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BHASKAR RAO DESIGNS FROM CYCLOTOMY

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Abstract

A Bhaskar Rao design is obtained from the incidence matrix of a partially balanced incomplete block design with m associate classes by negating some elements of the matrix in such a way that the inner product of rows α and β is c_i if α and β are ith associates. In this paper we use nested designs constructed from unions of cyclotomic classes to give Bhaskar Rao designs.

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1. Introduction

Let $X = A - B$, where A and B are $v \times b$ $(0, 1)$ matrices, and the Hadamard product of A and B, $A * B$, is zero. Then X is a *Bhaskar Rao design* if

(i) $XX^{\mathrm{T}} = rI + \sum_{i=1}^{m} c_i B_i$,

(ii) $N = A + B$ satisfies $NN^{\mathrm{T}} = rI + \sum_{i=1}^{m} \lambda_i B_i$ (that is, N is the incidence matrix of a PBIBD(m)).

Such a matrix X will be denoted by BRD$(v, b, r, k;\ \lambda_1, \ldots, \lambda_m;\ c_1, \ldots, c_m)$.

These designs have been considered by several authors. They are a generalization of weighing matrices (Geramita and Seberry (1979)) and are useful in the construction of PBIBDs (Bhaskar Rao (1970); Dey and Midha (1976); Street and Rodger (1979)).

We now introduce some notation useful in the remainder of the paper. As in Morgan et al. (1976): A & B is the collection of all the elements of A and B, preserving multiplicity; $A + B$ is the collection of nonzero sums $a + b$, $a \in A$, $b \in B$, preserving multiplicity; $A - B$ is similarly defined with nonzero differences; nA is the collection of n copies of A. As in Storer (1967), the cyclotomic number (i, j) is the number of ordered pairs s, t such that

$$x^{es+i} + 1 = x^{et+j} \quad (0 \leqslant s, t \leqslant f - 1),$$

where x is a primitive root of GF(p^n), $p^n = ef+1$. If there is doubt as to which factorization of p^n-1 is being used, it is specified by writing $(i,j)_e$.

Let G be an abelian group with its elements ordered as $g_1, g_2, \ldots, g_v$ in some way. Let T be a subset of G and suppose $T = R \cup S$, where $R \cap S = \emptyset$. We will denote by $(R, + : S, -)$ the matrix $A = (a_{ij})$ with

$$a_{ij} = \begin{cases} 1 & g_j - g_i \in R, \\ -1 & g_j - g_i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix obtained by squaring each element of A is the usual incidence matrix of T.

The following lemma appears in Street and Rodger (1979) for the case $m = 1$; the proof is analogous.

Lemma 1. *Let $T_1, T_2, \ldots, T_n$ be the initial blocks of a* PBIBD(m) *and assume the elements of the ith associate class occur λ_i times. Thus*

$$\mathop{\&}_{i=1}^{n} (T_i - T_i) = \mathop{\&}_{j=1}^{m} \lambda_j C_j,$$

where C_j is the jth associate class. Suppose $T_i = R_i \cup S_i$, $R_i \cap S_i = \emptyset$, $1 \leqslant i \leqslant n$, and

$$\left\{\mathop{\&}_{i=1}^{n} (R_i - R_i)\right\} \& \left\{\mathop{\&}_{i=1}^{n} (S_i - S_i)\right\} = \mathop{\&}_{j=1}^{m} \mu_j C_j.$$

Then $X = [(R_1, +; S_1, -) : (R_2, +; S_2, -) : \ldots : (R_n, +; S_n, -)]$ is a BRD$(v, nv, nk, k;$ $\lambda_1, \ldots, \lambda_m;$ $2\mu_1 - \lambda_1, \ldots, 2\mu_m - \lambda_m)$.

In the remainder of the paper we use results of Lehmer (1974), Homel and Robinson (1975) and Morgan et al. (1976) to construct sets which satisfy the conditions of this lemma.

2. Nested block designs

Lehmer (1974) has given a family of supplementary difference sets (sds). Her method is used to give related families of sds.

