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COEFFICIENT ESTIMATES, LANDAU'S THEOREM AND LIPSCHITZ-TYPE SPACES ON PLANAR HARMONIC MAPPINGS

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Abstract

In this paper, we investigate the properties of locally univalent and multivalent planar harmonic mappings. First, we discuss coefficient estimates and Landau's theorem for some classes of locally univalent harmonic mappings, and then we study some Lipschitz-type spaces for locally univalent and multivalent harmonic mappings.

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1. Introduction

Let D be a simply connected subdomain of the complex plane $\mathbb{C}$. A complex-valued function f defined in D is called a *harmonic mapping* if and only if both the real and the imaginary parts of f are real harmonic in D. It is known that every harmonic mapping f defined in D admits a decomposition $f = h + \overline{g}$, where h and g are analytic. Since the Jacobian J_f of f is given by

$$J_f = |f_z|^2 - |f_{\overline{z}}|^2 := |h'|^2 - |g'|^2,$$

f is locally univalent and sense-preserving in D if and only if $|g'(z)| < |h'(z)|$ in D; or equivalently if $h'(z) \neq 0$ and the dilatation $w = g'/h'$ has the property that $|w(z)| < 1$ in D (see [23]). We refer to [14, 17] for the theory of planar harmonic mappings.

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For $a \in \mathbb{C}$, let $\mathbb{D}(a, r) = \{z : |z - a| < r\}$. In particular, we use $\mathbb{D}_r$ to denote the disk $\mathbb{D}(0, r)$ and $\mathbb{D}$ the open unit disk $\mathbb{D}_1$. For harmonic mappings f defined in $\mathbb{D}$, we use the following standard notation:

$$\Lambda_f(z) = \max_{0\le\theta\le 2\pi} |f_z(z) + e^{-2i\theta} f_{\bar{z}}(z)| = |f_z(z)| + |f_{\bar{z}}(z)|$$

and

$$\lambda_f(z) = \min_{0\le\theta\le 2\pi} |f_z(z) + e^{-2i\theta} f_{\bar{z}}(z)| = ||f_z(z)| - |f_{\bar{z}}(z)||.$$

Thus, for a sense-preserving harmonic mapping f, one has $J_f(z) = \Lambda_f(z)\lambda_f(z)$.

2. Main results

For a harmonic mapping f in $\mathbb{D}$ and $r \in [0, 1)$, the *harmonic area function* $S_f(r)$ of f, counting multiplicity, is defined by

$$S_f(r) = \int_{\mathbb{D}_r} J_f(z)\, dA(z),$$

where dA denotes the normalized area measure on $\mathbb{D}$. In [21], the authors discussed some properties of harmonic area functions and proved that a harmonic self-homeomorphism of a disk does not increase the area of any concentric disk. In this paper, we discuss coefficients estimates and the Landau theorem for sense-preserving harmonic mappings having finite area. Let $\mathcal{H}$ denote the set of all sense-preserving harmonic mappings $f = h + \overline{g}$ in $\mathbb{D}$ satisfying the normalization $h(0) = g(0) = f_{\bar{z}}(0) = 0$, where h and g are analytic in $\mathbb{D}$. We denote by $\mathcal{H}(C)$ the class of all mappings $f = h + \overline{g} \in \mathcal{H}$ with the finiteness condition

$$C := \sup_{0<r<1} S_f(r) < \infty,$$

where h and g are analytic in $\mathbb{D}$. For any fixed $\alpha \ge 0$, let $\mathcal{H}_\alpha(C)$ denote all mappings $f \in \mathcal{H}(C)$ with $f_z(0) = \alpha$.

In order to state our main results, we first consider the class $\mathcal{H}(C)$. Throughout the discussion, we assume that h and g have the form

$$h(z) = \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=2}^{\infty} b_n z^n, \quad z \in \mathbb{D}, \tag{2.1}$$

whenever $f = h + \overline{g}$ belongs to $\mathcal{H}(C)$. Our results are as follows.

THEOREM 2.1. *Let $f \in \mathcal{H}(C)$, $r_0 = (\sqrt{5} - 1)/2 \approx 0.618$, and*

$$Q(r_0) = \left\{ \frac{(1 + r_0)}{r_0^2(1 - r_0)} C \right\}^{1/2} \approx 3.330\sqrt{C}.$$

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Then

$$\begin{cases} |a_1| \le \sqrt{2C} & \text{if } n = 1, \\ |a_n| + |b_n| \le \dfrac{4Q(r_0)}{\pi r_0^{n-1}}\Big(1 + \dfrac{1}{n-1}\Big)^{n-1} < \dfrac{4Q(r_0)e}{\pi r_0^{n-1}} & \text{if } n \ge 2, \end{cases}$$

where

$$e = \lim_{n\to\infty}\Big(1 + \frac{1}{n}\Big)^n. \tag{2.2}$$

In the special case where $C = 1/2$*, the estimate of* $|a_1|$ *is sharp and the extreme function is* $f(z) = z + \bar{z}^2/2$.

Let f be a sense-preserving harmonic mapping from $\mathbb{D}$ into $\mathbb{C}$. We say that f is a *K-quasiregular harmonic mapping* if and only if

$$\frac{\Lambda_f(z)}{\lambda_f(z)} \le K, \quad \text{that is } \frac{|f_{\bar{z}}(z)|}{|f_z(z)|} \le \frac{K-1}{K+1}, \text{ for } z \in \mathbb{D},$$

where $K \ge 1$. Moreover, if f is a univalent and K-quasiregular harmonic mapping, then f is called a K-quasiconformal harmonic mapping.

A harmonic mapping f is called a *harmonic Bloch mapping* if and only if

$$\sup_{z,w\in\mathbb{D},\ z\ne w} \frac{|f(z) - f(w)|}{\sigma(z, w)} < \infty,$$

where

$$\sigma(z, w) = \frac{1}{2}\log\Big(\frac{|1 - \bar{z}w| + |z - w|}{|1 - \bar{z}w| - |z - w|}\Big) = \operatorname{arctanh}\Big|\frac{z - w}{1 - \bar{z}w}\Big|$$

denotes the hyperbolic distance between z and w in $\mathbb{D}$.

