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THE RESOLVENT AND EXPECTED LOCAL TIMES FOR MARKOV-MODULATED BROWNIAN MOTION WITH PHASE-DEPENDENT TERMINATION RATES

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Abstract

We consider a Markov-modulated Brownian motion (MMBM) with phase-dependent termination rates, i.e. while in a phase i the process terminates with a constant hazard rate $r_i \geq 0$. For such a process, we determine the matrix of expected local times (at zero) before termination and hence the resolvent. The results are applied to some recent questions arising in the framework of insurance risk. We further provide expressions for the resolvent and the local times at zero of an MMBM reflected at its infimum.

Keywords: Markov-modulated Brownian motion; local time; resolvent; Markov-additive process

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1. Introduction

Let $\mathcal{J} = (J_t : t \geq 0)$ denote an irreducible Markov process with a finite state space $E = \{1, \dots, m\}$ and infinitesimal generator matrix $Q = (q_{ij})_{i,j\in E}$. We call J_t the phase at time t and $\mathcal{J}$ the phase process. Choosing parameters $\mu_i \in \mathbb{R}$ and $\sigma_i \geq 0$ for all $i \in E$, we define the level process $\mathcal{X} = (X_t : t \geq 0)$ by $X_t = X_0 + \int_0^t \mu_{J_s}\, \mathrm{d}s + \int_0^t \sigma_{J_s}\, \mathrm{d}W_s$ for all $t \geq 0$, where $\mathcal{W} = (W_t : t \geq 0)$ denotes a standard Wiener process that is independent of $\mathcal{J}$. Then $(\mathcal{X}, \mathcal{J})$ is called a Markov-modulated Brownian motion (MMBM). An MMBM is a Markov-additive process (MAP—see [2, Chapter XI]) without jumps.

Now add an absorbing phase, say Δ, to the phase space E to obtain $E' = E \cup \{\Delta\}$ and assume that this is entered from a phase i with a constant hazard rate $r_i \geq 0$. Denote the resulting process by $(\mathcal{X}, \mathcal{J})$ again. Define the exit rate vector by $\boldsymbol{r} := (r_i : i \in E)$. We shall assume throughout that Q is irreducible and that $\|\boldsymbol{r}\| := \sum_{i\in E} r_i > 0$. Then the absorption time $\tau_\Delta := \min\{t \geq 0 : J_t = \Delta\}$ has a phase-type distribution $\mathrm{PH}(\alpha, Q - \Delta_{\boldsymbol{r}})$ with $\alpha_i := \mathbb{P}(J_0 = i)$ for $i \in E$ and $\Delta_{\boldsymbol{v}}$ denoting the diagonal matrix with entries taken from the vector $\boldsymbol{v}$. We shall say that the MMBM $(\mathcal{X}, \mathcal{J})$ terminates at time τ_Δ and disregard any further evolution after this. The values r_i may be interpreted as state-dependent killing rates; see Section 3 of [7]. Similar results on terminating MMBMs have been obtained in Chapter 7 of [6]; cf. Example 2 in the present paper.

The present paper aims to determine the matrix of expected local times (at zero) for a terminating MMBM. Based on this, the resolvent is given as a corollary. A particular application of the resolvent is the determination of the transition probabilities over phase-type distributed time distances. The next section contains some preliminary results, while the main result is

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given in Section 3. Section 4 contains an application to insurance risk. The final section contains an extension of the results to reflected MMBMs.

2. Preliminaries: first passage times

Define the first passage times $\tau^+(x) := \inf\{t \geq 0 \colon X_t > x\}$ for all $x \geq 0$, and assume that $X_0 = 0$. Consider an E-dimensional row vector $\boldsymbol{r} = (r_i \colon i \in E)$ with nonnegative entries $r_i \geq 0$ for all $i \in E$. Define $\mathbb{E}(\exp[-\int_0^{\tau^+(x)} r_{J_s}\,\mathrm{d}s])$ as the $(E \times E)$-matrix with (i, j)th entry

$$\mathbb{E}_{ij}\biggl(\exp\biggl[-\int_0^{\tau^+(x)} r_{J_s}\,\mathrm{d}s\biggr]\biggr) := \mathbb{E}\biggl(\exp\biggl[-\int_0^{\tau^+(x)} r_{J_s}\,\mathrm{d}s\biggr];\, J_{\tau^+(x)} = j \biggm| J_0 = i,\ X_0 = 0\biggr).$$

In order to simplify the notation (and to ensure existence of the resolvent density later on), we shall from now on exclude the case of a phase $i \in E$ with $\mu_i = \sigma_i = 0$. We distinguish the phases by the subspaces $E_p := \{i \in E \colon \sigma_i = 0,\ \mu_i > 0\}$ as well as $E_n := \{i \in E \colon \sigma_i = 0,\ \mu_i < 0\}$ and $E_\sigma := \{i \in E \colon \sigma_i > 0\}$. The same arguments as in [4, Section 3] yield

$$\mathbb{E}\biggl(\exp\biggl[-\int_0^{\tau^+(x)} r_{J_s}\,\mathrm{d}s\biggr]\biggr) = \begin{pmatrix} I_a \\ A(\boldsymbol{r}) \end{pmatrix} (\mathrm{e}^{U(\boldsymbol{r})x}, \boldsymbol{0}),$$

where I_a denotes the identity matrix on $E_p \cup E_\sigma$ and $\boldsymbol{0}$ the zero matrix on $(E_p \cup E_\sigma) \times E_n$. The matrices $A = A(\boldsymbol{r})$ and $U = U(\boldsymbol{r})$ can be computed as in Section 3.1 of [5].

