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KNOTS WITH FREE PERIOD

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At the Georgia conference in 1961 Fox presented a paper, "Knots and periodic transformations", in which he asked which knots may be fixed by a periodic transformation of the 3-sphere. He distinguished eight cases according to the type of fixed point set of T and its relationship to the knot. Except for case a), all these cases have since received some attention and conditions have been given for knots to fall into each of these classes. In fact, the problem of deciding which knots fall into category d) (that is "periodic knots"; I will refer to them as *cyclically periodic* knots) has been the subject of at least six papers [**2**], [**3**], [**13**], [**15**], [**17**], [**22**], but measured by their effectiveness at determining the periods of the knots to nine crossings, the theorems contained in these papers are not entirely satisfactory.

Knots of type a), however, which are those knots fixed setwise by a periodic transformation, T, of the 3-sphere without fixed points have, to my knowledge, received no attention until now (although Fox observed that torus knots have infinitely many free periods). I call such knots *freely periodic* if as well as T being fixed point free, so is each power of T less than its period. The purpose of this paper is to give two necessary conditions for a knot to have a free period. These conditions turn out to be quite effective at determining the periods of the classical knots to nine crossings, the periods of only two knots being undecided. (In fact the possible periods of all knots to 10 crossings are determined here.) Knots in an arbitrary homology 3-sphere rather than just S^3 are considered, since this represents no added difficulty. The key observation is that knots with free period are algebraically very closely related to knots fixed pointwise by a periodic transformation of a homology 3-sphere. In fact both our conditions are also necessary conditions for a knot to be the fixed point set of a periodic transformation of a homology 3-sphere, and one of them was discovered by Kinoshita and Fox in considering this latter phenomenon. Thus, in fact most of this paper relates both to knots with a free period and to knots which are fixed point sets of periodic transformations.

1. The polynomial condition. A periodic transformation of period p of a space will be called *free* if T^i has no fixed points for $i = 1, 2, \ldots, p - 1$. Consider a (polygonal) knot, K, in a homology 3-sphere,

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X, and suppose that K is fixed (setwise) by a free periodic transformation, T, of X of period p. (A covering translation of the universal covering space of a lens space gives an example of such a periodic transformation.) K will be called *freely periodic*, and will be said to have *free period* p, or simply period p. Now, the orbit space of X under T is a 3-manifold, $\mathbf{X}$ and K lies over a knot, $\mathbf{K}$ in $\mathbf{X}$. The identification map, $\rho : X \rightarrow \mathbf{X}$ is a p-fold covering projection. I claim that $H_1(\mathbf{X}) \cong Z_p$. In fact, $\pi_1(X) < \pi_1(\mathbf{X})$, so

$$\pi_1(X) = [\pi_1(X), \pi_1(X)] < [\pi_1(\mathbf{X}), \pi_1(\mathbf{X})].$$

Therefore,

$$|\pi_1(\mathbf{X}) : [\pi_1(\mathbf{X}), \pi_1(\mathbf{X})]| < |\pi_1(\mathbf{X}) : \pi_1(X)| = p.$$

However, the group of covering translations is Z_p, and so $\pi_1(\mathbf{X})$ maps onto Z_p. So, $\pi_1(\mathbf{X})/[\pi_1(\mathbf{X}), \pi_1(\mathbf{X})]$ maps onto Z_p. This map must therefore be an isomorphism.