Lemma 2. *Let $p^n = 2mf+1$ be a prime power. Let f be odd and let x be a primitive root of* GF(p^n). *Denote the cyclotomic classes with $e = 2m$ by*

$$C_i = \{x^{2ms+i} \mid s = 0, 1, \ldots, f-1\}, \quad i = 0, 1, \ldots, 2m-1.$$

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Let $i_0 = 0, i_1, \ldots, i_{m-1}$ be a complete set of residues mod m such that $0 \leqslant i_j \leqslant 2m-1$ for every j and let A be a subset of $\{0, 1, \ldots, m-1\}$. Then the m sets

$$T_h = \bigcup_{j \in A} C_{i_j - i_h}, \quad h = 0, 1, \ldots, m-1$$

are $m-\{2mf+1;\ tf;\ t(tf-1)/2\}$ sds, *where $t = |A|$.*

PROOF. We see that

$$T_h - T_h = \mathop{\&}_{k=0}^{2m-1} \left(\sum_{j \in A} \sum_{i \in A} (i_i - i_j + m, k - i_j + i_h)\right) C_k$$

and

$$\mathop{\&}_{h=0}^{m-1} (T_h - T_h) = \mathop{\&}_{k=0}^{2m-1} \left(\sum_{h=0}^{m-1} \sum_{j \in A} \sum_{i \in A} (i_i - i_j + m, k - i_j + i_h)\right) C_k.$$

As f is odd,

$$(i_i - i_j + m, k - i_j + i_h) = (i_j - i_i + m, k + m + i_h - i_i)$$

and

$$\sum_{h=0}^{m-1} (i_i - i_j + m, k - i_j + i_h) + \sum_{h=0}^{m-1} (i_i - i_j + m, k + m + i_h - i_j) = f - \delta_{ij},$$

where δ_{ij} is the Kronecker delta. Summing over both i and j, we obtain

$$\begin{aligned} \lambda &= \sum_{j \in A} \sum_{i \in A} \sum_{h=0}^{m-1} (i_i - i_j + m, k - i_j + i_h) \\ &= \sum_{j \in A} \sum_{i \in A} \sum_{h=0}^{m-1} (i_j - i_i + m, k + m + i_h - i_i) \\ &= t(tf-1)/2, \quad \text{as required.} \end{aligned}$$

THEOREM 1. *Let $v = p^n = 2mf+1$ be a prime power, where f is odd. Then there exists a*

$$\mathrm{BRD}\,(v, mv, tf, mtf;\ t(tf-1)/2;\ (4fr^2 - 4tfr + t^2 f - t)/2)$$

for all $1 \leqslant r < t \leqslant m$.

PROOF. Let A be a subset of $\{0, 1, \ldots, m-1\}$ of order t such that $A = B \cup C$, where $B \cap C = \emptyset$ and B is of order r. Then let

$$T_h = \bigcup_{j \in A} C_{i_j - i_h}, \quad R_h = \bigcup_{j \in B} C_{i_j - i_h} \quad \text{and} \quad S_h = \bigcup_{j \in C} C_{i_j - i_h}$$

(where i_j and C_i are as defined in Lemma 2); the result follows from Lemmas 1 and 2.

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COROLLARY 1. *Let $v = p^n = 2mf+1$ be a prime power, where f is odd. Then there exists a* BRD$(v, mv, a^2f^2, ma^2f^2; a^2f(a^2f^2-1)/2; 0)$ *for all* $0 < a^2f \leqslant m$.

COROLLARY 2. *Let $v = p^n = 2mf+1$ be a prime power, where f is odd. Then there exists a* BRD$(v, mv, 2nf, 2mnf; n(2nf-1); -n)$ *for all* $0 < 2n \leqslant m$.

3. Nested PBIBDs

In this section we use the notation of Morgan et al. (1976); in addition let

$$M_i = \{x^{a2\alpha\beta+i} \mid a = 0, 1, \ldots, \gamma-1\}, \quad i = 0, 1, \ldots, 2\alpha\beta-1,$$

where x is a primitive root of GF(p^n) and $p^n = 2\alpha\beta\gamma+1$, where p is an odd prime and n, α, β, γ are positive integers with $\alpha, \beta, \gamma \geqslant 2$.