Remark 1. The generalised Laplace transforms of the first passage times $\tau^+(x)$ can be seen as transition probabilities among the transient phases $i, j \in E$ for the phase process $\mathcal{J}$, i.e.

$$\mathbb{E}_{ij}\biggl(\exp\biggl[-\int_0^{\tau^+(x)} r_{J_s}\,\mathrm{d}s\biggr]\biggr) = \mathbb{P}(\tau^+(x) < \tau_\Delta,\ J_{\tau^+(x)} = j \mid J_0 = i,\ X_0 = 0)$$

for $i, j \in E$ and $A_{ij} = \mathbb{P}(\tau^+(0) < \tau_\Delta,\ J_{\tau^+(0)} = j \mid J_0 = i,\ X_0 = 0)$ for $i \in E_n$, $j \in E_p \cup E_\sigma$.

Now define the downward first passage times $\tau^-(x) := \inf\{t \geq 0 \colon X_t < x\}$ for all $x \leq 0$ and assume that $X_0 = 0$. Let $(\mathcal{X}^+, \mathcal{J})$ denote the original MMBM, and define the process $(\mathcal{X}^-, \mathcal{J}) \overset{\mathrm{D}}{=} (-\mathcal{X}^+, \mathcal{J})$, where '$\overset{\mathrm{D}}{=}$' denotes equality in distribution. The two processes have the same generator matrix Q for $\mathcal{J}$, but the drift parameters are different. Denoting the variation and drift parameters for $\mathcal{X}^\pm$ by $\sigma_i^\pm$ and $\mu_i^\pm$, respectively, this means that $\sigma_i^- = \sigma_i^+$ and $\mu_i^- = -\mu_i^+$ for all $i \in E$.

The generalised Laplace transforms for $\tau^-(x)$ can of course be obtained by considering the upward first passage times for the process $(\mathcal{X}^-, \mathcal{J})$. Let $A^\pm = A^\pm(\boldsymbol{r})$ and $U^\pm = U^\pm(\boldsymbol{r})$ denote the matrices that determine the first passage times of $\mathcal{X}^\pm$. Furthermore, let I_d denote the identity matrix on $E_\sigma \cup E_n$ and $\boldsymbol{0}$ the zero matrix on $(E_\sigma \cup E_n) \times E_p$. Then

$$\mathbb{E}\biggl(\exp\biggl[-\int_0^{\tau^-(x)} r_{J_s}\,\mathrm{d}s\biggr]\biggr) = \begin{pmatrix} A^-(\boldsymbol{r}) \\ I_d \end{pmatrix} (\boldsymbol{0}, \mathrm{e}^{-U^-(\boldsymbol{r})x}) \quad \text{for all } x \leq 0.$$

Note that Remark 1 holds for $\tau^-(x)$ and $A^-(\boldsymbol{r})$ analogously.

3. The resolvent and expected local times for terminating MMBMs

Denote the indicator function of a set A by $\mathbb{I}_A$. Define the $\boldsymbol{r}$-resolvent of $(\mathcal{X}, \mathcal{J})$ as the matrix-valued function $S^{(\boldsymbol{r})}(x),\ x \in \mathbb{R}$, with entries

$$\begin{aligned} S_{ij}^{(\boldsymbol{r})}(x) &= \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \mathbb{E}\bigg(\int_0^{\tau_\Delta} \mathbb{I}_{\{|X_t - x| < \varepsilon,\ J_t = j\}} \,\mathrm{d}t \,\bigg|\, X_0 = 0,\ J_0 = i \bigg) \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^\infty \mathbb{P}(t < \tau_\Delta,\ X_t \in (x - \varepsilon, x + \varepsilon),\ J_t = j \mid X_0 = 0,\ J_0 = i) \,\mathrm{d}t \end{aligned}$$

for $i \in E$ and $j \in E_\sigma$, and

$$S_{ij}^{(\boldsymbol{r})}(x) = \frac{1}{|\mu_j|} \mathbb{E}(C(\{t < \tau_\Delta \colon X_t = x,\ J_t = j\}) \mid X_0 = 0,\ J_0 = i) \tag{1}$$

for $i \in E$ and $j \in E_p \cup E_n$, where $C(A)$ denotes the number of elements in a set A. We shall also write more briefly $S^{(\boldsymbol{r})}(x)\,\mathrm{d}x = \int_0^\infty \mathbb{P}(X_t \in \mathrm{d}x)\,\mathrm{d}t$. Note that the exponential time devaluation, which is the usual notion in the definition of a resolvent for Lévy processes, is now replaced by the phase-dependent termination rates contained in the vector $\boldsymbol{r}$. Regarding existence and basic properties of local times, see Chapter 5 of [3] for Lévy processes and Chapter 7 of [6] for MAPs.

