left[\begin{matrix}a, b \\ c\end{matrix}; z\right] = \frac{(abz/c)_\infty}{(z)_\infty}\, {}_2\phi_1\left[\begin{matrix}c/a, c/b \\ c\end{matrix}; abz/c\right].$$

Hence, and after setting $\sqrt{\alpha} = aq/b$, $\beta = b$ and replacing x by xb, (5.10) gives

(5.13)

$$\begin{aligned} &{}_2\phi_1\left[\begin{matrix}a, b \\ aq/b\end{matrix}; xq/b\right]\, {}_2\phi_1\left[\begin{matrix}aq, b \\ aq^2/b\end{matrix}; xq/b\right] \\ &= \frac{(xb, xaq^2/b)_\infty}{(xq/b, xb/aq)_\infty}(1+x)\, {}_5\phi_4\left[\begin{matrix}-aq/b, aq^2/b^2, aq^{3/2}/b, -aq^{3/2}/b, aq \\ a^2q^2/b^2, aq^2/b, aq^2/xb, xaq^2/b\end{matrix}; q\right] \\ &\quad+ \frac{(aq, aq^2/b^2, xaq/b, xq, xq)_\infty}{(aq/b, aq^2/b, aq/xb, xq/b, xq/b)_\infty}\, {}_5\phi_4\left[\begin{matrix}-x, xq/b, x\sqrt{q}, -x\sqrt{q}, xb \\ xaq/b, xq, xb/a, x^2q\end{matrix}; q\right]. \end{aligned}$$

Going back to (1.7), transform the ${}_8W_7$ on the *l. h. s.* using (1.8) again to get

$$(5.14)\quad \frac{(ACGq^{1+2k}/a^2, BCG/Fa^2, ABq^{1+k}/a^2, A^2q^{2+k}/Fa^2)_\infty}{(BCGq^k/a^2, ACGq^{1+k}/Fa^2, A^2q^{2+2k}/a^2, ABq/Fa^2)_\infty} \cdot {}_8W_7(A^2q^{1+2k}/a^2; Aq^{1+k}/B, Aq^{1+k}/C, Aq^{1+k}/G, Fq^k, Aq/a^2; BCG/Fa^2).$$

Now let $A = a^2/b$, $B = -aq/b$, $C = -G = a\sqrt{q}/b$, $F = a$, and replace the a^2 that appears in (1.7) and (5.14) by a^2q/xb^2. With these new values and (5.14), (1.7) becomes (5.15)

$$\sum_{k=0}^{\infty} \frac{(a^2/b, q\surd, -q\surd, D, E, a, -aq/b)_k}{(q, \surd, -\surd, a^2q/bD, a^2q/bE, xaq, -xaq)_k} \cdot \frac{(a\sqrt{q}/b, -a\sqrt{q}/b)_k (xa^2q)_{2k}}{(xa\sqrt{q}, -xa\sqrt{q})_k (a^2q/b)_{2k}} (xaq/DE)^k$$
$$\cdot {}_8W_7(xa^2q^{2k}; aq^k, -aq^k, a\sqrt{q}q^k, -a\sqrt{q}q^k, xb; xq/b)$$
$$= \frac{(xaq/b, xa^2q, -xq, xb, x\sqrt{q}, -x\sqrt{q})_\infty}{(xq/b, xaq, -xaq, xa\sqrt{q}, -xa\sqrt{q}, xb/a)_\infty} \cdot {}_5\phi_4\left[\begin{matrix} a^2q/bDE, a, -aq/b, a\sqrt{q}/b, -a\sqrt{q}/b \\ a^2q/bD, a^2q/bE, aq/xb, xaq/b \end{matrix}; q\right]$$
$$+ \frac{(xa^2q, a^2q/bDE, a, -aq/b, a\sqrt{q}/b, -a\sqrt{q}/b, xaq/D, xaq/E, x^2q)_\infty}{(xq/b, xaq, -xaq, xa\sqrt{q}, -xa\sqrt{q}, a^2q/bD, a^2q/bE, a/xb, xaq/DE)_\infty} \cdot {}_5\phi_4\left[\begin{matrix} xaq/DE, xb, -xq, x\sqrt{q}, -x\sqrt{q} \\ xaq/D, xaq/E, xbq/a, x^2q \end{matrix}; q\right].$$