In [15], Colonna proved that

$$\sup_{z,w\in\mathbb{D},\ z\ne w} \frac{|f(z) - f(w)|}{\sigma(z, w)} = \sup_{z\in\mathbb{D}}\{(1 - |z|^2)\Lambda_f(z)\}.$$

Moreover, the set of all harmonic Bloch mappings, denoted by the symbol $\mathcal{HB}$, forms a complex Banach space with the norm $\|\cdot\|$ given by

$$\|f\|_{\mathcal{HB}} = |f(0)| + \sup_{z\in\mathbb{D}}\{(1 - |z|^2)\Lambda_f(z)\}.$$

For K-quasiregular harmonic mappings, we have the following theorem.

THEOREM 2.2. *Let* $f = h + \bar{g}$ *be a* K*-quasiregular harmonic mapping in* $\mathbb{D}$ *satisfying* $C = \sup_{0<r<1} S_f(r) < \infty$*, where* $h(z) = \sum_{n=1}^{\infty} a_n z^n$ *and* $g(z) = \sum_{n=1}^{\infty} b_n z^n$*. Then* $f \in \mathcal{HB}$ *and*

$$|a_n| + |b_n| \le \begin{cases} \sqrt{CK} & \text{if } n = 1, \\ \dfrac{4\sqrt{CK}}{\pi}\Big(1 + \dfrac{1}{n-1}\Big)^{n-1} & \text{if } n \ge 2. \end{cases}$$

In the special case where $C = K = 1$*, the estimate of* $|a_1|$ *is sharp and the extreme function is* $f(z) = z$.

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The following result is obtained as an application of Theorem 2.1.

THEOREM 2.3. *Let $f \in \mathcal{H}_\alpha(C)$ with $0 < \alpha < Q(r_0)$, where r_0 and $Q(r_0)$ are the same as in Theorem 2.1. Then for $n \geq 2$,*

$$|a_n| + |b_n| \leq \frac{1}{n r_0^{n-1} Q(r_0)} \inf_{0<t<1} \left\{ \frac{Q(r_0)^2 - \alpha^2(1-t)^2}{t^{n-1}(1-t)} \right\} < \frac{Q(r_0)}{r_0^{n-1}} \left(1 + \frac{1}{n-1}\right)^{n-1}.$$

The classical theorem of Landau asserts the existence of an universal constant ρ such that every analytic function $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = f'(0) - 1 = 0$ is univalent in the disk $\mathbb{D}_\rho$ and, in addition, the range $f(\mathbb{D}_\rho)$ contains a disk of radius ρ^2. Recently, many authors have considered Landau's theorem for planar harmonic mappings (see, for example, [2–5, 8, 10, 11, 16, 20, 24, 32]). Applying Theorem 2.3, we obtain the following result, and since a bounded harmonic mapping in $\mathbb{D}$ has a finite area, we see that this result is indeed a generalization of [2, Theorem 3].

THEOREM 2.4. *Let $f \in \mathcal{H}_\alpha(C)$ with $0 < \alpha < Q(r_0)$, where r_0 and $Q(r_0)$ are the same as in Theorem 2.1. Define*

$$\rho = 1 - \frac{1}{\sqrt{1 + \alpha/(eQ(r_0))}} \quad \textit{and} \quad R_0 = r_0\rho\left(\alpha - \frac{eQ(r_0)\rho}{1-\rho}\right).$$

Then f is univalent in $\mathbb{D}_\rho$. Moreover, $f(r_0\mathbb{D}_\rho)$ contains a univalent disk $\mathbb{D}_{R_0}$.

A continuous increasing function $\omega : [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$ is called a *majorant* if $\omega(t)/t$ is nonincreasing for $t > 0$ (see [18]). Given a subset Ω of $\mathbb{C}$, a function $f : \Omega \to \mathbb{C}$ is said to belong to the *Lipschitz space* $L_\omega(\Omega)$ if there is a positive constant M such that

$$|f(z) - f(w)| \leq M\omega(|z - w|) \quad \text{for all } z, \quad w \in \Omega. \tag{2.3}$$

For $\delta_0 > 0$ and $0 < \delta < \delta_0$, we consider the following conditions on a majorant ω:

$$\int_0^\delta \frac{\omega(t)}{t}\, dt \leq M\omega(\delta) \tag{2.4}$$

and

$$\delta \int_\delta^{+\infty} \frac{\omega(t)}{t^2}\, dt \leq M\omega(\delta), \tag{2.5}$$

where M denotes a positive constant. A majorant ω is said to be *regular* if it satisfies conditions (2.4) and (2.5) (see [18]).

Dyakonov [18] characterized the holomorphic functions of class L_ω in terms of their modulus. Later, in [28, Theorems A and B], Pavlović came up with a relatively simple proof of Dyakonov's results. Recently, many authors have considered this topic and generalized Dyakonov's results to holomorphic functions and harmonic functions of one variable and several variables (see [1, 8, 18, 19, 25, 26, 28–31]). In this paper, we first extend [28, Theorems A and B] to planar harmonic mappings as follows.

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THEOREM 2.5. *Let ω be a majorant satisfying (2.4) and $f \in \mathcal{H}$. Then for all $r \in (0,1)$,*

$$f \in L_\omega(\mathbb{D}_r) \Longleftrightarrow |f| \in L_\omega(\mathbb{D}_r) \Longleftrightarrow |f| \in L_\omega(\mathbb{D}_r, \partial\mathbb{D}_r),$$

where $L_\omega(\mathbb{D}_r, \partial\mathbb{D}_r)$ denotes the class of continuous functions F on $\mathbb{D}_r \cup \partial\mathbb{D}_r$ which satisfy the condition (2.3) with some positive constant M, whenever $z \in \mathbb{D}_r$ and $w \in \partial\mathbb{D}_r$.

In [22], Korenblum proved the following result.