Remark 2. Since the termination rates do not depend on the level process, we can use the resolvent to determine the transition probabilities over a phase-type time distance. More precisely, $\mathbb{P}(X_{\tau_\Delta} \in \mathrm{d}y \mid X_0 = x) = \alpha S^{(\boldsymbol{r})}(y - x)\boldsymbol{r}\,\mathrm{d}y$ for all $x, y \in \mathbb{R}$, where $\boldsymbol{r}$ is seen as a column vector and $\alpha_i = \mathbb{P}(J_0 = i)$ for all $i \in E$. This property can be made more amenable to applications in the following way. Given a phase-type distributed time distance $Z \sim \mathrm{PH}(\alpha, T)$ of order m and an MMBM $(\mathcal{X}, \mathcal{J})$ with generator matrix Q for $\mathcal{J}$ and phase space E, we construct a phase space $E' := E \times \{1, \dots, m\}$ and a generator matrix $Q' := Q \oplus T + \Delta_{\mathbf{1}_E \otimes \eta}$, where '$\oplus$' and '$\otimes$' denote the Kronecker sum and product, respectively, $\eta = -T\mathbf{1}_m$, and $\mathbf{1}_m$ and $\mathbf{1}_E$ denote the column vectors on $\{1, \dots, m\}$ and E, respectively, with all entries being 1. Set further $\mu_{(i,j)} := \mu_i$ and $\sigma_{(i,j)} := \sigma_i$ for all $i \in E$ and $j \in \{1, \dots, m\}$, and denote the MMBM defined therewith by $(\mathcal{X}', \mathcal{J}')$. Then

$$\mathbb{P}(X_Z \in \mathrm{d}y,\ J_Z = j \mid X_0 = x,\ J_0 = i) = (e_i^\top \otimes \alpha) S'(y - x)(e_j \otimes \eta)\,\mathrm{d}y$$

for all $i, j \in E$ and $x, y \in \mathbb{R}$, where e_i denotes the ith canonical base column vector, $e_i^\top$ denotes its transpose, and S' denotes the $(\mathbf{1}_E \otimes \eta)$-resolvent of $(\mathcal{X}', \mathcal{J}')$. This identity is an application of the famous occupation density formula; see, e.g. Equation (V.2) of [3]. For an example of how to use it in a stochastic model, see Section 4.

Define the cumulant functions $\kappa_i(\beta) := \beta^2\sigma_i^2/2 + \beta\mu_i$ for all $i \in E$ and the cumulant matrix $K(\beta) := \Delta_{\kappa(\beta)} + Q$, where $\Delta_{\kappa(\beta)}$ denotes the diagonal matrix on E with entries $\kappa_i(\beta)$. A first observation is

$$\begin{aligned} \int_{\mathbb{R}} \mathrm{e}^{\beta x} S^{(\boldsymbol{r})}(x)\,\mathrm{d}x &= \int_0^\infty \int_{\mathbb{R}} \mathrm{e}^{\beta x} \mathbb{P}(X_t \in \mathrm{d}x)\,\mathrm{d}t \\ &= \int_0^\infty \mathrm{e}^{(K(\beta) - \Delta_{\boldsymbol{r}})t}\,\mathrm{d}t \\ &= (\Delta_{\boldsymbol{r}} - K(\beta))^{-1} \end{aligned} \tag{2}$$

for suitable values β; see Proposition 2.2 of [2] for the second equality or Theorem 7.11 of [6] for the whole statement.

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The local time $L^{(\boldsymbol{r})}$ at zero of the terminating MMBM $(\mathcal{X}, \mathcal{J})$ may then be defined as the resolvent at zero, i.e. $L^{(\boldsymbol{r})} := S^{(\boldsymbol{r})}(0)$. The spatial homogeneity of the level process $\mathcal{X}$, conditional on the phase process $\mathcal{J}$, further leads to the obvious relation

$$S^{(\boldsymbol{r})}(x) = \begin{cases} \begin{pmatrix} I_a \\ A^+ \end{pmatrix} (\mathrm{e}^{U^+ x}, \mathbf{0}) L^{(\boldsymbol{r})}, & x > 0, \\ \begin{pmatrix} A^- \\ I_d \end{pmatrix} (\mathbf{0}, \mathrm{e}^{-U^- x}) L^{(\boldsymbol{r})}, & x < 0, \end{cases} \tag{3}$$

between the resolvent and the local time; see Remark 1. Thus, the resolvent is continuous in $x \neq 0$, and, therefore, uniquely determined by its transform equation (2) on the one hand and by the local time on the other hand. Write

$$L^{(\boldsymbol{r})} = \begin{pmatrix} L^{(\boldsymbol{r})}_{pp} & L^{(\boldsymbol{r})}_{p\sigma} & L^{(\boldsymbol{r})}_{pn} \\ L^{(\boldsymbol{r})}_{\sigma p} & L^{(\boldsymbol{r})}_{\sigma\sigma} & L^{(\boldsymbol{r})}_{\sigma n} \\ L^{(\boldsymbol{r})}_{np} & L^{(\boldsymbol{r})}_{n\sigma} & L^{(\boldsymbol{r})}_{nn} \end{pmatrix}$$

in obvious block notation and use the same notation for $A^{\pm} = A^{\pm}(\boldsymbol{r})$ and $U^{\pm} = U^{\pm}(\boldsymbol{r})$. Furthermore, write Δ_{2/σ^2} for the diagonal matrix on E_σ with entries $2/\sigma_i^2$, $\Delta^p_{1/\mu}$ for the diagonal matrix on E_p with entries $1/\mu_i$, and $\Delta^n_{-1/\mu}$ for the diagonal matrix on E_n with entries $-1/\mu_i$.

