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THE GENERALIZED SHANNON SYSTEM IN WAVELET SPACE

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Abstract

The Shannon system is generalized and the expansion of a function in the generalized Shannon system is considered. No study of a wavelet expansion exists without the assumption of 'fast' decay of wavelets. The wavelet ψ which is associated with the generalized Shannon system has a 'slow' decay. The expansion of a function in the system is shown to converge at a point which satisfies the Lipschitz condition of order $\alpha > 0$. On the other hand, there is a continuous function whose wavelet expansion in the generalized Shannon system diverges. An observation of Gibbs' phenomenon is also given.

1. Introduction

The Shannon system is a prototype of wavelet systems. In the theory of signal and image processing, digital communications *etc.*, the Shannon function plays an important role, since it enables one to recover an analog signal from its sampled values at a discrete set for a band limited signal [12]. The Shannon system is a multiresolution analysis which is associated with the scaling function φ whose Fourier transform is the characteristic function of the interval $[-\pi, \pi]$, that is, $\widehat{\varphi}(w) = \chi_{[-\pi,\pi]}(w)$ [9, 15].

It is of interest to consider the scaling functions φ with $\widehat{\varphi}(w) = \chi_M(w)$, the characteristic function of the set M. Such scaling functions are called unimodular scaling functions and are studied in [6]. The multiresolution analysis which is generated by such a scaling function φ will here be called a *generalized Shannon system*. For example, if

$$M = [\alpha - 2\pi, -\pi] \cup [-\alpha, \alpha] \cup [\pi, 2\pi - \alpha],$$

where $2\pi/3 \leq \alpha < \pi$, then the function φ with $\widehat{\varphi}(w) = \chi_M(w)$ is a scaling function and it generates a generalized Shannon system.

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A multiresolution analysis $\{V_j\}$ is translation invariant if each space V_j is invariant under translation by arbitrary numbers. In [7], it is proved that all such multiresolution analyses have scaling functions whose Fourier transforms are characteristic functions. So the generalized Shannon system is the only translation invariant multiresolution analysis.

In this paper, the expansion of a function in the generalized Shannon system is considered. There are many results for the pointwise convergence of wavelet expansions [5, 13, 14]. But 'fast' decay conditions are usually assumed in these previous works. It is noted that the scaling function φ and the wavelet ψ in the generalized Shannon system have a 'slow' decay in the time domain. That is, $\varphi(t) = O(|t|)$, $\psi(t) = O(|t|)$ as $|t| \to \infty$.

In Section 2, some examples are given of the symmetric set M which gives scaling function φ with $\widehat{\varphi}(w) = \chi_M(w)$. The projection and its kernel in generalized Shannon systems is observed in Section 3. In Section 4, the kernel function is shown to generate a delta sequence. In Section 5, for a piecewise continuous function with a jump discontinuity which satisfies the left and right side Lipschitz condition of order $\alpha > 0$, the behavior of a wavelet expansion in generalized Shannon systems is studied. The pointwise convergence at a point which satisfies the Lipschitz condition of order $\alpha > 0$ follows. In contrast to Walter's work in [14], it is shown that there is a continuous function whose wavelet expansion in generalized Shannon systems diverge. Finally it is shown in Section 6 that a Gibbs phenomenon occurs in the wavelet expansion of a function with a jump discontinuity in generalized Shannon systems.

2. Some examples

In [6], it is known that if M is a finite disjoint union of closed intervals and it satisfies

(D1) $\bigcup_{k\in Z}(M + 2k\pi) = R$ (disjoint union except for end points),

(D2) $M \subset 2M$,

(D3) M contains a neighborhood of zero,

then the function φ with $\widehat{\varphi}(w) = \chi_M(w)$ is a scaling function for a generalized Shannon system. Some examples of such intervals M are given in [6].

We immediately see that the scaling function φ with $\widehat{\varphi}(w) = \chi_M(w)$ is real-valued if and only if M is symmetric. We now give some examples of such a symmetric M.

(a) The case where M consists of one interval: in [6], it is known that $M = [\alpha, \alpha + 2\pi]$ where $-2\pi < \alpha < 0$ if M consists of one interval. So the symmetric M is $[-\pi, \pi]$. It generates the well-known Shannon system.

(b) The case where M consists of three intervals: we see from [6] that M is of the form

$$M = [-\gamma, -\beta] \cup [-\alpha, 2\pi - \gamma] \cup [2\pi - \beta, 2\pi - \alpha],$$

where $0 < \alpha < \beta < \gamma \leq 2\alpha < 4\pi$. If M is symmetric about the origin then $2\pi = \alpha + \gamma$ and $\beta = \pi$. Hence it should be of the form

$$M = [\alpha - 2\pi, -\pi] \cup [-\alpha, \alpha] \cup [\pi, 2\pi - \alpha],$$

where $2\pi/3 \leq \alpha < \pi$.