THEOREM A. *Let u be a real harmonic function in $\mathbb{D}$ and $f_r(\theta) = u(re^{i\theta})$. Then*

$$\|f_r\|_{BMO} \leq \sqrt{1/2}\|u\|_B \sqrt{|\log(1-r^2)|} \quad (0 < r < 1),$$

where $\|u\|_B = \sup_{z\in\mathbb{D}}\{|\nabla u(z)|(1-|z|^2)\}$.

Let BMO_h be the complex Banach space of complex-valued and 2π-periodic functions $\psi \in L^2(0, 2\pi)$ modulo constants with norm

$$\|\psi\|_{BMO_h} = \sup_{z\in\mathbb{D}}\left\{\frac{1}{2\pi}\int_0^{2\pi} |\psi(e^{i\theta}) - f_\psi(z)|^2 P(e^{i\theta}, z)\, d\theta\right\}^{1/2} < \infty.$$

Here $P(e^{i\theta}, z)$ is the Poisson kernel given by

$$P(e^{i\theta}, z) = \frac{1-|z|^2}{|e^{i\theta}-z|^2} = \frac{\partial}{\partial n_\zeta}\log\left|\frac{\zeta - z}{1-\bar{z}\zeta}\right|, \tag{2.6}$$

where n_ζ is the outward normal to $\partial\mathbb{D}$ at $\zeta = e^{i\theta}$, and

$$f_\psi(z) = \frac{1}{2\pi}\int_0^{2\pi} P(e^{i\theta}, z)\psi(\theta)\, d\theta.$$

We have several extensions of Theorem A.

THEOREM 2.6. *Let ω be a majorant and f be a harmonic mapping in $\mathbb{D}$. If $\Lambda_f(z) \leq M\omega(1/(1-|z|))$ in $\mathbb{D}$ and $\psi_r(\theta) = f(re^{i\theta})$, then*

$$\|\psi_r\|_{BMO} \leq 2\sqrt{\omega(1)}Mr\sqrt{\int_0^1 \omega\left(\frac{1}{1-rt}\right) dt},$$

where $\theta \in [0, 2\pi)$ and $r \in (0,1)$.

REMARK. Let $f = u + iv$ be a complex-valued continuously differentiable function defined on $\mathbb{D}$. Then for $z = x + iy \in \mathbb{D}$,

$$\Lambda_f(z) \leq |\nabla u(x,y)| + |\nabla v(x,y)|, \tag{2.7}$$

where $\nabla u = (u_x, u_y)$ and $\nabla v = (v_x, v_y)$ (see [10, Lemma 2]). Therefore, the condition $\Lambda_f(z) \leq M\omega(1/(1-|z|))$ in Theorem 2.6 is weaker than the condition

$$|\nabla u(x,y)| + |\nabla v(x,y)| \leq M\omega\left(\frac{1}{1-|z|}\right).$$

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By taking $\omega(t) = t$ in Theorem 2.6, we get the following result which is also a generalization of [22, Theorem 1].

COROLLARY 2.7. *Let f be a harmonic mapping in $\mathbb{D}$. If $\psi_r(\theta) = f(re^{i\theta})$ and $\Lambda_f(z) \leq M/(1 - |z|)$ in $\mathbb{D}$, then*

$$\|\psi_r\|_{BMO_h} \leq 2\sqrt{r}M\sqrt{|\log(1 - r)|},$$

where $\theta \in [0, 2\pi)$ and $r \in (0, 1)$.

For K-quasiregular harmonic mappings with finite area, we have the following corollary.

COROLLARY 2.8. *Let f be a K-quasiregular harmonic mapping in $\mathbb{D}$ satisfying $C = \sup_{0<r<1} S_f(r) < \infty$. If $\psi_r(\theta) = f(re^{i\theta})$, then*

$$\|\psi_r\|_{BMO_h} \leq 2\sqrt{r}\sqrt{KC}\sqrt{|\log(1 - r)|},$$

where $\theta \in [0, 2\pi)$ and $r \in (0, 1)$.

A sense-preserving and univalent harmonic mapping f in $\mathbb{D}$ will be called a *fully convex harmonic mapping* if it maps every circle $|z| = r < 1$ onto a convex curve (see [12, page 138]). Clunie and Sheil-Small proved the following result.

THEOREM B [14, Corollary 5.8]. *Let $f = h + \overline{g}$ be an univalent and sense-preserving harmonic mapping in $\mathbb{D}$, where h and g are analytic in $\mathbb{D}$. If $f(\mathbb{D})$ is a convex domain, then for all $z_1, z_2 \in \mathbb{D}$ with $z_1 \neq z_2$,*

$$|g(z_1) - g(z_2)| < |h(z_1) - h(z_2)|.$$

In [13], Chuaqui and Hernández discussed the relationship between the images of the linear connectivity under harmonic mappings $f = h + \overline{g}$ and under their corresponding analytic counterparts h, where h and g are analytic in $\mathbb{D}$. For extensive discussions on this topic, see [6]. The following result is an analogous result of Theorem B.

THEOREM 2.9. *Let $f = h + \overline{g} \in \mathcal{H}$ be a fully convex harmonic mapping, where h and g are analytic in $\mathbb{D}$. Then for all $r \in (0, 1)$ and $z_1, z_2 \in \mathbb{D}_r$,*

$$\frac{|f(z_2) - f(z_1)|}{1 + r} \leq |h(z_2) - h(z_1)| \leq \frac{|f(z_2) - f(z_1)|}{1 - r}. \tag{2.8}$$

The following result, which is an improvement of Theorem B in the case of fully convex functions, easily follows from Theorem 2.9.

COROLLARY 2.10. *Let $f = h + \overline{g} \in \mathcal{H}$ be a fully convex harmonic mapping, where h and g are analytic in $\mathbb{D}$. Then h is univalent in $\mathbb{D}$.*

We also have the following theorem due to Clunie and Sheil-Small [14, Theorem 5.17] which helps to construct univalent close-to-convex harmonic functions.

