

ELLIPTIC MULTIPARAMETER EIGENVALUE PROBLEMS

by P. A. BINDING and K. SEDDIGHI

(Received 13th August 1985)

0. Preliminaries

We study the eigenproblem

$$W_m(\lambda)x_m = 0 \neq x_m \in H_m, \quad m = 1, \dots, k \quad (*)$$

where

$$W_m(\lambda) = T_m - \sum_{n=1}^k \lambda_n V_{mn}$$

and T_m, V_{mn} are self-adjoint operators on separable Hilbert spaces H_m . We assume the T_m to be bounded below with compact resolvents, and the V_{mn} to be bounded and to satisfy an “ellipticity” condition. If $k=1$ then ellipticity is automatic, and if each T_m is positive definite then the problem is “left definite”.

We show that certain eigenvectors x_m of (*) corresponding to $\lambda \in \mathbb{R}^k$ generate a basis X of decomposable tensors for a subspace G of $H = \bigotimes_{m=1}^k H_m$ of finite codimension. More specifically, we define k (in general indefinite) inner products, relative to which (i) X is positive and orthogonal and (ii) G has a common orthocomplement F in H . Bounds for $\dim F$ are given, and in particular if each T_m is nonnegative definite then $\dim F \leq 2 \prod_{n=1}^k \dim N(T_m)$.

1. Introduction

Completeness and expansion theorems have formed a central part of multiparameter spectral theory since early in this century. In particular, Hilbert’s work [14; Chapter 21] on a special pair of linked Sturm–Liouville equations has been successively generalised to what is nowadays called the “left-definite” (LD) case of the eigenproblem (*) of Section 0. Such problems include a number of systems of ordinary differential equations arising from separation of variables of classical boundary value problems into various coordinate systems—cf. [1; Chapter 1]. Indeed, early work on multiparameter theory was generally motivated by such problems, in the Sturm–Liouville form

$$(p_m y'_m)' + q_m y_m = \sum_{j=1}^k \lambda_j r_{mj} y_m, \quad (1.1)$$

for eigenfunctions y_m in suitable subspaces of $L_2[a_m, b_m]$. We remark that an alternative “right definite” condition can also be used, but the present expansion theory [21] cannot handle all such applications.

In the abstract framework used here, Källström and Sleeman have given a completeness theorem for the LD case—see [19: Chapter 5]. LD is the combination of “ellipticity” of the V_{mn} and the “definiteness” of the T_m . Ellipticity is automatic when $k=1$, and involves two definiteness conditions on combinations of the V_{mn} when $k=2$. The latter are automatically satisfied, for example, for the equations (1.1) whenever they arise from the plane Helmholtz equation in any of the four coordinate systems permitting separation of variables. The other, “definiteness”, condition can be a drawback of LD, since effectively it requires positive definiteness of each T_m —for more on this question, see Example 5.5 at the end. In the (elliptic) differential equation case, it is frequently the case that the T_m are *essentially* positive definite (e.g. for (1.2), p_m takes one sign) but the “lower order” terms and boundary conditions give T_m a finite number $\nu(T_m)$ of negative eigenvalues.

Even in the one parameter case

$$Tx = \lambda Vx \tag{1.2}$$

(where we drop subscripts for simplicity and where LD simply means positive definiteness of T), the results need modification when $\nu(T) > 0$. In particular, complex eigenvalues may occur, and the eigenvectors need not be dense in H , so root vectors may be required for completeness. In the finite dimensional case, such results can be found in Weierstrass’s 1880’s theory of canonical forms of pencils (see [17], §108 or [12], Chapter I.3 for modern accounts in the context of self-adjoint operators on indefinite inner product spaces). In the Sturm–Liouville case, Richardson was aware (apparently as a by-product of his 1910’s work on two-parameter problems) of the need for complex eigenvalues, and explicit examples requiring complex eigenvalues and/or root vectors can be found in recent work of Fleckinger and Mingarelli—cf. [10]. The abstract setting, which includes both the above cases, was investigated by Pontryagin in the 1940’s in the context of indefinite inner product spaces, and can be found in ([7], Chapter IX). An infinite dimensional completeness result involving compact operators was apparently treated first by Iohvidov in 1950, but without a published proof. We recently proved a completeness result for (1.2) [6], and we shall use it in a fundamental way in Section 3. Roughly, the LD result holds on a subspace G of codimension at most $3\nu(T)$, and the known finite dimensional result holds on a suitable complement of G . We remark that one can also approach (1.2) via Langer’s “spectral function”, but this is somewhat involved—see e.g. Daho and Langer [8] for the Sturm–Liouville case. Incidentally, all the above references treat *invertible* T .

Various generalisations of (1.2) can be treated via reduction to (*) (cf. Greenlee [13] and Roach and Sleeman [18] who use $k=1$ and $k=2$ respectively to treat quadratic eigenvalue problems). The only investigation we know of under ellipticity alone seems to be by Faierman [9] for the Sturm–Liouville case (1.1) with $k=2$. We are grateful to Dr. Faierman for his preprints which have enabled us to give the following comparison between his work and ours. Roughly, [9] takes advantage of a function space setting to derive uniform convergence of double Fourier-type series, but at the cost of continuity

conditions which can be checked only after rather detailed analysis of an elliptic partial differential equation

$$\Delta_n x = \lambda_n \Delta_0 x \tag{1.3}$$

for $n=2$. Our work uses k abstract equations (1.3) on $\bigotimes_{m=1}^k H_m$ for $n=1 \dots k$ to derive completeness results in abstract (but strong) metrics under weak continuity conditions—in fact L_∞ coefficients would suffice for (1.1). Faierman attacks (1.3), with Δ_n possibly noninvertible, using an *a posteriori* nondegeneracy condition ([9], Assumption 6.1). We use a more direct nondegeneracy condition, which we call “regularity”, permitting transformation to a case where Δ_n is invertible for each $n=1 \dots k$. Incidentally, regularity holds automatically if Δ_0 is $1-1$, and the latter can be checked on the original data either in the finite dimensional case ([2], Theorem 8.5.1) or in the Sturm–Liouville case (1.1) when Δ_0 is $1-1 \Leftrightarrow \det[r_{mn}]$ vanishes on a (k dimensional Lebesgue) null set. While Faierman’s nondegeneracy condition is removable (as he has informed us), he expands only functions with a common compact support, Ω say, on which $\det[r_{mn}] \neq 0$. It seems, then, that one could use a version of (1.3) where Δ_0 is $1-1$, defined via quadratic forms restricted to Ω .