(c) The case where M consists of five intervals: since M is symmetric and satisfies the condition (D3), M is of the form

$$M = [-\eta, -\delta] \cup [-\gamma, -\beta] \cup [-\alpha, \alpha] \cup [\beta, \gamma] \cup [\delta, \eta].$$

From the condition (D1), the intervals should match nicely:

$$\begin{aligned} [-\eta, -\delta] + 2\pi &= [\alpha, \beta], \\ [\delta, \eta] - 2\pi &= [-\beta, -\alpha], \\ [-\gamma, \gamma] &= [-\pi, \pi]. \end{aligned}$$

We leave the routine and tedious calculations as seen in [6] to the interested reader. Thus M should be of the form

$$M = [\alpha - 2\pi, \beta - 2\pi] \cup [-\pi, -\beta] \cup [-\alpha, \alpha] \cup [\beta, \pi] \cup [2\pi - \beta, 2\pi - \alpha],$$

where $\pi/2 \leq \alpha < \beta \leq 2\pi/3$. The restrictions on α and β follow from the condition (D2).

REMARK. It is seen in [6] that the associated wavelet in the generalized Shannon system is the unimodular wavelet and the support of its Fourier transform is given by $2M - M$.

3. Wavelet expansion in the generalized Shannon system

Let φ be a scaling function with $\widehat{\varphi}(w) = \chi_M(w)$. The generalized Shannon system is the multiresolution analysis V_j, where $\{\varphi_{j,k}(t) = 2^{j/2}\varphi(2^j t - k) : k \in Z\}$ is an orthonormal basis. The wavelet $\psi \in L^2(R)$ based on φ gives rise to the orthonormal basis $\{\psi_{j,k}\}$ of $L^2(R)$, where $\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)$, $j, k \in Z$. For each j, $f \in L^2(R)$ has the representation $f(t) = f_j(t) + r_j(t)$, where

$$f_j(t) = \sum_{k \in Z} a_{j,k}\varphi_{j,k}(t)$$

and

$$r_j(t) = \sum_{n=j}^{\infty} \sum_{k \in Z} c_{n,k} \psi_{n,k}(t).$$

The convergence is in the sense of L^2. The function $f_j \in V_j$ is in fact the orthogonal projection of f onto V_j [1, 2, 8, 9].

Suppose f is a compactly supported continuous function of bounded variation. Then we know that the following "Poisson Summation Formula" holds [1, 4]:

$$\sum_{k \in Z} f(t + 2k\pi) = \frac{1}{2\pi} \sum_{k \in Z} \left(\int_{-\infty}^{\infty} f(\eta) e^{ik\eta}\, d\eta \right) e^{-ikt}, \qquad \forall t \in R.$$

The orthogonal projection f_j of a function f in the dense subspace $C_0^\infty(R)$ of $L^2(R)$ onto V_j in the generalized Shannon system can be written as

$$f_j(t) = \sum_{k \in Z} \langle f, \varphi_{jk} \rangle \varphi_{jk}(t) \qquad \forall t \in R.$$

Since

$$\widehat{\varphi}_{jk}(w) = 2^{-j/2} \widehat{\varphi}(2^{-j} w) e^{-i(2^{-j}k)w},$$

we have

$$\begin{aligned}
\widehat{f_j}(w) &= \frac{1}{2\pi} \sum_{k \in Z} \langle \widehat{f}, \widehat{\varphi}_{jk} \rangle \widehat{\varphi}_{jk}(w) \\
&= \frac{1}{2\pi} \sum_{k \in Z} 2^{-j} \left(\int_{-\infty}^{\infty} \widehat{f}(\zeta) \overline{\widehat{\varphi}(2^{-j}\zeta)} e^{i(2^{-j}k)\zeta}\, d\zeta \right) e^{-i(2^{-j}k)w} \widehat{\varphi}(2^{-j} w) \\
&= \frac{1}{2\pi} \sum_{k \in Z} \left(\int_{-\infty}^{\infty} \widehat{f}(2^j \eta) \overline{\widehat{\varphi}(\eta)} e^{ik\eta}\, d\eta \right) e^{-i(2^{-j}k)w} \widehat{\varphi}(2^{-j} w) \\
&= \sum_{k \in Z} \widehat{f}(2^j(2^{-j} w + 2k\pi)) \overline{\widehat{\varphi}(2^{-j} w + 2k\pi)} \widehat{\varphi}(2^{-j} w) \quad \text{(by P. S. F)} \\
&= \sum_{k \in Z} \widehat{f}(w + 2^{j+1} k\pi) \overline{\widehat{\varphi}(2^{-j} w + 2k\pi)} \widehat{\varphi}(2^{-j} w) \\
&= \widehat{f}(w) |\widehat{\varphi}(2^{-j} w)|^2 = \widehat{f}(w) \chi_{2^j M}(w).
\end{aligned}$$