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THEOREM C. *Let $f = h + \overline{g}$ be a sense-preserving harmonic mapping in the unit disk $\mathbb{D}$, and suppose that $h + \epsilon g$ is convex for some $|\epsilon| \leq 1$. Then f is a univalent harmonic mapping from $\mathbb{D}$ onto a close-to-convex domain.*

In particular, Theorem C shows that a sense-preserving harmonic mapping in $\mathbb{D}$ is necessarily close to convex in $\mathbb{D}$ whenever the analytic part of it is convex. In contrast, under a mild restriction on f, namely $f_{\overline{z}}(0) = 0$, Corollary 2.10 shows that the analytic part h of a sense-preserving fully convex harmonic mapping $f = h + \overline{g}$ is necessarily univalent in the unit disk $\mathbb{D}$. On the other hand, another result of Clunie and Sheil-Small (see, for example, [14, Theorem 5.7 and Corollary 5.14]) shows that conclusion of Corollary 2.10 could be improved (see Ponnusamy and Kaliraj 'Constants and characterization for certain classes of univalent harmonic mappings', submitted for publication).

3. Coefficients estimates and the Landau–Bloch theorem for locally univalent harmonic mappings

LEMMA D [4, Lemma 1] or [7, Theorem 1.1]. *Let f be a harmonic mapping of $\mathbb{D}$ into $\mathbb{C}$ such that $|f(z)| \leq M$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n$. Then $|a_0| \leq M$ and, for all $n \geq 1$,*

$$|a_n| + |b_n| \leq \frac{4M}{\pi}.$$

Proof of Theorem 2.1. Let $f \in \mathcal{H}(C)$. For $z \in \mathbb{D}$, consider $w(z) = \overline{f_{\overline{z}}(z)}/f_z(z)$. Then $w(0) = 0$ and, by Schwarz's lemma, $|w(z)| < r$ for $|z| < r$, where $r \in (0, 1)$. This implies

$$\frac{\Lambda_f(z)}{\lambda_f(z)} = \frac{1 + |w(z)|}{1 - |w(z)|} \leq \frac{1 + r}{1 - r} := K(r). \tag{3.1}$$

Since $J_f(z) = \Lambda_f(z)\lambda_f(z)$ for the sense-preserving mapping f, we easily have

$$S_f(r) = \int_{\mathbb{D}_r} \Lambda_f(z)\lambda_f(z)\, dA(z) \geq \frac{1}{K(r)} \int_{\mathbb{D}_r} \Lambda_f^2(z)\, dA(z),$$

which implies

$$\int_{\mathbb{D}_r} \Lambda_f^2(z)\, dA(z) \leq CK(r).$$

For $\theta \in [0, 2\pi)$ and $z \in \mathbb{D}_r$, let $H_\theta(z) = (f_z(z) + e^{i\theta}\overline{f_{\overline{z}}(z)})^2$. Because $|H_\theta(z)|$ is subharmonic for $z \in \mathbb{D}_r$,

$$\begin{aligned}
|H_\theta(z)| &\leq \frac{\int_0^{r-|z|} \int_0^{2\pi} |H_\theta(z + \rho e^{i\beta})| \rho\, d\beta\, d\rho}{\pi(r - |z|)^2} \\
&\leq \frac{1}{(r - |z|)^2} \int_{\mathbb{D}_{r-|z|}} (|f_z(z)| + |f_{\overline{z}}(z)|)^2\, dA(z) \\
&\leq \frac{CK(r)}{(r - |z|)^2},
\end{aligned}$$

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and the arbitrariness of $\theta \in [0, 2\pi)$ gives the inequality

$$\Lambda_f^2(z) \le \frac{CK(r)}{(r-|z|)^2}. \tag{3.2}$$

For $\zeta \in \mathbb{D}$, let $F(\zeta) = r^{-1} f(r\zeta)$. Then $F(0) = 0$ and by (3.2) we see that

$$\Lambda_F(\zeta) = \Lambda_f(z) \le \frac{\sqrt{CK(r)}}{r} \frac{1}{1-|\zeta|}, \tag{3.3}$$

where $z = r\zeta$. By (3.1) and (3.3),

$$\Lambda_F(\zeta) \le \min_{0<r<1} \left\{ \frac{C(1+r)}{r^2(1-r)} \right\}^{1/2} \frac{1}{1-|\zeta|} = \frac{Q(r_0)}{1-|\zeta|},$$

where

$$Q(r_0) = \left[\frac{C(1+r_0)}{r_0^2(1-r_0)} \right]^{1/2} = \sqrt{(11+5\sqrt{5})C/2} \approx 3.330\sqrt{C}$$

and $r_0 = (\sqrt{5}-1)/2 \approx 0.618$. Again, for $w \in \mathbb{D}$ and a fixed $t \in (0, 1)$, let $G(w) = t^{-1} F(tw)$. Then $G(0) = 0$ and

$$\Lambda_G(w) = \Lambda_F(\zeta) \le \frac{Q(r_0)}{1-|\zeta|} = \frac{Q(r_0)}{1-t|w|} \le \frac{Q(r_0)}{1-t} := M(t), \tag{3.4}$$

and G has the form

$$G(w) = \sum_{n=1}^{\infty} A_n w^n + \sum_{n=1}^{\infty} \overline{B}_n \overline{w}^n,$$

where $A_n = a_n r_0^{n-1} t^{n-1}$, $B_n = b_n r_0^{n-1} t^{n-1}$ and $\zeta = wt$. For $\theta \in [0, 2\pi)$, we consider the function T defined by

$$T(w) = G_w(w) + e^{i\theta} G_{\overline{w}}(w).$$

Then $T(w)$ can be written in power series as

$$T(w) = \sum_{n=1}^{\infty} n(A_n w^{n-1} + e^{i\theta} \overline{B}_n \overline{w}^{n-1}).$$