As a singular 1-cycle, the knot $\mathbf{K}$ must represent a generator of $H_1(\mathbf{X})$, otherwise K would have more than one component. By Theorem 2.1 of [**11**], $H_1(\mathbf{X} - \mathbf{K}) \cong Z$. Let $N(\mathbf{K})$ be a regular neighbourhood of $\mathbf{K}$ and put $\mathbf{Y} = \mathrm{cl}(\mathbf{X} - N(\mathbf{K}))$. Then $H_1(\mathbf{Y}) \cong H_1(\mathbf{X} - \mathbf{K}) \cong Z$. We show that the generator of $H_1(\mathbf{Y})$ may be represented by a cycle in $\partial N(\mathbf{K})$. Let $\mathbf{m}$ be the boundary of a meridianal disc of $N(\mathbf{K})$. Let α generate $H_1(\mathbf{Y})$ and suppose that as an element of $H_1(\mathbf{X})$, $\alpha \sim j\mathbf{K}$. Let $\mathbf{l}$ be a curve on $\partial N(\mathbf{K})$ with $\mathbf{l} \sim \mathbf{K}$ in $N(\mathbf{K})$. Then, $\alpha - j\mathbf{K} \sim 0$ in $H_1(\mathbf{X})$. Thus, $\alpha - j\mathbf{l} \sim k\mathbf{m}$, and $\alpha \sim j\mathbf{l} + k\mathbf{m}$ in $H_1(\mathbf{Y})$. But j and k must be coprime, and so $j\mathbf{l} + k\mathbf{m}$ is represented by a curve in $\partial N(\mathbf{K})$. Now, let V be a solid torus with core, $\mathbf{K}'$. Let $\mathbf{X}'$ be the manifold obtained from the disjoint union $V \oplus \mathbf{Y}$ by sewing a meridian, $\mathbf{m}'$ of V to a curve $j\mathbf{l} + k\mathbf{m}$ generating $H_1(\mathbf{Y})$. Let $h : V \oplus \mathbf{Y} \rightarrow \mathbf{X}'$ be the identification map, call $h(\mathbf{K}')$ simply $\mathbf{K}'$, and let $V = N(\mathbf{K}')$. Then, $\mathbf{X}'$ is a homology 3-sphere and $\mathbf{X}' - N(\mathbf{K}')$ is homeomorphic to $\mathbf{X} - N(\mathbf{K})$. Let K' be the p-fold cyclic branched covering space of $\mathbf{X}'$ branched over $\mathbf{K}'$, let K' cover $\mathbf{K}'$ and $N(K')$ cover $N(\mathbf{K}')$. Then

$$X' - N(K') \cong X - N(K).$$

Therefore,

$$H_1(X' - N(K')) \cong H_1(X - N(K)) \cong Z.$$

Since X' is a cyclic branched covering space,

$$H_1(X') \oplus Z \cong H_1(X' - N(K')) \cong Z.$$

(See, for instance Example 3 of [**11**]), and so $H_1(X') \cong 0$. Since X' is a cyclic branched covering of $\mathbf{X}'$, the covering translations are periodic and fix the covering knot pointwise. Thus:

THEOREM 1.1. *If a knot K in a homology 3-sphere, X, is fixed setwise by a free transformation of X of period p, then there exists a knot, K', in a homology 3-sphere X' such that $X - K$ is homeomorphic to $X' - K'$, and K' is the set of fixed points of some periodic transformation of X' of period p.*

In fact, X' is the p-fold branched covering space of a homology 3-sphere, $\mathbf{X}'$, branched over a knot, $\mathbf{K}'$, and K' is the knot covering $\mathbf{K}'$.

Thus, if I is some knot invariant of a knot, K, in a homology 3-sphere, X, which depends only on the knot complement, then any condition on I necessary for K to be the fixed point set of a periodic transformation of X of period p is also a necessary condition for K to be fixed setwise by a free transformation of period p. Such invariants include (among many others) the Alexander polynomial and the homology groups of the cyclic branched coverings, both of which depend only on the knot group.

Now, the class of Alexander polynomials of knots in the 3-sphere is the same as the class of Alexander polynomials of knots in homology 3-spheres. (See [**1**].) Thus, we may adapt (with very slight modifications) the results of Kinoshita [**12**] and we have

THEOREM 1.2. *If K is a knot in a homology sphere, X, and K is fixed (setwise) by a free periodic transformation of X of period p, then*

$$(1.3) \quad \Delta_K(t^p) = \prod_{j=0}^{p-1} f(\xi^j t),$$

where ξ is a primitive p^{th} root of unity and f is a knot polynomial.

One can always test this condition for given Δ_K and p by factoring $\Delta_K(t^p)$ over the integers. It must have a factor of degree equal to the degree of Δ_K, namely f.

As pointed out by Fox [**6**], this condition has certain consequences:

(1.4) $\Delta_K(t)$ and $f(t)$ have the same degree.
(1.5) The leading coefficient of $\Delta_K(t)$ is a p^{th} power.
(1.6) The roots of Δ_K are the p^{th} powers of the roots of f.