We applied the Poisson summation formula to the function $\widehat{f}(2^j \eta) \overline{\widehat{\varphi}(\eta)}$, which is a piecewise smooth function, so the above equalities hold except for the finitely many end points of M. Since $\{\varphi(t-k)\}_{k \in Z}$ is an orthonormal basis for V_0, $\sum |\widehat{\varphi}(w + 2\pi k)|^2 = 1$. So the infinite sum of the above reduces to only one term, since $\widehat{\varphi}$ is a characteristic function.

By the above observation and the Fourier inversion formula, we now have a wavelet expansion of a function $f \in C_0^\infty(R)$ in the generalized Shannon system

$$\begin{aligned} f_j(t) &= \frac{1}{2\pi}\int_{2^j M} \widehat{f}(w)e^{iwt}\,dw \\ &= \int_{-\infty}^{\infty} f(\tau)\frac{1}{2\pi}\int_{-\infty}^{\infty} \chi_{2^j M}(w)e^{iw(t-\tau)}\,dw\,d\tau. \end{aligned}$$

Since the projection operator is continuous, the wavelet expansion in the dense subspace $C_0^\infty(R)$ can be extended to the space $L^2(R)$. The projection kernel onto the subspace V_j is

$$\delta_j(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi_{2^j M}(w)e^{iwt}\,dw.$$

It converges to the Dirac delta distribution $\delta(t)$ as $j \to \infty$ in the sense of tempered distributions, since $2^j M \nearrow R$ as $j \to \infty$. So it is a delta sequence.

The representation of the wavelet expansion in the generalized Shannon system is given by

$$f_j(t) = \int_{-\infty}^{\infty} f(\tau)\delta_j(t-\tau)\,d\tau = \int_{-\infty}^{\infty} f(t-\tau)\delta_j(\tau)\,d\tau \qquad \forall f \in L^2(R).$$

That is,

$$f_j(t) = f(t) * \delta_j(t) \quad \text{or} \quad \widehat{f_j}(w) = \widehat{f}(w)\chi_{2^j M}(w). \tag{3.1}$$

4. The delta sequences

In this section, some examples and properties of the delta sequence in the generalized Shannon system will be given. The delta sequence δ_j in the generalized Shannon system is a real-valued function if and only if M is symmetric. The delta sequence δ_j corresponding to the symmetric M given in Section 2 is real-valued and the explicit form for δ_0 can be calculated.

(a) If $M = [-\pi, \pi]$, then the delta sequence δ_0 is of the form

$$\delta_0(t) = \frac{1}{2\pi}\int_M e^{iwt}dw = \frac{\sin \pi t}{\pi t}.$$

Thus is the well-known Shannon function and generates the Shannon system.

(b) If $M = [\alpha - 2\pi, -\pi] \cup [-\alpha, \alpha] \cup [\pi, 2\pi - \alpha]$, where $2\pi/3 \le \alpha < \pi$, then the delta sequence δ_0 is given by

$$\delta_0(t) = \frac{\sin \pi t}{\pi t}(2\cos(\pi - \alpha)t - 1).$$

(c) If $M = [\alpha - 2\pi, \beta - 2\pi] \cup [-\pi, -\beta] \cup [-\alpha, \alpha] \cup [\beta, \pi] \cup [2\pi - \beta, 2\pi - \alpha]$ where $\pi/2 \le \alpha < \beta \le 2\pi/3$, then the delta sequence δ_0 is given by

$$\delta_0(t) = \frac{\sin \pi t}{\pi t}(2\cos(\pi - \alpha)t - 2\cos(\pi - \beta)t + 1).$$

We now summarize some properties of the delta sequence in the generalized Shannon system which will be used to analyze the behavior of the pointwise convergence of a wavelet expansion in the generalized Shannon system in the next section.

PROPOSITION 1. *The delta sequence*

$$\delta_j(t) = \frac{1}{2\pi}\int_{2^j M} e^{iwt}\, dw$$

is Riemann integrable as an improper integral and satisfies

(1) $\delta_j(t) = 2^j \delta_0(2^j t)$,
(2) $\int_{-\infty}^{\infty} \delta_j(t)dt = 1$, *for all* $j \in Z$,
(3) $\lim_{j\to\infty} \int_{|t|<1} \delta_j(t)dt = 1$,
(4) $\lim_{j\to\infty} \int_{1}^{\infty} \delta_j(t)dt = 0$,
(5) $\lim_{j\to\infty} \int_{-\infty}^{-1} \delta_j(t)dt = 0$.