As before, transform the ${}_8W_7$ on the *l. h. s.* of (5.15) into a ${}_2\phi_1$ using (5.1) then use (5.12) to get

$$(5.16)\quad \frac{(xa^2q, x^2q, xb)_\infty}{(x^2a^2q, xq, xq/b)_\infty} \sum_{k=0}^{\infty} \frac{(a^2/b, q\surd, -q\surd, D, E, b)_k}{(q, \surd, -\surd, a^2q/bD, a^2q/bE, a^2q/b^2)_k} \cdot \frac{(a^2q/b^2)_{2k}}{(a^2q/b)_{2k}} (xaq/DE)^k \, {}_2\phi_1\left[\begin{matrix} a^2q^{1+2k}/b^2, q/b \\ a^2q^{1+2k}/b \end{matrix}; xb\right].$$

Continuing with steps similar to (5.5)–(5.10), equation (5.15) yields

$$(5.17)\quad \sum_{k=0}^{\infty} \frac{(q/b, a^2q/b^2, aq/bD)_k}{(q, aq/b, a^2q/bD)_k} (xb)^k \cdot {}_4\phi_3\left[\begin{matrix} a^2q^{1+k}/b^2, Eq^{1+k}/b, a, D \\ a^2q^{1+k}/bD, aq^{1+k}/b, a^2q/bE \end{matrix}; axq/DE\right]$$
$$= \frac{(xaq/b)_\infty}{(xb/a)_\infty} \, {}_5\phi_4\left[\begin{matrix} a^2q/bDE, a, -aq/b, a\sqrt{q}/b, -a\sqrt{q}/b \\ a^2q/bD, a^2q/bE, aq/xb, xaq/b \end{matrix}; q\right]$$
$$+ \frac{(a, xq, a^2q/b^2, xaq/D, xaq/E, a^2q/bDE)_\infty}{(aq/b, xb, a/xb, a^2q/bD, a^2q/bE, xaq/DE)_\infty} \cdot {}_5\phi_4\left[\begin{matrix} xaq/DE, xb, -xq, x\sqrt{q}, -x\sqrt{q} \\ xaq/D, xaq/E, xbq/a, x^2q \end{matrix}; q\right].$$

An interesting case of (5.17) is when $D = b$, $E = a$. In this case, the $l.h.s.$ of (5.17) becomes a product of two ${}_2\phi_1$'s, one of which is

$$(5.18)\qquad {}_2\phi_1\left[\begin{matrix} aq/b^2, q/b \\ aq/b \end{matrix}; xb\right] = \frac{(xq/b)_\infty}{(xb)_\infty}\,{}_2\phi_1\left[\begin{matrix} a,b \\ aq/b \end{matrix}; xq/b\right],$$

by (5.12). Thus we get

$$(5.19)\qquad \begin{aligned} &\left\{{}_2\phi_1\left[\begin{matrix} a,b \\ aq/b \end{matrix}; xq/b\right]\right\}^2 \\ &= \frac{(xb, xaq/b)_\infty}{(xq/b, xb/a)_\infty}\,{}_5\phi_4\left[\begin{matrix} aq/b^2, a, -aq/b, a\sqrt{q}/b, -a\sqrt{q}/b \\ a^2q/b^2, aq/b, aq/xb, xaq/b \end{matrix}; q\right] \\ &+ \frac{(a, aq/b^2, xq, xq, xaq/b)_\infty}{(aq/b, aq/b, xq/b, xq/b, a/xb)_\infty} \\ &\cdot {}_5\phi_4\left[\begin{matrix} xq/b, xb, -xq, x\sqrt{q}, -x\sqrt{q} \\ xaq/b, xbq/a, xq, x^2q \end{matrix}; q\right]. \end{aligned}$$

This is due to Gasper and Rahman [6] which they used to obtain their q-Clausen formula as well as some other product formulas.

