Glasgow Math. J. **48** (2006) 247–249. © 2006 Glasgow Mathematical Journal Trust.
doi:10.1017/S001708950600303X. Printed in the United Kingdom

GROUPS WITH FINITELY MANY CONJUGACY CLASSES OF SUBGROUPS THAT ARE NOT NILPOTENT-BY-CHERNIKOV

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(Received 19 July, 2005; accepted 4 March, 2006)

Abstract. We prove some theorems on locally graded groups and nilpotent-by-Chernikov groups.

2000 *Mathematics Subject Classification.* 20F18, 20F19, 20F50.

In paper [**7**] the authors established that if G is a locally graded group in which every subgroup is nilpotent-by-Chernikov then either G is nilpotent-by-Chernikov or G is a perfect locally finite p-group for some prime p. In [**1**] it was pointed out that Theorem A of [**7**] contained an assertion about periodic groups that was not justified, but as the claim was only that the locally finite p-groups referred to above have all subgroups *nilpotent*, one sees that the following theorem is indeed a consequence of [**7**] and Theorem 1.3 of [**1**], an important result that deals with the case in which G is a locally finite p-group. (It is clear from the Introduction in [**1**] that the author was quite aware of this consequence.)

THEOREM 1. *Let G be a locally graded group in which every subgroup is nilpotent-by-Chernikov. Then G is nilpotent-by-Chernikov.*

We refer the reader to [**1**] for further references to articles along the lines of [**7**]. We remind the reader that a group G is locally graded if every non-trivial finitely generated subgroup of G has a non-trivial finite image; the above theorem presents a good indication as to why such a class of groups was introduced in the first place, as there exist infinite simple groups G that have all proper non-trivial subgroups of prime order (see [**8**]). An apparently weaker hypothesis than a group's having *all* proper subgroups satisfying a certain property has been addressed in several articles, including [**4**], [**11**] and [**12**]. The hypothesis in question is that, for some property P, the non-P subgroups of G fall into finitely many conjugacy classes; in particular, Theorem 4.1 of [**12**] states that an infinite, locally graded group G with just finitely many conjugacy classes of non-nilpotent subgroups is locally nilpotent and has only finitely many non-nilpotent subgroups. As was pointed out in [**12**], a non-nilpotent group with these properties is the direct product of a p-group K and a finite p'-group F, where p is a prime, and K (if nontrivial) has a "minimal non-nilpotent" subgroup H of finite index. By Theorem 1.3 of [**1**] we have that H is nilpotent-by-Chernikov, and so we may assert the following.

THEOREM 2. *Let G be a locally graded group and suppose that there are just finitely many conjugacy classes of non-nilpotent subgroups of G. Then G has just finitely many non-nilpotent subgroups, and if G is infinite and not nilpotent then G is the direct product of a nilpotent-by-Chernikov p-group K and a finite nilpotent p'-subgroup F, for some prime p. In any case, G is nilpotent-by-Chernikov.*

The main purpose of this note is to establish the following generalization of Theorem 1. We shall see that most of the work in proving this has already been carried out.

THEOREM 3. *Let G be a locally graded group that is not nilpotent-by-Chernikov. Then G has infinitely many non-(nilpotent-by-Chernikov) subgroups that are pairwise non-conjugate.*

There is an interesting "bounded version" of Theorem 1, proved in [**2**]. This states that a locally graded group in which every proper subgroup is (nilpotent of class at most n)-by-Chernikov (n a fixed positive integer) is (nilpotent of class at most n)-by-Chernikov. (Several results were established in [**3**] on groups in which all proper subgroups are (nilpotent (respectively, nilpotent of bounded class))-by-(finite rank), hypotheses that are in general weaker than those of being (nilpotent (or nil-n))-by-Chernikov. One consequence of Theorem 6 of that paper is that a locally finite p-group with all proper subgroups (nil-n)-by-Chernikov is soluble and nilpotent-by-Chernikov.) We remark that the case $n = 1$ of the above theorem from [**2**] was successfully dealt with in [**7**], subsequent to the proof for G periodic (and locally graded) in [**9**]. We shall denote by N_nC the class of groups under discussion (and by NC the class of nilpotent-by-Chernikov groups). Using this result from [**2**], our final theorem is a very easy consequence of Theorem 3.