PROOF. The proofs are routine. For example, to show (3), we note that

$$\int_{-1}^{1} \delta_j(t)dt = \int_{-1}^{1} \frac{1}{2\pi}\int_{2^j M} e^{iwt}\, dw\, dt$$
$$= \frac{1}{2\pi}\int_{2^j M}\int_{-1}^{1} e^{iwt}\, dt\, dw = \frac{1}{\pi}\int_{2^j M} \frac{\sin w}{w}\, dw.$$

Hence

$$\lim_{j\to\infty}\int_{-1}^{1} \delta_j(t)\, dt = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin w}{w}\, dw = 1.$$

This completes the proof.

PROPOSITION 2. *If* $f \in L^1(R)$, *then*

$$\int_{-\infty}^{\infty} f(t) t \delta_j(t)\, dt \to 0 \quad \text{as } j \to \infty.$$

PROOF. Let $M = [a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_n, b_n]$. Then by direct calculation

$$t\delta_j(t) = 2^j t\delta_0(2^j t) = 2^j t \frac{1}{2\pi}\int_M e^{iw2^j t}\, dw = \frac{1}{2\pi i}\sum_{k=1}^{n}\left[e^{i2^j b_k t} - e^{i2^j a_k t}\right].$$

As an example, let M be the Shannon scaling function, that is, $M = [-\pi, \pi]$. Then

$$t\delta_j(t) = \frac{1}{2\pi i}\left[e^{i2^j\pi t} - e^{-i2^j\pi t}\right] = \sin 2^j\pi t.$$

We must note that $2^j b_k$ and $2^j a_k$ go to either ∞ or $-\infty$ as $j \to \infty$, since $b_k \neq 0 \neq a_k$ for all $1 \leq k \leq n$. By using the Riemann-Lebesgue lemma, we have

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)t\delta_j(t)\,dt &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} f(t)\sum_{k=1}^{n}\left[e^{i2^j b_k t} - e^{i2^j a_k t}\right]dt \\ &= \frac{1}{2\pi i}\sum_{k=1}^{n}\int_{-\infty}^{\infty}\left[f(t)e^{i2^j b_k t} - f(t)e^{i2^j a_k t}\right]dt \to 0 \quad \text{as } j \to \infty.\end{aligned}$$

This completes the proof.

5. Pointwise convergence

In this section, we will consider the pointwise convergence of the wavelet expansion in the generalized Shannon system at a jump discontinuous point x_0 which satisfies the left and right side Lipschitz condition of order $\alpha > 0$. For a point which satisfies the Lipschitz condition of order $\alpha > 0$, the behaviors of the expansions of the regular wavelet, which assume certain 'fast' decay conditions, differ according to the position of the points [13]. But the behavior of the wavelet expansions in the generalized Shannon system is not affected by the position of points, since the generalized Shannon system is translation invariant [6, 7].

THEOREM 3. *Let $f \in C(R) \cap L^1(R) \cap L^2(R)$ have its Fourier transform $\widehat{f}$ in $L^1(R)$. Then f_j converges uniformly to f on R as $j \to \infty$.*

PROOF. Since $\widehat{f} \in L^1(R)$ and $\widehat{f_j} \in L^1(R)$ by (3.1), the error of the approximation is given by

$$f(t) - f_j(t) = \frac{1}{2\pi}\int\left[\widehat{f}(w) - \widehat{f_j}(w)\right]e^{itw}\,dw = \frac{1}{2\pi}\int_{R-2^jM}\widehat{f}(w)e^{itw}\,dw.$$

Hence we have

$$|f(t) - f_j(t)| \leq \frac{1}{2\pi}\int_{R-2^jM}\left|\widehat{f}(w)\right|\,dw,$$

which must converge to zero as $j \to \infty$. This completes the proof.

THEOREM 4. *Let $f \in L^1(R)$ be piecewise continuous. If f has a jump discontinuity at x_0 which satisfies the left and right side Lipschitz condition of order $\alpha > 0$, then*

$$f_j(x_0) \to (1-\Theta)f(x_0+) + \Theta f(x_0-) \quad \text{as } j \to \infty,$$

where $\Theta = \int_0^\infty \delta_0(\eta)d\eta$.