Theorem 1. *The block entries of the matrix $L^{(\boldsymbol{r})}$ of expected local times are given by*

$$\begin{aligned}
L^{(\boldsymbol{r})}_{pp} &= [I_p - A^-_{pn}A^+_{np} + (A^-_{p\sigma} + A^-_{pn}A^+_{n\sigma})(U^+_{\sigma\sigma} + U^-_{\sigma\sigma} + U^-_{\sigma n}A^+_{n\sigma})^{-1}(U^+_{\sigma p} + U^-_{\sigma n}A^+_{np})]^{-1} \\
&\qquad \times \Delta^p_{1/\mu}, \\
L^{(\boldsymbol{r})}_{\sigma p} &= -(U^+_{\sigma\sigma} + U^-_{\sigma\sigma} + U^-_{\sigma n}A^+_{n\sigma})^{-1}(U^+_{\sigma p} + U^-_{\sigma n}A^+_{np})L^{(\boldsymbol{r})}_{pp}, \\
L^{(\boldsymbol{r})}_{np} &= A^+_{np}L^{(\boldsymbol{r})}_{pp} + A^+_{n\sigma}L^{(\boldsymbol{r})}_{\sigma p}, \\
L^{(\boldsymbol{r})}_{p\sigma} &= (I_p - A^-_{pn}A^+_{np})^{-1}(A^-_{pn}A^+_{n\sigma} + A^-_{p\sigma})L^{(\boldsymbol{r})}_{\sigma\sigma}, \\
L^{(\boldsymbol{r})}_{\sigma\sigma} &= -[U^+_{\sigma p}(I_p - A^-_{pn}A^+_{np})^{-1}(A^-_{pn}A^+_{n\sigma} + A^-_{p\sigma}) + (U^+_{\sigma\sigma} + U^-_{\sigma\sigma}) \\
&\qquad + U^-_{\sigma n}(I_n - A^+_{np}A^-_{pn})^{-1}(A^+_{np}A^-_{p\sigma} + A^+_{n\sigma})]^{-1}\Delta_{2/\sigma^2}, \\
L^{(\boldsymbol{r})}_{n\sigma} &= (I_n - A^+_{np}A^-_{pn})^{-1}(A^+_{np}A^-_{p\sigma} + A^+_{n\sigma})L^{(\boldsymbol{r})}_{\sigma\sigma}, \\
L^{(\boldsymbol{r})}_{nn} &= [I_n - A^+_{np}A^-_{pn} + (A^+_{n\sigma} + A^+_{np}A^-_{p\sigma})(U^+_{\sigma\sigma} + U^-_{\sigma\sigma} + U^+_{\sigma p}A^-_{p\sigma})^{-1}(U^-_{\sigma n} + U^+_{\sigma p}A^-_{pn})]^{-1} \\
&\qquad \times \Delta^n_{-1/\mu}, \\
L^{(\boldsymbol{r})}_{\sigma n} &= -(U^+_{\sigma\sigma} + U^-_{\sigma\sigma} + U^+_{\sigma p}A^-_{p\sigma})^{-1}(U^-_{\sigma n} + U^+_{\sigma p}A^-_{pn})L^{(\boldsymbol{r})}_{nn}, \\
L^{(\boldsymbol{r})}_{pn} &= A^-_{pn}L^{(\boldsymbol{r})}_{nn} + A^-_{p\sigma}L^{(\boldsymbol{r})}_{\sigma n}.
\end{aligned}$$

Proof. We verify that the resolvent as proposed solves the transform equation

$$\int_{\mathbb{R}} \mathrm{e}^{\beta x} S^{(\boldsymbol{r})}(x)\,\mathrm{d}x = (\Delta_{\boldsymbol{r}} - K(\beta))^{-1}.$$

The same arguments as for Equation (5) of [4] yield

$$\Delta_{\boldsymbol{r}} \begin{pmatrix} I_a \\ A^+(\boldsymbol{r}) \end{pmatrix} = \Delta_{\sigma^2/2} \begin{pmatrix} I_a \\ A^+(\boldsymbol{r}) \end{pmatrix} U^+(\boldsymbol{r})^2 - \Delta_\mu \begin{pmatrix} I_a \\ A^+(\boldsymbol{r}) \end{pmatrix} U^+(\boldsymbol{r}) + Q \begin{pmatrix} I_a \\ A^+(\boldsymbol{r}) \end{pmatrix},$$

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where I_a denotes the identity matrix on $E_p \cup E_\sigma$ (use the function $\mathbb{E}_{ij}(\exp[-\int_0^{\tau^+(x)} r_{J_s}\, \mathrm{d}s])$ instead of $f_{ij}(x)$ as defined in Section 3 of [4]). Note that I_a is denoted by I_σ in [4] and there is a typo in Equation (6) of [4], where it should state $-\Delta_\mu$ instead of $+\Delta_\mu$. The cumulant matrix can be written as $K(\beta) = \Delta_{\sigma^2/2}\beta^2 + \Delta_\mu \beta + Q$, whence we obtain

$$(K(\beta) - \Delta_{\boldsymbol{r}})\begin{pmatrix} I_a \\ A^+ \end{pmatrix} = \left(\Delta_{\sigma^2/2}\begin{pmatrix} I_a \\ A^+ \end{pmatrix}(\beta I_a - U^+) + \Delta_\mu \begin{pmatrix} I_a \\ A^+ \end{pmatrix}\right)(\beta I_a + U^+).$$

For the negative process $(\mathcal{X}^-, \mathcal{J})$, we obtain in the same way

$$\Delta_{\boldsymbol{r}}\begin{pmatrix} A^-(\boldsymbol{r}) \\ I_d \end{pmatrix} = \Delta_{\sigma^2/2}\begin{pmatrix} A^-(\boldsymbol{r}) \\ I_d \end{pmatrix} U^-(\boldsymbol{r})^2 + \Delta_\mu \begin{pmatrix} A^-(\boldsymbol{r}) \\ I_d \end{pmatrix} U^-(\boldsymbol{r}) + Q\begin{pmatrix} A^-(\boldsymbol{r}) \\ I_d \end{pmatrix},$$

where I_d denotes the identity matrix on $E_\sigma \cup E_n$, and, hence,

$$(K(\beta) - \Delta_{\boldsymbol{r}})\begin{pmatrix} A^- \\ I_d \end{pmatrix} = \left(\Delta_{\sigma^2/2}\begin{pmatrix} A^- \\ I_d \end{pmatrix}(\beta I_d + U^-) + \Delta_\mu \begin{pmatrix} A^- \\ I_d \end{pmatrix}\right)(\beta I_d - U^-).$$