Applying (3.4), for $w \in \mathbb{D}$,

$$|T(w)| \le \Lambda_G(w) < M(t).$$

By Lemma D, for $n \ge 2$,

$$n(|A_n + e^{i\theta} \overline{B}_n|) \le \frac{4M(t)}{\pi},$$

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which gives

$$\begin{aligned}|a_n| + |b_n| &\le \frac{4Q(r_0)}{\pi n r_0^{n-1}} \inf_{0<t<1}\left\{\frac{1}{t^{n-1}(1-t)}\right\}\\ &= \frac{4Q(r_0)}{\pi r_0^{n-1}}\left(1+\frac{1}{n-1}\right)^{n-1}\\ &< \frac{4Q(r_0)e}{\pi r_0^{n-1}},\end{aligned}$$

where e is defined by (2.2). Finally, we come to prove that $|a_1| \le \sqrt{2C}$. Without loss of generality, we assume that

$$C = \iint_{\mathbb{D}} J_f(z)\, dA(z).$$

Then, since the dilatation ω of $f = h + \overline{g}$ satisfies the relation $g'(z) = \omega(z)h'(z)$, by the definition of the Jacobian,

$$\begin{aligned}C &= \iint_{\mathbb{D}} J_f(z)\, dA(z)\\ &= \iint_{\mathbb{D}} (1 - |\omega(z)|^2)|h'(z)|^2\, dA(z)\\ &\ge \iint_{\mathbb{D}} (1 - |z|^2)|h'(z)|^2\, dA(z) \quad \text{(by Schwarz' lemma } |\omega(z)| \le |z|)\\ &= \sum_{n=1}^{\infty} \frac{n}{n+1}|a_n|^2\\ &\ge \frac{|a_1|^2}{2},\end{aligned}$$

which shows that $|a_1| \le \sqrt{2C}$. In particular, if $C = 1/2$, then the estimate of $|a_1|$ is sharp and the extreme function is $f(z) = z + \overline{z}^2/2$. The proof of the theorem is complete. □

Proof of Theorem 2.2. By the hypotheses, $f = h + \overline{g}$ is a K-quasiregular harmonic mapping in $\mathbb{D}$. Therefore, as in the proof of Theorem 2.1, we see that

$$\int_{\mathbb{D}} \Lambda_f^2(z)\, dA(z) \le CK$$

and

$$\Lambda_f^2(z) \le \frac{CK}{(1-|z|)^2} \quad \text{for } z \in \mathbb{D},$$

so that

$$\Lambda_f(z) \le \frac{\sqrt{CK}}{1-|z|} \quad \text{for } z \in \mathbb{D}. \tag{3.5}$$

Thus, $f \in \mathcal{HB}$.

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On the other hand, for $\zeta \in \mathbb{D}$, let $F(\zeta) = r^{-1} f(r\zeta)$. For $w \in \mathbb{D}$ and $\theta \in [0, 2\pi)$, let

$$T(\zeta) = F_{\zeta}(\zeta) + e^{i\theta} F_{\overline{\zeta}}(\zeta).$$

Then

$$T(\zeta) = \sum_{n=1}^{\infty} n(a_n \zeta^{n-1} + e^{i\theta} \overline{b}_n \overline{\zeta}^{n-1}) r^{n-1} \quad \text{for } \zeta \in \mathbb{D},$$

and $|T(\zeta)| < \sqrt{CK}/(1-r)$ for $\zeta \in \mathbb{D}$.

We see that

$$|T(0)| = |a_1 + e^{i\theta}\overline{b}_1| < \frac{\sqrt{CK}}{1-r},$$

which implies that

$$|a_1| + |b_1| \le \sqrt{CK} \inf_{0<r<1} \left\{ \frac{1}{1-r} \right\} = \sqrt{CK}.$$

By Lemma D, for $n \ge 2$, we easily have

$$n(|a_n| + |b_n|) \le \frac{4\sqrt{CK}}{\pi} \inf_{0<r<1} \left\{ \frac{1}{r^{n-1}(1-r)} \right\}. \tag{3.6}$$

Then by (3.6),

$$|a_n| + |b_n| \le \begin{cases} \sqrt{CK} & \text{if } n = 1, \\ \dfrac{4\sqrt{CK}}{\pi}\left(1 + \dfrac{1}{n-1}\right)^{n-1} & \text{if } n \ge 2. \end{cases}$$

The proof of the theorem is complete. □

The following result is well known (see, for example, [27]).

Lemma E. *Let ψ be an analytic function in $\mathbb{D}$ with $\psi(z) = \sum_{n=0}^{\infty} c_n z^n$. If $|\psi(z)| \le 1$, then for each $n \ge 1$, $|c_0|^2 + |c_n| \le 1$.*

Proof of Theorem 2.3. Let $f = h + \overline{g} \in \mathcal{H}_{\alpha}(C)$ with $0 < \alpha < Q(r_0)$, where r_0 and $Q(r_0)$ are the same as in Theorem 2.1.

Following the proof of Theorem 2.1, for $w \in \mathbb{D}$ and $\theta \in [0, 2\pi)$, we let

$$H(w) = \frac{G_w(w) + e^{i\theta}\overline{G_{\overline{w}}(w)}}{M(t)},$$

where G and $M(t)$ are the same as in the proof of Theorem 2.1. Then

$$H(w) = \frac{1}{M(t)} \sum_{n=1}^{\infty} n(A_n + e^{i\theta} B_n) w^{n-1}$$

and $|H(w)| < 1$ for $w \in \mathbb{D}$, where $A_n = a_n r_0^{n-1} t^{n-1}$ and $B_n = b_n r_0^{n-1} t^{n-1}$. By Lemma E,

$$\frac{n|A_n + e^{i\theta} B_n|}{M(t)} \le 1 - \frac{\lambda_G^2(0)}{M^2(t)} = 1 - \frac{\alpha^2}{M^2(t)}.$$