PROOF. By Proposition 1, we have that for any $j \in Z$,

$$\int_0^\infty \delta_j(\tau)\,d\tau = \int_0^\infty \delta_0(\tau)\,d\tau = \Theta \quad \text{and} \quad \int_{-\infty}^0 \delta_j(\tau)\,d\tau = \int_{-\infty}^0 \delta_0(\tau)\,d\tau = 1-\Theta.$$

The following expression is possible by Proposition 1,

$$\begin{aligned}
&[(1-\Theta)f(x_0+) + \Theta f(x_0-)] - f_j(x_0) \qquad (5.1)\\
&\quad = [(1-\Theta)f(x_0+) + \Theta f(x_0-)] - \int_{-\infty}^\infty f(x_0-\tau)\delta_j(\tau)\,d\tau\\
&\quad = \int_{-\infty}^0 [f(x_0+) - f(x_0-\tau)]\,\delta_j(\tau)\,d\tau + \int_0^\infty [f(x_0-) - f(x_0-\tau)]\,\delta_j(\tau)\,d\tau.
\end{aligned}$$

We show that the first integral of (5.1) converges to zero as $j \to \infty$. For this purpose, we write

$$\begin{aligned}
&\int_{-\infty}^0 [f(x_0+) - f(x_0-\tau)]\,\delta_j(\tau)\,d\tau\\
&\quad = \int_{-1}^0 [f(x_0+) - f(x_0-\tau)]\,\delta_j(\tau)\,d\tau + \int_{-\infty}^{-1} f(x_0+)\,\delta_j(\tau)\,d\tau\\
&\qquad - \int_{-\infty}^{-1} f(x_0-\tau)\,\delta_j(\tau)\,d\tau\\
&\quad = I_1 + I_2 + I_3.
\end{aligned}$$

We want to show that $I_i \to 0 \quad$ as $j \to \infty$ for $i = 1, 2, 3$. Indeed, we have

$$I_1 = \int_{-1}^0 \left[\frac{f(x_0+) - f(x_0-\tau)}{(-\tau)^\alpha}\right](-\tau)^{\alpha-1}(-\tau)\delta_j(\tau)\,d\tau.$$

Since f satisfies the Lipschitz condition of order $\alpha > 0$ at the right-hand side of x_0, there is a constant C_α such that

$$\left|\frac{f(x_0+) - f(x_0-\tau)}{(-\tau)^\alpha}\right| \le C_\alpha \quad (-1 < \tau < 0).$$

Since $(-\tau)^{\alpha-1}$ is an L^1-function in $[-1, 0]$, $\left[\frac{f(x_0+)-f(x_0-\tau)}{(-\tau)^\alpha}\right](-\tau)^{\alpha-1}\chi_{[-1,0]} \in L^1(R)$. By Proposition 2, we have $I_1 \to 0$ as $j \to \infty$.

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From Proposition 1, we immediately have

$$I_2 = f(x_0+) \int_1^{\infty} \delta_j(\tau)\, d\tau \to 0 \quad \text{as } j \to \infty.$$

Since $\frac{f(x_0-\tau)}{\tau}\chi_{[-\infty,-1]} \in L^1(R)$,

$$I_3 = \int_{-\infty}^{-1} f(x_0-\tau)\delta_j(\tau)\, d\tau = \int_{-\infty}^{-1} \frac{f(x_0-\tau)}{\tau}\tau\delta_j(\tau)\, d\tau$$

tends to zero as $j \to \infty$ by Proposition 2. Therefore we have seen that the first integral of (5.1) converges to zero as $j \to \infty$.

By the same reason, the second integral of (5.1) also converges to zero as $j \to \infty$, so we have the following convergence:

$$\begin{aligned} f_j(x_0) &\to f(x_0+) \int_{-\infty}^{0} \delta_0(\eta)\, d\eta + f(x_0-) \int_0^{\infty} \delta_0(\eta)\, d\eta \\ &= (1-\Theta) f(x_0+) + \Theta f(x_0-), \end{aligned}$$

where $\Theta = \int_0^{\infty} \delta_0(\eta)\, d\eta = 1 - \int_{-\infty}^{0} \delta_0(\eta)\, d\eta$. This completes the proof.

