

## ON TWO LEMMAS OF BROWN AND SHEPP HAVING APPLICATION TO SUM SETS AND FRACTALS, III

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### Abstract

We improve some results of [17], which relate to key tools given in [7] for establishing canonical inequalities used in the analysis of sum sets and fractals.

### 1. Introduction

In recent years a number of striking results have been derived for probability measures of certain sum sets involving fractals (see, for example, [2, 4, 5, 8, 9, 13]). By the algebraic sum set  $E + F$  of two sets  $E, F$  we signify the set

$$E + F = \{x + y : x \in E, y \in F\}.$$

The earliest and simplest of these results, established in [5], is the following.

**THEOREM A.** *Suppose  $m$  denotes Lebesgue measure and  $\mu_c$  Cantor–Lebesgue measure, that is, the uniform distribution on the Cantor subset  $C$  of  $[0, 1]$  formed by repeated removal of middle thirds. If  $E, F$  are Borel subsets of  $[0, 1]$ , then*

$$m(E + F) \geq 2\mu_c(E)^\alpha \mu_c(F)^\alpha,$$

where  $\alpha = \log 3 / \log 4$ .

These results depend on novel inequalities involving nonintegral powers of real numbers. Thus Theorem A uses the inequality

$$x^\alpha y^\alpha + \max\{x^\alpha(1-y)^\alpha, y^\alpha(1-x)^\alpha\} + (1-x)^\alpha(1-y)^\alpha \geq 1 \quad (0 \leq x, y \leq 1) \quad (1.1)$$

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established by Woodall [19].

Woodall and subsequently Hajela and Seymour [10] derived a variety of interesting results in combinatorial geometry from the latter inequality, which is therefore of some interest in its own right. The history of these results and related ideas pertaining to a multivariate extension of (1.1) (see [4, 12]) is quite colourful. A brief account is given by Brown [2].

Inequalities like (1.1) may be shown to depend on canonical univariate inequalities (see Brown [2, 3]). Thus (1.1) can be deduced from the following result.

**PROPOSITION B.** *Suppose that  $s, t \geq 1$  and  $s^{-1} + t^{-1} = \log 3 / \log 2$ . Then*

$$1 + x + x^2 \geq (1 + x^s)^{1/s} (1 + x^t)^{1/t} \quad (1.2)$$

for all for  $0 \leq x \leq 1$  if and only if  $3(s + t) \leq 8$ .

These inequalities are central. For example, (1.2) also leads to more general multivariate inequalities than (1.1) and consequent new measure-theoretic inequalities. A comprehensive account is given in [6].

Establishing the canonical univariate inequalities can be quite tricky and some effort has been put into sharpening techniques for their derivation (see [7, 11, 14–17] and most recently [1]). The ‘two lemmas’ of the title are special cases of the two parts of Theorem 3.2 below, the earliest versions of which are due to Brown and Shepp [7] and have influenced further work in the area.

This paper is a continuation of [17], from which we take the terminology and notation. We are able to improve the main theorems of [17], due to improvements in the auxiliary results and lemmas. In Section 2 we give a strengthened form of the main lemma underlying the results of [1]. Improved forms of several existing theorems follow immediately and are given in Section 3.

## 2. Key results

In [1] the following theorem is proved.

**THEOREM C.** *Let  $p_i$  ( $i = 1, \dots, n; n \geq 2$ ) be positive real numbers with  $\sum_{i=1}^n p_i = 1$ . Then for all real numbers  $a_1, \dots, a_n$  we have*

$$\max_{1 \leq i \leq n} a_i - \sum_{i=1}^n p_i a_i \geq \frac{p}{n-1} \sum_{1 \leq i < j \leq n} \left( \sqrt{|a_i|} - \sqrt{|a_j|} \right)^2,$$

where

$$p = \min_{1 \leq i \leq n} p_i. \quad (2.1)$$

The proof of this theorem is rather complicated, especially for the case when the quantities  $a_i$  are not all of the same sign.

We shall give a very simple proof of the following improvement of this theorem.

**THEOREM 2.1.** *Let  $p_i$  ( $i = 1, \dots, n; n \geq 2$ ) be positive real numbers with  $\sum_{i=1}^n p_i = 1$  and  $a_1, \dots, a_n$  real numbers. Suppose*

$$a_{i'} = \max_{1 \leq i \leq n} a_i.$$

Then we have

$$a_{i'} - \sum_{i=1}^n p_i a_i \geq \frac{p'}{n-1} \sum_{1 \leq i < j \leq n} |a_i - a_j|,$$

where  $p' = \min_{i \neq i'} p_i$ .