THEOREM 4. *Let n be a positive integer and let G be a locally graded group that is not in the class* N_nC. *Then G has infinitely many non-*(N_nC)*-subgroups that are pairwise non-conjugate.*

The main requirement for our proof is the following lemma. Its proof is very similar to that of Lemma 4.5 of [**12**], and that proof in turn makes reference at one stage to the proof of Proposition 1 of [**11**]. Accordingly, we present a sketch proof only of our lemma, and invite the interested reader to supplement this sketch with the details from [**11**] and [**12**].

LEMMA. *Let G be a locally graded group with finitely many conjugacy classes of subgroups that are not nilpotent-by-Chernikov. Then G is locally (soluble-by-finite).*

Proof. Suppose the result false and let S be a finitely generated subgroup of G that is not soluble-by-finite. Denote by R the finite residual of S. If r is the maximal number of elements required to generate a finitely generated subgroup of G that is not soluble-by-finite (note that such an r exists) then one shows that S/R has rank at most r and consequently is soluble-by-finite. The soluble radical H/R of S/R is not periodic, since it is infinite and finitely generated. Let K denote the product of all normal subgroups of S that are locally (nilpotent-by-finite) (or $L(NF)$, say); then K is $L(NF)$, by the same argument as that used for establishing the Hirsch-Plotkin theorem (see, for example, [**10**; 12.1.2]). Moreover, K contains all of the subnormal $L(NF)$-subgroups of S. If S/K is not soluble-by-finite and H/M is an arbitrary soluble image of H then M is not $L(NF)$ and we may argue as in [**12**] to obtain a contradiction. So S/K is soluble-by-finite; let J/K denote its soluble radical and let U/K be the Fitting radical of J/K. If $u \in U \setminus K$ then $\langle u\rangle K$ is subnormal in S and so is not $L(NF)$. The set of all such $\langle u\rangle K$ is a union of finitely many conjugacy classes and so U/K is not periodic since it is infinite. We may therefore choose $x \in U$ that has infinite order modulo K. For each prime p and positive integer i the subgroup $X_i := \langle x^{p^i}\rangle K$ is not $L(NF)$, and K is the $L(NF)$-radical of X_i. Again we may follow the argument from [**12**] to obtain our final contradiction.

Proof of Theorem 3. Again suppose that G satisfies the hypotheses of the lemma. We shall show that G is nilpotent-by-Chernikov, thereby establishing Theorem 3. If G is $L(NF)$ (that is, locally (nilpotent-by-finite)) then G satisfies the maximal condition locally and hence, by Lemma 3.2 of **[4]**, satisfies the minimal condition for non-(NC)-subgroups. Theorem 1 provides us with the result in this case. Assume for a contradiction that $G \notin L(NF)$, so that G has a local system consisting of finitely generated non-$L(NF)$-subgroups. Every member of this local system is soluble-by-finite and hence, by the conjugacy class hypothesis, (soluble of derived length d)-by-(order at most e) for some integers d, e and, by Proposition 1.K.2 of **[6]**, G too has this structure, so G is soluble-by-finite. Since the hypotheses on G are inherited by subgroups of finite index, we may assume that G is soluble.

Let G be of minimal derived length subject to being non-$L(NF)$. If G' is not $L(NF)$ then G/G' has just finitely many conjugacy classes of subgroups and is therefore finite, so G' inherits our hypotheses and is $L(NF)$. By this contradiction G' is $L(NF)$ and so the $L(NF)$-radical D of G contains G'. By the same argument as used for G' we see that every normal subgroup of G that is not $L(NF)$ has finite index in G, and so D is $L(NF)$ and G/D is infinite. Furthermore, every subgroup that contains D properly is not $L(NF)$ and therefore has finite index in G, so G/D is infinite cyclic, say $G = D\langle g\rangle$. But, for each prime p, the subgroup $\langle x^p\rangle D$ is not $L(NF)$, and since there are infinitely many such subgroups and no two are conjugate in G we have a contradiction that shows that $G \in L(NF)$ and hence completes the proof of Theorem 3.

Proof of Theorem 4. Suppose that G satisfies the hypotheses but not the conclusion of Theorem 4. By Theorem 3, G is nilpotent-by-Chernikov; in particular G is $L(NF)$, and another application of Lemma 3.2 of **[4]**, using the main result of **[2]** in place of Theorem 1 above, gives the desired contradiction.

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