Equation (3) yields (for small enough $|\beta|$)

$$\begin{aligned}\int_{\mathbb{R}} \mathrm{e}^{\beta x} S^{(\boldsymbol{r})}(x)\,\mathrm{d}x &= \int_0^\infty \mathrm{e}^{\beta x}\begin{pmatrix} I_a \\ A^+ \end{pmatrix}(\mathrm{e}^{U^+ x}, \mathbf{0})\,\mathrm{d}x\, L^{(\boldsymbol{r})} + \int_0^\infty \mathrm{e}^{-\beta x}\begin{pmatrix} A^- \\ I_d \end{pmatrix}(\mathbf{0}, \mathrm{e}^{U^- x})\,\mathrm{d}x\, L^{(\boldsymbol{r})} \\ &= \begin{pmatrix} I_a \\ A^+ \end{pmatrix}(-(\beta I_a + U^+)^{-1}, \mathbf{0})L^{(\boldsymbol{r})} + \begin{pmatrix} A^- \\ I_d \end{pmatrix}(\mathbf{0}, (\beta I_d - U^-)^{-1})L^{(\boldsymbol{r})}.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}&(\Delta_{\boldsymbol{r}} - K(\beta))\int_{\mathbb{R}} \mathrm{e}^{\beta x} S^{(\boldsymbol{r})}(x)\,\mathrm{d}x \\ &\quad= \left(\Delta_{\sigma^2/2}\begin{pmatrix} I_a \\ A^+ \end{pmatrix}(\beta I_a - U^+) + \Delta_\mu \begin{pmatrix} I_a \\ A^+ \end{pmatrix}, \mathbf{0}\right)L^{(\boldsymbol{r})} \\ &\qquad - \left(\mathbf{0}, \Delta_{\sigma^2/2}\begin{pmatrix} A^- \\ I_d \end{pmatrix}(\beta I_d + U^-) + \Delta_\mu \begin{pmatrix} A^- \\ I_d \end{pmatrix}\right)L^{(\boldsymbol{r})} \\ &\quad= \begin{pmatrix} \Delta_\mu^{pp} & -\Delta_\mu^{pp} A^-_{p\sigma} & -\Delta_\mu^{pp} A^-_{pn} \\ -\Delta_{\sigma^2/2} U^+_{\sigma p} & -\Delta_{\sigma^2/2}(U^+_{\sigma\sigma} + U^-_{\sigma\sigma}) & -\Delta_{\sigma^2/2} U^-_{\sigma n} \\ \Delta_\mu^{nn} A^+_{np} & \Delta_\mu^{nn} A^+_{n\sigma} & -\Delta_\mu^{nn} \end{pmatrix} L^{(\boldsymbol{r})}.\end{aligned} \tag{4}$$

The facts that $L^{(\boldsymbol{r})}_{n\sigma} = A^+_{np} L^{(\boldsymbol{r})}_{p\sigma} + A^+_{n\sigma} L^{(\boldsymbol{r})}_{\sigma\sigma}$ and $L^{(\boldsymbol{r})}_{p\sigma} = A^-_{pn} L^{(\boldsymbol{r})}_{n\sigma} + A^-_{p\sigma} L^{(\boldsymbol{r})}_{\sigma\sigma}$ follow from Remark 1. It now simply remains to verify that $L^{(\boldsymbol{r})}$ as given in the statement yields the result.

Example 1. The case of a Brownian motion with variance σ^2 and drift μ is covered by Exercise 2 of [3, Section VII]. There the resolvent density is given by $u^q(x) = \Phi'(q)\mathrm{e}^{-\Phi(q)x}$ for $x > 0$, where $\Phi(q)$ is the positive inverse of the cumulant function $\psi(\beta) := \beta^2\sigma^2/2 + \beta\mu$. Thus, $\Phi(q) = -\sigma^{-2}(\mu - \sqrt{\mu^2 + 2q\sigma^2})$ and because of $U^\pm(q) = \sigma^{-2}(\pm\mu - \sqrt{\mu^2 + 2q\sigma^2})$ we obtain $\Phi(q) = -U^+(q)$. Since further $U^+(q) + U^-(q) = -2/\sigma^2\sqrt{\mu^2 + 2q\sigma^2}$ and, hence, $\Phi'(q) = (\sqrt{\mu^2 + 2q\sigma^2})^{-1} = -2\sigma^{-2}(U^+(q) + U^-(q))^{-1}$, we obtain $L^q = \Phi'(q)$ and $S^q(x) = u^q(x)$ according to Theorem 1 and (3), respectively.