**PROOF.** Let  $b_n \leq b_{n-1} \leq \dots \leq b_1$  be a permutation of  $a_1, \dots, a_n$  and  $r_1, \dots, r_n$  the corresponding permutation of  $p_1, \dots, p_n$ . Then

$$a_{i'} - \sum_{j=1}^n p_j a_j = b_1 - \sum_{j=1}^n r_j b_j = \sum_{j=2}^n r_j (b_1 - b_j).$$

For each  $i \in \{1, \dots, n-1\}$  we have

$$\sum_{j=2}^n r_j (b_1 - b_j) \geq \sum_{j=i+1}^n r_j (b_1 - b_j) \geq \sum_{j=i+1}^n r_j (b_i - b_j) \geq p' \sum_{j=i+1}^n (b_i - b_j).$$

Hence

$$\begin{aligned} a_{i'} - \sum_{j=1}^n p_j a_j &= \frac{1}{n-1} \sum_{i=1}^{n-1} \sum_{j=2}^n r_j (b_1 - b_j) \\ &\geq \frac{1}{n-1} p' \sum_{i=1}^{n-1} \sum_{j=i+1}^n (b_i - b_j) \\ &= \frac{p'}{n-1} \sum_{i=1}^{n-1} \sum_{j=i+1}^n |a_i - a_j| \end{aligned}$$

and we are done.

**REMARK 2.2.** We have  $p' \geq p$ , where  $p$  is defined by (2.1). Hence Theorem 2.1 is in fact stronger than Theorem C, since

$$|a_i - a_j| \geq \left( \sqrt{|a_i|} - \sqrt{|a_j|} \right)^2.$$

Also, the remark in [1] that the constant factor  $p/(n - 1)$  cannot be replaced by a greater number is not completely correct. As we see, this factor can be replaced by  $p'/(n - 1)$ , which may be greater if  $p' = p$ .

In fact, we can establish an even better estimate of difference between  $\max a_i$  and the weighted mean, in terms of the symmetric function  $\sum_{i < j} |a_i - a_j|$ .

Let us prove first following minimax lemma, which may be of independent interest.

LEMMA 2.3. *Let  $p_1, \dots, p_n$  be positive numbers. Then*

$$M := \sup_{\substack{q_1, \dots, q_n \\ q_1 + \dots + q_n = 1}} \min_{\substack{1 \leq i \leq n \\ i \leq j \leq n}} q_i p_j$$

is attained and is given by

$$M = \frac{1}{\sum_{i=1}^n 1/p'_i}, \tag{2.2}$$

where  $p'_1, \dots, p'_n$  are defined by  $p'_i = \min_{i \leq j \leq n} p_j$ .

PROOF. The minimum of  $q_i p_j$  taken over  $1 \leq i \leq n$  and  $i \leq j \leq n$  will be nonnegative if and only if each  $q_i$  is nonnegative, so without loss of generality we may restrict attention to the situation in which  $\{q_1, \dots, q_n\}$  is a set of nonnegative probabilities. In this case, that  $M$  exists and is attained follows by compactness.

For transparency of reference, the quantities  $q_i p_j$  may be arranged in rows as

$$\begin{matrix} q_1 p_1 & q_1 p_2 & \cdots & q_1 p_i & \cdots & q_1 p_n \\ & q_2 p_2 & \cdots & q_2 p_i & \cdots & q_2 p_n \\ & & & & & \vdots \\ & & & q_i p_i & \cdots & q_i p_n \\ & & & & & \vdots \\ & & & & & q_n p_n \end{matrix}$$

For each member in the  $i$ -th row we have  $q_i p_j \geq q_i p'_i$ , since  $j \geq i$ . Suppose that the supremum  $M$  is attained for some  $n$ -tuple  $(q_1, \dots, q_n)$  for which  $q_i p'_i < q_j p'_j$  for some  $i \neq j$ . Then we can replace  $q_i$  by  $q'_i$  and  $q_j$  by  $q'_j$  such that  $q'_i p'_i = q'_j p'_j$  and  $q'_i + q'_j = q_i + q_j$ . Doing this will increase the value of the minimum, a contradiction to the definition of  $M$ .

Hence we have

$$q_1 p'_1 = q_2 p'_2 = \cdots = q_n p'_n = M.$$

Since

$$\sum_{i=1}^n \frac{M}{p'_i} = \sum_{i=1}^n q_i = 1,$$

(2.2) follows.