Choose an integer t such that $0 < t \leqslant 2\alpha\beta$, and t distinct integers $a_1, a_2, \ldots, a_t$ such that $0 \leqslant a_1 < a_2 < \ldots < a_t \leqslant 2\alpha\beta-1$. Define

$$G_i = \bigcup_{i=1}^{t} M_{a_h+i}, \quad i = 0, 1, \ldots, 2\alpha\beta-1$$

and

$$H_i = \{0\} \cup G_i, \quad i = 0, 1, \ldots, 2\alpha\beta-1.$$

LEMMA 3. *If γ is odd, $-M_i = M_{i+\alpha\beta}$ and*

$$G_i - G_i = \mathop{\&}_{k=0}^{2\alpha\beta-1} g_k M_{k+i}, \quad i = 0, 1, \ldots, 2\alpha\beta-1,$$

where

$$g_k = \sum_{h=1}^{t}\sum_{j=1}^{t} (a_j - a_h + \alpha\beta, k - a_h)_{2\alpha\beta},$$

$$H_i - H_i = \mathop{\&}_{k=0}^{2\alpha\beta-1} h_k M_{k+i}, \quad i = 0, 1, \ldots, 2\alpha\beta-1$$

where

$$h_k = g_k + \sum_{j=1}^{t} \delta_{a_j,k} + \sum_{j=1}^{t} \delta_{a_j+\alpha\beta,k}.$$

The designs constructed in the following theorem generalize Theorem 2.7 of Homel (1972) (see also Homel and Robinson (1975)).

THEOREM 2. *Let $\beta\gamma$ be odd, and let $p^n = 2\alpha\beta\gamma+1$ be a prime power. Then the β sets $G_{2\alpha j}$, $j = 0, 1, \ldots, \beta-1$, or the β sets $H_{2\alpha j}$, $j = 0, 1, \ldots, \beta-1$, may be used as the initial* *blocks of a* PBIBD(m) *where* $v = p^n, b = \beta v, m \leqslant \alpha$ *and each associate class consists of a cyclotomic class*

$$C_k = \{x^{a\alpha+k} \mid a = 0, 1, \ldots, 2\beta\gamma - 1\} \quad k = 0, 1, \ldots, \alpha - 1$$

or a union of such classes. Table 1 contains the parameters of the design when $m = \alpha$.

TABLE 1.

Initial blocks	k	r	Parameters when $m = \alpha$		
			n_i	λ_i	p_{ij}^k
(i) $G_{2\alpha j}$	$t\gamma$	βk	$2\beta\gamma$	$\sum_{j=0}^{\beta-1} g_{i+2\alpha j}$	$(k-j, i-j)_\alpha$
(ii) $H_{2\alpha j}$	$t\gamma+1$	βk	$2\beta\gamma$	$\sum_{j=0}^{\beta-1} h_{i+2\alpha j}$	$(k-j, i-j)_\alpha$

PROOF. (i) From Lemma 3 we have

$$\underset{j=0}{\overset{\beta-1}{\&}} (G_{2\alpha j} - G_{2\alpha j}) = \underset{k=0}{\overset{2\alpha\beta-1}{\&}} g_k \underset{j=0}{\overset{\beta-1}{\&}} M_{2\alpha j+k}$$

$$= \underset{k=0}{\overset{2\alpha-1}{\&}} \left(\sum_{i=0}^{\beta-1} g_{k+2\alpha i} \right) C_k^{(2\alpha)},$$

where $C_k^{(2\alpha)}$ is the kth cyclotomic class with $e = 2\alpha$. Now

$$C_k = C_k^{(2\alpha)} \cup C_{k+\alpha}^{(2\alpha)}$$

and

$$\sum_{i=0}^{\beta-1} g_{k+2\alpha i} = \sum_{i=0}^{\beta-1} \left(\sum_{h=1}^{t} \sum_{j=1}^{t} (a_j - a_h + \alpha\beta, k + 2\alpha i - a_h)_{2\alpha\beta} \right)$$

$$= \sum_{i=0}^{\beta-1} \left(\sum_{h=1}^{t} \sum_{j=1}^{t} (a_h - a_j + \alpha\beta, k + \alpha(2i+\beta) - a_j)_{2\alpha\beta} \right)$$

$$= \sum_{i=0}^{\beta-1} g_{k+\alpha(2i+1)},$$

if β is odd. Thus the associate classes are as claimed. As in Morgan et al. (1976), $p_{ij}^k = p_{ji}^k$ as $2\beta\gamma$ is even.