Example 2. For the case $E_n = \varnothing$, the matrix $L^{(\boldsymbol{r})}$ plays a role in the investigation of the scale function as in Section 7.5 of [6]. In particular, Section 7.7 therein states that in this case $L^{(\boldsymbol{r})} = \Xi^{-1}$, where

$$\Xi = \Delta_\mu(\Pi^+ - \Pi^-\Pi^+_-) - \tfrac{1}{2}\Delta^2_\sigma(\Pi^+(\Lambda^+ - \alpha I) + \Pi^-(\Lambda^- + \alpha I)\Pi^+_-)$$

in the notation of [6]. This translates for the case $E_n = \varnothing$ as $\Pi^+ = I_a$, $\Pi^- = \binom{A^-}{I_\sigma}$, $\Pi^+_- = (\mathbf{0}, I_\sigma)$, and $\Lambda^\pm = U^\pm$. Thus, in our notation

$$\begin{aligned}\Xi &= \Delta_\mu \begin{pmatrix} I_p & -A^- \\ \mathbf{0} & \mathbf{0} \end{pmatrix} - \Delta_{\sigma^2/2} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ U^+_{\sigma p} & U^+_{\sigma\sigma} + U^- \end{pmatrix} \\ &= \begin{pmatrix} \Delta^p_\mu & -\Delta^p_\mu A^-_{p\sigma} \\ -\Delta_{\sigma^2/2} U^+_{\sigma p} & -\Delta_{\sigma^2/2}(U^+_{\sigma\sigma} + U^-_{\sigma\sigma}) \end{pmatrix},\end{aligned}$$

since $E_n = \varnothing$ implies that $A^- = A^-_{p\sigma}$ and $U^- = U^-_{\sigma\sigma}$. This is the upper left-hand part of the matrix in (4).

4. An application to insurance risk

The authors of [1] considered a compound Poisson model, which is observed at the times of a renewal process with Erlang-distributed renewal intervals. We shall consider a compound Poisson model with phase-type claim sizes and interobservation times. The rate of the exponential interclaim times is denoted by $\lambda > 0$. Let $T \sim \mathrm{PH}(\alpha^{(o)}, T^{(o)})$ denote a generic observation interval and $C \sim \mathrm{PH}(\alpha^{(c)}, T^{(c)})$ a generic claim size. Denote the dimensions of $\alpha^{(o)}$ and $\alpha^{(c)}$ by $m^{(o)}$ and $m^{(c)}$, respectively. Furthermore, denote the exit rate vectors by $\eta^{(o)} := -T^{(o)}\mathbf{1}$ and $\eta^{(c)} := -T^{(c)}\mathbf{1}$, where $\mathbf{1}$ denotes column vectors of appropriate dimension with all entries being 1.

In order to determine the values of interest, we employ the construction described in Remark 2 and consider an MMBM with phase space $E = E_p \cup E_n$, where $E_p = \{1, \ldots, m^{(o)}\}$ and $E_n = \{(i, k) \colon i \in E_p,\ 1 \le k \le m^{(c)}\}$. The positive drift phases from E_p simply store the phase of the observation period. The second variable in a negative drift phase from E_n stores the phase of the claim size distribution, while the first variable remembers the phase of the observation period into which to jump back after the claim is paid out. Thus, the generator matrix of the phase process is given as the block matrix

$$Q = \begin{pmatrix} -\lambda I + T^{(o)} + \Delta_{\eta^{(o)}} & \lambda I \otimes \alpha^{(c)} \\ I \otimes \eta^{(c)} & I \otimes T^{(c)} \end{pmatrix}.$$

The phase-dependent parameters are given as $\mu_i = c$ and $\sigma_i = 0$ for $i \in E_p$, where $c > 0$ denotes the rate of premium income. For $i \in E_n$, we set $\mu_i = -1$ and $\sigma_i = 0$.

Let $p(x, y) := \mathbb{P}(X_T \in \mathrm{d}y \mid X_0 = x)$ denote the transition densities between observation points. Setting $r_i := \eta^{(o)}_i$ for all $i \in E_p$ and $r_i := 0$ for all $i \in E_n$, and noting that we observe only phases in E_p (for which $\mu_i = c$), we can evaluate those as

$$\begin{aligned} p(x, y) &= \alpha^{(o)} S^{(\boldsymbol{r})}_{pp}(y - x)\eta^{(o)} \\ &= \begin{cases} \dfrac{1}{c}\alpha^{(o)} \mathrm{e}^{U^+\cdot(y-x)}(I_p - A^-_{pn}A^+_{np})^{-1}\eta^{(o)}, & y \ge x, \\ \dfrac{1}{c}\alpha^{(o)} A^-_{pn}\mathrm{e}^{U^-\cdot(x-y)}A^+_{np}(I_p - A^-_{pn}A^+_{np})^{-1}\eta^{(o)}, & y < x, \end{cases} \end{aligned}$$

for all $x, y \in \mathbb{R}$, where we write $U^\pm = U^\pm(\boldsymbol{r})$ and $A^\pm = A^\pm(\boldsymbol{r})$.

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In particular, if the claim sizes and the interobservation times are exponential with parameters $\nu > 0$ and $\gamma > 0$, respectively, then we obtain $E = \{1, 2\}$ and

$$Q = \begin{pmatrix} -\lambda & \lambda \\ \nu & -\nu \end{pmatrix}$$

as well as $\boldsymbol{r} = (\gamma, 0)$. Example 3 of [5] with $\beta = \nu$ yields

$$A^-(\boldsymbol{r}) = \frac{\nu + U^-(\boldsymbol{r})}{\nu}, \qquad A^+(\boldsymbol{r}) = \frac{\nu}{\nu - U^+(\boldsymbol{r})},$$
$$\text{and} \quad U^\pm(\boldsymbol{r}) = \frac{1}{2c}\Big(\pm(c\nu - \gamma - \lambda) - \sqrt{(c\nu - \gamma - \lambda)^2 + 4c\nu\gamma}\Big).$$