Our plan is as follows. In Section 2 we state the assumptions, including definitions of ellipticity and regularity, and we show how to transform the eigenvalues so that each operator Δ_n is self-adjoint and bounded below with compact inverse. In particular, $\nu(\Delta_n)$ is finite, and (1.3) is equivalent to

$$B_n x = \lambda_n^{-1} x,$$

where $B_n = \Delta_n^{-1} \Delta_0$ is compact. In Section 3 we prove that the B_n commute, and that bases for joint eigenspaces of the B_n can be constructed out of the eigenvectors of (*). In Section 4 we combine these results with those of [6] to give a general completeness theorem. Roughly, the LD result holds on a subspace $G \subseteq H$ of codimension at most $3 \sum_{n=1}^k \nu(\Delta_n)$, and on a suitable complement of G the problem reduces to one studied by Atkinson ([2], Chapter 6). We also show how to bound $\text{codim } G$ in terms of the original data (*). Finally in Section 5 we discuss the situation when each T_m is nonnegative definite—in this case the bound on $\text{codim } G$ improves to 2ν where $\nu = \prod_{n=1}^k \dim N(T_m)$. In particular, if $\nu=1$ (e.g. in the Sturm–Liouville case (1.1) with separated end conditions) then $\text{codim } G=1$ or 2 , and we characterize both cases directly in terms of the original data.

2. Transformations of the problem

Of the self-adjoint operators on a Hilbert space H , we shall distinguish by $BS(H)$ those which are bounded, by $PI(H)$ those which have a bounded positive definite inverse, and by $CR(H)$ those which are bounded below with compact resolvent. Necessarily

$$T \in CR(H) \Rightarrow T + \zeta I \in PI(H) \Rightarrow (T + \zeta I)^{-1} \in BS(H)$$

for large positive ζ . Recall the standing

Assumption 2.1. $T_m \in CR(H_m), V_{mn} \in BS(H_m), 1 \leq m, n \leq k.$

From now on, H will denote the (Hilbert) tensor product $\bigotimes_{m=1}^k H_m$. We start by inducing V_{mn} into H , yielding an operator $V_{mn}^\dagger \in BS(H)$, and we note that V_{mn}^\dagger and V_{ji}^\dagger commute if $m \neq j$. Thus we can define Δ_0 as the $k \times k$ determinant $\det[V_{mn}^\dagger]$, with Δ_{0mn} as the corresponding (m, n) th cofactor. If $u = u_1 \otimes \dots \otimes u_k$ is a decomposable tensor with $\|u_m\| = 1, m = 1 \dots k$, then the expression $\delta_{0mn}(u) = (u, \Delta_{0mn}u)$ is a $(k-1) \times (k-1)$ determinant with elements of the form $(u_j, V_{ji}u_j)$. The original definition of ‘‘ellipticity’’ [15] requires k linear combinations of the $\delta_{0mn}(u)$ to be positively bounded below, but ([4], p. 480) shows that this positivity may be assumed of each $\delta_{0mn}(u)$, after a nonsingular linear transformation of the eigenvalues λ . Then ([4], Theorem 3.1) shows that we may assume ‘‘ellipticity’’ in the form:

Assumption 2.2. $\Delta_{0mn} \in PI(H), 1 \leq m, n \leq k$.

(If $k=1$ then we write $\Delta_{0mn} = I$, so ellipticity is automatic). We emphasize that the original assumption is on the original data (*).

We induce T_m similarly into H as a self-adjoint operator T_m^\dagger . Let D denote the intersection $\bigcap_{m=1}^k D(T_m^\dagger)$, which is easily seen to be dense in H . For $n = 1 \dots k$, Δ_n is then defined as the closure in H of

$$\Delta_n|_D = \sum_{m=1}^k \Delta_{0mn} T_m^\dagger. \tag{2.1}$$

Using the fact that Δ_{0mn} and T_m^\dagger commute on $D(T_m^\dagger)$, one easily shows that $\Delta_n|_D$ is symmetric, and it will turn out that Δ_n is self-adjoint.

We are now in a position to state our ‘‘regularity’’ condition:

Assumption 2.3. For $n = 1 \dots k, \Delta_0 y = 0 \neq y \in H$ implies

$$(y, \Delta_n x_n) \neq 0 \text{ for some } x_n \in N(\Delta_0) \cap D(\Delta_n).$$

This is satisfied vacuously if Δ_0 is 1 – 1. First we shall discuss the consequence for (1.2), i.e. for $k=1$. Then $\Delta_0 = V$ and $\Delta_1 = T$, and the following result shows that ‘‘regularity’’ gives the usual definition ([11], p. 25) for pencils in finite dimensions.