Condition (1.5) rules out many knot types for any value of p. Fox drew several results from these conditions. His condition that Δ_K can not be a reducible quadratic can be greatly strengthened as follows:

PROPOSITION 1.7. *If a knot K has a quadratic Alexander polynomial, other than $1 - t + t^2$, then it does not have any free period.*

Proof. We show first that no cyclic covering space of any knot with a quadratic Alexander polynomial other than $1 - t + t^2$ can be a homology sphere. Since the order of the cyclic covering homology groups depend only on the Alexander polynomial, we may assume that $\mathbf{K}$ is a genus one knot in the 3-sphere with Seifert matrix $V = \begin{pmatrix} \gamma & \gamma \\ \gamma - 1 & \gamma \end{pmatrix}$

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having polynomial $f(t) = \gamma + (1 - 2\gamma)t + \gamma t^2$ with $\gamma \neq 0, 1$. If $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\Gamma = VJ$, then $F_q = \Gamma^q - (\Gamma - I)^q$ is a relation matrix for $H_1(M_q)$, where M_q is the q-fold branched covering. It is enough to show that $|H_1(M_q)| \neq 1$ for q a prime, since if q divides h, then $|H_1(M_q)|$ divides $|H_1(M_h)|$. Now certainly $H_1(M_2)$ can not be trivial, so we assume that q is an odd prime.

Plans [**18**] showed that for odd q, F_q is then of the form

$$\begin{pmatrix} n_q & 0 \\ 0 & n_q \end{pmatrix}$$

where $n_q = (p_q - 2\gamma p_{q-1})$ and p_q is defined recursively by

$$p_0 = 0, p_1 = 1, p_{q+1} = p_q - p_{q-1}\gamma.$$

One easily sees that $n_q(\gamma)$ is then a polynomial in γ of the form $1 + a_1\gamma + a_2\gamma^2 + \ldots + a_n\gamma^n$ where $n = (q - 1)/2$. Now expanding $\Gamma^q - (\Gamma - I)^q$, one sees that $F_q \equiv I$ (modulo q). Thus, for all γ, $n_q(\gamma) \equiv 1$ (modulo q). In particular, for $\gamma = 1, 2, \ldots, n$, we have

$$a_1 + a_2\gamma + \ldots + a_n\gamma^{n-1} = 0 \quad \text{in } Z_q.$$

This gives a set of equations

$$\begin{pmatrix} 1 & 1 & \cdots\cdots & 1 \\ 1 & 2 & 2^2 \quad \cdots & 2^{n-1} \\ & & & \\ & & & \\ 1 & n & \cdots\cdots & n^{n-1} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$$

written symbolically, $A\mathbf{a} = \mathbf{0}$. Since

$$\det A = \prod_{1 \leqq i < j \leqq n} (j - i) \neq 0 \quad \text{in } Z_q,$$

the only solution is $a_i = 0$ in Z_q. Therefore,

$$n_q(\gamma) = 1 + q\gamma(a_1' + a_2'\gamma + \ldots + a_n'\gamma^{n-1}).$$

However, looking more closely at the form of the polynomial $r(\gamma) = a_1' + a_2'\gamma + \ldots + a_n'\gamma^{n-1}$, we see that $a_1' = -1$ and $a_n' = \pm 1$. Furthermore, it is an alternating polynomial (that is, the signs of the a_i' alternate). In order for $n_q(\gamma)$ to equal ± 1, $\gamma = 0$ or γ must be a root of r. However, the only possible integer root is $\gamma = 1$. Thus F_q can not be a relation matrix for the trivial group unless $\gamma = 1$ or $\gamma = 0$ and then either $f(t) = 1 - t + t^2$ or $f(t) = 1$. Now if $f(t) = 1 - t + t^2$ then from the periodicity of

$H_1(M_q)$ (see [**9**]), $|H_1(M_q)| = 1$ if and only if $q = \pm 1$ (modulo 6). Then one verifies that

$$\Delta_k(t^q) = \prod_{i=1}^{q} f(\xi^i t) = 1 - t^q + t^{2q}$$

(see also (4.4)). This completes the proof.

2. The homology group of the 2-fold covering space. We consider now the consequences for the structure of the homology group of the 2-fold covering space of a knot in a homology 3-sphere of the assumption that K has a free period (or equally well, of the assumption that K is the fixed point set of a periodic transformation).