Since $\alpha^{(o)} = 1$ and $\eta^{(o)} = \gamma$, we obtain

$$p(0, y) = \begin{cases} \dfrac{\gamma}{c} \mathrm{e}^{U^+ x} \dfrac{\nu - U^+}{-U^+ - U^-}, & y > 0, \\ \dfrac{\gamma}{c} \mathrm{e}^{U^- y} \dfrac{\nu + U^-}{-U^+ - U^-}, & y < 0. \end{cases}$$

Setting $\delta = 0$ in [1], we see from (13) therein that the notation translates as $U^+(\boldsymbol{r}) = -\rho_\gamma$ and $U^-(\boldsymbol{r}) = -R_\gamma$. Noting that in [1] the authors considered the net claim process, the increments of which are the negative increments of the risk reserve, we find that the result for the density function above coincides with $g_\delta(y)$ as given in Section 3.1 of [1].

Coming back to the more general setting of phase-type claim sizes and interobservation times, we now consider the expected number of ruin events between observation times. This may be considered as a risk measure in order to determine a suitable distribution for the interobservation times. Given that the risk reserve starts with $x > 0$ at the beginning of an observation interval, denote this as $R(\alpha^{(o)}, T^{(o)} \mid x)$. Recalling (1) and $|\mu_i| = 1$ for all $i \in E_n$, we obtain $R(\alpha^{(o)}, T^{(o)} \mid x) = \alpha^{(o)} A^-_{pn} \mathrm{e}^{U^- x}(I_p - A^+_{np} A^-_{pn})^{-1}$ for all $x > 0$.

5. The resolvent and the local times at zero for reflected MMBMs

Now we consider an MMBM that is reflected upwards at zero. Define the infimum process $\mathcal{I} = (I_t : t \geq 0)$ by $I_t := \inf_{s \leq t} X_s \wedge 0$ for all $t \geq 0$, and the reflected process by $\mathcal{Y} := \mathcal{X} - \mathcal{I}$. Again, we assume throughout that $X_0 = 0$. Define the $\boldsymbol{r}$-resolvent of $(\mathcal{Y}, \mathcal{J})$ as the matrix-valued function $R^{(\boldsymbol{r})}(x)$ with entries $\lim_{\varepsilon \downarrow 0}(2\varepsilon)^{-1}\mathbb{E}(\int_0^{\tau_\Delta} \mathbb{I}_{\{|Y_t - x| < \varepsilon,\ J_t = j\}}\, \mathrm{d}t \mid Y_0 = 0,\ J_0 = i)$ for $x > 0$ and $i, j \in E$. Note that the (scalar) local time at zero, say $l^{(\boldsymbol{r})}(0)$, can be defined as the absolute infimum before τ_Δ, i.e. $l^{(\boldsymbol{r})}(0) = -I_{\tau_\Delta}$; cf. Chapter IX.2 of [2].

In order to state the result, we first introduce some abbreviations to simplify the notation. Define the matrices

$$C^+ := \begin{pmatrix} \mathbf{0} & I_\sigma \\ \multicolumn{2}{c}{A^+(\boldsymbol{r})} \end{pmatrix} \quad \text{and} \quad C^- := \begin{pmatrix} \multicolumn{2}{c}{A^-(\boldsymbol{r})} \\ I_\sigma & \mathbf{0} \end{pmatrix}$$

of dimensions $E_d \times E_a$ and $E_a \times E_d$ as well as the matrix $W^- := \binom{A^-(\boldsymbol{r})}{I_d}$ of dimension $E \times E_d$. Denote the first exit time from the interval $[0, x]$ by $\tau := \inf\{t \geq 0\colon X_t \notin [0, x]\}$. Define the matrices $\Psi^+_{\boldsymbol{r}}(x) := \mathbb{P}(\tau < \tau_\Delta,\ X_\tau = x \mid X_0 = 0,\ J_0 \in E_p)$ and $\Psi^-_{\boldsymbol{r}}(x) := \mathbb{P}(\tau < \tau_\Delta,\ X_\tau = 0 \mid X_0 = 0,\ J_0 \in E_p)$ of dimensions $E_p \times E_a$ and $E_p \times E_d$, respectively. According to [7], we obtain for these the expressions

$$\Psi^+_{\boldsymbol{r}}(x) = (I_p, \mathbf{0})(\mathrm{e}^{U^+ x} - C^- \mathrm{e}^{U^- x} C^+ \mathrm{e}^{U^+ x})(I - C^- \mathrm{e}^{U^- x} C^+ \mathrm{e}^{U^+ x})^{-1}$$

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and

$$\Psi_{\boldsymbol{r}}^{-}(x) = (A^{-} - (I_p, \mathbf{0})\mathrm{e}^{U^{+}x}C^{-}\mathrm{e}^{U^{-}x})(I - C^{+}\mathrm{e}^{U^{+}x}C^{-}\mathrm{e}^{U^{-}x})^{-1}.$$