Theorem 2.4 *If $k=1$ then there is an open real interval $J =]-\rho, \rho[$ such that $T + \varepsilon V$ is 1 – 1 for all nonzero $\varepsilon \in J$.*

Proof. For each $\zeta \in \mathbb{C}$, write $T(\zeta) = T + \zeta V$. If no interval J exists as stated, then

$$0 \in \sigma_p(T + \zeta_j V) \quad j = 1, 2, \dots$$

for a sequence of real $\zeta_j \rightarrow 0$ as $j \rightarrow \infty$. From the properties of self-adjoint holomorphic families ([16], p. 386) we conclude that the eigenvalues of $T(\zeta)$ are holomorphic at $\zeta = 0$, and therefore $0 \in c_p(T(\zeta))$ for all small $|\zeta|$. In particular, we can find $y^\varepsilon \in H$ of unit norm such that

$$(T + \varepsilon i V)y^\varepsilon = 0 \text{ for all small real } \varepsilon.$$

Elementary manipulations now give

$$(y^\varepsilon, Vy^\eta) = (y^\varepsilon, Ty^\eta) = 0 \quad \text{for all small nonzero real } \varepsilon \text{ and } \eta.$$

In particular, if S is the linear span of such y^ε , then

$$(y, Ty) = 0 \quad \text{for all } y \in S.$$

It follows from ([7], Lemma IX.1.1) that $\dim S$ is finite, so let $y_l = y^{\varepsilon_l}$, $l = 1 \dots m$, be a basis for S . Suppose

$$y_n = \sum_{l=1}^m \alpha_l y_l \tag{2.2}$$

where $\varepsilon_n \neq \varepsilon_l$, $l = 1 \dots m$. Then

$$\begin{aligned} \varepsilon_n Vy_n &= iTy_n = i \sum_{l=1}^m \alpha_l Ty_l \\ &= \sum_{l=1}^m \varepsilon_l \alpha_l Vy_l. \end{aligned}$$

It follows from (2.2) that $Vy = 0$, where

$$y = \sum_{l=1}^m (\varepsilon_n - \varepsilon_l) \alpha_l y_l. \tag{2.3}$$

Now for any $x \in N(V) \cap D(T)$, we have

$$(y_l, Tx) = (-\varepsilon_l Vy_l, x) = (-\varepsilon_l iy_l, Vx) = 0$$

so it follows from (2.3) that $(y, Tx) = 0$. Thus $y = 0$ by Assumption 2.3, and so each $\alpha_l = 0$ by (2.3). But then $y_n = 0$ by (2.2) and we contradict $\|y_n\| = 1$. □

We now proceed to general k , and we shall repeatedly use a construction based on translations of $\sigma(T_m)$. To motivate this, we note that

$$T_m + \alpha I_m \in PI(H_m) \quad \text{for large positive } \alpha. \tag{2.4}$$

Hence although $T_m^\dagger \notin CR(H)$ in general, it is true that $T_m^\dagger + \zeta I \in PI(H)$ for large positive ζ . Thus if T_m is replaced by $T_m + \zeta I_m$, then the analogue of Δ_n , which we denote by $\Delta_n(\zeta)$, belongs to $PI(H)$. Specifically, we take $\Delta_n(\zeta)$ as the closure of $\sum_{m=1}^k \Delta_{0mn}(T_m^\dagger + \zeta I)$, and so

$$\Delta_n(\zeta) = \Delta_n + \zeta \Sigma_n, \quad \Sigma_n = \sum_{m=1}^k \Delta_{0mn}, \quad n = 1 \dots k. \tag{2.5}$$

Theorem 2.5. *After a possible translation of the eigenvalues, we may assume, in addition to Assumptions 2.1–3, that Δ_n is 1–1 and belongs to $CR(H)$, for $n = 1 \dots k$.*

Proof. Choose large enough positive α to satisfy (2.4). Throughout this proof, n will denote an integer between 1 and k . Since $D(T_m + \alpha I_m) = D(T_m)$, $\Delta_n(\alpha)|_D$ is essentially self-adjoint by ([20], Lemma 4.4). Since $\Sigma_n \in BS(H)$, $\Delta_n(\alpha)$ is self-adjoint, so $\Delta_n(\alpha) \in CR(H)$ by ([4], Theorem 3.1). Thus $\Delta_n = \Delta_n(\alpha) - \alpha \Sigma_n \in CR(H)$ by ([5], Lemma 1).

If we replace λ by $\lambda + \varepsilon$ where $\varepsilon \in \mathbb{R}^k$, then we must replace T_m by $W_m(\varepsilon)$ in (*). Thus Δ_n is replaced by the closure of

$$\sum_{m=1}^k \Delta_{0mn} W_m(\varepsilon)^\dagger = \Delta_n|_D + \varepsilon_n \Delta_0,$$

i.e. by $\Delta_n + \varepsilon_n \Delta_0$. Here we use standard identities ([15], equation 2.2) and $\Delta_0 \in BS(H)$.

Since $\Delta_n \in CR(H)$, we may apply Theorem 2.4, with (T, V) replaced by (Δ_n, Δ_0) , to conclude that $\Delta_n + \varepsilon_n \Delta_0$ is 1–1 for all translations with ε_n small and nonzero. Moreover $\Delta_n + \varepsilon_n \Delta_0 \in CR(H)$ by ([5], Lemma 1), and if $\Delta_0 y = 0 \neq y$ then

$$(y, (\Delta_n + \varepsilon_n \Delta_0)x) = (y, \Delta_n x) \neq 0 \quad \text{for some } x \in N(\Delta_0) \cap D(\Delta_n + \varepsilon_n \Delta_0).$$

Thus Assumption 2.3 remains valid, and the other assumptions are unaffected by the translation. □

3. The operators B_n

By Theorem 2.5, we may define compact operators

$$B_n = \Delta_n^{-1} \Delta_0, \quad n = 1, \dots, k$$

and as in [4], they will be the main tool for the completeness theorem when $k > 1$. In this section we shall extend two crucial properties of the B_n from the LD case, by means of analytic perturbation theory. The first property is as follows.