LEMMA 2.1. *If $H_1(M_i)$ represents the homology group of the i-fold cyclic covering space of a knot in a homology 3-sphere, there is a homomorphism of $H_1(M_{2n})$ onto $H_1(M_2)$ with kernel a direct double.*

Proof. A. Plans [**18**] considered the structure of the homology groups of covering spaces of knots in the 3-sphere. For a good summary of his results, see the review by Fox [**5**], or [**7**]. Plans used a relation matrix for these homology groups derived from a Seifert matrix of the knot. The results hold equally well for the homology groups of covering spaces of knots in a homology sphere. In fact, one can adapt the appropriate proofs on page 127 and pages 201–215 of [**20**] to obtain the connexion between the Seifert matrix and the covering space homology. Then, Plans's arguments apply unchanged.

In particular, Plans proved that $F_{2g} = F_2.W$ where F_i represents a certain $2n \times 2n$ relation matrix for $H_1(M_i)$ and

$$W = \sum \begin{pmatrix} 0 & a_j \\ -a_j & 0 \end{pmatrix}.$$

Here, n is the genus of the knot. Denoting by $W'\colon A_{2n} \to B_{2n}$ and $F_2'\colon B_{2n} \to C_{2n}$ the homomorphisms represented by W and F_2 respectively (here, A_{2n}, B_{2n} and C_{2n} are free abelian groups of rank $2n$), we have

$$C_{2n}/\mathrm{Im}(F_2') \cong (C_{2n}/\mathrm{Im}(F_2'.W'))/(\mathrm{Im}(F_2')/\mathrm{Im}(F_2'.W')).$$

However,

$$C_{2n}/\mathrm{Im}(F_2') \cong H_1(M_2) \text{ and } C_{2n}/\mathrm{Im}(F_2'.W') \cong H_1(M_{2g}).$$

Further, F_2 has rank $2n$ (since $H_1(M_2)$ is finite), so

$$\mathrm{Im}(F_2')/\mathrm{Im}(F_2'.W') \cong B_{2n}/\mathrm{Im}(W'),$$

which is a direct double because of the form of W.

THEOREM 2.2. *Let K be a knot in a homology sphere, X, and T a transformation of period p which either fixes K pointwise, or else which is free*

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and fixes K setwise. Then there exists some knot polynomial, f, such that

$$\Delta_K(t^p) = \prod_{i=1}^{p} f(\xi^i t)$$

and there is a homomorphism of $H_1(M_2)$ onto some group, E, of order $|f(-1)|$ and the kernel is a direct double. Hence, $H_1(M_2)$ is not cyclic unless for some such f, $|f(-1)| = |\Delta(-1)|$. In particular, if $p = 2$, then $H_1(M_2)$ is always a direct double.

Proof. We consider the case where K is fixed pointwise. The other case is then covered by Theorem 1.1. The 2-fold cyclic branched covering, M_2, of X branched along K is a $2p$-fold branched covering of the orbit space, $\mathbf{X}$, branched along the factor knot, $\mathbf{K}$. Then, f is the knot polynomial of $\mathbf{K}$ in the homology sphere, $\mathbf{X}$. We put $E = H_1(\mathbf{M}_2)$, where $\mathbf{M}_2$ is the 2-fold covering space of $\mathbf{X}$ branched over $\mathbf{K}$. Then $|f(-1)| = |E|$ and $|\Delta(-1)| = |H_1(M_2)|$. If $p = 2$, then $\mathbf{M}_2 = X$, and so $|f(-1)| = 1$. The theorem now follows from Lemma 2.1.

3. Examples of freely periodic knots. We begin by considering torus knots.

THEOREM 3.1. *The torus knot of type (r, s) has free period p if and only if p is coprime with rs.*

Proof. Fox's Corollary 1 in [**6**] along with our Theorem 1.1 show this condition to be necessary. Fox apparently also observed that this was sufficient [**8**]. To prove it, consider the 3-sphere as a Seifert fibre space over the 2-sphere with two exceptional fibres of types (r, s) and (s, r). A non-exceptional fibre is a torus knot of type (r, s) which is fixed setwise by the periodic transformation, T, of period p which moves each point on an exceptional fibre along that fibre a distance $1/p$ times its length. (This mapping may be extended continuously to the exceptional fibres.) Now T^i; $i = 1, \ldots, p$ fixes no point on any non-exceptional fibre, and the condition that p and rs are coprime ensures that there is no fixed point on either exceptional fibre. (This sort of transformation is described in [**21**], Section 14.)