6. **Special cases of (5.13) and (5.19).** To see the importance of formulas (5.13) and (5.19), we shall look into the special cases when they give q-analogues of products of ordinary hypergeometric series. To start with, using (5.1) and then (1.8) we have

$$\begin{aligned} &{}_2\phi_1\left[\begin{matrix} a,b \\ aq/b \end{matrix}; xq/b\right] \\ &= \frac{(xq, x^2aq)_\infty}{(xaq, x^2q)_\infty}\,{}_8W_7(ax; \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, xb; xq/b) \\ (6.1)\qquad &= \frac{(x^2aq, xq, -xq, xq\sqrt{a}/b, -xq\sqrt{a}/b)_\infty}{(x^2q, xq/b, -xaq/b, xq\sqrt{a}, -xq\sqrt{a})_\infty} \\ &\cdot {}_8W_7(-xa/b; \sqrt{a}, -\sqrt{a}, \sqrt{aq}/b, -\sqrt{aq}/b, -x; -xq), \\ (6.2)\qquad &= \frac{(x^2aq, xq, -x, xq\sqrt{aq}/b, -xq\sqrt{aq}/b)_\infty}{(x^2q, xq/b, -xaq^2/b, x\sqrt{aq}, -x\sqrt{aq})_\infty} \\ &\cdot {}_8W_7(-xaq/b; \sqrt{aq}, -\sqrt{aq}, q\sqrt{a}/b, -q\sqrt{a}/b, -xq; -x). \end{aligned}$$

On the other hand, we can use Jackson's [7]

$$(6.3)\qquad {}_2\phi_1\left[\begin{matrix} a,b \\ c \end{matrix}; x\right] = \frac{(ax)_\infty}{(x)_\infty}\,{}_2\phi_2\left[\begin{matrix} a, c/b \\ c, ax \end{matrix}; bx\right]$$

and thus obtain

$$(6.4)\qquad {}_2\phi_1\left[\begin{matrix} a,b \\ aq/b \end{matrix}; xq/b\right] = \frac{(xaq/b)_\infty}{(xq/b)_\infty}\,{}_2\phi_2\left[\begin{matrix} a, aq/b^2 \\ aq/b, xaq/b \end{matrix}; xq\right],$$

$$(6.5)\qquad = \frac{(xq)_\infty}{(xq/b)_\infty}\,{}_2\phi_2\left[\begin{matrix} q/b, b \\ aq/b, xq \end{matrix}; xaq/b\right].$$

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At this point, we proceed to use (5.13), (5.19), (6.1), (6.2), (6.4) and (6.5) to give, as mentioned, q-analogues of products of ordinary hypergeometric series. Replacing a, b by q^a, q^b and letting $q \longrightarrow 1^-$, we can show that (5.13) and (5.19) are q-analogues of

$$\begin{aligned}{}_2F_1\left[\begin{matrix}a,b\\1+a-b\end{matrix};x\right]{}_2F_1\left[\begin{matrix}a+1,b\\2+a-b\end{matrix};x\right]\\=\frac{(1+x)}{(1-x)^{2a+2}}\,{}_3F_2\left[\begin{matrix}a+2-2b,a+3/2-b,a+1\\2(a+1-b),a+2-b\end{matrix};\frac{-4x}{(1-x)^2}\right]\end{aligned}\tag{6.6}$$

and

$$\left\{{}_2F_1\left[\begin{matrix}a,b\\1+a-b\end{matrix};x\right]\right\}^2={}_3F_2\left[\begin{matrix}1+a-2b,a,a+1/2-b\\2a+1-2b,1+a-b\end{matrix};\frac{-4x}{(1-x)^2}\right],\tag{6.7}$$

respectively. If we use (6.1), (6.4) and (6.5) in (5.19), we get the results given in [6]. And if (6.2) is used, then (5.19) gives