Theorem 3.1. *The B_n are pairwise commutative.*

Proof. Using (2.5) for arbitrary $\zeta \in \mathbb{C}$, we define R_n as the resolvent set for

$$E_n = -\Sigma_n^{-1/2} \Delta_n \Sigma_n^{-1/2}$$

and we claim that $\zeta \in R_n$ if and only if $\Delta_n(\zeta)$ is 1–1. Indeed

$$\begin{aligned} \Delta_n(\zeta)x = 0 &\neq x \in H \\ \Leftrightarrow (\Delta_n + \zeta \Sigma_n)x = 0 &\neq x \\ \Leftrightarrow E_n y = \zeta y, \quad y = \Sigma_n^{1/2} x &\neq 0. \end{aligned}$$

Since both $\Delta_n(\zeta)$ and E_n have purely point spectrum, our claim is established.

Choose large enough $\alpha > 0$ to satisfy (2.4). Then $\Delta_n(\alpha)$ is positive definite, so by Theorem 2.5, $\{0, \alpha\} \subset R_n$. Since E_n is self-adjoint, we find

$$Z_\varepsilon = \{\zeta: |\zeta| < \varepsilon\} \cup \{\zeta: |\zeta - \alpha| < \varepsilon\} \cup \{\zeta: \alpha - 2\varepsilon < |2\zeta - \alpha| < \alpha + 2\varepsilon, \text{Im } \zeta > 0\} \tag{3.1}$$

is a connected and simply connected open subset of R_n for small enough $\varepsilon > 0$.

Define $B_n(\zeta) = \Delta_n(\zeta)^{-1} \Delta_0$ for $\zeta \in R_n$. By ([4], Theorem 4.2),

$$B_m(\zeta)B_n(\zeta) = B_n(\zeta)B_m(\zeta) \tag{3.2}$$

for each ζ belonging to the real interval $] \alpha, \alpha + \varepsilon[\subset R_n$. It follows that the holomorphic operator valued functions on both sides of (3.2) agree for all $\zeta \in Z_\varepsilon$ and in particular for $\zeta = 0$ —cf. ([16], pp. 367–368). □

The second property we need expresses a connection between common eigenvectors of the B_n and eigentensors $x^\otimes = x_1 \otimes \dots \otimes x_k$ where x_m satisfy (*) for $\lambda \in \mathbb{R}^k$. The set of x satisfying $B_n x = \mu_n x$, $n = 1 \dots k$, is denoted by $E(\mu)$ whenever $\mu \in \mathbb{R}^k$ and the resulting subspace is nontrivial. We define $k + 1$ sesquilinear forms on $E(\mu)$ by $(x, y)_l = (x, \Delta_l y)$, $l = 0 \dots k$.

Theorem 3.2. *Any eigentensor of (*), with $\lambda \in \mathbb{R}^k$, belongs to $E(\mu)$ where $\mu_n = \lambda_n^{-1}$, $n = 1 \dots k$. Conversely if $0 \neq \mu \in \mathbb{R}^k$ and, for some n , $(x, x)_n$ is of one sign for all nonzero $x \in E(\mu)$, then $E(\mu)$ has a (finite) basis of eigentensors which are $(\ ,)_l$ -orthogonal for each $l = 0 \dots k$.*

Proof. The first sentence is standard—cf. ([4], Theorem 6.1). The converse is proved via the technique of Theorem 3.1. Note that $\mu_n \neq 0$ for each $n = 1 \dots k$. By ([4], p. 485), we have the identity

$$\prod_{l=1}^k \lambda_l B_l(\zeta) = (T_m^\dagger + \zeta I)^{-1} \sum_{n=1}^k V_{mn}^\dagger \lambda_n \prod_{l=1, l \neq n}^k \lambda_l B_l(\zeta)$$

for $\lambda \in \mathbb{C}^k$ and $\zeta \in] \alpha, \alpha + \varepsilon[$, the notation being as in (3.1) and (3.2).

Rearranging, we obtain

$$(I + (\zeta - \alpha)(T_m^\dagger + \alpha I)^{-1}) \prod_{l=1}^k \lambda_l B_l(\zeta) = (T_m^\dagger + \alpha I)^{-1} \sum_{n=1}^k V_{mn}^\dagger \lambda_n \prod_{l=1, l \neq n}^k \lambda_l B_l(\zeta). \tag{3.3}$$

Since the homomorphic operator-valued functions on both sides of (3.3) agree for all $\zeta \in] \alpha, \alpha + \varepsilon[$, and hence for all $\zeta \in Z_\varepsilon$ (3.1), we may set $\zeta = 0$, apply both sides to

$$x = \lambda_n B_n x \quad \text{where} \quad \lambda_n = \mu_n^{-1}, \quad n = 1 \dots k,$$

and rearrange to give

$$W_m(\lambda)^\dagger x = 0, \quad m = 1 \dots k.$$

Let P_m be the orthoprojector of H_m onto $N(W_m(\lambda))$. Replacing T_m, V_{mn} by $P_m T_m, P_m V_{mn}$ restricted to $N(W_m(\lambda))$, we obtain a finite dimensional version of (*) with one eigenvalue, viz. λ . The replacement for H is just $E(\mu)$, and so the conclusion follows directly from ([2], Theorem 7.9.1). □

4. Completeness in the general case

Although our results lead to fundamental sets of vectors in H , it is more convenient to express them in terms of complete orthogonal bases of vectors in the subspaces $D_n = D(|\Delta_n|^{1/2})$, $n = 1 \dots k$. We shall endow D_n with two sesquilinear forms,

$$(x, y)_n = (x, \Delta_n y) \quad \text{and} \quad [x, y]_n = (x, |\Delta_n| y)$$

for $y \in D(\Delta_n)$, with completion for $y \in D_n$. For more on these constructions, see e.g. ([6], §3). In particular D_n is a Hilbert space under $[\ , \]_n$, and topological concepts (e.g. closure) will be understood in the $[\ , \]_n$ sense. Algebraic concepts (e.g. orthogonality) will however be understood in the $(\ , \)_n$ sense. Note that the number v_n of negative eigenvalues of Δ_n is finite by Theorem 2.5, so D_n is a Pontryagin space under $(\ , \)_n$.