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Since $|A_n| + |B_n| = (|a_n| + |b_n|)r_0^{n-1}t^{n-1}$, the arbitrariness of $\theta \in [0, 2\pi)$ gives

$$(|a_n| + |b_n|)r_0^{n-1}t^{n-1} \leq \frac{1}{n}\left(M(t) - \frac{\alpha^2}{M(t)}\right) = \frac{1}{n}\left(\frac{Q^2(r_0) - \alpha^2(1-t)^2}{(1-t)Q(r_0)}\right)$$

which implies that

$$\begin{aligned}
|a_n| + |b_n| &\leq \frac{1}{nr_0^{n-1}t^{n-1}}\left(\frac{Q^2(r_0) - \alpha^2(1-t)^2}{(1-t)Q(r_0)}\right) \\
&\leq \frac{1}{nr_0^{n-1}Q(r_0)} \inf_{0<t<1}\left\{\frac{Q^2(r_0) - \alpha^2(1-t)^2}{t^{n-1}(1-t)}\right\} \\
&< \frac{Q(r_0)}{nr_0^{n-1}} \inf_{0<t<1}\left\{\frac{1}{t^{n-1}(1-t)}\right\} = \frac{Q(r_0)}{r_0^{n-1}}\left(1 + \frac{1}{n-1}\right)^{n-1} \\
&< \frac{Q(r_0)e}{r_0^{n-1}},
\end{aligned}$$

where e is defined by (2.2). The proof of the theorem is complete.

Proof of Theorem 2.4. As in Theorem 2.3, let $f = h + \overline{g}$, where g and h are analytic in $\mathbb{D}$ and have the form (2.1). For $\zeta \in \mathbb{D}$, let $F(\zeta) = f(r_0\zeta)/r_0$, where r_0 is the same as in Theorem 2.1. From the proof of Theorem 2.3, for $n \geq 2$,

$$|a_n| + |b_n| < \frac{Q(r_0)e}{r_0^{n-1}}. \tag{3.7}$$

To prove the univalence of F, we choose two distinct points $\zeta_1, \zeta_2 \in \mathbb{D}_\rho$, where

$$\rho = 1 - \frac{1}{\sqrt{1 + \frac{\alpha}{eQ(r_0)}}}. \tag{3.8}$$

Then (3.7) and (3.8) yield that

$$\begin{aligned}
|F(\zeta_2) - F(\zeta_1)| &= \left|\int_{[\zeta_1,\zeta_2]} F_\zeta(\zeta)\,d\zeta + F_{\overline{\zeta}}(\zeta)\,d\overline{\zeta}\right| \\
&\geq \left|\int_{[\zeta_1,\zeta_2]} F_\zeta(0)\,d\zeta + F_{\overline{\zeta}}(0)\,d\overline{\zeta}\right| \\
&\quad - \left|\int_{[\zeta_1,\zeta_2]} (F_\zeta(\zeta) - F_\zeta(0))\,d\zeta + (F_{\overline{\zeta}}(\zeta) - F_{\overline{\zeta}}(0))\,d\overline{\zeta}\right| \\
&> |\zeta_1 - \zeta_2|\left[\lambda_F(0) - \sum_{n=2}^{\infty}(|a_n| + |b_n|)nr_0^{n-1}\rho^{n-1}\right] \\
&\geq |\zeta_1 - \zeta_2|\left[\alpha - Q(r_0)\left(1 + \frac{1}{n-1}\right)^{n-1} \cdot \frac{\rho(2-\rho)}{(1-\rho)^2}\right] \\
&> |\zeta_1 - \zeta_2|\left[\alpha - eQ(r_0) \cdot \frac{\rho(2-\rho)}{(1-\rho)^2}\right] = 0, \quad \text{by (3.8)}.
\end{aligned}$$

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Thus, $F(\zeta_2) \neq F(\zeta_1)$. The univalence of F follows from the arbitrariness of ζ_1 and ζ_2. This implies that f is univalent in $\mathbb{D}_{r_0\rho}$.

Now, for any $\zeta' = \rho e^{i\theta} \in \partial\mathbb{D}_\rho$, we easily obtain that

$$\begin{aligned}|F(\zeta')| &\geq \alpha\rho - \sum_{n=2}^{\infty}(|a_n| + |b_n|)r_0^{n-1}\rho^n \\ &\geq \alpha\rho - eQ(r_0)\sum_{n=2}^{\infty}\rho^n \\ &= \rho\Big(\alpha - \frac{eQ(r_0)\rho}{1-\rho}\Big) = \frac{R_0}{r_0}.\end{aligned}$$

Therefore, $f(\mathbb{D}_{r_0\rho})$ contains a univalent disk of radius R_0. The proof of the theorem is complete. □

4. Lipschitz-type spaces of harmonic mappings

LEMMA F [9, Lemma 1]. *Let f be a K-quasiregular harmonic mapping in $\mathbb{D}$ with $f(\mathbb{D}) \subset \mathbb{D}$. Then for all $z \in \mathbb{D}$,*

$$\Lambda_f(z) \leq K\frac{1 - |f(z)|^2}{1 - |z|^2}. \tag{4.1}$$

Moreover, (4.1) is sharp when $K = 1$.

Proof of Theorem 2.5. The implications $f \in L_\omega(\mathbb{D}_r) \Rightarrow |f| \in L_\omega(\mathbb{D}_r) \Rightarrow |f| \in L_\omega(\mathbb{D}_r, \partial\mathbb{D}_r)$ are obvious. We only need to prove $|f| \in L_\omega(\mathbb{D}_r, \partial\mathbb{D}_r) \Rightarrow f \in L_\omega(\mathbb{D}_r)$. For $z \in \mathbb{D}$, let $w(z) = \overline{f_{\overline{z}}(z)}/f_z(z)$. Then $w(0) = 0$ and for any fixed $r \in (0, 1)$ and $z \in \mathbb{D}_r$, $|w(z)| < r$. This gives

$$\frac{\Lambda_f(z)}{\lambda_f(z)} = \frac{1 + |w(z)|}{1 - |w(z)|} \leq \frac{1 + r}{1 - r} = K(r).$$