COROLLARY 5. *Suppose $f \in L^1(R)$ satisfies the Lipschitz condition of order $\alpha > 0$ at x_0. Then*

$$f_j(x_0) = \int_{-\infty}^{\infty} f(\tau)\delta_j(x_0-\tau)\, d\tau \to f(x_0) \quad \text{as } j \to \infty.$$

THEOREM 6. *Let $M = [a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_n, b_n]$ and Θ be given as in Theorem 4. Then the real and imaginary parts of Θ are*

$$\Re(\Theta) = \frac{1}{2} \quad \text{and} \quad \Im(\Theta) = \frac{1}{2\pi} \log \left| \frac{b_1 b_2 \cdots b_n}{a_1 a_2 \cdots a_n} \right|.$$

PROOF. We have that

$$\begin{aligned} \Re(\Theta) &= \Re\left(\int_0^{\infty} \delta_0(\eta)\, d\eta \right) \\ &= \Re\left(\frac{1}{2\pi} \int_0^{\infty} \int_M e^{iw\eta}\, dw\, d\eta \right) \\ &= \frac{1}{2\pi} \int_0^{\infty} \int_M \cos w\eta\, dw\, d\eta \\ &= \frac{1}{2\pi} \int_0^{\infty} \sum_{i=1}^{n} \left(\frac{\sin b_i\eta}{\eta} - \frac{\sin a_i\eta}{\eta} \right) d\eta. \end{aligned}$$

Since M contains a neighborhood of zero, there is one interval $[a_{i_0}, b_{i_0}]$ such that $a_{i_0} < 0 < b_{i_0}$. Thus the end points of the other intervals $[a_i, b_i]$, with $i \neq i_0$, have the same sign. Since for any $c > 0$,

$$\int_0^\infty \frac{\sin c\eta}{\eta} d\eta = \frac{\pi}{2} \quad \text{and} \quad \frac{\sin(-c\eta)}{\eta} d\eta = -\frac{\sin c\eta}{\eta} d\eta,$$

we have

$$\Re(\Theta) = \frac{1}{2\pi} \int_0^\infty \sum_{i=1}^n \left(\frac{\sin b_i\eta}{\eta} - \frac{\sin a_i\eta}{\eta} \right) d\eta = \frac{1}{2}.$$

The imaginary part is given by

$$\begin{aligned} \Im(\Theta) &= \Im\left(\int_0^\infty \delta_0(\eta)\, d\eta \right) = \Im\left(\frac{1}{2\pi} \int_0^\infty \int_M e^{iw\eta}\, dw\, d\eta \right) \\ &= \frac{1}{2\pi} \int_0^\infty \int_M \sin w\eta\, dw\, d\eta = \frac{1}{2\pi} \int_0^\infty \sum_{i=1}^n \left(\frac{\cos a_i\eta}{\eta} - \frac{\cos b_i\eta}{\eta} \right) d\eta \\ &= \frac{1}{2\pi} \sum_{i=1}^n \int_0^\infty \frac{\cos a_i\eta - \cos b_i\eta}{\eta} d\eta. \end{aligned}$$

If we set

$$F(t) = \int_0^\infty e^{-\eta t} \left(\frac{\cos a\eta - \cos b\eta}{\eta} \right) d\eta, \quad t > 0,$$

then F is differentiable and

$$\begin{aligned} F'(t) &= \int_0^\infty e^{-\eta t} (\cos b\eta - \cos a\eta)\, d\eta = \Re \int_0^\infty \left(e^{(ib-t)\eta} - e^{(ia-t)\eta} \right) d\eta \\ &= \Re\left(\frac{1}{t-ib} - \frac{1}{t-ia} \right) = \frac{t}{b^2+t^2} - \frac{t}{a^2+t^2}. \end{aligned}$$

Therefore

$$F(t) = \frac{1}{2} \log\left(\frac{b^2+t^2}{a^2+t^2} \right)$$

and

$$\int_0^\infty \frac{\cos a\eta - \cos b\eta}{\eta} d\eta = \lim_{t\to 0} F(t) = \log\left| \frac{b}{a} \right|.$$

Hence we have

$$\Im\left(\int_0^\infty \delta_0(\eta) d\eta \right) = \frac{1}{2\pi} \log\left| \frac{b_1 b_2 \cdots b_n}{a_1 a_2 \cdots a_n} \right|.$$

This completes the proof.

By Theorems 4 and 6, we see that

$$\Re\big(f_j(x_0)\big) \to \frac{1}{2}\{f(x_0+) + f(x_0-)\}.$$

It is well known that the Fourier series expansion of $f(x)$ converges to $\frac{1}{2}\{f(x_0+) + f(x_0-)\}$ at the jump discontinuity x_0 [16]. If the delta sequence in the generalized Shannon system is real-valued then we can derive a similar result.

COROLLARY 7. *Suppose the delta sequence δ_j in the generalized Shannon system is a real-valued function and f is a function as in Theorem* 4. *Then*

$$f_j(x_0) \to \frac{1}{2}\{f(x_0+) + f(x_0-)\} \quad \text{as } j \to \infty.$$

REMARK. We consider a convergence problem of continuous functions: is it true that for every continuous function f the wavelet expansion of f converges to $f(t)$ at every point t? It is known that the expansion of a function in regular wavelets converges uniformly on compact subsets of intervals of continuity [14]. Things are not so nice in the generalized Shannon system. The wavelet ψ in the generalized Shannon system has slow decay. By the standard argument using the uniform boundedness theorem, we can show that there is a continuous function whose wavelet expansion diverges in the generalized Shannon system.