We shall give a preliminary completeness result for n fixed between 1 and k . If μ is an eigenvalue of B_n then we write $N_n^j(\mu) = N(B_n - \mu I)^j$. The case $j = 1$ corresponds to the eigenspace $N(B_n - \mu I)$, which we denote by $E_n(\mu)$. Given $j \geq 2v_n + 1$, it can be shown ([7], p. 191) that $N_n^j(\mu) = N_n^{j+1}(\mu)$, and we write

$$F_n(\mu) = N_n^j(\mu) + N_n^j(\bar{\mu})$$

for such j .

Lemma 4.1. $D_n = N(\Delta_0) \oplus F_n \oplus G_n$ where $N(\Delta_0) = F_n(0)$, F_n is an $(\ , \)_n$ -orthogonal direct sum of the $F_n(\mu)$ and $\dim F_n \leq 3v_n$. G_n is the closed linear span of the remaining $F_n(\mu)$ and is a Hilbert space under $(\ , \)_n = [\ . \]_n$. The summands in G_n are $(\ , \)_n$ -orthogonal and satisfy $F_n(\mu) = E_n(\mu)$ and $\mu \in \mathbb{R}$ in each case.

Proof. The results follow from ([6], Corollaries 4 and 5). In particular, $F_n(0) = E_n(0) = N(\Delta_0)$ follows from ([6], Corollary 1). □

We now remove some of the dependence on n by means of the constructions

$$E(\mu) = \bigcap_{n=1}^k E_n(\mu_n) \quad \text{and} \quad F(\mu) = \bigcap_{n=1}^k F_n(\mu_n)$$

for $\mu_n \in \sigma_p(B_n)$, $n = 1 \dots k$. It will turn out that $E(\mu)$ is needed only when $0 \neq \mu \in \mathbb{R}^k$, as in Theorem 3.2.

Lemma 4.2. For each $n = 1 \dots k$, $D_n = N(\Delta_0) \oplus F \oplus G$, where $N(\Delta_0) = F(0)$, F is a finite direct sum of the $F(\mu)$, and G is the closed linear span of the remaining $F(\mu)$, for each of which $0 \neq \mu \in \mathbb{R}^k$ and $F(\mu) = E(\mu)$.

Remark. It is an easy calculation to show that the $(\cdot, \cdot)_n$ are topologically equivalent, so G does not depend on n .

Proof. Set $j \geq 2v_n + 1$ for each $n = 1 \dots k$. If $x \in F_n(\mu_n)$ then $x = y + z$ where $y \in N_n^j(\mu_n)$ and $z \in N_n^j(\bar{\mu}_n)$. By Theorem 3.1,

$$B_m x = B_m y + B_m z \in N_n^j(\mu_n) + N_n^j(\bar{\mu}_n) = F_n(\mu_n)$$

so $F_n(\mu_n)$ is B_m -invariant for each $m = 1 \dots k$. We now decompose the summands of Lemma 4.1 into root subspaces of B_m for each $m \neq n$ in turn. Evidently $N(\Delta_0) = F_n(0)$ remains unchanged, and D_n is decomposed into a direct sum of the $F(\mu)$ —cf. ([2], Theorem 3.6.1).

We claim that $F(\mu) = E(\mu)$ and $\mu \in \mathbb{R}^k$ hold for all but finitely many μ . Indeed if we take a Jordan basis for any B_m in any finite sum of the $F(\mu)$ then the Jordan chain lengths are prescribed by Lemma 4.1 with n replaced by m . In particular, all but finitely many chains have length one and correspond to real eigenvalues. \square

The main completeness result is a refinement of the above.

Theorem 4.3. *The direct sums in Lemma 4.2 are $(\cdot, \cdot)_l$ -orthogonal for each $l = 0 \dots k$, as are the summands in G for different μ . Moreover F can be chosen so that G is a Hilbert space under $(\cdot, \cdot)_n = [\cdot, \cdot]_n$ for each $n = 1 \dots k$, and a set of eigentensors for $(*)$ exists forming a complete orthogonal basis of G in all of these inner products.*

Proof. Suppose $x \in N(\Delta_0)$ and $y \in F(\mu)$, $0 \neq \mu \in \mathbb{C}^k$. Then $(x, y)_0 = 0$ is trivial, and $\mu_n \neq 0$ implies $(x, y)_n = 0$ by ([7], Theorem II.2.5). Now suppose $u \in E(\mu)$, $0 \neq \mu \in \mathbb{R}^k$ and $v \in F(\rho)$, $\rho \neq \mu$. If $\rho_n \neq \mu_n$ then $(u, v)_n = 0$, again by ([7], Theorem II.2.5). Thus for $l = 1 \dots k$

$$(u, v)_l = (\Delta_l u, v) = \mu_l^{-1} (\Delta_0 u, v) = \mu_n \mu_l^{-1} (\Delta_n u, v) = \mu_n \mu_l^{-1} (u, v)_n \tag{4.1}$$

must vanish. The claims of the first sentence are now established.