**THEOREM 2.4.** *Let  $p_i$  ( $i = 1, \dots, n; n \geq 2$ ) be positive real numbers with  $\sum_{i=1}^n p_i = 1$  and  $a_1, \dots, a_n$  real numbers. Let  $b_n \leq \dots \leq b_1$  be a permutation of  $a_1, \dots, a_n$ , and  $r_1, \dots, r_n$  the corresponding permutation of  $p_1, \dots, p_n$ . Define*

$$p'_i = \min_{1 \leq j \leq n} r_j, \quad i = 2, \dots, n.$$

Then

$$\max_{1 \leq i \leq n} a_i - \sum_{i=1}^n p_i a_i \geq \frac{1}{\sum_{i=2}^n 1/p'_i} \cdot \sum_{1 \leq i < j \leq n} |a_i - a_j|. \tag{2.3}$$

**PROOF.** For each  $i \geq 1$  we have

$$\max_{1 \leq i \leq n} a_i - \sum_{i=1}^n p_i a_i = b_1 - \sum_{j=1}^n r_j b_j = \sum_{j=2}^n r_j (b_1 - b_j) \geq \sum_{j=i+1}^n r_j (b_i - b_j). \tag{2.4}$$

Let  $q_1, \dots, q_n$  be positive numbers satisfying  $q_1 + \dots + q_n = 1$ . Multiplying (2.4) by  $q_i$  and summing over  $i = 1, \dots, n - 1$  gives

$$\begin{aligned} \max_{1 \leq i \leq n} a_i - \sum_{i=1}^n p_i a_i &= \frac{1}{1 - q_n} \sum_{i=1}^{n-1} \sum_{j=2}^n q_i r_j (b_1 - b_j) \\ &= \sum_{i=1}^{n-1} \sum_{j=2}^n q'_i r_j (b_1 - b_j) \\ &\geq \sum_{i=1}^{n-1} \sum_{j=i+1}^n q'_i r_j (b_i - b_j), \end{aligned}$$

where  $q'_i = q_i / (1 - q_n)$  ( $i = 1, \dots, n - 1$ ). Since  $q'_1 + \dots + q'_{n-1} = 1$ , Lemma 2.3 yields

$$\begin{aligned} \max_{1 \leq i \leq n} a_i - \sum_{i=1}^n p_i a_i &\geq \max_{q_1, \dots, q_{n-1}} \min_{1 \leq i < j \leq n} q'_i r_j \sum_{i=1}^{n-1} \sum_{j=i+1}^n (b_i - b_j) \\ &= \frac{1}{\sum_{i=2}^n 1/p'_i} \sum_{1 \leq i < j \leq n} |a_i - a_j|. \end{aligned}$$

REMARK 2.5. Suppose that  $a_n \leq a_{n-1} \leq \dots \leq a_1$ . If  $p'_i = p_i$  for all  $i \geq 2$ , that is,  $p_2 \leq p_3 \leq \dots \leq p_n$ , then the multiplying factor in (2.3) is  $[1/p_2 + \dots + 1/p_n]^{-1}$  which can be much better than the value  $p_2/(n - 1)$  in Theorem 2.1.

COROLLARY 2.6. For  $n = 2$ , we have

$$a_1p_1 + a_2p_2 = \max_{i=1,2} a_i - p'_2|a_1 - a_2|,$$

where  $p'_2 = p_1$  if  $a_1 < a_2$  and  $p'_2 = p_2$  if  $a_1 > a_2$ . In either case, we have

$$a_1p_1 + a_2p_2 \leq \max_{i=1,2} a_i - \min(p_1, p_2)|a_1 - a_2|.$$

Therefore in the case  $n = 2$ , (2.4) and (2.3) hold with equality.

### 3. Applications

The improvements to the basic lemma given in the previous section have direct consequences for several theorems given in [17]. The proofs of the first two involve only straightforward modifications of those used in earlier improvements presented in [1] and so are omitted.

THEOREM 3.1. Let  $a, b, s_i$  and  $t_i$  ( $i = 0, 1, 2$ ) be positive real numbers with  $a/s_i + b/t_i = 1$  ( $i = 0, 1, 2$ ) and  $s_1 < s_0 < s_2$ , and let  $f, g : (0, \infty) \rightarrow \mathbf{R}$  be convex functions. Then

$$h_0 \leq \max(h_1, h_2) - \alpha|h_1 - h_2|,$$

where

$$h_i = \frac{f(s_i)}{s_i} + \frac{g(t_i)}{t_i} \quad (i = 0, 1, 2)$$

and

$$\alpha = \min\left(\frac{(s_2 - s_0)s_1}{(s_2 - s_1)s_0}, \frac{(s_0 - s_1)s_2}{(s_2 - s_1)s_0}\right) \in (0, 1).$$

Similarly, we have the following for  $L^p$  norms.