Theorem 2. *The* $\boldsymbol{r}$*-resolvent of the reflected MMBM* $(\mathcal{Y}, \mathcal{J})$ *for* $x > 0$ *and* $\|\boldsymbol{r}\| > 0$ *is*

$$R^{(\boldsymbol{r})}(x) = \begin{pmatrix} \Psi_{\boldsymbol{r}}^{+}(x) - \Psi_{\boldsymbol{r}}^{-}(x)G_{\boldsymbol{r}}(x)^{-1}H_{\boldsymbol{r}}(x) \\ -G_{\boldsymbol{r}}(x)^{-1}H_{\boldsymbol{r}}(x) \end{pmatrix} (I_a + C^{-}\mathrm{e}^{U^{-}x}G_{\boldsymbol{r}}(x)^{-1}H_{\boldsymbol{r}}(x))^{-1} \times (I_a - C^{-}\mathrm{e}^{U^{-}x}C^{+}\mathrm{e}^{U^{+}x})L_{(a,.)}^{(\boldsymbol{r})},$$

where we write

$$G_{\boldsymbol{r}}(x) = (U^{-}\mathrm{e}^{-U^{-}x} + C^{+}\mathrm{e}^{U^{+}x}U^{+}C^{-})(\mathrm{e}^{-U^{-}x} - C^{+}\mathrm{e}^{U^{+}x}C^{-})^{-1}$$

and

$$H_{\boldsymbol{r}}(x) = (C^{+}U^{+} + U^{-}C^{+})(C^{-}\mathrm{e}^{U^{-}x}C^{+} - \mathrm{e}^{-U^{+}x})^{-1}.$$

The distribution of the local time at zero is given by $\mathbb{P}(l^{(\boldsymbol{r})}(0) > y \mid Y_0 = i) = e_i^{\top}W^{-}\mathrm{e}^{U^{-}y}\mathbf{1}$ *for all* $y \geq 0$.

Proof. We first observe that, starting at $x > 0$, the local times of $(\mathcal{Y}, \mathcal{J})$ at x before hitting the zero level coincide with the local times of the free MMBM $(\mathcal{X}, \mathcal{J})$ before $\tau^{-}(0)$. Considering that the level x must have been reached from below, i.e. in an ascending phase, these are given by the term $(I - C^{-}\mathrm{e}^{U^{-}x}C^{+}\mathrm{e}^{U^{+}x})L_{(a,.)}^{(\boldsymbol{r})}$; cf. Equation (7.4) of [6]. Using the results from Theorem 1 of [5] (adapted to the present reflection at the infimum), we can determine the probabilities to go from level x (in an ascending phase) to level 0 and then back to x before τ_{Δ} as $-C^{-}\mathrm{e}^{U^{-}x}G_{\boldsymbol{r}}(x)^{-1}H_{\boldsymbol{r}}(x)$. Thus, the expected number of such down and up crossings is given by $(I_a + C^{-}\mathrm{e}^{U^{-}x}G_{\boldsymbol{r}}(x)^{-1}H_{\boldsymbol{r}}(x))^{-1}$. Premultiplying by the probabilities of reaching level x from $X_0 = 0$ before τ_{Δ} for the first time, namely by $-G_{\boldsymbol{r}}(x)^{-1}H_{\boldsymbol{r}}(x)$ if $J_0 \in E_d$ and by $\Psi_{\boldsymbol{r}}^{+}(x) - \Psi_{\boldsymbol{r}}^{-}(x)G_{\boldsymbol{r}}(x)^{-1}H_{\boldsymbol{r}}(x)$ if $J_0 \in E_p$, yields the stated formula for the resolvent. The statement for the local time at zero follows immediately from the definition $l^{(\boldsymbol{r})}(0) = -I_{\tau_{\Delta}}$.

Example 3. We consider a Brownian motion with variation σ^2 and drift μ that is reflected at zero. In this case, $\boldsymbol{r}$ is a number, $U^{\pm}$ are given as in Example 1, and

$$G_{\boldsymbol{r}}(x) = \frac{U^{-} + U^{+}\mathrm{e}^{(U^{+}+U^{-})x}}{1 - \mathrm{e}^{(U^{+}+U^{-})x}}$$

as well as

$$H_{\boldsymbol{r}}(x) = \frac{-(U^{+} + U^{-})\mathrm{e}^{U^{+}x}}{1 - \mathrm{e}^{(U^{-}+U^{+})x}}.$$

Thus,

$$-G_{\boldsymbol{r}}(x)^{-1}H_{\boldsymbol{r}}(x) = \frac{U^{+} + U^{-}}{U^{-} + U^{+}\mathrm{e}^{(U^{+}+U^{-})x}}\mathrm{e}^{U^{+}x}$$

and

$$1 + \mathrm{e}^{U^{-}x}G_{\boldsymbol{r}}(x)^{-1}H_{\boldsymbol{r}}(x) = U^{-}\frac{1 - \mathrm{e}^{(U^{-}+U^{+})x}}{U^{-} + U^{+}\mathrm{e}^{(U^{-}+U^{+})x}}.$$

Furthermore, $L^{(\boldsymbol{r})} = -U^+U^-/(U^+ + U^-)\boldsymbol{r}^{-1}$ and, hence,

$$R^{(\boldsymbol{r})}(x) = \frac{(U^+ + U^-)\mathrm{e}^{U^+x}}{U^- + U^+\mathrm{e}^{(U^+ + U^-)x}} \, \frac{U^- + U^+\mathrm{e}^{(U^- + U^+)x}}{U^-(1 - \mathrm{e}^{(U^- + U^+)x})} \, \frac{U^+U^-(1 - \mathrm{e}^{(U^- + U^+)x})}{-\boldsymbol{r}(U^+ + U^-)}$$
$$= \frac{-U^+\mathrm{e}^{U^+x}}{\boldsymbol{r}}.$$

Thus, a reflected Brownian motion observed at an exponential time has an exponential distribution with parameter $-U^+ = (\sqrt{\mu^2 + 2q\sigma^2} - \mu)/\sigma^2$. This result is part of the statement in [2, Problem 3.3, Chapter IX].

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