NOTE ON A PAPER BY ROBINSON

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1. Introduction. In a recent paper Robinson[1] has obtained an explicit formula for the expression of an invariant matrix of an invariant matrix as a direct sum of invariant matrices. The object of the present note is to show that this formula may be deduced from known properties of Schur functions, with the aid of a result which the author has proved elsewhere.[2]

2. The reduction of $\{\alpha\}\otimes\{\beta\}$. Let $\{\alpha\}$ and $\{\beta\}$ be Schur functions, of respective degrees m and n, corresponding to partitions $[\alpha]$ of m and $[\beta]$ of n. Our problem is to express $\{\alpha\}\otimes\{\beta\}$ in terms of Schur functions $\{\lambda\}$ of degree mn, where the symbol $\otimes$ denotes the so-called "new multiplication" of Littlewood.[3] A property of this operation which is important from our point of view is that it is distributive on the right with respect to addition and (ordinary) multiplication.[4] Consequently, the expression $\{\alpha\}\otimes f$ has a meaning for any rational symmetric function f (in which the number of indeterminates is at least equal to the degree). Suppose, now, that $S_0 = 1$, and that, for positive integral k, S_k is the sum of the kth powers of the indeterminates involved; and that for any partition $[\tau] = [\tau_1, \tau_2, \ldots, \tau_t]$ of a positive integer t into t non-negative parts we write

2.1 $$S_\tau = \prod_{i=1}^{t} S_{\tau_i}.$$

Then the author has shown[5] that

2.2 $$m!\,\{\alpha\} \otimes S_k = \sum_\rho h_\rho \chi_\rho^{(\alpha)} S_{[k\rho]},$$

where h_ρ is the order of the class of the symmetric group of degree m corresponding to the partition $[\rho]$, $\chi_\rho^{(\alpha)}$ is the characteristic of this class in the irreducible representation of the group corresponding to the partition $[\alpha]$, $[k\rho]$ denotes that partition of mk whose elements are k times the corresponding elements of $[\rho]$, and the summation extends over all partitions $[\rho]$ of m.

Since the operation $\otimes$ is distributive with respect to multiplication on the right, it follows from 2.2 that, if $[\nu] = [\nu_1, \nu_2, \ldots, \nu_n]$ is any partition of n into n non-negative parts,[6] we have

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[1][4].

[2][5].

[3][1], p. 206, or [2].

[4][2], [3], p. 286.

[5][5].

[6]The introduction of zero parts merely effects a slight formal simplification.

2.3 $$(m!)^n\{\alpha\} \otimes S_\nu = \sum_{\rho_1,\ldots,\rho_n} \big(\prod_{i=1} h_{\rho_i}\chi_{\rho_i}^{(\alpha)}\big) S_{[\nu,\rho]},$$

where the partitions $[\rho_i] = [\rho_{i1}, \rho_{i2}, \ldots, \rho_{im}]$ range independently over all partitions of m into m non-negative parts, and $[\nu, \rho]$ denotes the partition of mn into the mn non-negative parts $\nu_i\rho_{ij}(i = 1, \ldots, n; j = 1, \ldots, m)$.

Now, if $[\beta]$, $[\nu]$ are partitions of n, and if $[\lambda']$, $[\lambda]$ are partitions of mn, we have[7]

2.4 $$n!\,\{\beta\} = \sum_\nu h_\nu\chi_\nu^{(\beta)} S_\nu,$$

2.5 $$S_{\lambda'} = \sum_\lambda \chi_{\lambda'}^{(\lambda)} \;\{\lambda\}.$$

Accordingly, since the operation $\otimes$ is distributive with respect to addition on the right,

$$n!\,\{\alpha\} \otimes \{\beta\} = \sum h_\nu\chi_\nu^{(\beta)} [\{\alpha\} \otimes S_\nu],$$

whence, by 2.3 and 2.5,

2.6 $$(m!)^n n!\{\alpha\} \otimes \{\beta\} = \sum_{\nu,\rho_1,\ldots,\rho_n} h_\nu\chi_\nu^{(\beta)} \big(\prod_{i=1}^{n} h_{\rho_i}\chi_{\rho_i}^{(\alpha)}\big)\; \chi_{[\nu,\rho]}^{(\lambda)}\{\lambda\},$$

where $[\nu, \rho]$ is the partition of mn defined above.

3. The symmetric group. The formula 2.6 is equivalent to that given in 5.1 of Robinson's paper.[8] To see this it is necessary to explain Robinson's notation. Let $a_{l1}, \ldots, a_{lm}$ for $l = 1, \ldots, n$, be n sets of m symbols, let Σ_l be the symmetric group of degree m on the lth set of symbols, and let Σ be the symmetric group of degree mn on the entire set of mn symbols. Then Σ possesses a subgroup H which is the direct product of the n symmetric groups Σ_l, and a subgroup Σ', simply isomorphic with the symmetric group of degree n, whose operations permute the first suffixes of the symbols a_{lj} but leave the second suffixes fixed; the operations of Σ' thus permute the n sets of symbols but leave the order of the symbols in the individual sets unchanged. Let h be an operation of H which commutes with an operation s of Σ', let $\chi_\beta(s)$ be the characteristic of s in the representation of Σ' corresponding to the partition $[\beta]$ of n, and let $\phi_\lambda(hs)$ and $\chi(hs)$ be the characteristics of hs in the representation of Σ corresponding to the partition $[\lambda]$ of mn, and in the representation of the normalizer of H generated by $[\alpha]^n$. Then Robinson's expression for the coefficient of $\{\lambda\}$ in $\{\alpha\} \otimes \{\beta\}$ is

3.1 $$\frac{1}{(m!)^n n!} \sum_{h,s} \chi_\beta(s)\chi(hs)\phi_\lambda(hs).$$

[7][1] p. 86.
[8][4], p. 172.

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Suppose, now, that s belongs to the class of the symmetric group of degree n corresponding to the partition $[\nu] = [\nu_1, \ldots, \nu_n]$. Then $\chi_\beta(s)$ is the character $\chi_\nu^{(\beta)}$ of 2.6. If h is any operation of H, it leaves fixed each of the n sets of m symbols, and effects a certain permutation p_l of the second suffixes of the symbols of the lth set. It is easily seen that a necessary and sufficient condition that h commutes with s is that the permutations p_l should be the same for any pair of sets belonging to a cycle in the permutation of these sets effected by s. If, then, $[\rho_i]$ denote the class of the permutation effected by h on the sets permuted in the cycle (of ν_i sets) corresponding to the integer ν_i in $[\nu]$, it is easily seen that, in $\sum$, hs belongs to the class $[\nu, \rho]$ defined previously, so that $\phi_\lambda(hs) = \chi_{[\nu,\rho]}^{(\lambda)}$. Finally, the relation

$$\chi(hs) = \prod_{i=1} \chi_{\rho_i}^{(\alpha)}$$

is easily seen to be a consequence of section 4 of Robinson's paper. The coefficients h_ν, h_{ρ_i} in 2.6 are accounted for by the fact that the summations in 2.4 extend over classes while those in 3.1 extend over operations. Thus the two formulae are really the same.

It is perhaps pertinent to point out that, from the point of view of applications to invariant theory, the value of these explicit formulae is severely limited by the fact that tables of characters of the symmetric group are only available for comparatively low degrees. A practical method for evaluating $\{\alpha\} \otimes \{\beta\}$ when mn exceeds the limit of existing tables is still much to be desired.

References

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