Let X be the space of continuous functions supported in $[-1, 1]$. Then X is a Banach space relative to the supremum norm $\|f\|_\infty$ for $f \in X$. Let $\Lambda_j f = f_j(0)$ for all $f \in X$. It can be written as

$$\Lambda_j f = f_j(0) = \int_{-\infty}^{\infty} f(-\tau)\delta_j(\tau)\,d\tau.$$

It can be shown that

$$\|\Lambda_j\| = \int_{-1}^{1} |\delta_j(\tau)|\,d\tau = 2^{j+1}.$$

Then the uniform boundedness theorem asserts that there is a continuous function $f \in X$ such that

$$\|\Lambda_j f\| = |f(0)| \to \infty \quad \text{as } j \to \infty;$$

so the wavelet expansion of $f(t)$ in generalized Shannon system diverges at $t = 0$.

6. On Gibbs' phenomenon

In this section, we will look at a Gibbs phenomenon on the wavelet expansion of a function in the generalized Shannon system. When a Fourier series is used to approximate a function with a jump discontinuity, an overshoot or an undershoot at the jump discontinuity occurs. If S_n denotes the nth partial sum of the Fourier series expansion of the 2π-periodic function defined by

$$F(x) = \begin{cases} 1 & \text{for} \quad 0 \le x < 1 \\ -1 & \text{for} \quad -1 \le x < 0, \end{cases}$$

then, of course, $S_n(x) \to 1$ for all $0 < x < 1$. However, there is a sequence $\{x_n\}$ of positive numbers converging to 0 such that $S_n(x_n)$ converges to a number greater than 1. Indeed

$$\lim_{n\to\infty} S_n(a/n) = \frac{2}{\pi}\int_0^{\pi a} \frac{\sin t}{t}\, dt,$$

so that

$$\lim_{n\to\infty} S_n(1/n) = \frac{2}{\pi}\int_0^{\pi} \frac{\sin t}{t} i\, dt = 1.17898\ldots > 1.$$

This phenomenon was noticed by A. Michelson [10] and explained by J. W. Gibbs [3] in 1899. It has been shown that a Gibbs phenomenon occurs in the wavelet expansion of a function with a jump discontinuity for a wide class of wavelets [5, 11, 13].

To begin with looking for Gibbs effects in the generalized Shannon system, we will first study a function f which has a jump discontinuity at zero. We remark that $f_j(x-\alpha) = g_j(x)$ if $g(x) = f(x-\alpha)$ for a given number $\alpha \in R$. Since the generalized Shannon system is translation invariant [6, 7], $g_j(x)$ and $f_j(x)$ belong to the same resolution space V_j if $g(x) = f(x-\alpha)$.

To study the Gibbs phenomenon of functions with a jump discontinuity at zero, it suffices to look at wavelet expansions of the function

$$f(x) = \begin{cases} 1, & 0 \le x < 1, \\ -1, & -1 < x < 0, \\ 0, & \text{otherwise}, \end{cases} \tag{6.1}$$

since other functions with a jump discontinuity at zero can be written in terms of f plus a function which is continuous at the origin.

THEOREM 8. *Let δ_j be a real-valued delta sequence in the generalized Shannon system. For f defined in* (6.1), *there is a Gibbs phenomenon at the right-hand side of*

0 *if there exists an* $a > 0$ *such that*

$$2\int_0^a \delta_0(\eta)\, d\eta > 1.$$

PROOF. In the generalized Shannon system, the orthogonal projection of f defined in (6.1) onto V_j is given by

$$f_j(t) = \int_{-\infty}^{\infty} f(\tau)\delta_j(t-\tau)\, d\tau = \int_{-1}^{1} \delta_j(t-\tau)\, d\tau.$$

If we take $t = 2^{-j}a$, we have

$$\begin{aligned} f_j(2^{-j}a) &= \int_0^1 \delta_j(2^{-j}a-\tau)\, d\tau - \int_{-1}^0 \delta_j(2^{-j}a-\tau)\, d\tau \\ &= 2^j \int_0^1 \delta_0(a-2^j\tau)\, d\tau - 2^j \int_{-1}^0 \delta_0(a-2^j\tau)\, d\tau \\ &= \int_{a-2^j}^{a} \delta_0(\eta)\, d\eta - \int_a^{a+2^j} \delta_0(\eta)\, d\eta. \end{aligned}$$

Since M is symmetric, we have

$$f_j(2^{-j}a) = \int_{-a}^{a} \delta_0(\eta)\, d\eta - \int_{-a+2^j}^{a+2^j} \delta_0(\eta)\, d\eta.$$

Thus

$$\lim_{j\to\infty} f_j(2^{-j}a) = \int_{-a}^{a} \delta_0(\eta)\, d\eta.$$

If the value $\lim_{j\to\infty} f_j(2^{-j}a)$ is greater than one then there exists a Gibbs phenomenon in the sense that $f_j(2^{-j}a)$ converges to a number greater than one for a positive number a. This completes the proof.