$$\begin{aligned}&\{{}_8W_7(-xaq/b;\sqrt{aq},-\sqrt{aq},q\sqrt{a}/b,-q\sqrt{a}/b,-xq;-x)\}^2\\&=\left[\frac{(x^2q,xq/b,-xaq^2/b,x\sqrt{aq},-x\sqrt{aq})_\infty}{(x^2aq,xq,-x,xq\sqrt{aq}/b,-xq\sqrt{aq}/b)_\infty}\right]^2\cdot\{r.h.s.\text{ of }(5.19)\},\end{aligned}\tag{6.8}$$

which upon replacing $\sqrt{a}$, $\sqrt{aq}/b$ by q^α, q^β, respectively, and then taking the limit $q \longrightarrow 1^-$ give

$$\begin{aligned}&\left\{{}_2F_1\left[\begin{matrix}\alpha+1/2,\beta+1/2\\\alpha+\beta+1/2\end{matrix};z\right]\right\}^2\\&=(1-z)^{-1}\,{}_3F_2\left[\begin{matrix}2\alpha,2\beta,\alpha+\beta\\\alpha+\beta+1/2,2\alpha+2\beta\end{matrix};z\right].\end{aligned}\tag{6.9}$$

If we use both (6.1) and (6.2), then (5.19) gives

$$\begin{aligned}&{}_8W_7(-xa/b;\sqrt{a},-\sqrt{a},\sqrt{aq}/b,-\sqrt{aq}/b,-x;-xq)\\&\cdot{}_8W_7(-xaq/b;\sqrt{aq},-\sqrt{aq},q\sqrt{a}/b,-q\sqrt{a}/b,-xq;-x)\\&=\frac{(-xaq/b,-xaq^2/b,x\sqrt{aq},-x\sqrt{aq},xq\sqrt{a},-xq\sqrt{a})_\infty}{(-x,-xq,xq\sqrt{a}/b,-xq\sqrt{a}/b,xq\sqrt{aq}/b,-xq\sqrt{aq}/b)_\infty}\\&\cdot\left[\frac{x^2q,xq/b)_\infty}{(x^2aq,xq)_\infty}\right]^2\cdot\{r.h.s.\text{ of }(5.19)\}.\end{aligned}\tag{6.10}$$

This is the q-analogue of

$$\begin{aligned}&{}_2F_1\left[\begin{matrix}\alpha,\beta\\\alpha+b+1/2\end{matrix};z\right]{}_2F_1\left[\begin{matrix}\alpha+1/2,\beta+1/2\\\alpha+\beta+1/2\end{matrix};z\right]\\&=(1-z)^{-1/2}\,{}_3F_2\left[\begin{matrix}2\alpha,2\beta,\alpha+\beta\\\alpha+\beta+1/2,2\alpha+2\beta\end{matrix};z\right].\end{aligned}\tag{6.11}$$

We should mention that (6.9) and (6.11) are easily obtained from Clausen's formula by Euler's transformation [4, 1.2 (2)].

We now turn back to equation (5.13). Transforming the first ${}_2\phi_1$ by (6.2) and the second by (6.1) yields

$$\begin{aligned}&{}_8W_7(-xaq/b;\sqrt{aq},-\sqrt{aq},q\sqrt{a}/b,-q\sqrt{a}/b,-xq;-x)\\ &\quad\cdot{}_8W_7(-xaq/b;\sqrt{aq},-\sqrt{aq},q\sqrt{a}/b,-q\sqrt{a}/b,-x;-xq)\\ &=\frac{(xq\sqrt{aq},-xq\sqrt{aq},x\sqrt{aq},-x\sqrt{aq})_\infty}{(x^2aq,x^2aq^2,-x,-xq)_\infty}\\ &\quad\cdot\left[\frac{(x^2q,xq/b,-xaq^2/b)_\infty}{(xq,xq\sqrt{aq}/b,-xq\sqrt{aq}/b)_\infty}\right]^2\cdot\{r.h.s.\text{ of }(5.13)\}.\end{aligned}\tag{6.12}$$