Thus there exist fibred knots with infinitely many free periods in contrast to the situation with cyclic periods where a fibred knot has only finitely many cyclic periods [**15**]. However, it seems likely that torus knots are the only knots with this property.

In fact, if a knot has infinitely many free periods then it results from (1.6) that the roots of its Alexander polynomial must all be roots of unity. This was pointed out to me by J. A. Hillman, and it will become clear in the next section.

Fox asked (question 7 of [**8**]) which knots may be fixed by periodic transformations of all periods. Here is a partial answer. If a knot has infinitely many prime periods, then it must have either infinitely many prime cyclic periods (in which case by Corollary 4 of [**15**], $\Delta_K = 1$) or infinitely many prime free periods or it is the fixed point set of periodic transformations of arbitrarily large period (in both of which latter cases, the roots of its Alexander polynomial are roots of unity).

Regarding cable knots, Fox showed in [**6**] (given our Theorem 1.1) that a 2 strand cable knot can not have free period 2. In fact, this can be improved using Theorem 3.1 of [**10**] to the following statement: If k is a (p, q) cable knot with carrier K, then k has free period 2 if and only if K has free period 2 and pq is odd.

We now construct further examples of freely periodic knots and for this we need a generalisation of Conway's definition of a tangle [**4**]. An *n-tangle* is a portion of a knot diagram enclosed in a square with n strings emerging from the left edge and n strings emerging from the right edge of the square. Two tangles may be multiplied by placing them side by side and joining up the adjacent strings. (This is called addition by Conway.) Given a tangle, t, the tangle obtained by flipping it over top to bottom is called t_h (see Conway). The n-braids are a special sort of n-tangle and the multiplication of tangles corresponds to the usual braid multiplication. We will therefore use the usual braid notation. In particular, $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ are n-tangles which generate the braid group, and the centre of the n-braid group is generated by an element, Δ^2, where

$$\Delta = s_1 s_2 \ldots s_{n-1} \quad \text{and} \quad s_i = \sigma_1 \sigma_2 \ldots \sigma_{n-i} \quad ([\mathbf{16}]).$$

Let a tangle, t, be called *symmetric* if it is identical with t_h (not just isotopic to t_h). As with braids, one obtains a link from an n-tangle by joining the right hand strings to the left hand strings with n disjoint arcs. The resultant link will be called the *closure* of the n-tangle. Claude Bourrain has studied n-tangles in some detail. Their relevance here is in the following result.

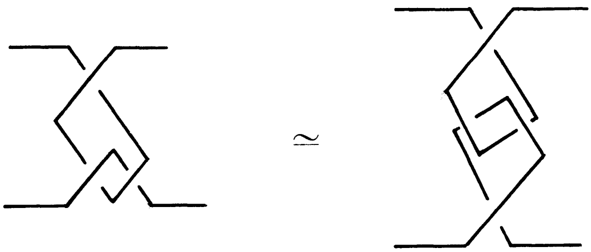
PROPOSITION 3.2. *If t is any n-tangle and $m = 2n$, then the closure of $\Delta^m t^p$ is a link with free period p if $(n, p) = 1$.*

If t is any symmetric n-tangle and m is any integer, then the closure of $\Delta^m t^p$ has free period p if $(2m, p) = 1$.

Proof. One way to visualise this is to draw the tangle t^p on a strip of paper and then to join the two ends of the strip together with m half twists to form a twisted annulus or Möbius band, A. The resulting link is the closure of $\Delta^m t^p$. The 3-sphere may be Seifert fibred in such a way that A is a union of fibres and the centre line of A is an exceptional fibre of type $(2, m)$ if m is odd and of type $(1, n)$ if $m = 2n$ is even. Then the transformation previously described taking each fibre $1/p^{\text{th}}$ of a revolution along its length takes the link onto itself and is free under the stated conditions.

Since any tame periodic transformation of S^3 with fixed point set an unknotted circle is a revolution [**14**] it is easy to see that a knot with cyclic period, p, must be the closure of some n-tangle t^p. Any sort of similar statement about knots with free periods must presumably wait until free periodic transformations have been classified entirely.