If $(\cdot, \cdot)_n \neq [\cdot, \cdot]_n$ for some n for infinitely many $E(\mu)$, then we contradict Lemma 4.1. Thus F can be chosen to include all $E(\mu)$ for which $(\cdot, \cdot)_n \neq [\cdot, \cdot]_n$ for some n , and so $(\cdot, \cdot)_n = [\cdot, \cdot]_n$ on G for all n by the orthogonality already established. Theorem 3.2 now completes the proof. \square

We shall now discuss some questions related to the dimension of F . One way of reducing $\dim F$ is to include in G those $E(\mu)$ for which $(\cdot, \cdot)_n$ is definite for some n (and hence for all n by (4.1)). The price is that G need no longer be a Hilbert space, although it will be a Pontryagin space, under each $(\cdot, \cdot)_n$. Another way of reducing $\dim F$ is to remove subspaces of $E(\mu)$ from certain $F(\mu) \neq E(\mu)$. More precisely, we may decompose $F(\mu)$ into $L_n(\mu) = F_n(\mu) \cap F_n$ and $M_n(\mu) = F_n(\mu) \cap G_n \subseteq E_n(\mu_n)$ for $\mu \neq 0$ by Lemma 4.1. Let $M(\mu) = \bigcap_{n=1}^k M_n(\mu)$ and let $L(\mu)$ be the $(\cdot, \cdot)_l$ -orthogocomplement of $M(\mu)$ in $F(\mu)$ for some l (and hence for all $l = 1 \dots k$ by (4.1)). By the rules of dimension (cf. ([17], p. 51)),

$$\dim L(\mu) = \dim F(\mu) - \dim M(\mu) \leq \sum_{n=1}^k \dim L_n(\mu). \tag{4.2}$$

We now sum the $L(\mu)$ and $M(\mu)$ over those μ for which $F(\mu) \subseteq F$, yielding subspaces L and M of F . By Lemma 4.1, $(\cdot, \cdot)_n = [\cdot, \cdot]_n$ on M , and by (4.2) and Lemma 4.1,

$$\dim L \leq \sum_{n=1}^k 3v_n. \tag{4.3}$$

Thus $\dim F$ can be reduced to $\dim L$, and G remains a Hilbert space in each inner product $(\cdot, \cdot)_n, n=1 \dots k$. The price is that we cannot assert the existence of a decomposable basis for M .

Finally let us consider bounds for $\dim F$ in terms of the original data (*). Since the various possibilities for $\dim F$ (e.g. (4.3)) can all be expressed in terms of the v_n , it will suffice to estimate the latter. For simplicity we shall bound v_1 in the case $k=2$ —the general case is analogous but involves extra terms. By Assumptions 2.1–2, we have

$$\Delta_1 = T_1^\dagger S_1 + T_2^\dagger S_2, \quad H = (H_1^- \oplus H_1^+) \oplus (H_2^- \oplus H_2^+)$$

where $S_2 = V_{12}^\dagger$ and $S_1 = -V_{22}^\dagger \in SP(H)$ and H_m^+, H_m^- are the positive and negative spectral subspace of T_m . In particular, $\dim H_m^- = v(T_m) < \infty$. We seek an upper bound for the dimension of a $(\cdot, \cdot)_1$ -negative subspace in $H_1^- \otimes H_2^+$. Suppose $T_1 \geq \alpha I_1, V_{12} \geq \beta I_1$ and $V_{22} \geq \gamma I_2$ with $\alpha, \gamma < 0$ and $\beta > 0$. Let T_2 possess δ eigenvalues (repeated according to multiplicity) in the interval $[0, \alpha\gamma/\beta]$. Any subspace K of $H_1^- \otimes H_2^+$ of dimension $> \delta v(T_1)$ must contain an element x satisfying

$$(x, T_2 x) > \alpha\gamma(x, x)/\beta.$$

Thus $S_2^{-1/2}K$ contains $y = S_2^{-1/2}x$ satisfying

$$(y, T_2^\dagger S_2 y) > \alpha\gamma(y, y) \geq (y, T_1^\dagger S_1 y)$$

i.e. $(y, y)_1 > 0$. Since S_2 is an isomorphism, we have $\delta v(T_1)$ as our required bound. A similar bound (say $\epsilon v(T_2)$) can be found for the maximal dimension of $(\cdot, \cdot)_1$ -negative subspaces of $H_1^+ \otimes H_2^-$, and so our final bound for v_1 is

$$(v(T_1) + \epsilon)(v(T_2) + \delta) - \epsilon\delta.$$

5. The nonnegative definite case

Throughout this section we shall employ:

Assumption 5.1. T_m is nonnegative definite for each $m = 1 \dots k$,

in addition to Assumptions 2.1–3. It is convenient to discuss the one parameter case (1.2) first. By Theorem 2.5 we may replace λ by $\lambda + \epsilon$, where $\epsilon \in \mathbb{R}$ is small, so that T is replaced by an invertible operator $T' = T + \epsilon V$. Then we may define a compact operator $B' = (T')^{-1}V$ and associated subspaces $E'(\mu)$ and $F'(\mu)$ for $\mu \in \sigma_p(B')$, as in the prelude to Lemma 4.1. It is convenient to write $\eta = \epsilon^{-1}$.

Lemma 5.2.

- (i) *The eigenvalues λ of (1.2) and μ of B' are all real.*
- (ii) *$F'(\mu) = E'(\mu)$ for all $\mu \neq \eta$.*
- (iii) *$E'(\eta) = N(B' - \eta I) = N(T)$ and $F'(\eta) = N(B' - \eta I)^2$.*

Proof. (i) $B'x = \mu x$ is equivalent to

$$\mu Tx = (1 - \varepsilon\mu)Vx \tag{5.1}$$

so if $\mu = 0$ then $\mu \in \mathbb{R}$ and $x \in N(V)$. From now on we assume $\mu \neq 0$. Then (5.1) shows that (1.2) is satisfied with $\lambda = \mu^{-1} - \varepsilon$. Conversely, (1.2) implies

$$x = (\lambda + \varepsilon)B'x$$

so if $x \neq 0$ then again we have $\mu = (\lambda + \varepsilon)^{-1} \neq 0$, i.e. $\lambda = \mu^{-1} - \varepsilon$.