THEOREM 3.2. We adopt the assumptions and notation of the previous theorem.

(i) Suppose the quantities  $P_i = \|f\|_{s_i} \|g\|_{t_i}$  ( $i = 0, 1, 2$ ) all exist. Then

$$P_0 \leq \max(P_1, P_2) - \alpha|P_1 - P_2|.$$

(ii) Suppose  $x = (x_i)$ ,  $u = (u_i)$  ( $i = 1, \dots, n$ ) and  $y = (y_j)$ ,  $v = (v_j)$  ( $j = 1, \dots, m$ ) are sequences of positive real numbers. Then

$$Q_0 \leq \max(Q_1, Q_2) - \alpha|Q_1 - Q_2|,$$

where

$$Q_i = S_n^{[s_i]}(x, u) S_m^{[t_i]}(y, v) \quad (i = 0, 1, 2) \quad \text{and}$$

$$S_n^{[t]} = \left( \sum_{i=1}^n u_i x_i^t \right)^{1/t}.$$

The following result is an improvement of Theorem 1 from [17].

**THEOREM 3.3.** Suppose that positive numbers  $u_{i,j}$  ( $i = 0, 1, 2; j = 1, \dots, n$ ) satisfy  $u_{1,j} \geq u_{0,j} \geq u_{2,j}$  ( $1 \leq j \leq n$ ) and

$$a_j u_{i,1} + b_j u_{i,j} = 1 \quad (i = 0, 1, 2; 2 \leq j \leq n) \quad (3.1)$$

for positive constants  $a_j, b_j$  ( $2 \leq j \leq n$ ). If  $F_j : (0, \infty) \rightarrow \mathbf{R}$  ( $1 \leq j \leq n$ ) are convex functions, then

$$\sum_{j=1}^n F_j(u_{0,j}) \leq \max_{i=1,2} \left\{ \sum_{j=1}^n F_j(u_{i,j}) \right\} - \min(\lambda, 1 - \lambda) \left| \sum_{j=1}^n [F_j(u_{1,j}) - F_j(u_{2,j})] \right|,$$

where  $\lambda$  is defined by

$$\lambda := \frac{u_{2,1} - u_{0,1}}{u_{2,1} - u_{1,1}}.$$

**PROOF.** From (3.1), we conclude as in Theorem 1 of [17] that

$$\lambda = \frac{u_{0,j} - u_{2,j}}{u_{1,j} - u_{2,j}} = \frac{u_{2,1} - u_{0,1}}{u_{2,1} - u_{1,1}},$$

$$1 - \lambda = \frac{u_{1,j} - u_{0,j}}{u_{1,j} - u_{2,j}} = \frac{u_{0,1} - u_{1,1}}{u_{2,1} - u_{1,1}}.$$

Also, by the convexity of  $F_j$ ,

$$F_j(u_{0,j}) \leq \frac{u_{0,j} - u_{2,j}}{u_{1,j} - u_{2,j}} F_j(u_{1,j}) + \frac{u_{1,j} - u_{0,j}}{u_{1,j} - u_{2,j}} F_j(u_{2,j}),$$

that is,

$$F_j(u_{0,j}) \leq \lambda F_j(u_{1,j}) + (1 - \lambda) F_j(u_{2,j}).$$

After summation, we obtain by Corollary 2.6 that

$$\begin{aligned} \sum_{j=1}^n F_j(u_{0,j}) &\leq \lambda \sum_{j=1}^n F_j(u_{1,j}) + (1 - \lambda) \sum_{j=1}^n F_j(u_{2,j}) \\ &\leq \max_{i=1,2} \left\{ \sum_{j=1}^n F_j(u_{i,j}) \right\} - \min(\lambda, 1 - \lambda) \left| \sum_{j=1}^n [F_j(u_{1,j}) - F_j(u_{2,j})] \right|. \end{aligned}$$

REMARK 3.4. By Corollary 2.6, the factor  $\min(\lambda, 1 - \lambda)$  can be replaced by  $\lambda$  or  $1 - \lambda$ , depending on the value of the maximum. The same is true for the factor  $\alpha$  in Theorems 3.1 and 3.2.

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