Now the knot 10_{155} is the closed 3-braid $\Delta^2\sigma_1{}^{-3}\sigma_2\sigma_1{}^{-3}\sigma_2$ and 10_{157} is the closure of $\Delta^2(\sigma_1{}^{-1}\sigma_2)^4$. So 10_{155} has period 2 and 10_{157} has periods 2 and 4. Furthermore, Conway's notation for knots 9_{48} and 10_{75} is 21, 21, 21− and 21, 21, 21+ respectively. In our notation this is $t^3\Delta^{-1}$ and $t^3\Delta$ where t represents the tangle 210 which has a symmetric form (see diagram). So 9_{48} and 10_{75} both have period 3.



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Conway points out that any rational tangle, t, is equal to t_h. Presumably, more is true, namely that any rational tangle has a symmetric form. This would allow us to write down in Conway's notation any number of periodic knots. For instance $a, a, \ldots, a+$ and $a, a, \ldots, a-$ (p a's) have free period p where a is any rational tangle, and p is odd.

4. Free periods of knots with ten crossings or less. We now set about finding the various possible free periods of the tabled knots to 10 crossings. First of all we obtain a bound for the possible periods of a knot. For a polynomial f, define

$$\text{max mod } (f) = \max (|r_i|)$$

as r_i runs over all roots of f. I must thank J. A. Hillman for pointing out the following useful lemma.

LEMMA 4.1. *For each n there exists a number $A(n)$ such that if f is an integral monic polynomial of degree n, then either all roots of f are roots of unity, or* max mod $(f) \geqq A(n)$.

The proof of this is a minor adaptation of the proof of Lemma 11.5 of [**19**]. Now let $A'(n) = \min_{f \in S}(\max\operatorname{mod}(f))$ where S is the set of all knot polynomials of degree n which have constant term ± 1, and whose roots are not all roots of unity. The constant term of a knot of period p must be a p^{th} power. This provides a bound for the possible periods if this constant term is not ± 1. Otherwise, we can obtain a bound as follows.

COROLLARY TO LEMMA 4.1. *Let Δ_K be a knot polynomial of degree n and with constant term ± 1, and suppose Δ_K has a root which is not a root of unity. If a is the largest integer such that $A'(n)^a \leqq \max\operatorname{mod}(\Delta_K)$, then K has no free period greater than a.*

Proof. If K has free period $b > a$, then the corresponding f in (1.3) has roots which are b^{th} roots of those of Δ_K. Hence

$$\max\operatorname{mod}(f) = \max\operatorname{mod}(\Delta_K)^{1/b} < A'(n).$$

Thus, all roots of f are roots of unity, and so are the roots of Δ_K.

Thus, to find a bound for the periods of the knots in the tables to 10 crossings, one must find the values of $A'(n)$ for $n = 4, 6$ and 8. For given A and n, the proof of Lemma 4.1 places bounds on the coefficients of monic polynomials, f, of degree n satisfying $\max\operatorname{mod}(f) < A$. Thus a computer search for $A'(n)$ is possible. It was necessary to take advantage of the properties of knot polynomials in order to refine these bounds for the computation to be feasible. The following values were found:

$$A'(4) = 1 \cdot 618033989, A'(6) = 1 \cdot 321663152, \qquad A'(8) = 1 \cdot 169283030.$$

Now, using the corollary to Lemma 4.1 it was found that no knot of ten crossings or less has period greater than 5, unless all of the roots of its Alexander polynomial are roots of unity. Given this, it is a simple task to find those polynomials, Δ_K, which satisfy the polynomial condition. The formulae of Kinoshita ([**12**], pp. 49, 50) were useful here. The following knots have possible free periods: For $p = 2$: 10_{123}, 10_{155}, 10_{157}; $p = 3$: 9_{48}, 10_{75}; $p = 4$: 10_{157}. Apart from these, the following knots satisfied the polynomial condition, but were eliminated on the grounds of Theorem 2.2: $p = 2$: 8_9, 10_{137}; $p = 3$: 9_{27}, 10_{143}. This list does not include those knots the roots of whose polynomials are roots of unity. These we treat next.

If the roots of a polynomial are roots of unity, then it is a product of cyclotomic polynomials. Denote the d^{th} cyclotomic polynomial by Φ_d and its degree by ϕ_d. The reader is invited to verify the following properties.