If $(x, Tx) \neq 0$ then $(x, Vx) \neq 0$ so λ is a ratio of reals, whence λ and μ are both real. If $(x, Tx) = 0$ then $Tx = 0$ by Assumption 5.1 so $\mu = \eta$ and $\lambda = 0$ are both real.

(ii) It suffices to prove

$$y \in N((B' - \mu I)^2) \Rightarrow y \in N(B' - \mu I).$$

The left side can be written $(B' - \mu I)y = x$, i.e.

$$[(1 - \varepsilon\mu)V - \mu T]y = T'x \tag{5.2}$$

where x satisfies (5.1), i.e. $[(1 - \varepsilon\mu)V - \mu T]x = 0$. Using (5.2) we obtain

$$0 = (x, T'x). \tag{5.3}$$

On the other hand, (5.1) can also be written $\mu T'x = Vx$, so

$$(x, Vx) = \mu(x, T'x).$$

Using (5.3) twice, we have $(x, Vx) = 0$ and $(x, Tx) = 0$ in turn, and so $Tx = 0$ by Assumption 5.1. As in (i) $x \neq 0$ forces $\mu = \eta$, and so in fact $x = 0$, i.e. $y \in N(B' - \mu I)$.

(iii) The first contention is immediate from (5.1). For the second, we shall prove

$$z \in N((B' - \eta I)^3) \Rightarrow z \in N((B' - \eta I)^2).$$

The left side can be written $(B' - \eta I)z = y$, i.e.

$$-\eta Tz = T'y \tag{5.4}$$

where y and x satisfy (5.2) and (5.1) for $\mu = \eta$, viz.

$$-\eta Ty = T'x \quad \text{and} \quad Tx = 0. \tag{5.5}$$

Thus

$$0 = (x, Tz) = -\varepsilon(x, T'y) = (y, Ty)$$

whence $Ty=0$. Comparing (5.4) and $Ty=0$ with (5.5) we see that $y \in N(B' - \eta I)$ and so indeed $z \in N((B' - \eta I)^2)$. □

By setting $T = \Delta_n$ and $V = \Delta_0$ we obtain an improvement of Lemma 4.1 that nevertheless still depends on n . To remove that dependence we need one more result.

Lemma 5.3. $N(\Delta_n) = N := \bigotimes_{m=1}^k N(T_m)$, for each $n = 1 \dots k$.

Proof. $(x, \Delta_n x) = \sum_{m=1}^k (x, T_m^\dagger \Delta_{0mn} x)$ for all $x \in D$ so Δ_n is nonnegative definite. Thus if $x \in N(\Delta_n)$ then there is a sequence $x_j \in D$ such that $x_j \rightarrow x$ and

$$(x_j, \Delta_n x_j) \rightarrow (x, \Delta_n x) = 0 \quad \text{as } j \rightarrow \infty.$$

In particular, the nonnegative sequence

$$\begin{aligned} \|(\Delta_{0mn}^{1/2} (T_m^\dagger)^{1/2} x_j)\|^2 &= (x_j, (\Delta_{0mn}^{1/2} (T_m^\dagger)^{1/2})^2 x_j) \\ &= (x_j, \Delta_{0mn} T_m^\dagger x_j) \rightarrow 0 \quad \text{as } j \rightarrow \infty. \end{aligned}$$

Thus $(T_m^\dagger)^{1/2} x_j \rightarrow 0$, and since $(T_m^\dagger)^{1/2}$ is closed we have $T_m^\dagger x = 0$ for $m = 1 \dots k$. Thus $x \in N$ —see ([4], p. 485), and so $N(\Delta_n) \subseteq N$. The reverse implication is obvious. □

We are now ready for the main completeness result. It improves Theorem 4.3 sufficiently to make the subsequent remarks in Section 4 unnecessary in this case. Using Theorem 2.5, we translate the eigenvalues λ by ε and we write

$$\eta_n = \varepsilon_n^{-1}, \quad B'_n = (\Delta_n + \varepsilon_n \Delta_0)^{-1} \Delta_0, \quad n = 1 \dots k,$$

with associated subspaces $F'_n(\mu_n)$, $E'(\mu)$ etc. It is convenient to write $v = \dim N$ —see Lemma 5.3.

Theorem 5.4. *It is possible to choose $F = F'(\eta)$ in Theorem 4.3, with $\dim F \leq 2v$.*

Proof. By Lemma 5.2, it is possible to choose $F_n = F'_n(\eta_n)$ in Lemma 4.1. Now choose a Jordan basis for B'_m , $m \neq n$, in the B'_n -invariant subspace $F'_n(\mu_n)$. Any Jordan chain for B'_m of length 2 must correspond to eigenvalue η_m by Lemma 4.1 with n replaced by m . By Lemmas 5.2 and 5.3, the first element of such a chain belongs to $N \subset F'_n(\eta_n)$, and so $F'(\eta) = F_n(\eta_n)$ and $F'(\mu) = E'(\mu)$ whenever $\mu_m \neq \eta_m$ for any $m = 1 \dots k$. Finally $\dim F \leq 2v$ because each Jordan chain for $B_m|_F$ starts in N and has length at most two. □

One can specify $\dim F$ precisely in terms of the behaviour of $(\cdot, \cdot)_0$ on N . Let M_0 be the (Gram) matrix with (m, n) th element $(u^m, u^n)_0$ for a suitable basis u^1, \dots, u^v of N . Appealing to the canonical form ([12], Theorem I.3.3) for B'_m on N , in the $(\cdot, \cdot)_m$ inner

product, we see that there is a Jordan basis for B'_m relative to which M_0 is diagonal with nonzero (resp. zero) elements corresponding to 1×1 (resp. 2×2) Jordan blocks for B'_m . Thus

$$\dim F = v + v_0$$

where v_0 denotes the number of zero eigenvalues of M_0 . On the other hand, we note that decomposable bases of N also exist ([2], Theorem 4.3.1) and the elements of M_0 are then determinants with elements of the form $(x_j, V_{jl}x_j)$. By the law of inertia ([17], §92), we can determine v_0 , and hence $\dim F$, by means of these simpler matrices, expressible directly in terms of the original data (*).