Now, for a fixed point $z \in \mathbb{D}_r$, we consider the function

$$F(\eta) = f(z + d(z)\eta)/M_z, \quad \eta \in \mathbb{D},$$

where $d(z) := d(z, \partial\mathbb{D}_r)$ denotes the Euclidean distance from z to the boundary $\partial\mathbb{D}_r$ of $\mathbb{D}_r$ and $M_z := \sup\{|f(\zeta)| : |\zeta - z| < d(z)\}$. By an elementary calculation, we obtain that

$$\frac{\Lambda_F(\eta)}{\lambda_F(\eta)} = \frac{\Lambda_f(\xi)}{\lambda_f(\xi)} \leq K(r),$$

where $\xi = z + d(z)\eta$. Then F is a $K(r)$-quasiregular harmonic mapping of $\mathbb{D}$ into itself. By Lemma F, we see that

$$\Lambda_F(0) \leq K(r)(1 - |F(0)|^2)$$

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which may be written as

$$d(z)\Lambda_f(z) \le 2K(r)(M_z - |f(z)|). \tag{4.2}$$

Without loss of generality, we let $\zeta \in \partial\mathbb{D}_r$ with $|\zeta - z| = d(z)$, and let $w \in \mathbb{D}(z, d(z))$. Then

$$\begin{aligned} |f(w)| - |f(z)| &\le ||f(w)| - |f(\zeta)|| + ||f(\zeta)| - |f(z)|| \\ &\le M\omega(d(z)) + M\omega(2d(z)) \\ &\le 3M\omega(d(z)), \end{aligned}$$

and thus

$$\sup_{w\in\mathbb{D}(z,d(z))} (|f(w)| - |f(z)|) \le 3M\omega(d(z)),$$

which implies that $M_z - |f(z)| \le 3M\omega(d(z))$. This inequality together with (4.2) shows that

$$\Lambda_f(z) \le 6MK(r) \cdot \frac{\omega(d(z))}{d(z)}, \quad z \in \mathbb{D}_r. \tag{4.3}$$

Finally, given any two points $z_1, z_2 \in \mathbb{D}_r$, let $\gamma \subset \mathbb{D}_r$ be a rectifiable curve joining z_1 and z_2. Integrating (4.3) along γ,

$$|f(z_1) - f(z_2)| \le \int_\gamma (|f_z(z)| + |f_{\bar{z}}(z)|)\, ds(z) \le 6MK(r) \int_\gamma \frac{\omega(d(z))}{d(z)}\, ds(z). \tag{4.4}$$

Therefore, (4.4) yields

$$|f(z_1) - f(z_2)| \le M_1 \cdot \omega(|z_1 - z_2|),$$

where M_1 is a positive constant. The proof of the theorem is complete.

Proof of Theorem 2.6. By (2.6) and integrating by parts,

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta}) - f(rz)|^2 P(e^{i\theta}, z)\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta}) - f(rz)|^2 \frac{\partial}{\partial n_\zeta} \log\left|\frac{\zeta - z}{1 - \bar{z}\zeta}\right| d\theta \\ &= -\frac{2r^2}{\pi}\int_{\mathbb{D}} \log\left|\frac{w - z}{1 - \bar{z}w}\right| (|f_\xi(rw)|^2 + |f_{\bar{\xi}}(rw)|^2)\, dA(w) \\ &\le -\frac{2r^2}{\pi}\int_{\mathbb{D}} \log\left|\frac{w - z}{1 - \bar{z}w}\right| \Lambda_f^2(rw)\, dA(w) \\ &\le -\frac{2r^2M^2}{\pi}\int_{\mathbb{D}} \log\left|\frac{w - z}{1 - \bar{z}w}\right| \omega^2\Big(\frac{1}{1 - r|w|}\Big)\, dA(w) \\ &\le \frac{2r^2M^2}{\pi}\int_{\mathbb{D}} \log\frac{1}{|w|}\, \omega^2\Big(\frac{1}{1 - r|w|}\Big)\, dA(w) \end{aligned}$$

$$
\begin{aligned}
&= 4r^2M^2 \int_0^1 \rho|\log\rho|\omega^2\Big(\frac{1}{1-r\rho}\Big)\,d\rho\\
&= 4r^2M^2 \int_0^1 \Big[|\log\rho|\frac{d}{d\rho}\int_0^\rho t\omega^2\Big(\frac{1}{1-rt}\Big)\,dt\Big]\,d\rho\\
&= 4r^2M^2\Big[|\log\rho|\int_0^\rho t\omega^2\Big(\frac{1}{1-rt}\Big)\,dt\Big|_0^1 + \int_0^1 \frac{\int_0^\rho t\omega^2\big(\frac{1}{1-rt}\big)\,dt}{\rho}\,d\rho\Big]\\
&= 4r^2M^2 \int_0^1 \frac{\int_0^\rho t\omega^2\big(\frac{1}{1-rt}\big)\,dt}{\rho}\,d\rho\\
&\le 4r^2M^2 \int_0^1 \int_0^\rho \omega^2\Big(\frac{1}{1-rt}\Big)\,dt\,d\rho\\
&= 4r^2M^2 \int_0^1 \Big(\int_t^1 d\rho\Big)\omega^2\Big(\frac{1}{1-rt}\Big)\,dt\\
&= 4r^2M^2 \int_0^1 (1-t)\omega^2\Big(\frac{1}{1-rt}\Big)\,dt\\
&= 4r^2M^2 \int_0^1 \frac{1-t}{1-rt}\Big[(1-rt)\omega\Big(\frac{1}{1-rt}\Big)\Big]\omega\Big(\frac{1}{1-rt}\Big)\,dt\\
&\le 4r^2M^2\omega(1) \int_0^1 \omega\Big(\frac{1}{1-rt}\Big)\,dt,
\end{aligned}
$$

which implies

$$
\|\psi_r\|_{BMO_h} \le 2\sqrt{\omega(1)}Mr\sqrt{\int_0^1 \omega\Big(\frac{1}{1-rt}\Big)\,dt},
$$

where $\xi = rw$. The proof is complete. □

Proof of Corollary 2.8. Corollary 2.8 easily follows from (3.5) and Corollary 2.7. □