The proof of (ii) is similar.

Let A and B be two disjoint sets of t and s distinct integers between 0 and $\alpha\beta - 1$. Let $C = A \cup B$ and let $E'_{j\alpha}$, $E''_{j\alpha}$, and $E_{j\alpha}$, $0 \leqslant j \leqslant \beta - 1$, be the initial blocks for the construction of Morgan et al. (1976), Theorem 2, using the sets A, B and C respectively (thus $\beta\gamma$ is even). Let the parameters of these designs be n_i, λ_i, p_{ij}^k; n_i, μ_i, p_{ij}^k; and n_i, ν_i, p_{ij}^k respectively and note that $E_{j\alpha} = E'_{j\alpha} \cup E''_{j\alpha}$. Similarly let $F'_{j\alpha} = E'_{j\alpha} \cup \{0\}$, $F''_{j\alpha} = E''_{j\alpha} \cup \{0\}$ and $F_{j\alpha} = E_{j\alpha} \cup \{0\}$ (so $F_{j\alpha} = E'_{j\alpha} \cup F''_{j\alpha}$), with parameters $n_i, \lambda'_i, p_{ij}^k$; n_i, μ'_i, p_{ij}^k; and n_i, ν'_i, p_{ij}^k respectively.

THEOREM 3. *Let $v = p^n = \alpha\beta\gamma + 1$ be a prime power, where $\beta\gamma$ is even. Then there exist:*

(*i*) $\mathrm{BRD}(v, \beta v, \beta k, (t+s)\gamma;\ v_1, \ldots, v_\alpha;\ 2(\lambda_1 + \mu_1) - v_1, \ldots, 2(\lambda_\alpha + \mu_\alpha) - v_\alpha)$;

(*ii*) $\mathrm{BRD}(v, \beta v, \beta k, (t+s)\gamma + 1;\ v'_1, \ldots, v'_\alpha;\ 2(\lambda'_1 + \mu'_1) - v'_1, \ldots, 2(\lambda'_\alpha + \mu'_\alpha) - v'_\alpha)$.

PROOF. Apply Lemma 1 to the designs discussed above.

We may use the designs of Theorem 2 in a similar construction; here

$$v_i = \sum_{j=0}^{\beta-1} \Big(\sum_{x \in C} \sum_{y \in C} (y - x + \alpha\beta, i + 2\alpha j - x)_{2\alpha\beta}\Big),$$

$$v'_i = v_i + \sum_{j=0}^{\beta-1} \Big(\sum_{x \in C} (\delta_{x, i+2\alpha j} + \delta_{x+\alpha\beta, i+2\alpha j})\Big) \quad \text{and so on.}$$

THEOREM 4. *Let $v = p^n = 2\alpha\beta\gamma + 1$ be a prime power, where $\beta\gamma$ is odd. Then there exist:*

(*i*) $\mathrm{BRD}(v, \beta v, \beta k, (t+s)\gamma;\ v_1, \ldots, v_\alpha;\ 2(\lambda_1 + \mu_1) - v_1, \ldots, 2(\lambda_\alpha + \mu_\alpha) - v_\alpha)$;

(*ii*) $\mathrm{BRD}(v, \beta v, \beta k, (t+s)\gamma + 1;\ v'_1, \ldots, v'_\alpha;\ 2(\lambda'_1 + \mu'_1) - v'_1, \ldots, 2(\lambda'_\alpha + \mu'_\alpha) - v'_\alpha)$.

Homel and Robinson (1975) give a number of nested designs which may also be used in a similar construction.

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