Finally, let us illustrate some of the above with a special case.

Example 5.5. Suppose the eigenspace corresponding to the minimal eigenvalue of each T_m has dimension one. In particular, this is the case for the Sturm–Liouville case (1.1) with separated end conditions—cf. [9] for $k=2$.

Three cases can arise. First, if T_m is positive definite for some m , then $v=0$ and so $\dim F=0$. The eigentensors are complete in H , and we have the situation of LD—in fact the problem is transformable to one satisfying LD ([3], Corollary 2.2 and subsequent references).

Assume now that $T_m x_m = 0$ for some nonzero $x_m \in H_m$, $m=1, \dots, k$. Then N is spanned by $x^\otimes = x_1 \otimes \dots \otimes x_k$. Evidently $v=1$, so $\dim F$ depends on v_0 . We note that $(x^\otimes, x^\otimes)_0$ is a determinant δ_0 with (m, n) th entry $(x_m, V_{mn}x_m)$. If $\delta_0 \neq 0$ then $v_0=0$, so $\dim F=1$, i.e. $F=N$ and again the eigentensors are complete.

The third case arises when $\delta_0=0$, and then $v_0=1$, so $\dim F=2$ and the eigentensors are not complete. F is spanned by x^\otimes and a common root vector y , say, for the B'_m .

Acknowledgements. The research of the first author was supported by NSERC of Canada, and the research of the second author was supported by a Post Doctoral Fellowship at the University of Calgary.

REFERENCES

1. F. M. ARSCOTT, *Periodic Differential Equations: An Introduction to Mathieu, Lamé and Allied Functions* (Macmillan Press, 1964).
2. F. V. ATKINSON, *Multiparameter Eigenvalue Problems, Vol. 1* (Academic Press, 1972).
3. P. A. BINDING, Multiparameter definiteness conditions, *Proc. Roy. Soc. Edinburgh* **89A** (1981), 319–332.
4. P. A. BINDING, Left definite multiparameter eigenvalue problems, *Trans. Amer. Math. Soc.* **272** (1982), 475–486.
5. P. A. BINDING and P. J. BROWNE, A variational approach to multiparameter eigenvalue problems in Hilbert space, *SIAM J. Math. Anal.* **9** (1978), 1054–1067.

6. P. A. BINDING and K. SEDDIGHI, On root vectors of self-adjoint pencils, *J. Funct. Anal.*, to appear.
7. J. BOGNÁR, *Indefinite Inner Product Spaces* (Springer-Verlag, 1974).
8. K. DAHO and H. LANGER, Sturm–Liouville operators with an indefinite weight function, *Proc. Roy. Soc. Edinburgh* **78A** (1977), 161–191.
9. M. FAIERMAN, Expansions in eigenfunctions of a two-parameter system of differential equations I–IV, to appear.
10. J. FLECKINGER and A. B. MINGARELLI, On the eigenvalues of non-definite elliptic operators, *Differential Equations* (ed. I. W. Knowles, R. T. Lewis) (North-Holland, 1984), 219–227.
11. F. R. GANTMACHER, *The Theory of Matrices, Vol. 2* (Chelsea, 1959).
12. I. GOHBERG, P. LANCASTER and L. RODMAN, *Matrices and Indefinite Scalar Products* (Birkhäuser-Verlag, 1983).
13. W. M. GREENLEE, Double unconditional bases associated with a quadratic characteristic parameter problem, *J. Funct. Anal.* **15** (1974), 306–339.
14. D. HILBERT, *Grundzüge Einer Allgemeinen Theorie der Linearen Integralgleichungen*, 1912 (reprinted by Chelsea, 1953).
15. A. KÄLLSTRÖM and B. D. SLEEMAN, A left definite multiparameter eigenvalue problem in ordinary differential equations, *Proc. Roy. Soc. Edinburgh* **74A** (1976), 145–155.
16. T. KATO, *Perturbation Theory for Linear Operators* (Springer-Verlag, 1966).
17. A. I. MAL'CEV, *Foundations of Linear Algebra* (Freeman, 1963).
18. G. F. ROACH and B. D. SLEEMAN, On the spectral theory of operator bundles II, *Applicable Anal.* **9** (1979), 29–36.
19. B. D. SLEEMAN, *Multiparameter Spectral Theory in Hilbert Space* (Pitman, 1978).
20. H. VOLKMER, On multiparameter theory, *J. Math. Anal. Appl.* **86** (1982), 44–53.
21. H. VOLKMER, On the completeness of eigenvectors of right definite multiparameter problems, *Proc. Roy. Soc. Edinburgh* **96A** (1984), 69–78.

DEPARTMENT OF MATHEMATICS AND STATISTICS
 UNIVERSITY OF CALGARY
 CALGARY, ALBERTA
 CANADA T2N 1N4

DEPARTMENT OF MATHEMATICS
 UNIVERSITY OF SHIRAZ
 IRAN