Proof of Theorem 2.9. Differentiating both sides of the equation $f^{-1}(f(z)) = z$ yields the relations

$$
(f^{-1})_\zeta h' + (f^{-1})_{\overline{\zeta}} g' = 1 \quad \text{and} \quad (f^{-1})_\zeta \overline{g'} + (f^{-1})_{\overline{\zeta}} \overline{h'} = 0,
$$

where $\zeta = f(z)$. This gives

$$
(f^{-1})_\zeta = \frac{\overline{h'}}{J_f} \quad \text{and} \quad (f^{-1})_{\overline{\zeta}} = -\frac{\overline{g'}}{J_f}. \tag{4.5}
$$

Since $\Omega = f(\mathbb{D}_r)$ is convex, for any two distinct points $z_1,\ z_2 \in \mathbb{D}_r$ and $t \in [0, 1]$,

$$
\varphi(t) = (f(z_2) - f(z_1))t + f(z_1) \in \Omega.
$$

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Let $\gamma = f^{-1} \circ \varphi$ and $f(z_2) - f(z_1) = |f(z_2) - f(z_1)|e^{i\theta_0}$. For $z \in \mathbb{D}$, let $w(z) = \overline{f_{\bar{z}}(z)}/f_z(z)$. Then $w(0) = 0$ and for $z \in \mathbb{D}_r$, $|w(z)| < r$, where $r \in (0, 1)$. This implies that f is $K(r)$-quasiconformal harmonic mapping in $\mathbb{D}_r$, where $K(r) = (1 + r)/(1 - r)$. By calculations and (4.5),

$$\begin{aligned}
|h(z_2) - h(z_1)| &= \Big|\int_\gamma h'(z)\, dz\Big| \\
&= \Big|\int_0^1 h'(\gamma(t))\, \frac{d}{dt}\gamma(t)\, dt\Big| \\
&= \Big|\int_0^1 h'(\gamma(t))\Big[\varphi'(t)\frac{\partial}{\partial\zeta} f^{-1}(\varphi(t)) + \overline{\varphi'(t)}\frac{\partial}{\partial\bar{\zeta}} f^{-1}(\varphi(t))\Big]\, dt\Big| \\
&= |f(z_2) - f(z_1)|\Big|\int_0^1 h'(\gamma(t))\Big(\frac{\overline{h'(\gamma(t))}}{J_f(\gamma(t))}e^{i\theta_0} - \frac{\overline{g'(\gamma(t))}}{J_f(\gamma(t))}e^{-i\theta_0}\Big)\, dt\Big| \\
&\leq |f(z_2) - f(z_1)| \int_0^1 |h'(\gamma(t))|\frac{|h'(\gamma(t))| + |g'(\gamma(t))|}{J_f(\gamma(t))}\, dt \\
&= |f(z_2) - f(z_1)| \int_0^1 \frac{1}{1 - |w(\gamma(t))|}\, dt \\
&\leq \frac{|f(z_2) - f(z_1)|}{1 - r}.
\end{aligned}$$

This gives the second inequality in (2.8). Next we prove the first inequality. Applying (4.5), we see that

$$\begin{aligned}
&\mathrm{Re}\,[e^{-i\theta_0}(\overline{g(z_2) - g(z_1)})] \\
&= \mathrm{Re}\Big[e^{-i\theta_0}\Big(\overline{\int_0^1 g'(\gamma(t))\, \frac{d}{dt}\gamma(t)\, dt}\Big)\Big] \\
&= \mathrm{Re}\Big\{e^{-i\theta_0}\Big[\overline{\int_0^1 g'(\gamma(t))\Big(\varphi'(t)\frac{\partial}{\partial\zeta} f^{-1}(\varphi(t)) + \overline{\varphi'(t)}\frac{\partial}{\partial\bar{\zeta}} f^{-1}(\varphi(t))\Big)\, dt}\Big]\Big\} \\
&= |f(z_2) - f(z_1)|\, \mathrm{Re}\Big\{e^{-i\theta_0}\Big[\overline{\int_0^1 \frac{g'(\gamma(t))\overline{h'(\gamma(t))}e^{i\theta_0} - |g'(\gamma(t))|^2 e^{-i\theta_0}}{J_f(\gamma(t))}\, dt}\Big]\Big\} \\
&\leq |f(z_2) - f(z_1)| \int_0^1 \frac{|h'(\gamma(t))\overline{g'(\gamma(t))}e^{-2i\theta_0}| - |g'(\gamma(t))|^2}{J_f(\gamma(t))}\, dt \\
&\leq |f(z_2) - f(z_1)| \int_0^1 \frac{|w(\gamma(t))|}{1 + |w(\gamma(t))|}\, dt \\
&\leq \frac{r|f(z_2) - f(z_1)|}{1 + r},
\end{aligned}$$

which gives

$$\mathrm{Re}\Big\{\frac{\overline{g(z_2) - g(z_1)}}{f(z_2) - f(z_1)}\Big\} \leq \frac{r}{1 + r}. \tag{4.6}$$

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It is not difficult to see that

$$\mathrm{Re}\left\{\frac{h(z_2)-h(z_1)}{f(z_2)-f(z_1)}\right\} = 1 - \mathrm{Re}\left\{\frac{\overline{g(z_2)-g(z_1)}}{f(z_2)-f(z_1)}\right\}. \tag{4.7}$$

By (4.6) and (4.7),

$$\begin{aligned}\frac{|h(z_2)-h(z_1)|}{|f(z_2)-f(z_1)|} &\geq \mathrm{Re}\left\{\frac{h(z_2)-h(z_1)}{f(z_2)-f(z_1)}\right\}\\ &= 1 - \mathrm{Re}\left\{\frac{\overline{g(z_2)-g(z_1)}}{f(z_2)-f(z_1)}\right\}\\ &\geq 1 - \frac{r}{1+r}\\ &= \frac{1}{1+r}.\end{aligned}$$

The proof of this theorem is complete. □

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