(4.2) $\Phi_d(1) = \pm 1$ unless d is a prime power, q^α, in which case, $\Phi_d(1) = q$.

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(4.3) $\Phi_d(-1) = \pm 1$ unless $d = 2q^\alpha$ where q is a prime, in which case, $\Phi_d(-1) = q$.

(4.4) $\prod_{i=1}^{p} \Phi_d(\xi^i t) = (\Phi_n(t^p))^m$ where ξ is a primitive p^{th} root of unity, $n = d/(p, d)$ and $m = \phi(d)/\phi(n)$.

PROPOSITION 4.5. *Consider a knot polynomial $\Delta_K = \Phi_{a_1}{}^{m_1} \ldots \Phi_{a_r}{}^{m_r}$. By (4.2), a_i can not be a prime power. If the knot has free period p, then for each $i = 1, \ldots, r$ there are integers b_i and p_i such that $b_i p_i = p$, where $(a_i, p_i) = 1$ and b_i divides a_i.*

This is equivalent to the condition of Fox [**6**], and is the best one can do with the polynomial condition, for this condition is satisfied by the polynomial

$$f(t) = \prod_{i=1}^{r} \Phi_{a_i}(t^{b_i})^{m_i/b_i}.$$

However, a better result can be expected using Theorem 2.2. If we compare $\Delta(-1)$ with $f(-1)$ as just defined, then $|f(-1)| < |\Delta(-1)|$ unless for all i, $|\Phi_{a_i}(-1)| = 1$, or $b_i = 1$. In view of (4.3) we are led to the additional condition:

PROPOSITION 4.6. *If, further to the conditions of Proposition 4.5, $H_1(M_2)$ is cyclic, then $b_i = 1$ for all i such that $a_i = 2.q^\alpha$. That is, $(p, a_i) = 1$.*

The above considerations do not suffice to prove this proposition, since the f mentioned above may not be the only f satisfying the condition (1.3). What is true, however, is that any such f must be a product of cyclotomic polynomials. So, suppose that a_i is of the form $2.q^\alpha$ where q is a prime other than 2. There must exist d such that (4.4) is satisfied with $n = a_i = 2q^\alpha$ and $m \leqq m_i$. Now, n divides d, so it follows that $\Phi_d(-1) \leqq \Phi_n(-1) = q$. Thus, $|f(-1)| < |\Delta(-1)|$ unless $m = 1$. However, in that case, $\phi(d) = \phi(n) = q^{\alpha-1}(q-1)$, and so $d = 2.q^\alpha = n$. Finally, $(p, a_i) = (p, n) = (p, d) = 1$.

We now give a complete list of possible free periods of the knots to 10 crossings. The following knots definitely have the given periods. Those with an infinite number of periods are torus knots.

3_1	$(p, 6) = 1$	9_{48}	$p = 3$
5_1	$(p, 10) = 1$	10_{75}	$p = 3$
7_1	$(p, 14) = 1$	10_{124}	$(p, 15) = 1$
8_{19}	$(p, 12) = 1$	10_{155}	$p = 2$
9_1	$(p, 18) = 1$	10_{157}	$p = 2, p = 4$

The following knots may have the given periods on the evidence of our theorems.

8_{10} $(p, 6) = 1$	10_{123} $p = 2$
8_{20} $(p, 6) = 1$	10_{132} $(p, 10) = 1$
10_{62} $(p, 30) = 1$	10_{140} $(p, 6) = 1$
10_{99} $p = \alpha . r$ where	10_{143} $(p, 6) = 1$
$(r, 6) = 1$ and $\alpha = 1, 2$ or 4	

Observe that Kinoshita [**12**] made a slight error in stating his Theorems 4 and 6. On the evidence of the polynomial condition the following knots with up to 9 crossings may be the fixed point set of a periodic transformation of period three: 5_1, 7_1, 8_{10}, 9_{27}, 9_{48}. He mixed up the knots 9_{48} and 9_{47} (no doubt because their Alexander polynomials are interchanged in the Reidemeister table) and omitted the knot 9_{27}.

Although the Smith conjecture has apparently been proved by Thurston, Meeks, Yau and others, it is perhaps of some slight interest to remark on purely algebraic grounds that the knots listed above are the only ones of ten crossings or less which may be the fixed point set of a periodic transformation of S^3.

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