Theorem 8 gives a criterion for the existence of Gibbs phenomenon in the generalized Shannon system: a Gibbs effect occurs near the right-hand side of the origin if there is a number $a > 0$ such that

$$\int_0^a \delta_0(\eta)\, d\eta > \frac{1}{2}.$$

It is known that there is a Gibbs phenomenon for the Shannon system.

In the case where $M = [\alpha - 2\pi, -\pi] \cup [-\alpha, \alpha] \cup [\pi, 2\pi - \alpha]$, we observe the numerical values of

$$F(\alpha) = \int_0^1 \frac{\sin \pi t}{\pi t}(2\cos(\pi-\alpha)t - 1)\, dt$$

TABLE 1. The numerical values of $F(\alpha)$ and $G(\alpha)$.

α	$F(\alpha)$	$G(\alpha)$
$\frac{2}{3}\pi$	0.482284	0.611170
$\frac{3}{4}\pi$	0.528236	0.555786
$\frac{4}{5}\pi$	0.550002	0.522794
$\frac{5}{6}\pi$	0.561960	0.502766
$\frac{6}{7}\pi$	0.569216	0.489946
$\frac{7}{8}\pi$	0.573943	0.481318
$\frac{8}{9}\pi$	0.577193	0.475262
$\frac{9}{10}\pi$	0.579522	0.470859

and

$$G(\alpha) = \int_0^2 \frac{\sin \pi t}{\pi t}(2\cos(\pi - \alpha)t - 1)\, dt$$

in Table 1.

We can guess that $F(\alpha)$ is increasing and $G(\alpha)$ is decreasing in $2\pi/3 \le \alpha < \pi$. We now show that this guess is actually true.

THEOREM 9. *In the generalized Shannon system which has the scaling function φ with $\widehat{\varphi} = \chi_M$, where M is symmetric and consists of three intervals, the corresponding wavelet expansion shows a Gibbs phenomenon.*

PROOF. This will be proved by showing that $F(\alpha)$ is increasing and $G(\alpha)$ is decreasing in $2\pi/3 \le \alpha < \pi$. We show that $F'(\alpha) > 0$ and $G'(\alpha) < 0$ in $2\pi/3 \le \alpha < \pi$.

F is differentiable and

$$\begin{aligned} F'(\alpha) &= \frac{1}{\pi}\int_0^1 2\sin \pi t \sin(\pi - \alpha)t\, dt \\ &= \frac{1}{\pi}\int_0^1 \big(\cos \alpha t - \cos(2\pi - \alpha)t\big)\, dt \\ &= \frac{1}{\pi}\left(\frac{\sin\alpha}{\alpha} - \frac{\sin(2\pi - \alpha)}{2\pi - \alpha}\right). \end{aligned}$$

Since $2\pi/3 \le \alpha < 2\pi - \alpha \le 4\pi/3$, $(\sin\alpha)/\alpha > (\sin(2\pi - \alpha))/(2\pi - \alpha)$. Thus $F'(\alpha) > 0$. Therefore $F(\alpha)$ is increasing in $2\pi/3 \le \alpha < \pi$.

Now G is differentiable and

$$\begin{aligned} G'(\alpha) &= \frac{1}{\pi}\int_0^2 2\sin\pi t\sin(\pi-\alpha)t\,dt \\ &= \frac{1}{\pi}\int_0^2 \big(\cos\alpha t-\cos(2\pi-\alpha)t\big)\,dt \\ &= \frac{1}{\pi}\left(\frac{\sin 2\alpha}{2\alpha}-\frac{\sin 2(2\pi-\alpha)}{2(2\pi-\alpha)}\right). \end{aligned}$$

For $2\pi/3 \le \alpha < \pi$, $(\sin 2\alpha)/(2\alpha) < 0$ and $(\sin 2(2\pi-\alpha))/(2(2\pi-\alpha)) > 0$. So $G'(\alpha) < 0$ in $2\pi/3 \le \alpha < \pi$. Therefore $G(\alpha)$ is decreasing in $2\pi/3 \le \alpha < \pi$. This completes the proof.

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