Replacing $\sqrt{aq}$, $q\sqrt{a}/b$ by q^α, q^β, respectively, and then letting $q\longrightarrow 1^-$ give

$$\begin{aligned}&{}_2F_1\left[\begin{matrix}\alpha,\beta\\ \alpha+\beta-1/2\end{matrix};z\right]{}_2F_1\left[\begin{matrix}\alpha,\beta\\ \alpha+\beta+1/2\end{matrix};z\right]\\ &={}_3F_2\left[\begin{matrix}2\alpha,2\beta,\alpha+\beta\\ 2\alpha+2\beta-1,\alpha+\beta+1/2\end{matrix};z\right],\end{aligned}\tag{6.13}$$

a result due to Orr [10]. On the other hand, if we transform the first ${}_2\phi_1$ by (6.1) and the second by (6.2), we get

$$\begin{aligned}&{}_8W_7(-xa/b;\sqrt{a},-\sqrt{a},\sqrt{aq}/b,-\sqrt{aq}/b,-x;-xq)\\ &\quad\cdot{}_8W_7(-xaq^2/b;q\sqrt{a},-q\sqrt{a},q\sqrt{aq}/b,-q\sqrt{aq}/b,-xq;-x)\\ &=\frac{(x^2q,xq/b,xq\sqrt{a},-xq\sqrt{a})^2_\infty(-xaq/b,-xaq^3/b)_\infty}{(xq)^2_\infty(x^2aq,x^2aq^2,-x,-xq,xq\sqrt{a}/b,xq^2\sqrt{a}/b)_\infty}\\ &\quad\cdot\frac{1}{(-xq\sqrt{a}/b,-xq^2\sqrt{a}/b)_\infty}\cdot\{r.h.s.\text{ of }(5.13)\},\end{aligned}\tag{6.14}$$

which is the q-analogue of

$$\begin{aligned}&{}_2F_1\left[\begin{matrix}\alpha,\beta\\ \alpha+\beta+1/2\end{matrix};z\right]{}_2F_1\left[\begin{matrix}\alpha+1,\beta+1\\ \alpha+\beta+3/2\end{matrix};z\right]\\ &={}_3F_2\left[\begin{matrix}2\alpha+1,2\beta+1,\alpha+\beta+1\\ 2\alpha+2\beta+1,\alpha+\beta+3/2\end{matrix};z\right].\end{aligned}\tag{6.15}$$

Two more formulas can be obtained from (5.13) via (6.1) and (6.2). They are q-analogues of formulas equivalent to (6.13) and (6.15).

Finally, if we use (6.4) and (6.5) as we have done with (6.1) and (6.2), we get q-analogues of

$$\begin{aligned}&{}_2F_1\left[\begin{matrix}\alpha,\beta\\ (\alpha+\beta+1)/2\end{matrix};z\right]{}_2F_1\left[\begin{matrix}\alpha+1,\beta+1\\ (\alpha+\beta+3)/2\end{matrix};z\right]\\ &=(1-2z)\,{}_3F_2\left[\begin{matrix}\alpha+1,\beta+1,1+(\alpha+\beta)/2\\ \alpha+\beta+1,(\alpha+\beta+3)/2\end{matrix};4z(1-z)\right]\end{aligned}\tag{6.16}$$

and

$$\begin{aligned}&{}_2F_1\left[\begin{matrix}\alpha,1-\alpha\\ \beta\end{matrix};z\right]{}_2F_1\left[\begin{matrix}\alpha,1-\alpha\\ \beta+1\end{matrix};z\right]\\ &=(1-2z)(1-z)^{2\beta-1}\,{}_3F_2\left[\begin{matrix}1+\beta-\alpha,\beta+1/2,\alpha+\beta\\ 2\beta,\beta+1\end{matrix};4z(1-z)\right].\end{aligned}\tag{6.17}$$

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Department of Mathematics
University of Ottawa
Ottawa, Ontario K1N 6N5